Vibrations of Ideal Circular Membranes (*e.g.* Drums) and Circular Plates:

Solution(s) to the wave equation in 2 dimensions – this problem has cylindrical symmetry ⇒ Bessel function solutions for the radial (r) wave equation, harmonic {sine/cosine-type} solutions for the azimuthal (φ) portion of wave equation. Please see/read "Mathematical Musical Physics of Wave Equation – Part II" *p.* 16-20 for further details…

Boundary condition: Ideal circular membrane (drum head) is ***clamped*** at radius a ⇒ must have transverse displacement ***node*** at $r = a$.

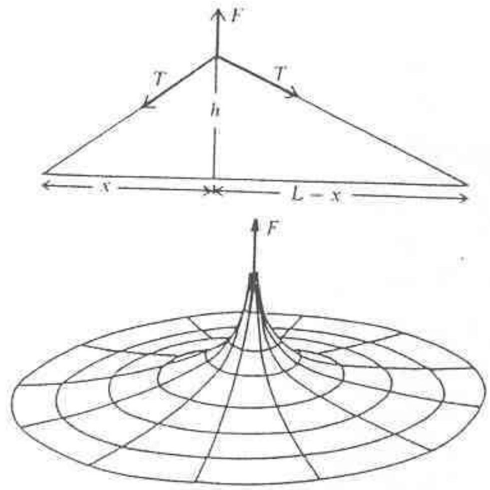


FIGURE 3.4. Reaction of a string and membrane to a force applied at a point.

The 2-D wave equation for transverse waves on a drum head – approximated as a cylindrical membrane has Bessel function solutions in the radial (r) direction and cosine-type functions in the azimuthal (φ) direction (see P406 Lect. Notes "Mathematical Musical Physics of the Wave Equation – Part II", *p.* 16-20): $\psi_{m,n}^{disp}(r,\varphi,t) = A_{m,n} J_m(k_{m,n} r)\cos(m\varphi)\cos(\omega_{m,n} t)$ where $J_m(x_{mn}) = J_m(k_{mn} r)$, $x_{mn} = k_{mn} r$ (*n.b.* dimensionless quantity), k_{mn} = wavenumber = $2\pi/\lambda_{nm}$. The integer index m = 0,1,2,3... refers both to the order # of the {ordinary} Bessel function (in the radial, r-direction) ***and*** also the azimuthal (φ-direction) node #. The index n = 1,2,3,4... refers to the n^{th} non-trivial zero of the Bessel function $J_m(x_{mn})$, *i.e.* when $x_{mn} = k_{mn} a = 0.0$. The boundary condition that the circular membrane is rigidly attached at its outer radius $r = a$ requires that

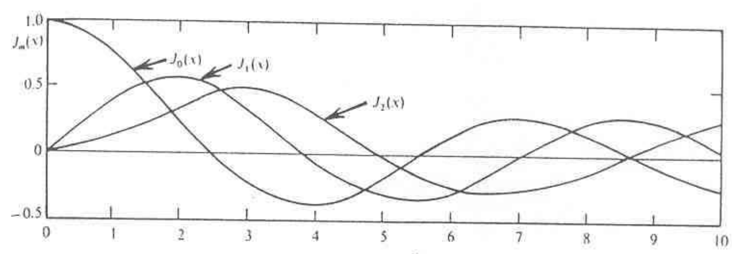


FIGURE 3.5. First three Bessel functions.

there be a transverse displacement ***node*** at $r = a$, *i.e.* $\psi_{m,n}^{disp}(r = a,\varphi,t) = 0$. This gives rise to distinct modes of vibration of the drum head (see 2-D and 3-D pix on next page):

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

Thus, we need ***two*** indices (m, n) to fully specify the 2-D modal vibration harmonics of the circular membrane ***because*** it is a 2-dimensional object. Low-lying eigenmodes of 2-D ***transverse*** displacement amplitudes are shown in the figures below:

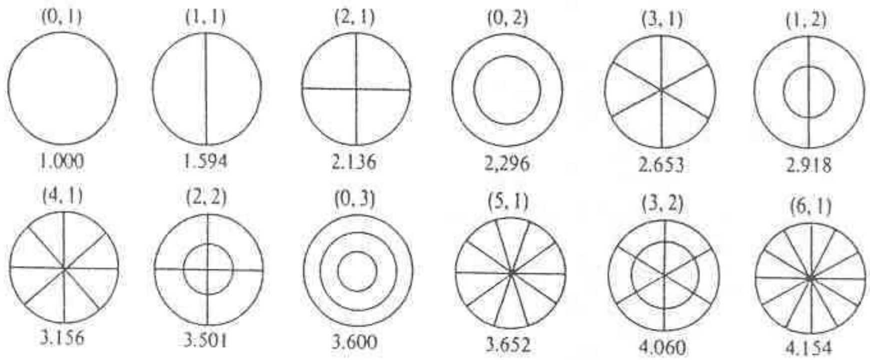
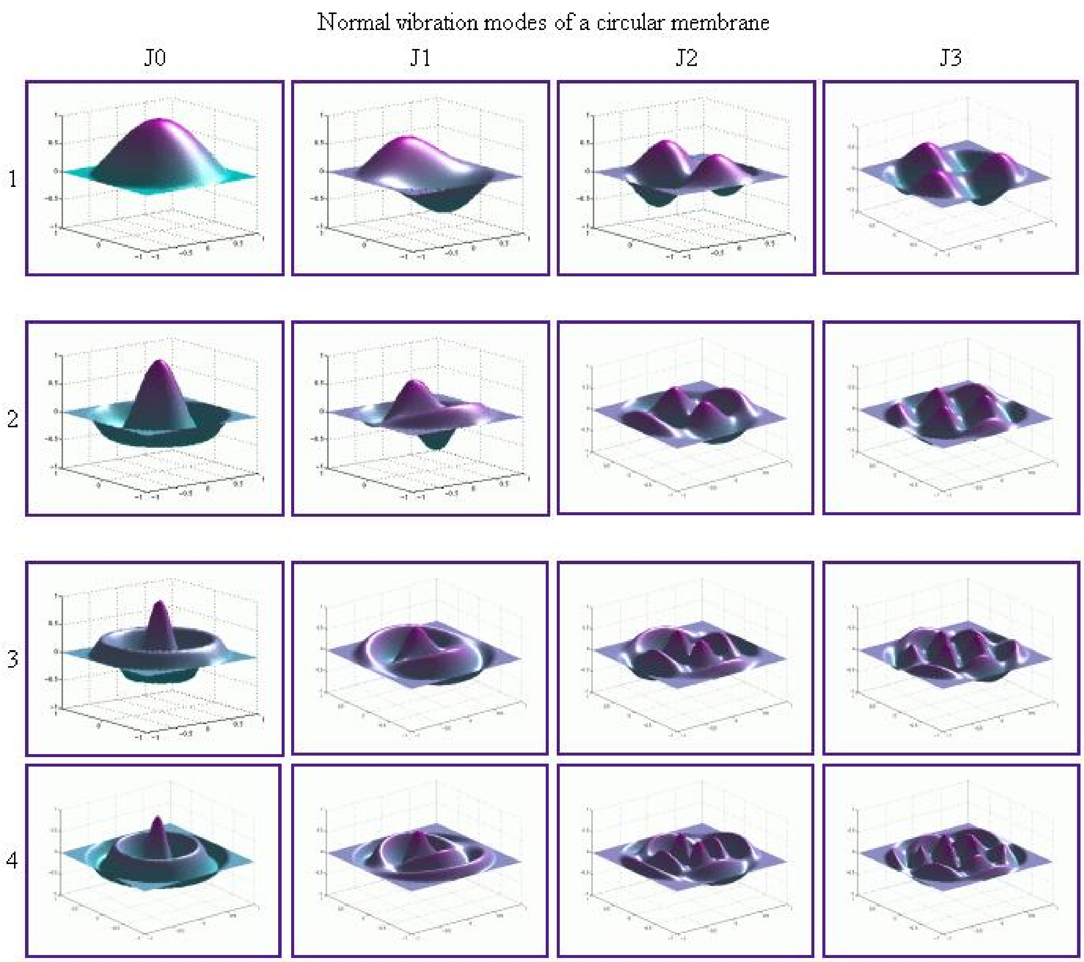


FIGURE 3.6. First 14 modes of an ideal membrane. The mode designation (m, n) is given above each figure and the relative frequency below. To convert these to actual frequencies, multiply by $(2.405/2\pi a)\sqrt{T/\sigma}$, where a is the membrane radius.



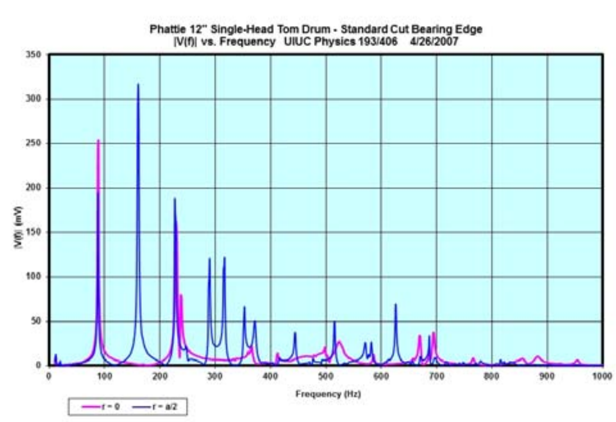
©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

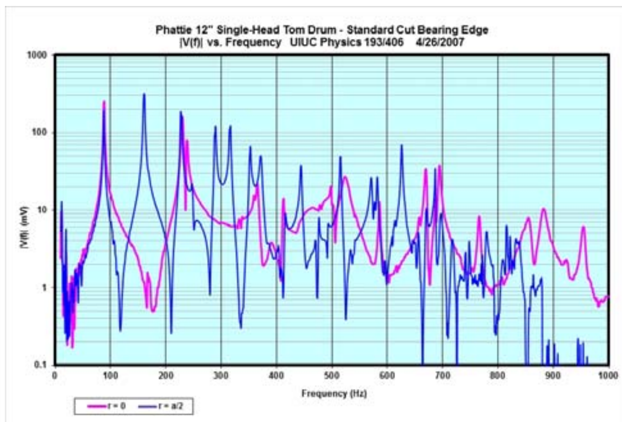
The modal frequencies of a circular membrane are $f_{m,n} = \omega_{m,n}/2\pi = vk_{m,n}/2\pi$, but we also have the relation $k_{m,n} = x_{m,n}/a$ where $x_{m,n}$ is the value of the n^{th} non-trivial zero of the m^{th}–order Bessel function $J_m\left(x_{m,n}\right) = J_m\left(k_{m,n}a\right) = 0$, *e.g.* for m = 0 and n = 1,2,3,4,5… then $J_0\left(x_{0,n}\right) = J_0\left(k_{0,n}a\right) = 0$ when $x_{0,n} = k_{0,n}a = 2.405, 5.520, 8.654, 11.793, 14.931, \ldots$ respectively.

The speed of propagation of transverse waves on a (perfectly-compliant) circular membrane clamped at its outer edge is $v = \sqrt{T_\ell/\sigma}$ where $T_\ell\left(N/m\right)$ is the surface tension (per unit length) of the membrane and $\sigma\left(kg/m^2\right)$ is the areal mass density of the membrane/drum head. Thus:

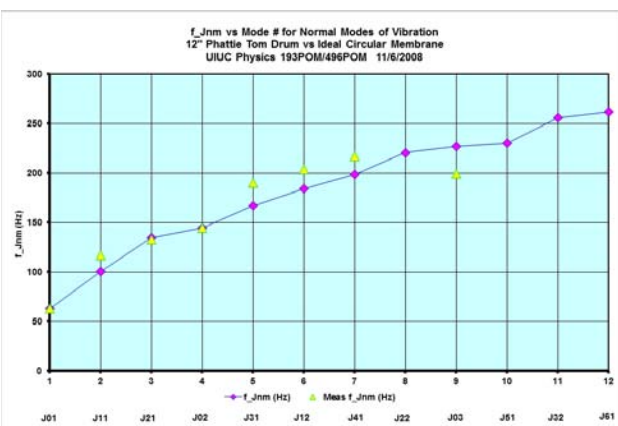
$$f_{m,n} = \frac{\omega_{m,n}}{2\pi} = \frac{vk_{m,n}}{2\pi} = \frac{k_{m,n}}{2\pi}\sqrt{\frac{T_\ell}{\sigma}} = \frac{k_{m,n}a}{2\pi a}\sqrt{\frac{T_\ell}{\sigma}} = \frac{x_{m,n}}{2\pi a}\sqrt{\frac{T_\ell}{\sigma}} \quad (Hz)$$

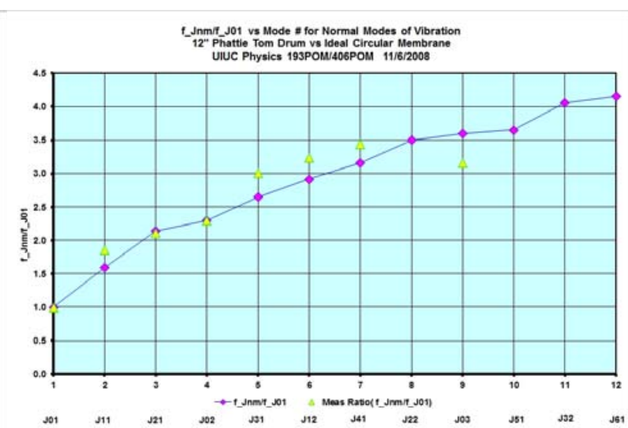
<u>Example</u>: A frequency scan of the resonances associated with the modal vibrations of a Phattie 12" single-head tom drum using the UIUC Physics 193/406POM modal vibrations PC-based data acquisition system is shown in the figures below:





<u>Data *vs.* Theory Comparison of Phattie 12" Tom Drum J_{nm} Modal Frequencies</u>:





n.b. The clear mylar drum head on the Phattie 12" tom drum does have ***<u>finite</u>*** ***<u>stiffness</u>***, *i.e.* it is ***<u>not</u>*** ***<u>perfectly</u>*** compliant, as for an ***<u>ideal</u>*** circular membrane… which affects/alters the resonance frequencies of modes of vibration of drum head….

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

Vibrations of Circular Plates - clamped *vs*. free *vs*. simply supported edges:

TABLE 3.1. Vibration frequencies of a circular plate with clamped edge.

$f_{01} = 0.4694 c_L h/a^2$	$f_{11} = 2.08 f_{01}$	$f_{21} = 3.41 f_{01}$	$f_{31} = 5.00 f_{01}$	$f_{41} = 6.82 f_{01}$
$f_{02} = 3.89 f_{01}$	$f_{12} = 5.95 f_{01}$	$f_{22} = 8.28 f_{01}$	$f_{32} = 10.87 f_{01}$	$f_{42} = 13.71 f_{01}$
$f_{03} = 8.72 f_{01}$	$f_{13} = 11.75 f_{01}$	$f_{23} = 15.06 f_{01}$	$f_{33} = 18.63 f_{01}$	$f_{43} = 22.47 f_{01}$

TABLE 3.2. Vibration frequencies of a circular plate with free edge.

—	—	$f_{20} = 0.2413 c_L h/a^2$	$f_{30} = 2.328 f_{20}$	$f_{40} = 4.11 f_{20}$	$f_{50} = 6.30 f_{20}$
$f_{01} = 1.73 f_{20}$	$f_{11} = 3.91 f_{20}$	$f_{21} = 6.71 f_{20}$	$f_{31} = 10.07 f_{20}$	$f_{41} = 13.92 f_{20}$	$f_{51} = 18.24 f_{20}$
$f_{02} = 7.34 f_{20}$	$f_{12} = 11.40 f_{20}$	$f_{22} = 15.97 f_{20}$	$f_{32} = 21.19 f_{20}$	$f_{42} = 27.18 f_{20}$	$f_{52} = 33.31 f_{20}$

TABLE 3.3. Vibration frequencies of a circular plate with a simply supported edge.

$f_{01} = 0.2287 c_L h/a^2$	$f_{11} = 2.80 f_{01}$	$f_{21} = 5.15 f_{01}$
$f_{02} = 5.98 f_{01}$	$f_{12} = 9.75 f_{01}$	$f_{22} = 14.09 f_{01}$
$f_{03} = 14.91 f_{01}$	$f_{13} = 20.66 f_{01}$	$f_{23} = 26.99 f_{01}$

Vibrations of a Circular Plate: Free Edge

Chladni's Law (1802):

$$f_{m,n} = v(m+2n)^p$$

Mode # (n, m) are (φ, r) integers (*e.g.* = 0,1,2,3, … *etc*.)

For flat circular plates: $p = 2$
For *non*-flat circular plates: $p < 2$ (*e.g*. cymbals)

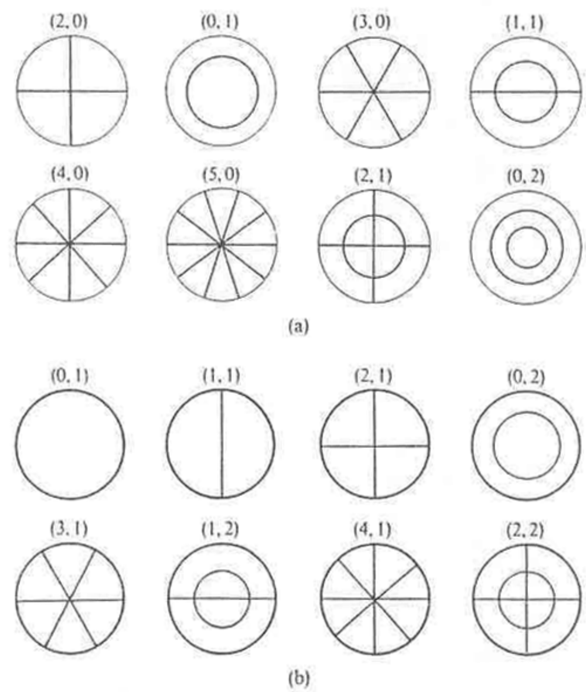
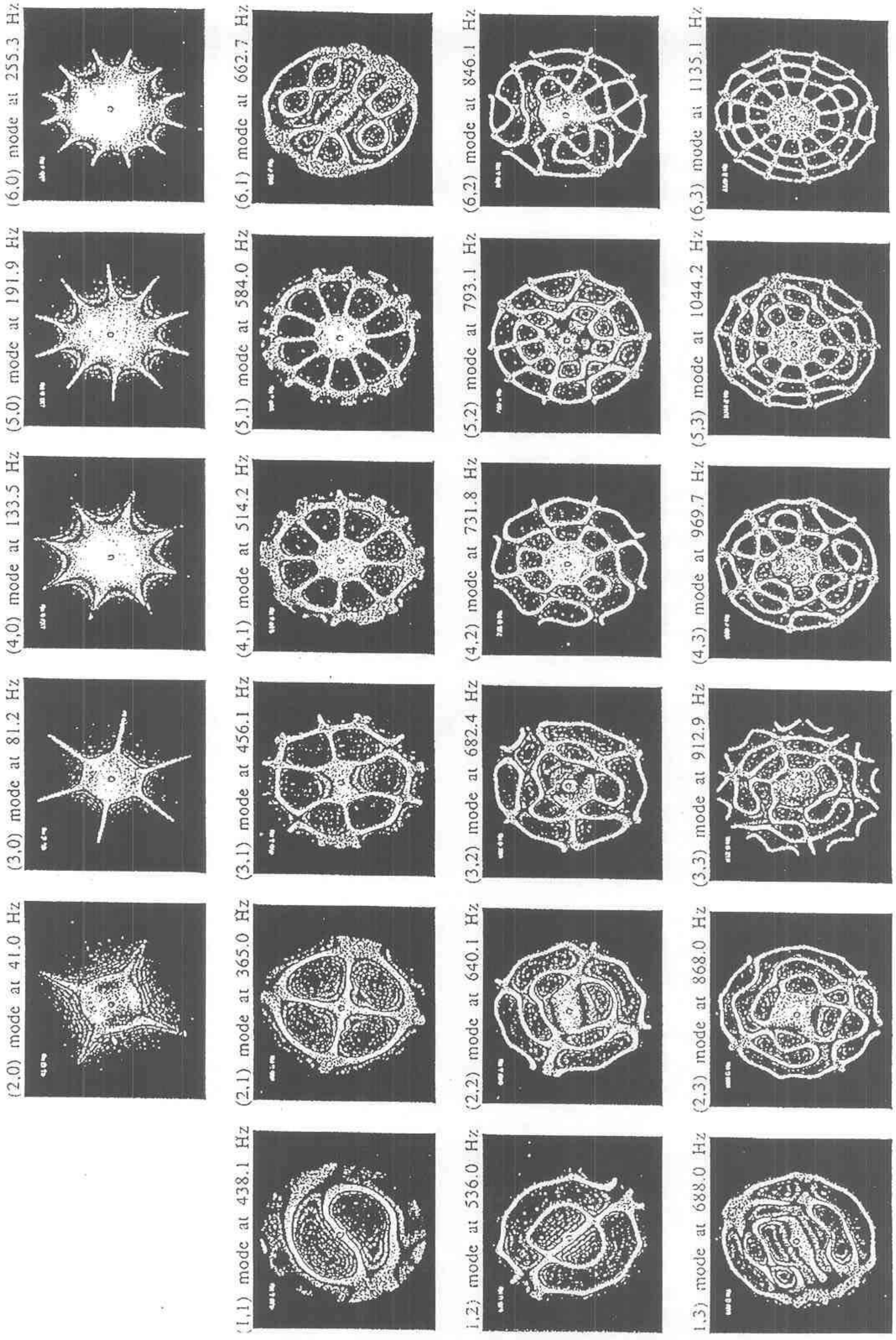


FIGURE 3.8. Vibrational modes of circular plates: (a) free edge and (b) clamped or simply supported edge. The mode number (n, m) gives the number of nodal diameters and circles, respectively.

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.



Modes of an 18 inch Medium Crash Cymbal

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

Modal Vibrations of Cymbals: (continued)

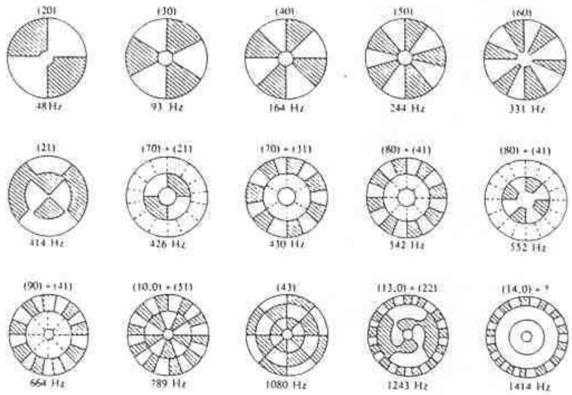
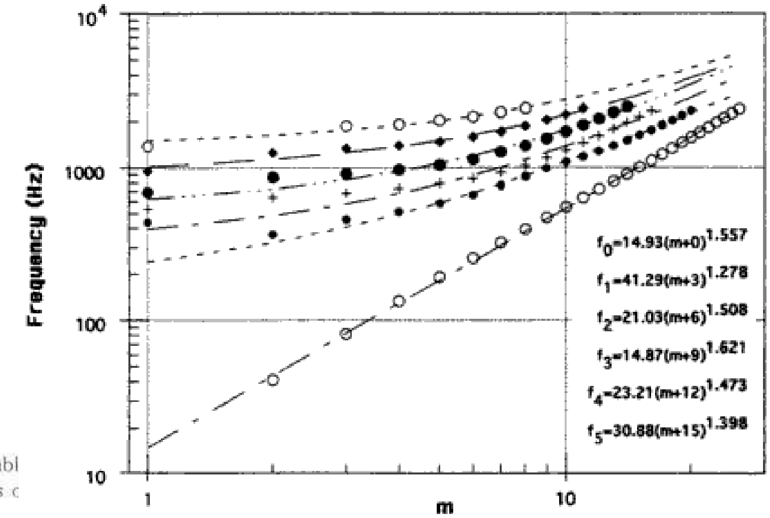
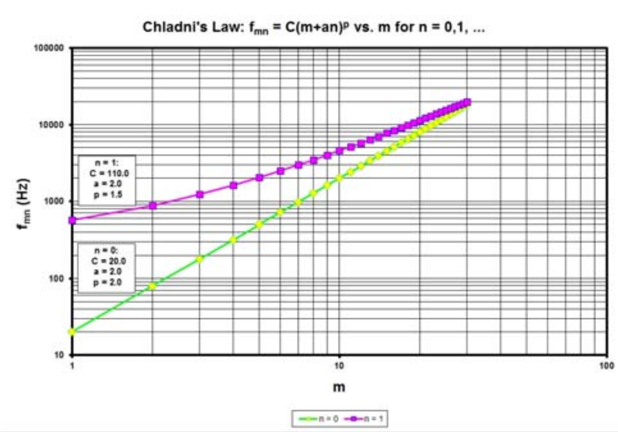


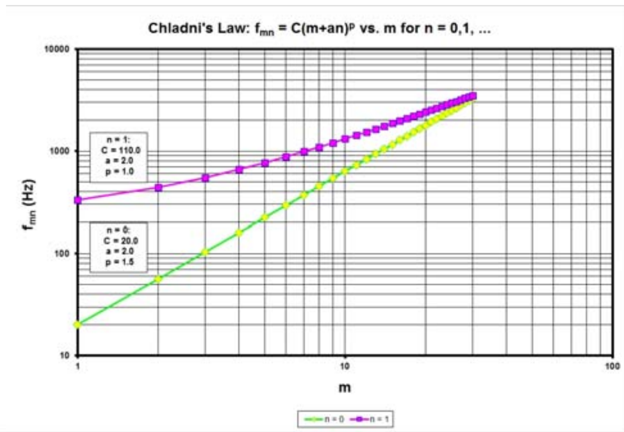
FIGURE 20.2. Modes of vibration of a 38-cm cymbal. The first six modes resembl those of a flat plate, but after that the resonances tend to be combinations c two or more modes (Rossing and Peterson, 1982).



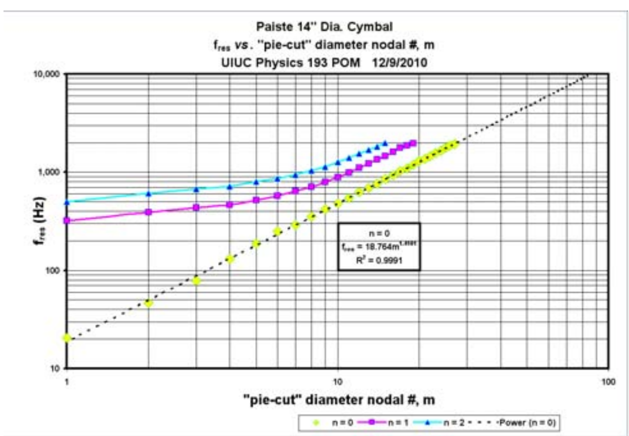
Modal frequencies of a 46-cm-diameter medium crash cymbal as a function of *m* and *n*

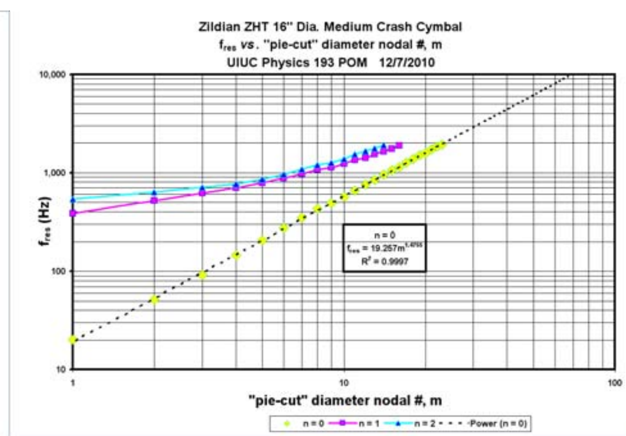
Theory:





Data:





©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

Modal Vibrations of Flat 2-D Rectangular Plates & Stretched 2-D Rectangular Membranes:

B.C.'s: Edges of a flat rectangular plate can be ***fixed*** or ***free***, or ***simply supported***…
⇒ different boundary conditions for 2-D wave equation on rectangular plate…
⇒ different allowed solutions for vibrational modes – again, two indices *m*, *n*

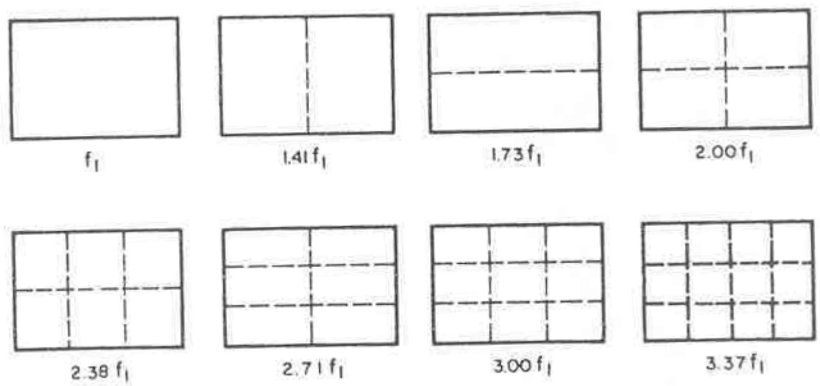


FIG. 15. Some of the modes of vibration of a stretched rectangular membrane. The length of the membrane is 1.41 times the width.

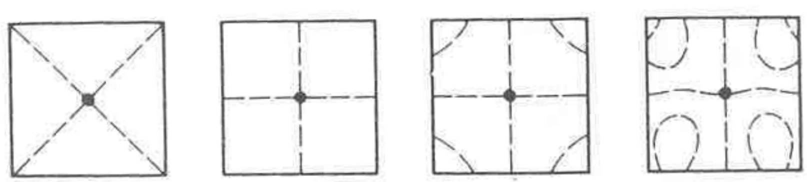


FIG. 16. Some of the modes of vibration of a square plate fixed at the center and set into oscillation by bowing.

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

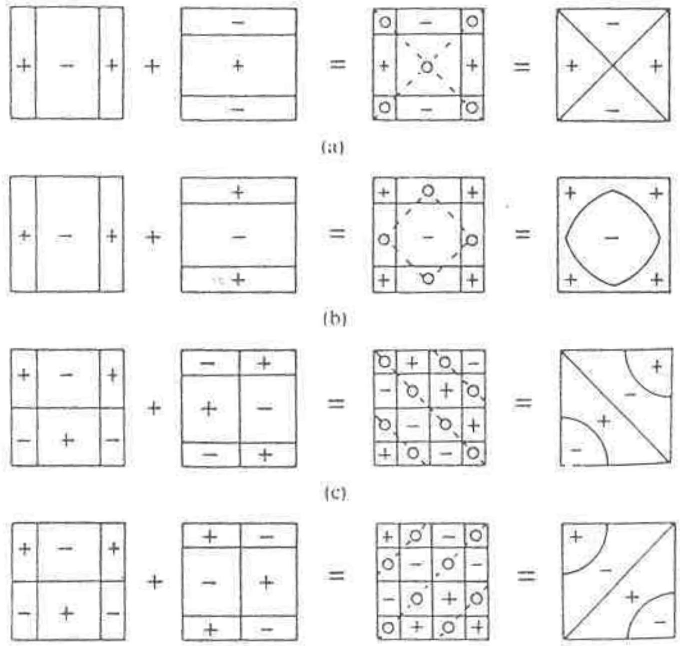


FIGURE 3.12. Graphical construction of combination modes in a square isotropic plate: (a) (2, 0) − (0, 2), x mode; (b) (2, 0) + (0, 2), ring mode; (c) (2, 1) − (1, 2) mode: and (d) (2, 1) + (1, 2) mode.

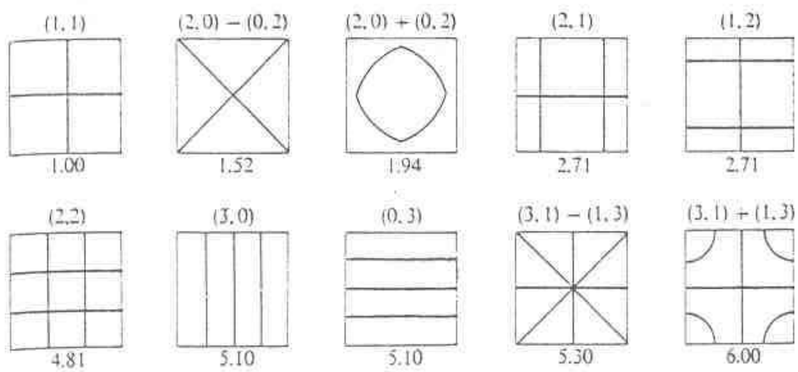


FIGURE 3.13. The first 10 modes of an isotropic square plate with free edges. The modes are designated by m and n, the numbers of nodal lines in the two directions, and the relative frequencies for a plate with $\nu = 0.3$ are given below the figures.

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

Chladni Patterns of {Real} 2-D Vibrating Plates:

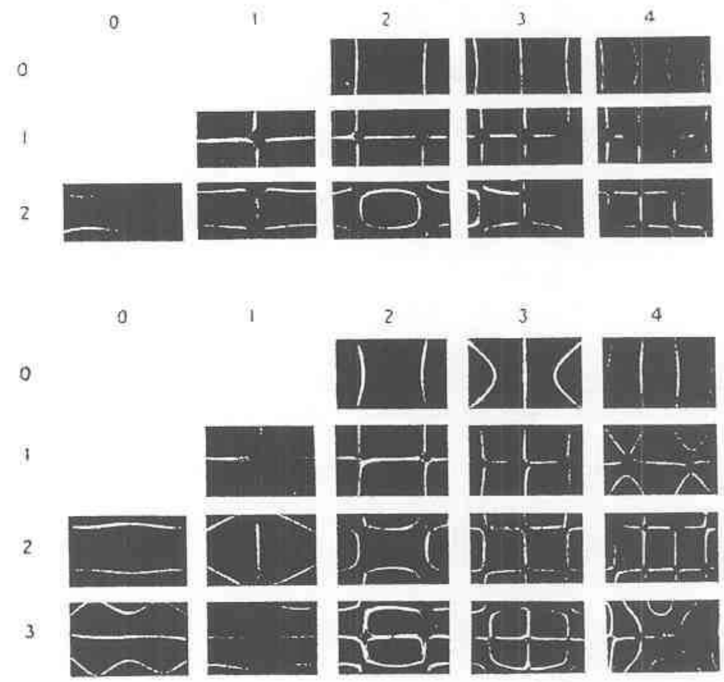


FIGURE 3.9. Chladni patterns showing the vibrational modes of rectangula plates of different shapes: (a) $L_x/L_y = 2$; (b) $L_x/L_y = 3/2$ (Waller, 1949).

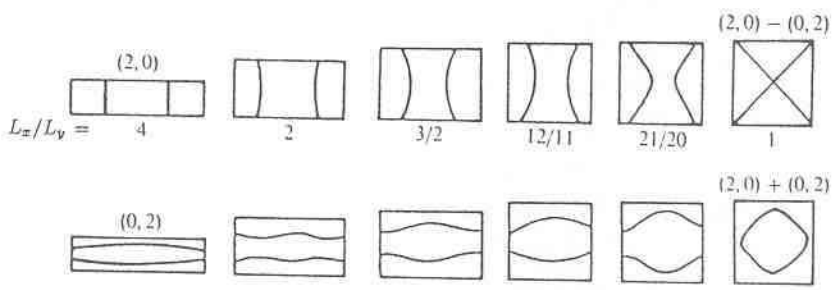


FIGURE 3.10. Mixing of the (2, 0) and (0, 2) modes in rectangular plates with different L_x/L_y ratios (after Waller, 1961).

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

Modal Vibrations of Handbells & Church Bells

21.1. Modes of Vibration of Church Bells 677

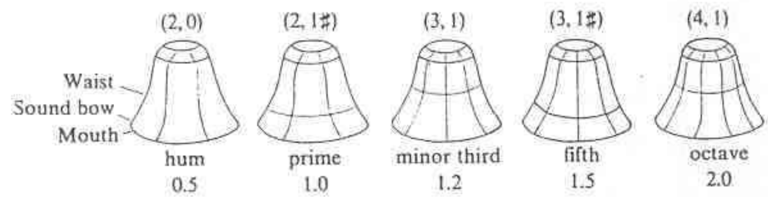


FIGURE 21.1. The first five vibrational modes of a tuned church bell or carillon bell. Dashed lines indicate the nodes. Frequencies (Hz) relative to the prime and names of the corresponding partials are given below each diagram (Rossing, 1984b).

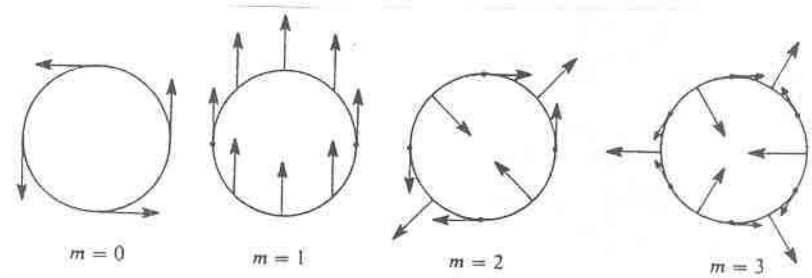
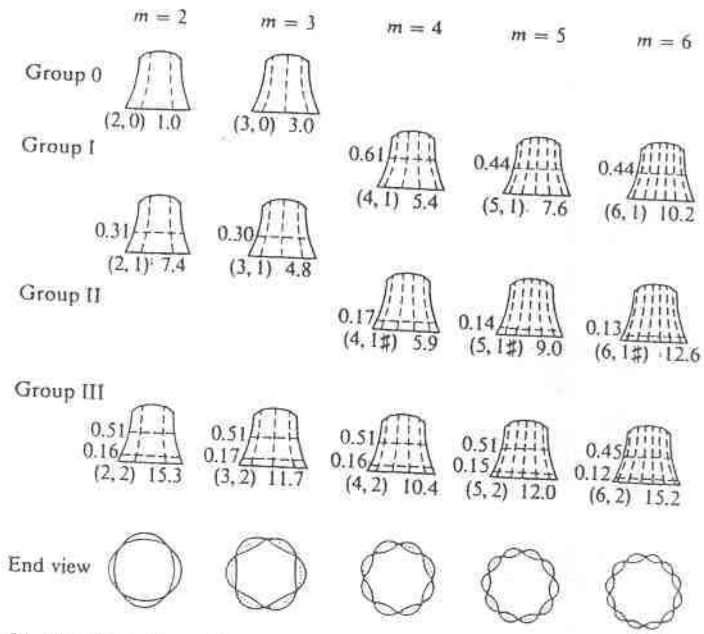


FIGURE 21.2. Motion of a bell for inextensional modes of small m. Modes with $m = 0$ and $m = 1$ require one or more nodal circles ($n > 0$).

The two integers (m, n) respectively denote the number of complete nodal (m) ***azimuthal*** **meridians** extending over the top of bell (*n.b.* = ½ of such nodes along a circumference) and n = number of nodal **circles**. Note that since have two integers, the handbell/churchbell effectively vibrates as a 2-D object – it is simply bent/deformed into a 3-D spatial object…

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

Modal Vibrations of Handbells/Churchbells: (continued)



= 21.15. Periodic table of vibrational modes in a handbell. Below each g are the relative modal frequencies in a Malmark C_5 handbell. At lower . n) gives the number of nodal meridians $2m$ and nodal circles n (Rossing rrin, 1987).

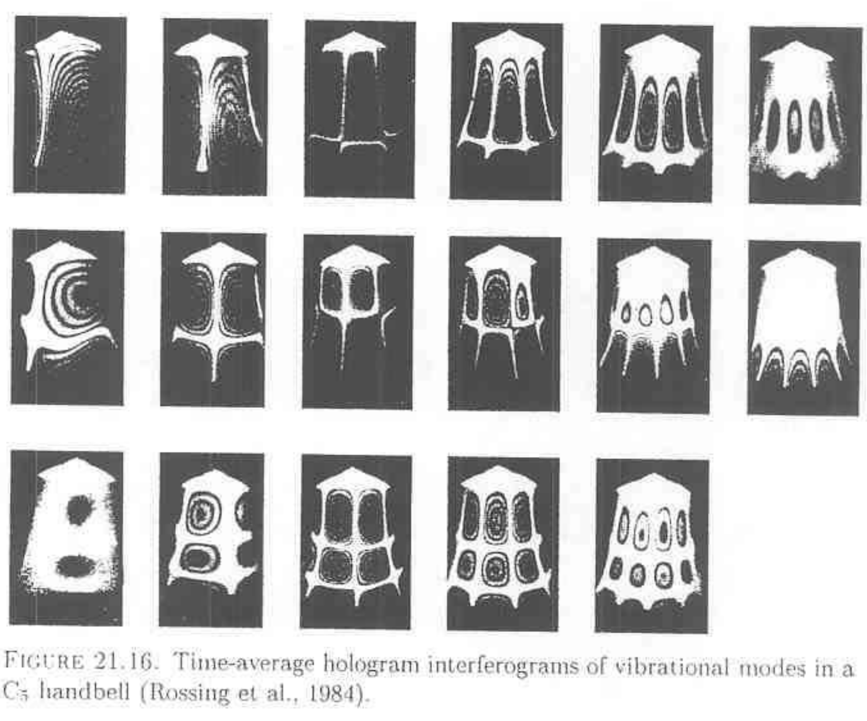


FIGURE 21.16. Time-average hologram interferograms of vibrational modes in a C_5 handbell (Rossing et al., 1984).

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

Vibrational Modes of an Acoustic Guitar:

Top surface, all by itself:

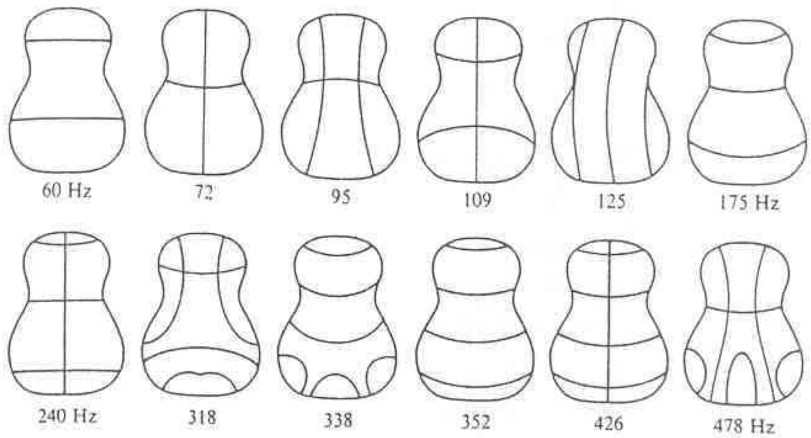


FIGURE 9.6. Vibration modes of a guitar plate blank (without braces) with a free edge (adapted from Rossing, 1982b).

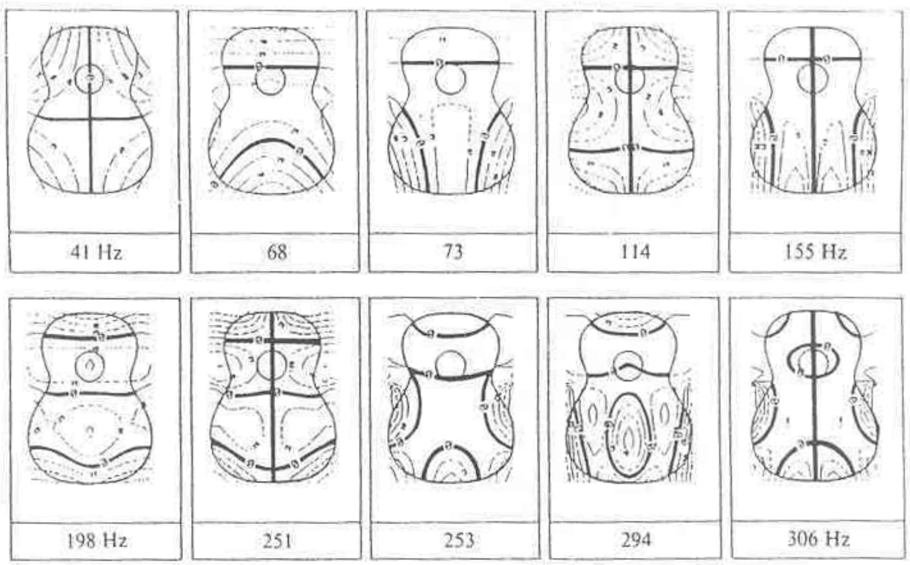


FIGURE 9.7. Vibration modes of a classical guitar top plate with traditional fan bracing (adapted from Richardson and Roberts, 1985).

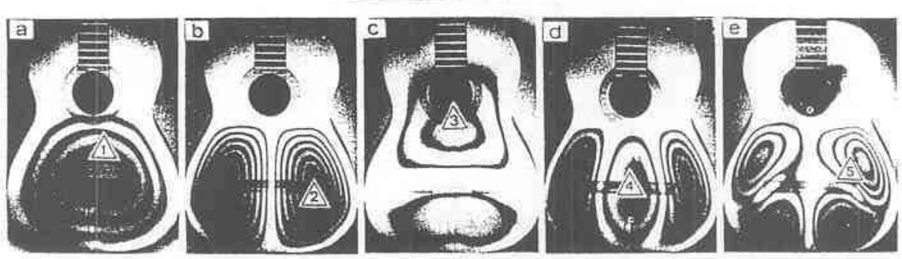


FIGURE 9.8. Vibration modes of a classical guitar top plate glued to fixed ribs but without the back (Jansson, 1971).

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

Modal Vibrations of Acoustic/Classical Guitars:

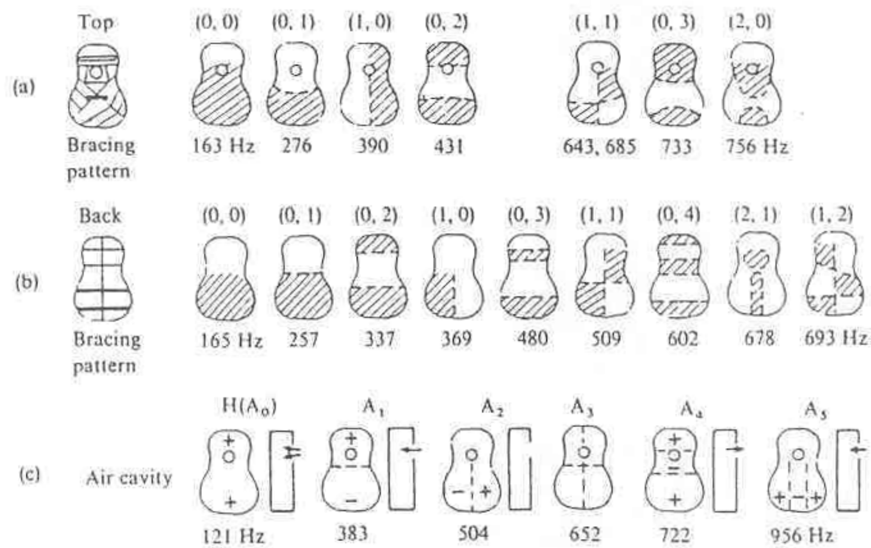


FIGURE 9.9. (a) Modes of a folk guitar top (Martin D-28) with the back and ribs in sand. (b) Modes of the back with the top and ribs in sand. (c) Modes of the air cavity with the guitar body in sand. Modal designations are given above the figures and modal frequencies below (Rossing et al., 1985).

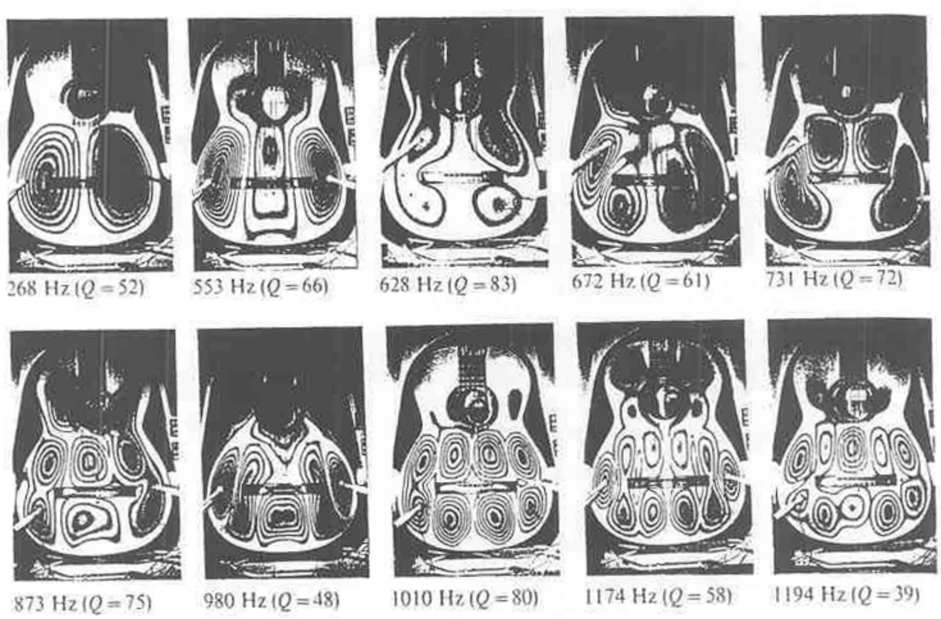


FIGURE 9.16. Time-averaged holographic interferograms of top-plate modes of a guitar (Guitar BR11). The resonant frequencies and Q values of each mode are shown below the interferograms (Richardson and Roberts, 1985).

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

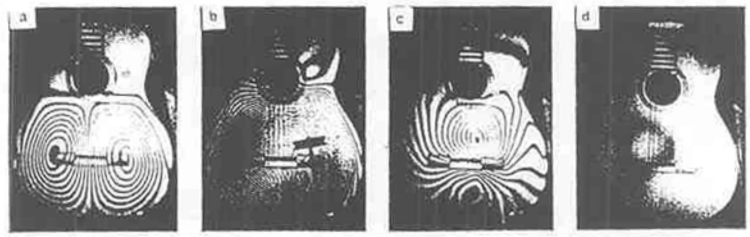
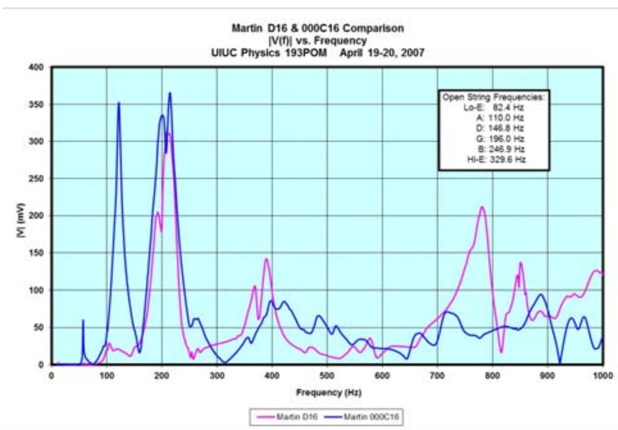
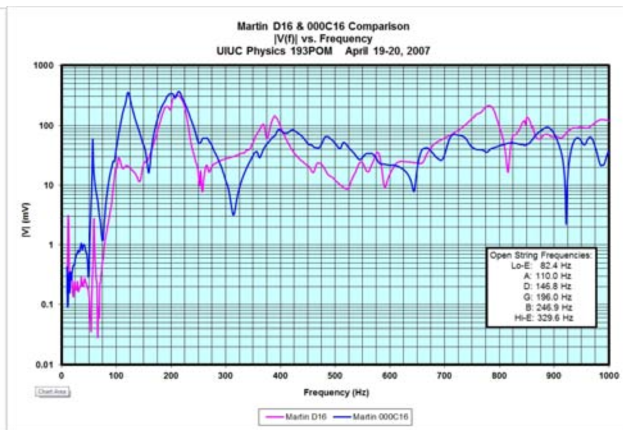


FIGURE 9.17. Holographic interferograms showing top plate distortions for (a) a static force at 1.0 N applied to the sixth string parallel to the bridge, (b) a force of 0.5 N applied to the first string perpendicular to the bridge, (c) a longitudinal force of 2.0 N applied to the first string, and (d) a torque caused by twisting the third string one full turn (Jansson, 1982).

<u>Example</u>: Frequency scan comparison of the mechanical resonances associated with the modal vibrations of a Martin D16 *vs*. a Martin 000C16 guitar using the UIUC Physics 193/406 POM modal vibrations PC-based data acquisition system:





©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois
2002 - 2017. All rights reserved.

Modal Vibrations of Violins/Violas/Cellos, *etc*.

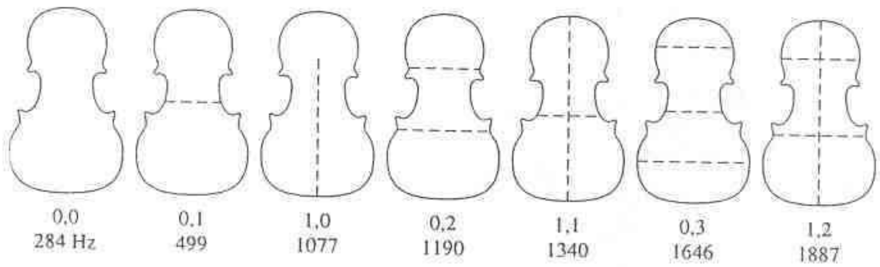


FIGURE 10.13. Modes of a violin air cavity. Mode frequencies are from Roberts and Rossing (1997).

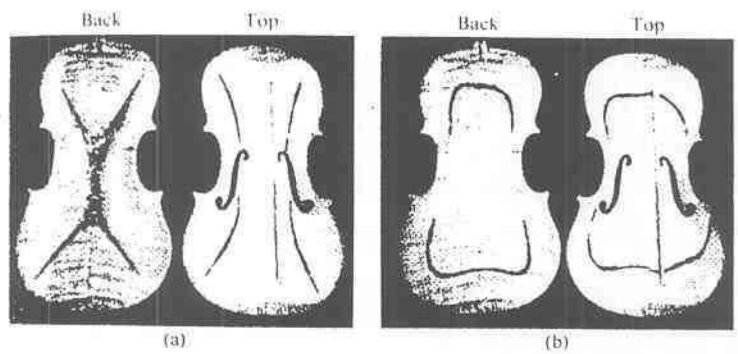


FIGURE 3.18. Chladni patterns showing two modes of vibration in the top and back of a viola (Hutchins, 1977).

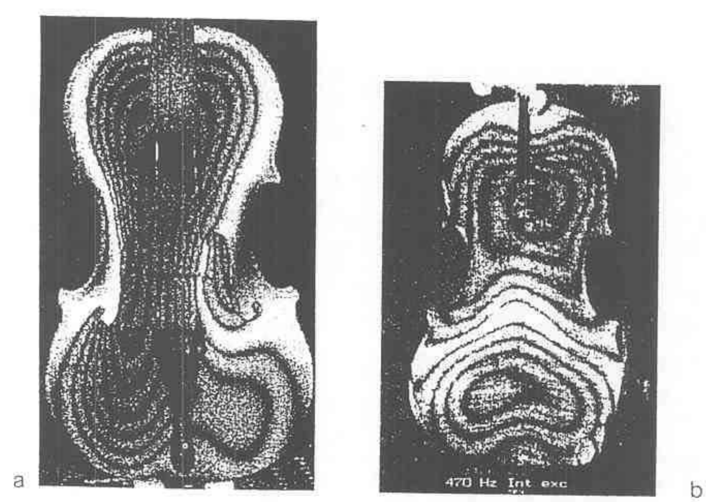


FIGURE 10.12. Interferograms of two air modes in violins using electronic TV holography. (a) A_0 mode excited by sound from a loudspeaker (from Saldner et al., 1996); (b) A_1 mode excited by applying a sound pressure internally (Roberts and Rossing, 1997).

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

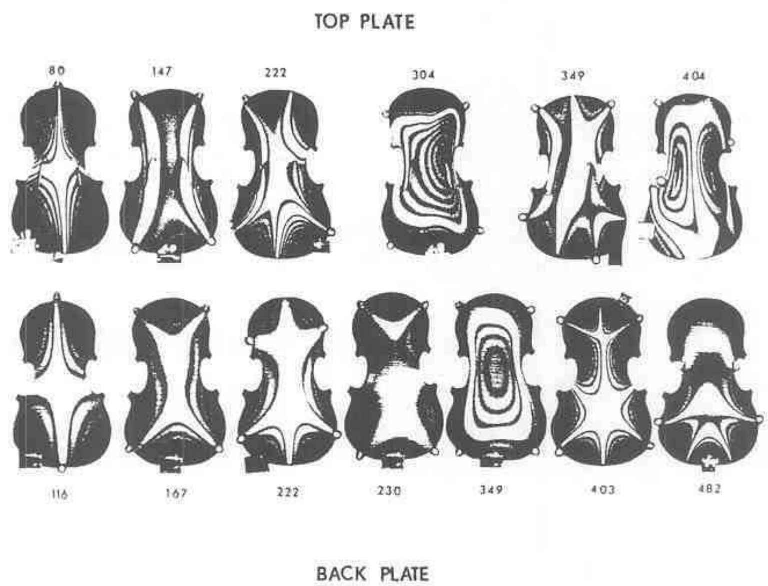


FIGURE 10.14. Time-average holographic interferograms of a free violin top plate and back plate (Hutchins et al., 1971).

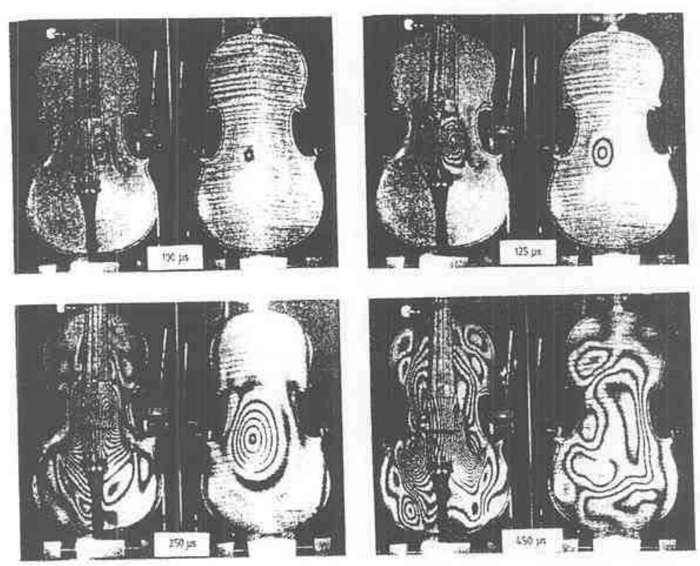


FIGURE 10.15. Interferograms of the top and back plates of a violin at 100 μs, 125 μs, 250 μs, and 450 μs after application of a bridge impulse parallel to the top plate. Note the wave propagation in the top plate is outward from both bridge feet and in the back plate it is outward from the soundpost (Molin et al., 1990).

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois 2002 - 2017. All rights reserved.

Legal Disclaimer and Copyright Notice:

Legal Disclaimer:

The author specifically disclaims legal responsibility for any loss of profit, or any consequential, incidental, and/or other damages resulting from the mis-use of information contained in this document. The author has made every effort possible to ensure that the information contained in this document is factually and technically accurate and correct.

Copyright Notice:

The contents of this document are protected under both United States of America and International Copyright Laws. No portion of this document may be reproduced in any manner for commercial use without prior *written* permission from the author of this document. The author grants permission for the use of information contained in this document for *private*, *non-commercial* purposes only.

©Professor Steven Errede, Department of Physics, University of Illinois at Urbana-Champaign, Illinois
2002 - 2017. All rights reserved.