Two Remarks on Counting Propositional Logic*

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Abstract

Counting propositional logic was recently introduced in relation to randomized computation and shown able to logically characterize the full counting hierarchy [1]. This paper aims to clarify the nature and expressive power of its univariate fragment. On the one hand, we make the connection of our logic with stochastic experiments explicit, proving that *any* (and only) event(s) associated with dyadic distribution can be simulated in this formalism. On the other, we provide an effective procedure to measure the probability of counting formulas.

Keywords

Randomized Computation, Probability Logic, Dyadic Distributions.

1. Introduction

The need for reasoning about uncertain knowledge and probability has come out in several areas of research, from AI to economics, from linguistics to theoretical computer science (TCS, for short). For example, probabilistic models are crucial when considering randomized programs and algorithms or dealing with partial information, e.g. in expert systems. It was this concrete demand that led to the first attempts to analyze probabilistic reasoning *in a formal way*, and to the development of a few logical systems, starting with Nilsson's pioneering proposal in 1986:

> Because many artificial intelligence applications require the ability to reason with uncertain knowledge, it is important to seek appropriate generalizations of logic from this case. [2, p. 71]

For probabilistic algorithms behavioral properties, like termination or equivalence, have *quantitative* nature, that is computation terminates *with a certain probability*, and programs simulate the desired function *up to some probability of error* (for instance, when dealing with learning algorithms). Then, how can such properties be studied within a logical system?

In a series of recent works [3, 1, 4], we introduce logics with counting and measure quantifiers, providing a new formal framework to study probability, and show them strongly related to several aspects of randomized computation. In particular, our counting logics are shown able to logically characterize probabilistic complexity classes, while counting-quantified formulas can be seen as expressing that a program behaves in a certain way with a given probability. The main goal of this paper is to clarify what the expressive power (and limit) of the simple, univariate

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fragment of counting propositional logic [1] ($\mathbf{CPL}_0$, for short) is, so to better understand its connection with both randomized computation and other probability systems.

2. On Logic and Randomized Computation

The development of counting logics is part of an overall study aiming to analyze interactions between (quantitative) logic and probabilistic computation, in order to deepen our knowledge of both.

2.1. Overview

Our project was motivated by two main considerations. On the one hand, since their appearance in the 1970s, probabilistic computational models have become more and more pervasive in several fast-growing areas of computer science and technology, from statistical learning to approximate computing. On the other, the development of different computational models has considerably benefitted from the mutual interchanges existing between logic and TCS. Nevertheless, there is at least one crucial aspect of the theory of computation which was only marginally touched by such fruitful interactions, namely *randomized* computation. The global purpose of our study is to lay the foundation for a new approach to bridge this gap, and its key ingredient consists in considering new *inherently quantitative* logics, the language of which includes non-standard quantifiers able to "measure" the probability of their argument formulas.

So far, we have mostly focussed on a few specific aspects of the interaction between quantitative logics and randomized computation:

- Complexity theory: classical propositional logic (**PL**, for short) provides the first example of an $\mathbf{NP}$-complete problem [5], while its quantified version characterizes the full polynomial hierarchy [6]. Instead, no analogous logical counterpart was known for probabilistic and counting classes [7, 8, 9]. In [1], we introduce a counting propositional system, called **CPL**, which offers a logical characterization of Wagner's counting hierarchy.
- Programming language theory: type systems for randomized λ-calculi, also guaranteeing various forms of termination properties, were introduced in the last decades, for example in [10]. Yet, these systems are not "logically oriented" and no Curry-Howard correspondence [11] is known for them. In [4], we define an intuitionistic version of **CPL**, called $\mathbf{iCPL}_0$, which is able to capture quantitative behavioral properties and provides (the logical side of) a probabilistic correspondence in the style of Curry and Howard.
- Computation theory: arithmetics and the theory of deterministic computation are linked by numerous, deep results from logic and recursion theory. In [3], we present a quantitative extension of the language of Peano Arithmetic (**PA**, for short). This new language, called **MQPA**, allows us to formalize basic results from probability theory, which are not expressible in **PA** (for example, the so-called "infinite monkey theorem" or the " random walk theorem"), and to establish a probabilistic version of Gödel's arithmetization [12].

2.2. (Univariate) Counting Propositional Logic in a Nutshell

In standard **PL** formulas are interpreted as single truth-values. The core idea of our counting semantics consists in modifying this intuition in a quantitative sense, associating formulas with *measurable* sets of (satisfying) valuations. Specifically, given a counting formula F, its interpretation is taken to correspond to the set of all maps $f \in 2^{\mathbb{N}}$ "making F true". Any such set belongs to the standard Borel algebra over $2^{\mathbb{N}}$, $\mathscr{B}(2^{\mathbb{N}})$, yielding a genuinely quantitative semantics. In particular, atomic propositions correspond to *cylinder sets* [13] of the form

$$Cyl(i) = \{f \in 2^{\mathbb{N}} \mid f(i) = 1\},$$

with $i \in \mathbb{N}$, while molecular expressions are interpreted in the natural way as standard operations of complementation, finite intersection and union. Clearly, such "interpretation sets" are measurable and can be associated with the unique cylinder measure $\mu_{\mathscr{C}}$, where for any $i \in \mathbb{N}$, $\mu_{\mathscr{C}}(Cyl(i)) = \frac{1}{2}$ [13].

We can then enrich our language with new formulas expressing the measure of such sets. By adapting Wagner's notion of counting operator [9], we introduce two non-standard quantifiers, $\mathbf{C}^q$ and $\mathbf{D}^q$, with $q \in \mathbb{Q}_{[0,1]}$. Basically, counting-quantified formulas $\mathbf{C}^q F$ and $\mathbf{D}^q F$ express that F is satisfied in a certain portion of all its possible interpretations to be (resp.) greater or strictly smaller than the given q. For example, the formula $\mathbf{C}^{1/2} F$ intuitively says that F is satisfied by *at least* one half of its valuations. Semantically, this amounts at checking $\mu_{\mathscr{C}}(\llbracket F \rrbracket) \geq \frac{1}{2}$.

Definition 1. *Formulas of* ***CPL***$_0$ are defined by the grammar below:

$$F := \mathbf{i} \;\big|\; \neg F \;\big|\; F \wedge F \;\big|\; F \vee F \;\big|\; \mathbf{C}^q F \;\big|\; \mathbf{D}^q F,$$

where $i \in \mathbb{N}$ and $q \in \mathbb{Q}_{[0,1]}$. Given the standard cylinder space $\mathscr{P} = (2^{\mathbb{N}}, \sigma(\mathscr{C}), \mu_{\mathscr{C}})$, for each formula of **CPL**$_0$ F, its *interpretation* is the measurable set $\llbracket F \rrbracket \in \mathscr{B}(2^{\mathbb{N}})$ defined as follows:

$$\llbracket \mathbf{i} \rrbracket = Cyl(i) \qquad \llbracket \mathbf{C}^q G \rrbracket = \begin{cases} 2^{\mathbb{N}} & \text{if } \mu_{\mathscr{C}}(\llbracket G \rrbracket) \geq q \\ \emptyset & \text{otherwise} \end{cases}$$

$$\llbracket \neg G \rrbracket = 2^{\mathbb{N}} - \llbracket G \rrbracket$$

$$\llbracket G_1 \wedge G_2 \rrbracket = \llbracket G_1 \rrbracket \cap \llbracket G_2 \rrbracket \qquad \llbracket \mathbf{D}^q G \rrbracket = \begin{cases} 2^{\mathbb{N}} & \text{if } \mu_{\mathscr{C}}(\llbracket G \rrbracket) < q \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\llbracket G_1 \vee G_2 \rrbracket = \llbracket G_1 \rrbracket \cup \llbracket G_2 \rrbracket$$

Notice that, in [1], we even introduce a labelled calculus, which is proved sound and complete for the semantics above.

3. On the Expressive Power of CPL$_0$

Our counting logics are strongly related to probabilistic reasoning and, indeed, **CPL**$_0$ offers a natural model to express the probability of events associated with Bernoulli distributions. In fact, we show that counting formulas can simulate experiments associated to *any* dyadic probability distribution.

3.1. Expressing Exact Probability

In $\mathbf{CPL}_0$, we can easily express that a formula is true with *precisely* a certain probability. For clarity's sake, we do so by means of auxiliary quantifiers, $\mathbb{C}^q$ and $\mathbb{D}^q$, intuitively saying that their argument formula is true with probability (resp.) strictly greater or smaller (or equal) than the given index.

Notation 1. So-called *white counting quantifiers* are interpreted as follows:

$$[\![\mathbb{C}^q F]\!] := \begin{cases} 2^{\mathbb{N}} & \text{if } \mu_{\mathscr{C}}([\![F]\!]) > q \\ \emptyset & \text{otherwise} \end{cases} \qquad [\![\mathbb{D}^q F]\!] := \begin{cases} 2^{\mathbb{N}} & \text{if } \mu_{\mathscr{C}}([\![F]\!]) \leq q \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly, these quantifiers do not extend the expressive power of $\mathbf{CPL}_0$, as definable in terms of primitive $\mathbf{C}^q$ and $\mathbf{D}^q$,[1] but due to them exact probability is expressed *in a compact way.*

Example 1. *For example, we formalize that the formula $F = \mathbf{1} \wedge \mathbf{2}$ is true with probability $\frac{1}{4}$ as*

$$F_{ex} = \mathbf{C}^{1/4}(\mathbf{1} \wedge \mathbf{2}) \wedge \mathbb{D}^{1/4}(\mathbf{1} \wedge \mathbf{2}).$$

3.2. Simulating Dyadic Distributions

By their semantic definition, it is natural to interpret atomic formulas of $\mathbf{CPL}_0$ as infinite sequences of independently and uniformly distributed random bits – that is, more concretely, as infinite sequences of (independent) *fair* coin tosses – and, in general, counting formulas as formally representing experiments associated with *specific* probability distributions. For instance, the fact that, when tossing an unbiased coin twice, the probability that it returns HEAD both times is $\frac{1}{4}$ can be expressed in $\mathbf{CPL}_0$ by the formula F_{ex} above (which is also easily proved valid in our semantics). Generally speaking, counting formulas can simulate events associated with *any* dyadic probability distribution, but those related to non-dyadic ones only approximately.

In fact, we capture atomic sampling from a Bernoulli distribution of *non-reducible* parameter $p = \frac{m}{2^n}$ by molecular formulas of $\mathbf{CPL}_0$, while corersponding complex events are expressed combining such formulas in the usual way. To make this intuition less vague, let us consider the following simple example.

Example 2. *Let a* biased *coin return* HEAD *only 25% of the time. In this case, it is clear that a single toss cannot be formalized by an atomic formula of* $\mathbf{CPL}_0$*. Yet, it can be easily expressed using a molecular formula, namely one in the form* $(\mathbf{i} \wedge \mathbf{j})$ *(with $i, j \in \mathbb{N}$ fresh). Consequently, also properties concerning complex events can be captured in* $\mathbf{CPL}_0$*. For instance, that the probability for at least one of two subsequent biased tosses to return* HEAD *is greater than $\frac{1}{3}$ is formalized by the (valid) formula:*

$$F_{bias} = \mathbf{C}^{1/3}\big((\mathbf{1} \wedge \mathbf{2}) \vee (\mathbf{3} \wedge \mathbf{4})\big).$$

[1] For further details, see Appendix B.1, and, in particular, Proposition 2.

In the same way, we can (quantitatively) simulate any discrete distribution with $\sharp X = 2^n$.[2] Something different happens when considering experiments related to non-dyadic distributions. Indeed, by Lemma 1 below, formulas of $\mathbf{CPL}_0$ can simulate these events – such as tossing a biased coin returning HEAD with probability $\frac{1}{3}$ – only *in an approximate way*.[3]

Lemma 1. *For any formula of $\mathbf{CPL}_0$ F, there exist $n, m \in \mathbb{N}$, such that $\mu_{\mathscr{C}}(\llbracket F \rrbracket) = \frac{m}{2^n}$.*

4. Measuring Formulas of $\mathbf{CPL}_0$

In [1], the validity of counting formulas is decided accessing an oracle for $\sharp$`SAT`, counting the satisfying models of Boolean formulas.[4] Here, we provide an effective procedure to measure formulas of $\mathbf{CPL}_0$, *without appealing for an external source*, thus making the task done by the oracle *explicit*. In our opinion, this could make the comparison with other probability logics, for example [14], more clear and support a better understanding of the "nature" of our non-standard quantifiers. We hope this is also the first step to shed new lights on the complexity of deciding formulas of $\mathbf{CPL}_0$ (and of proofs in the corresponding calculus $\mathbf{LK}_{\mathbf{CPL}_0}$ [1, Sec. 2.2]).[5]

The skeleton of our procedure is as follows. Given a formula of $\mathbf{CPL}_0$, we first consider its (inner) *not-quantified* formulas. We do so by passing through a special form, the measure of which can be computed in a straightforward way (Lemma 3). We prove that any formula of $\mathbf{CPL}_0$ *without quantifiers* can be converted into such measurable form (Lemma 4). Notably, the procedure we offer is effective, but not necessarily "feasible" as requiring argument formulas to be put in disjunctive normal form (DNF, for short). Finally, we can deal with nested quantifications: by measuring argument formulas, one can substitute the corresponding inner counting-quantified expression with either $\top$ or $\bot$.

4.1. Measurable Normal Form

For simplicity, before defining so-called *measurable normal forms*, we introduce notational conventions and the so-called auxiliary, polite forms.

Notation 2. We use $L_1, L_2 \ldots$ for *literals*, i.e. atomic formulas or their negations. Given $L_i =$ literal, we define $\overline{L_i}$ as follows:

$$\overline{L_i} = \begin{cases} \mathbf{j} & \text{if } L_i = \neg \mathbf{j} \\ \neg \mathbf{j} & \text{if } L_i = \mathbf{j}, \end{cases}$$

with $j \in \mathbb{N}$. We use $\bot$ as a shorthand for $L_j \wedge \overline{L_j}$ and $\top$ for $L_j \vee \overline{L_j}$.

[2] Of course, any information concerning the nature of variables involved in the experiment is lost, but quantitative aspects, that is events' probability, are all preserved through the formalization. For further details, see Appendix B.2.

[3] Generalizations of $\mathbf{CPL}_0$ associated with a probability space $(2^{\mathbb{N}}, \sigma(\mathscr{C}), \mu)$, where μ is not necessarily the measure of i.i.d. sequences of random bits, are cursorily presented in Appendix B.2. Observe that, differently from standard $\mathbf{CPL}_0$, the logics presented in the quoted Appendix can naturally formalize also events related to (any) non-dyadic Bernoulli distributions.

[4] In [1], a $\mathbf{CPL}_0$-formula F is said to be *valid* when $\llbracket F \rrbracket = 2^{\mathbb{N}}$ and *invalid* when $\llbracket F \rrbracket = \emptyset$.

[5] Indeed, one can even introduce syntactical expressions of the form $\mathsf{meas}(F) = q$, where F is a counting formula *without quantifiers* and $q \in \mathbb{Q}_{[0,1]}$, to be interpreted as predictable: $\llbracket \mathsf{meas}(F) = q \rrbracket$ is true when $\mu_{\mathscr{C}}(\llbracket F \rrbracket) = \llbracket q \rrbracket_{\mathbb{Q}}$. Basing on them, $\mathbf{LK}_{\mathbf{CPL}_0}$ can be modified so to become *purely syntactical* (without loosing completeness).

Definition 2 (Polite Normal Forms)**.** A formula of $\mathbf{CPL}_0$ in conjunctive normal form (CNF, for short) $C = \bigwedge_{i\in\{1,\ldots,n\}} L_i$, is said to be in *conjunctive polite form* (CPF, for short) if either $C \in \{\bot, \top\}$ or both $L_k \neq L_{k'}$ and $L_k \neq \overline{L_{k'}}$, for any $k \neq k' \in \{1, \ldots, n\}$. A formula of $\mathbf{CPL}_0$ in DNF $D = \bigvee_{j\in\{1,\ldots,m\}} C_j$, is said to be in *disjunctive polite form* (DPF, for short) if either $D \in \{\bot, \top\}$ or for each $k \in \{1, \ldots, m\}$, C_k is in CPF and $C_k \notin \{\bot, \top\}$.

Lemma 2. *Given a formula of* $\mathbf{CPL}_0$ *in DNF* D*, there is a* D^* *such that* D^* *is in DPF and* $D \equiv D^*$.

4.2. Conversion into MNF

Now, as anticipated, we introduce a special form, such that expressions in this form can be "measured" in a straightforward way. We start by defining *contradictory pairs*, that is formulas (in CPF) the conjunction of which is invalid.

Definition 3 (Contradictory Pair)**.** Two formulas of $\mathbf{CPL}_0$ in CPF, $C_i = \bigwedge_{j\in\{1,\ldots,n\}} L_j$ and $C_j = \bigwedge_{k\in\{1,\ldots,m\}} L_k$, are said to be *mutually contradictory* when there exist a $j \in \{1, \ldots, n\}$ and a $k \in \{1, \ldots, m\}$ such that $L_j = \overline{L_k}$ (or $L_k = \overline{L_j}$).

By Definition 1 plus basic measure theory, it is easy to see that the measure of the disjunction of two contradictory formulas in CPF is the sum of the measure of each disjunct (which, being themselves in CPF, are measurable as well). The generalization of this intuition leads to the definition below.

Definition 4 (Measurable Normal Form)**.** A formula of $\mathbf{CPL}_0$ $F = \bigvee_{i\in\{1,\ldots,n\}} C_i$ is in *measurable normal form* (MNF, for short), if either $F \in \{\bot, \top\}$ or F is in DPF and for each $j, k \in \{1, \ldots, n\}$, C_j and C_k are mutually contradictory.

Lemma 3. *Given a formula of* $\mathbf{CPL}_0$ *in MNF, say* $F = \bigvee_{i\in\{1,\ldots,n\}} C_i$*: i. if* $F = \top$*, then* $\mu_{\mathscr{C}}(\llbracket F \rrbracket) = 1$*, ii. if* $F = \bot$*, then* $\mu_{\mathscr{C}}(\llbracket F \rrbracket) = 0$*, iii. otherwise* $\mu_{\mathscr{C}}(\llbracket F \rrbracket) = \sum_{i\in\{1,\ldots,n\}} \mu_{\mathscr{C}}(\llbracket C_i \rrbracket)$.

Observe that, as said, each disjunct is in CPF, so, again by basic measure theory, its measure is easily computable as well and any $\llbracket C_i \rrbracket$ can effectively be measured.[6]

We conclude our proof showing that each formula in DPF can actually be "converted" into MNF. To do so, we notice that two disjuncts can be mutually related in three ways only: (1) if one is a sub-formula of the other, the former is simply removed;[7] (2) if they are a contradictory pair, the form is already as desired and the next pair is considered; (3) if one of the two disjuncts, say C_i, contains a literal L_k, such that neither L_k or $\overline{L_k}$ occurs in the other disjunct, say C_j, we substitute C_j with $C'_j = C_j \wedge L_k$ and $C''_j = C_j \wedge \overline{L_k}$ to be both taken into account again.

Lemma 4. *For each* $\mathbf{CPL}_0$*-formula in DPF* F*, there is a formula in MNF* F^{**} *such that* $F \equiv F^{**}$.

We conclude by putting Lemmas 4 and 2 and Lemma 3 together, so to obtain the desired procedure that, for any formula of $\mathbf{CPL}_0$, effectively computes the measure of its probability.

[6]For further details, see Corollary 1.

[7]A formula in CPF, say $C_i = \bigwedge_{k\in\{1,\ldots,n\}} L_k$, is said to be a *sub-formula* of another formula in CPF, say $C_j = \bigwedge_{k'\in\{1,\ldots,m\}} L_{k'}$, when (they are not a contradictory pairs or $C_j \in \{\bot, \top\}$ or $C_i \in \{\bot, \top\}$ and) for each $k' \in \{1, \ldots, m\}$, there is a $k \in \{1, \ldots, n\}$, such that $L_k = L_{k'}$. For example, a formula $\mathbf{1} \wedge \mathbf{2} \wedge \mathbf{3}$ is a sub-formula of $\mathbf{1} \wedge \mathbf{2}$.

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Appendix

A. Section 2

Preliminaries. In order to avoid clash in terminology, we briefly recap a few standard notions from probability theory. An *outcome* or *point* is the result of a single execution of an experiment, the *sample space* Ω is the set of all possible outcomes, and an event is a subset of Ω. Two events, say E_1 and E_2, are *disjoint* or *mutually exclusive*, when they cannot happen at the same time, that is $E_1 \cap E_2 = \emptyset$. Two events are *independent* when the occurrence of one does not affect the probability for the other to occur. A class of subsets of Ω is a (σ-)*field* if containing Ω itself and being closed under the formation of complements and (in)finite unions. A probability measure $\textsc{Prob}(\cdot)$ is a real-valued function defined on a field satisfying Kolmogorov's axioms [15]. So, in particular, given two disjoint events, E_1 and E_2, $\textsc{Prob}(E_1 \cup E_2) = \textsc{Prob}(E_1) + \textsc{Prob}(E_2)$, while for two independent events, E'_1 and E'_2, $\textsc{Prob}(E'_1 \cap E'_2) = \textsc{Prob}(E'_1) \cdot \textsc{Prob}(E'_2)$.

Following [13], infinite sequences of (random) tosses can be represented as $\omega = (\omega(1), \omega(2), \omega(3), \dots)$, where for any $\omega \in 2^{\mathbb{N}}$ and $k \geq 1$, $\omega(k) \in 2$, being $2 = \{0, 1\}$. Then, 2^n is the Cartesian product consisting of the n-long sequences $u_1, \dots, u_n$, with $u_i \in 2$ for any $i \in \{1, \dots, n\}$ and $2^{\mathbb{N}}$ is the set of all infinite sequences of elements in 2. A *cylinder of rank k* is a set of the form $\mathrm{C}_H = \{\omega : (\omega(1), \dots, \omega(k)) \in H\}$, where $H \subset 2^k$. When H is a singleton, C_H is a *thin cylinder*, corresponding to the fact that the first k repetitions of the experiment have outcomes $u_1, \dots, u_k$ in sequence. The class of cylinders of all ranks, denoted as $\mathscr{C}$, is a field and a standard probability measure can be defined over it. In particular, we assign a probability measure $\mu_{\mathscr{C}}$ to any cylinder of rank k, such that for any $j \in \{1, \dots, k\}$, $u_j \in 2$, and corresponding probability p_{u_j} of getting u_j,

$$\mu_{\mathscr{C}}(\mathrm{C}_H) = \sum_H p_{u_1} \cdots p_{u_k}.$$

So, as a special case, when C_H is a thin cylinder, $\mu_{\mathscr{C}}\{\omega : (\omega(1), \dots, \omega(n)) = (u_1, \dots, u_n)\} = p_{u_1} \cdots p_{u_n}$. Notice also that if the coin is fair, for each tossing $p_0 = p_1 = \frac{1}{2}$. In this case (since for any $k > 0$, H is finite) the Proposition 1 below follows.

Proposition 1. *If $p_0 = p_1 = \frac{1}{2}$, then for any cylinder of rank k, call it C_H, there exist $m, n \in \mathbb{N}$, such that $\mu_{\mathscr{C}}(C_H) = \frac{m}{2^n}$.*

Going back to $\sigma(\mathscr{C})$, i.e. the smallest σ-algebra including $\mathscr{C}$ and which is Borel, a well-defined probability measure is assigned to it by generalizing in the natural way the measure defined above for a cylinder of rank k.

B. Section 3

B.1. Section 3.1

Proofs from Section 3.1.

Lemma 5. *For every formula of **CPL**$_0$ F, and $q \in \mathbb{Q}_{[0,1]}$,*

$$\mu_{\mathscr{C}}(\llbracket F \rrbracket) \triangleright q \quad \Leftrightarrow \quad \mu_{\mathscr{C}}(\llbracket \neg F \rrbracket) \triangleleft 1 - q,$$

with $(\triangleright, \triangleleft) \in \{(\geq, \leq), (\leq, \geq), (>, <), (<, >)\}$.

Proof. Let us consider the case $\leq, \geq$. By Definition 1 $\mu_{\mathscr{C}}(\llbracket \neg F \rrbracket) = \mu_{\mathscr{C}}(2^{\mathbb{N}} - \llbracket F \rrbracket) = 1 - \mu_{\mathscr{C}}(\llbracket F \rrbracket)$. So, trivially, $\mu_{\mathscr{C}}(\llbracket F \rrbracket) \geq q$ iff $1 - \mu_{\mathscr{C}}(\llbracket F \rrbracket) \leq 1 - q$ iff $\mu_{\mathscr{C}}(\llbracket \neg F \rrbracket) \leq 1 - q$. All the other cases are proved in a similar way. □

Proposition 2. *For any formula of **CPL**$_0$ F, and $q \in \mathbb{Q}_{[0,1]}$:*

$$\mathbf{C}^q \neg F \equiv \mathbb{D}^{1-q} F \qquad \mathbf{D}^q \neg F \equiv \mathbb{C}^{1-q} F$$
$$\mathbf{C}^q \neg F \equiv \neg \mathbb{C}^{1-q} F \qquad \mathbf{D}^q \neg F \equiv \neg \mathbb{D}^{1-q} F.$$

Proof. The proof relies on Definition 1 and Lemma 5:

$$\llbracket \mathbf{C}^q \neg F \rrbracket = \begin{cases} 2^{\mathbb{N}} & \text{if } \mu_{\mathscr{C}}(\llbracket \neg F \rrbracket) \geq q \\ \emptyset & \text{otherwise} \end{cases} \qquad \llbracket \mathbf{D}^q \neg F \rrbracket = \begin{cases} 2^{\mathbb{N}} & \text{if } \mu_{\mathscr{C}}(\llbracket \neg F \rrbracket) < q \\ \emptyset & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2^{\mathbb{N}} & \text{if } \mu_{\mathscr{C}}(\llbracket F \rrbracket) \leq 1-q \\ \emptyset & \text{otherwise} \end{cases} \qquad = \begin{cases} 2^{\mathbb{N}} & \text{if } \mu_{\mathscr{C}}(\llbracket F \rrbracket) > 1-q \\ \emptyset & \text{otherwise} \end{cases}$$

$$= \llbracket \mathbb{D}^{1-q} F \rrbracket \qquad = \llbracket \mathbb{C}^{1-q} F \rrbracket$$

$$\llbracket \mathbf{C}^q \neg F \rrbracket = \llbracket \neg \mathbf{D}^q \neg F \rrbracket \qquad \llbracket \mathbf{D}^q \neg F \rrbracket = \llbracket \neg \mathbf{C}^q \neg F \rrbracket$$

$$= \llbracket \neg \mathbb{C}^{1-q} F \rrbracket \qquad = \llbracket \neg \mathbb{D}^{1-q} F \rrbracket.$$

□

B.2. Section 3.2

Proof from Section 3.2. We prove that formulas of $\mathbf{CPL}_0$ are interpreted as events associated with *dyadic* distributions relying on Proposition 1. To do so, we show that any counting formula is interpreted as a cylinder of rank k, for some $k \in \mathbb{N}$.

Lemma 6. *For any formula of* $\boldsymbol{CPL}_0$ F*, there is a cylinder of rank* k C_H*, such that* $\llbracket F \rrbracket = C_H$*.*

Proof. The proof is by induction on the structure of F:

- $F = \mathbf{i}$ for some $i \in \mathbb{N}$. Then $\llbracket \mathbf{i} \rrbracket = Cyl(i)$, which is a thin cylinder.
- $F = \neg G$. By IH, there is a k and a cylinder of rank k, C_H, such that $\llbracket G \rrbracket = \mathsf{C}_H$. Let $H' = 2^k - \mathsf{C}_H (\equiv 2^{\mathbb{N}} - \mathsf{C}_H)$. Then, $\llbracket \neg G \rrbracket = 2^{\mathbb{N}} - \llbracket G \rrbracket = 2^{\mathbb{N}} - \mathsf{C}_H = \mathsf{C}_{H'}$, which is clearly a cylinder of rank k as well.
- $F = G_1 \wedge G_2$. By IH, there exist $k_1, k_2 \in \mathbb{N}$ and cylinders of rank k_1, k_2, C_{H_1} and C_{H_2}, such that (resp.) $\llbracket G_1 \rrbracket = \mathsf{C}_{H_1}$ and $\llbracket G_2 \rrbracket = \mathsf{C}_{H_2}$. Then, if $k_1 = k_2$, $\llbracket F \rrbracket = \llbracket G_1 \rrbracket \cap \llbracket G_2 \rrbracket = \mathsf{C}_{H_1} \cap \mathsf{C}_{H_2} = \mathsf{C}_{H_1 \cap H_2}$, which is a cylinder of rank k_1 as well. Otherwise, assume $k_1 > k_2$ (the case $k_2 > k_1$ is equivalent). Let H_2' consists of the sequences $(u_1, \dots, u_{k_1})$ in 2^{k_1} such that the truncated sequence $(u_1, \dots, u_{k_2})$ is in H_2. Then, $\mathsf{C}_{H_2} \equiv \mathsf{C}'_{H_2} = \{\omega : (\omega(1), \dots, \omega(k_1)) \in H_2'\}$. We conclude that $\llbracket F \rrbracket = \llbracket G_1 \rrbracket \cap \llbracket G_2 \rrbracket = \mathsf{C}_{H_1} \cap \mathsf{C}'_{H_2} = \mathsf{C}_{H_1 \cap H_2'}$, which is a cylinder of rank k_1.
- $F = G_1 \wedge G_2$. Similar to the case above.
- $F = \mathbf{C}^q G$ and $F = \mathbf{D}^q G$. Then, either $\llbracket F \rrbracket = 2^{\mathbb{N}}$ or $\llbracket F \rrbracket = \emptyset$, which are both cylinders of rank k (in particular, in the former case $k = 0$).

□

Proof of Lemma 1. By putting Proposition 1 and Lemma 6 together. □

Notice also that a "syntactic" proof of Lemma 1 is obtained as a corollary of the results provided in Section 4.

Non-Dyadic Bernoulli Distributions. As said, one can simulate events associated with non-dyadic distributions *in an approximate way* only.

Example 3. *Let us consider a biased coin returning* HEAD *only* $\frac{1}{3}$ *of the time. We cannot simulate this event in* $\textbf{\textit{CPL}}_0$*, but "approximate" it with* $n = 2m$ *variables of* $\textbf{\textit{CPL}}_0$ *in the following sense. If* $m = 2$ *we can* down-approximate *a single toss of the biased coin as:*

$$F_{ndy} = (\mathbf{1} \wedge \mathbf{2}) \vee \big((\neg\mathbf{1} \wedge \mathbf{2}) \wedge (\mathbf{3} \wedge \mathbf{4})\big).$$

Observe that disjuncts are mutually contradictory, so $\mu_{\mathscr{C}}([\![F_{ndy}]\!]) = \frac{5}{16}$*. If* $m = 3$*, (down-)approximation is obtained as,*

$$F'_{ndy} = (\mathbf{1} \wedge \mathbf{2}) \vee \big((\neg\mathbf{1} \wedge \mathbf{2}) \wedge (\mathbf{3} \wedge \mathbf{4})\big) \vee \big((\neg\mathbf{1} \wedge \mathbf{2}) \wedge (\neg\mathbf{3} \wedge \mathbf{4}) \wedge (\mathbf{5} \wedge \mathbf{6})\big).$$

Then, $\mu_{\mathscr{C}}([\![F'_{ndy}]\!]) = \frac{21}{64}$*. In general, the more* n *is increased, the more precise is the approximation of the desired event.*

Although events associated with non-dyadic distributions cannot be expressed in $\mathbf{CPL}_0$ in a precise way, when switching to ($\mathbf{CPL}_0^\star$ or) the measure-quantified language **MQPA** [3] such formalization becomes possible.

Generalizing $\mathbf{CPL}_0$. As seen, the semantics for $\mathbf{CPL}_0$ is associated with a canonical cylinder space $(2^{\mathbb{N}}, \sigma(\mathscr{C}), \mu_{\mathscr{C}})$, where $\mu_{\mathscr{C}}$ is the standard measure $\mu_{\mathscr{C}}(Cyl(i)) = \frac{1}{2}$ for any $i \in \mathbb{N}$, corresponding to tossing *fair* coins [13]. It is possible to generalize this framework in a straightforward way, so to allow the measure to be associated with distributions other than dyadic ones. Indeed, we can define extended $\mathbf{CPL}_0^\star$ associated with a probability space $\mathscr{P}^\star = (2^{\mathbb{N}}, \sigma(\mathscr{C}), \mu^\star)$, where $\mu^\star$ is *any* properly-defined probability measure over $\sigma(\mathscr{C})$. Then, the grammar and semantics for $\mathbf{CPL}_0^\star$ is as for $\mathbf{CPL}_0$ except for counting-quantified formulas.

Definition 5. Extended formulas are defined by substituting standard counting quantifiers with $\mathbf{C}^q_{\mu^\star}$ and $\mathbf{D}^q_{\mu^\star}$, the interpretation of which is now based on $\mathscr{P}^*$:

$$[\![\mathbf{C}^q_{\mu^\star} F]\!] = \begin{cases} 2^{\mathbb{N}} & \text{if } \mu^\star([\![F]\!]) \geq q \\ \emptyset & \text{otherwise} \end{cases} \qquad [\![\mathbf{D}^q_{\mu^\star} F]\!] = \begin{cases} 2^{\mathbb{N}} & \text{if } \mu^\star([\![F]\!]) < q \\ \emptyset & \text{otherwise.} \end{cases}$$

Then, $\mathbf{CPL}_0$-formulas $\mathbf{C}^q F$ and $\mathbf{D}^q F$ become special cases of extended ones, namely $\mathbf{C}^q_{\mu^\star} F$ and $\mathbf{D}^q_{\mu^\star} F$ (resp.), where $\mu^\star = \mu_{\mathscr{C}}$. On the other hand, in $\mathbf{CPL}_0^\star$ we can simulate experiments corresponding to tossing (arbitrarily) *biased* coins.

Example 4. *Let us consider a biased coin, which returns* HEAD *only* $\frac{1}{3}$ *of the time. Then, letting* $\mu^\star(Cyl(i)) = \frac{1}{3}$ *(for any* $i \in \mathbb{N}$*), we can express that the probability for subsequent tosses to be successful is greater than* $\frac{1}{9}$ *as*

$$F_{star} = \mathbf{C}^{1/9}_{\mu^\star}(\mathbf{1} \wedge \mathbf{2}).$$

Clearly, since $\mu^\star\big([\![\mathbf{1} \wedge \mathbf{2}]\!]\big) = \frac{1}{3} \cdot \frac{1}{3}$*, the formula is valid, i.e.* $[\![\mathbf{C}^{1/9}_{\mu^*}(\mathbf{1} \vee \mathbf{2})]\!] = 2^{\mathbb{N}}$.

One can even define a calculus $\mathbf{LK}_{\mathbf{CPL}_0^\star}$ for this extended semantics, with no substantial change with respect to the proof system $\mathbf{LK}_{\mathbf{CPL}_0}$, introduced in [1]. Indeed, only so-called external hypotheses are related to probability measure and, consequently, no rule, except those involving such measuring conditions, needs to be modified. Generalizations are obtained in the following way:

$$\frac{\mu^\star(\llbracket b \rrbracket) = 0}{\vdash b \rightarrowtail F}\, R^{\rightarrowtail}_{\mu^\star} \qquad\qquad \frac{\vdash c \rightarrowtail F \qquad \mu^\star(\llbracket c \rrbracket) \geq q}{\vdash b \rightarrowtail \mathbf{C}^q_{\mu^\star} F}\, R^{\rightarrowtail}_{\mathbf{C}^\star}$$

C. Proofs from Section 4

C.1. Section 4.1

Measurable Normal Form.

Proof of Lemma 2. Let $D = \bigvee_{i \in \{1,\dots,n\}} C_i$ be in DNF. For any $C_i = \bigwedge_{j \in \{1,\dots,m\}} L_j$, with $i \in \{1,\dots,n\}$, we define C_i^* applying the transformations below:

- $*$ if $C_i = \top$, then $C_i^* = \top$.
- $*$ otherwise, consider each $j \in \{1,\dots,m\}$, starting with $j = 1$:
  - $i.$ if $L_j = \bot$, then $C_i^* = \bot$.
  - $ii.$ if $L_j = \top$, then L_j is removed and $j+1$ is considered.
  - $iii.$ if $L_j \notin \{\bot, \top\}$, we consider each pedex $k \neq j \in \{1,\dots,m\}$, starting with the first:
    - $a.$ if $L_j = L_k$, then L_k is removed and the subsequent pedex (different from j and k) is considered.
    - $b.$ if $L_j = \overline{L_k}$, then $C_i^* = \bot$.[8]
    - $c.$ otherwise, L_j is left unchanged and $k+1$ is considered.

It is clear that $C_i \equiv C_i^*$. We now consider $D' = \bigvee_{i \in \{1,\dots,n'\}} C_i^*$ and define D^* applying the following transformations:

- $*$ if $C_i^* = \bot$ for any $i \in \{1,\dots,n'\}$, then $D^* = \bot$.
- $*$ otherwise, we consider each $i \in \{1,\dots,n'\}$ starting with $i = 1$:
  - $i.$ if $C_i = \top$, then $D^* = \top$.
  - $ii.$ if $C_i = \bot$, then C_i is removed and $i+1$ is considered.
  - $iii.$ if $C_i \in \{\top, \bot\}$, we consider each pedex starting with the first $k' \neq i \in \{1,\dots,n'\}$:
    - $a.$ if C_i and $C_{k'}$ contain exactly the same literals, then $C_{k'}$ is removed and the subsequent pedex (different from both k', i) is considered.
    - $b.$ otherwise, C_i is (at least temporarily) left unchanged and the subsequent pedex (different from both k', i) is considered.

Again it is clear that $D \equiv D^*$. □

[8] Actually, due to Notation 2, case $b.$ should already be considered as the first case $*$.

Observe that for any formula F in DPF, either $F \in \{\bot, \top\}$ or no instance of $\bot, \top$ occurs in it. As anticipated in Section 4, it is easy to measure the probability of a formula in CPF.

Proposition 3. *Given a formula C in CPF: $i.$ if $C = \top$, then $\mu_{\mathscr{C}}(\llbracket C \rrbracket) = 1$, $ii.$ if $C = \bot$, then $\mu_{\mathscr{C}}(\llbracket C \rrbracket) = 0$, $iii.$ otherwise, $C = \bigwedge_{i\in\{1,\dots,n\}} L_i$ and $\mu_{\mathscr{C}}(\llbracket C \rrbracket) = \frac{1}{2^n}$.*

Proof. Case $i., ii.$ are trivial consequences of Definition 1 and basic measure theory. Case $iii.$ relies on Definition 2. Since C does not contain $\bot, \top$ (or contradictions) or repetitions, by semantic definition, its literals have to be interpreted as *independent* events, the measure of which is known. Thus, for basic measure theory, $\mu_{\mathscr{C}}\big(\llbracket \bigwedge_{i\in\{1,\dots,n\}} L_i \rrbracket\big) = \mu_{\mathscr{C}}\big(\bigcap_{i\in\{1,\dots,n\}} \llbracket L_i \rrbracket\big) = \frac{1}{2^n}$. □

Proof of Lemma 3. As before, cases $i., ii.$ hold by Definition 1 and basic measure theory. Case $iii.$ is proved relying on Definition 4: for any $j \neq k \in \{1, \dots, n\}$, (C_j, C_k) is a contradictory pair. Then, by Definition 3, $\llbracket C_j \rrbracket \cap \llbracket C_k \rrbracket = \emptyset$ for any i, k. So, we conclude $\mu_{\mathscr{C}}\big(\llbracket \bigvee_{i\in\{1,\dots,n\}} C_i \rrbracket\big) = \mu_{\mathscr{C}}\big(\bigcup_{i\in\{1,\dots,n\}} \llbracket C_i \rrbracket\big) = \sum_{i\in\{1,\dots,n\}} \mu_{\mathscr{C}}(\llbracket C_i \rrbracket)$. □

Corollary 1. *Given a formula of **CPL**$_0$ in MNF, $F = \underbrace{\bigwedge_{i\in\{1,\dots,m_1\}} L_i \vee \dots \vee \bigwedge_{j\in\{1,\dots,m_n\}} L_j}_{n \text{ times}}$:*

$i.$ *if $F = \top$, then $\mu_{\mathscr{C}}(\llbracket F \rrbracket) = 1$,*
$ii.$ *if $F = \bot$, then $\mu_{\mathscr{C}}(\llbracket F \rrbracket) = 0$,*
$iii.$ *otherwise $\mu_{\mathscr{C}}(\llbracket F \rrbracket) = \underbrace{1/2^{m_1} + \dots + 1/2^{m_n}}_{n \text{ times}}$.*

Proof. By Proposition 3 and Lemma 3. □

C.2. Section 4.2

Conversion into MNF. We show how to convert a **CPL**$_0$-formula in DPF into an equivalent formula in MNF.

Proof of Lemma 4. Given a formula of **CPL**$_0$ in DPF $F = \bigvee_{i\in\{1,\dots,n\}} C_i$, we define a formula F^{**} in MNF such that $F \equiv F^{**}$ as follows:

- if $F \in \{\bot, \top\}$, then $F^{**} = F$.
- otherwise, we consider each $i \in \{1, \dots, n\}$, starting with $i = 1$:
  - $i.$ if there is a $j \neq i \in \{1, \dots, n\}$ such that C_i is a sub-formula of C_j, then C_i is removed and $i + 1$ is considered.
  - $ii.$ otherwise, we consider each pedex $j \neq i \in \{1, \dots, n\}$ starting from the first one:
    - $a.$ if C_i and C_j are mutually contradictory, then $j + 1$ is considered.
    - $b.$ otherwise, for $C_i = \bigwedge_{k\in\{1,\dots,l\}} L_k$ and $C_j = \bigwedge_{k'\in\{1,\dots,l'\}} L_{k'}$, we consider each $k \in \{1, \dots, l\}$ starting with $k = 1$:
      - if there is a $k' \in \{1, \dots, l'\}$ such that $L_k = L_{k'}$, then L_k is left unchanged and $k + 1$ is considered.

- if there is no $k' \in \{1, \ldots, l'\}$ such that $L_k = L_{k'}$, then C_i is replaced by two formulas $C_i' = C_i \wedge L_{k'}$ and $C_i'' = C_i \wedge \overline{L_{k'}}$ and $a.$-$b.$ are applied again to both.

We consider each $k' \in \{1, \ldots, l'\}$ starting with $k' = 1$:

- if there is a $k \in \{1, \ldots, l\}$ such that $L_k = L_{k'}$, then $L_{k'}$ is left unchanged and $k' + 1$ is considered.
- if there is no $k \in \{1, \ldots, l\}$ such that $L_k = L_{k'}$, then C_j is replaced by $C_j' = C_j \wedge L_{k'}$ and $C_j'' = C_j \wedge \overline{L_{k'}}$ and $a.$-$b.$ are applied again to both.

Then, we consider $j + 1$.

When $j + 1 = n$, $i + 1$ is considered (until also $i + 1 = n$).

$\square$