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Superadditivity of Fisher's Information and Logarithmic Sobolev Inequalities

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We prove a theorem characterizing Gaussian functions and we prove a strict superaddivity property of the Fisher information. We use these results to determine the cases of equality in the logarithmic Sobolev inequality on R^n equipped with Lebesgue measure and with Gauss measure. We also prove a strengthened form of Gross's logarithmic Sobolev inequality with a "remainder term" added to the left side. Finally we show that the strict form of Gross's inequality is a direct consequence of an inequality due to Blachman and Stam, and that this in turn is a direct consequence of strict superadditivity of the Fisher information.

INTRODUCTION

The main theme of this paper is the fact that logarithmic Sobolev inequalities on R^n equipped with either Lebesgue measure or Gauss measure are direct consequences of strict superadditivity of the Fisher information, defined in (10), which we prove here. In fact this result not only implies the sharp inequalities; it also determines the cases of equality in them, and these results, Theorems 4 and 5, are new.

In order to apply the strict superadditivity of the Fisher information, Theorem 3, we prove a characterization of Gaussian functions, Theorem 1, which is interesting in its own right, and generalizes a classic theorem of Skitovich. Our proof moreover uses only real variable methods, and is simpler than Skitovich's proof of his theorem. Strict superadditivity of Fisher's information is itself a consequence of Theorem 2 which is a Sobolev space analog of Minkowski's inequality.

Theorem 6 is a slight digression from the main theme; here we show how an inequality of Beckner [3] and Hirschman [14] implies Gross's [12]

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logarithmic Sobolev inequality with a "remainder term" on the left side which yields another proof of Theorem 5.

Finally we show how the sharp form of an interesting inequality of Blachman [5] and Stam [23] is an immediate consequence of Theorem 2, and then we show that Gross's inequality, together with the statement about cases of equality, is a simple consequence of the Blachman–Stam inequality.

We have applied [8] our result concerning cases of equality in the Lebesgue measure logarithmic Sobolev inequality (4) to prove a conjecture of Lieb [19] concerning cases of equality in his entropy inequality for coherent states.

A secondary theme of this paper is the interplay between contrasting properties of the Gauss measure logarithmic Sobolev inequality (1) and its Lebesgue measure equivalent (4). In [9], among other things, this interplay was exploited to prove (1). This simple equivalence could be used, for example, to simplify some of the analysis in [24], and deserves to be more widely known.

I am very grateful to Elliott Lieb for discussing with me his beautiful work [20] on sharp bounds for operators with Gaussian kernels as it developed. I learned from him the doubling trick I use to prove Theorem 4. He combines the doubling trick with Minkowski's inequality and the Hadamard factorization theorem to prove that maximizing functions in his problem must be Gaussian. The main point here about Minkowski's inequality is that it is a strict convexity inequality which is saturated only by functions whose absolute values are product functions. This motivated our formulation of Theorem 2.

I also thank Leonard Gross, Dan Stroock, and Gerhard Hegerfeldt for valuable advice and references.

Theorems and Proofs

Here dm_t always denotes the measure $t^{-n/2}e^{-\pi|x|^2/t}d^nx$ on R^n. We will simply write dm for dm_1.

Gross' logarithmic Sobolev inequality [9] is

$$\int |f|^2 \log |f|^2 \, dm \leqslant \frac{1}{\pi} \int |\nabla f|^2 \, dm, \qquad \int f|^2 \, dm = 1. \tag{1}$$

Of course we may state this for $f \geqslant 0$ without loss of generality. Because dm is a probability measure, the left side is always well defined, infinity admitted as a possible value, and is finite with (1) holding whenever the right side is finite.

A simple rescaling transforms this into

$$\int |f|^2 \log |f|^2 \, dm_t \leqslant \frac{t}{\pi} \int |\nabla f|^2 \, dm_t, \qquad \int |f|^2 \, dm_t = 1 \tag{2}$$

for all $t > 0$, and the same remarks on its interpretation apply.

Now suppose g belongs to Schwartz space on R^n and $\int |g|^2 \, d^n x = 1$. Then if we write $f(x) = t^{n/4} e^{\pi |x|^2/2t} g(x)$ and insert f into (2) we obtain

$$\int |g|^2 \log |g|^2 \, d^n x + \frac{\pi}{t} \int |x|^2 |g|^2 \, d^n x + \frac{n}{2} \int |g|^2 \, d^n x \log t$$

$$\leqslant \frac{t}{\pi} \int \left(\left(\frac{\pi}{t} \right)^2 |x|^2 |g|^2 + \frac{\pi x}{t} \cdot \nabla |g|^2 + |\nabla g|^2 \right) d^n x. \tag{3}$$

Since $\int |x|^2 |g|^2 \, d^n x$ is finite, we may cancel it from both sides so we have

$$\int |g|^2 \log |g|^2 \, d^n x \leqslant \frac{t}{\pi} \int |\nabla g|^2 \, d^n x$$

$$- \left(n + \frac{n}{2} \log t \right), \qquad \int |g|^2 \, d^n x = 1. \tag{4}$$

It is known that one has equality in (1) for the functions

$$f_a(x) = e^{\pi(a \cdot x - |a|^2/2t)}, \qquad a \in R^n. \tag{5}$$

From this it is easy to see that one has equality in (4) for the functions

$$g_a(x) = t^{-n/4} e^{-\pi |x - a|^2/2t}, \qquad a \in R^n. \tag{6}$$

We will prove here that these are the only possible cases of equality. Notice that we have cancelled a second moment in our passage from (3) to (4), so the two results are not immediately equivalent. We will first settle the cases of equality in (4), and will then use that to settle the cases of equality in (1). We will use two results of independent interest. The first is a characterization of Gaussian functions. By a Gaussian function f on R^n we mean any function of the form

$$f(x) = e^{-(1/2) x \cdot C^{-1} x + b \cdot x + a},$$

where C is a positive definite $n \times n$ matrix, here called the covariance of f, $b \in \mathbf{C}^n$, and $a \in \mathbf{C}$.

THEOREM 1. *Let $f(x, y)$ be a function on R^{2n} which belongs to $L^p(R^{2n}, d^{2n}x)$ for some p with $1 \leqslant p \leqslant \infty$. Suppose f is a product function* *in the coordinates* (x, y) *as well as in the coordinates* $((x+y)/\sqrt{2}, (x-y)/\sqrt{2})$. *That is, suppose there exist functions* ϕ_1, ϕ_2 *and* ψ_1, ψ_2 *in* $L^p(R^n, d^nx)$ *so that*

$$f(x, y) = \phi_1(x)\,\phi_2(y) = \psi_1\left(\frac{x+y}{\sqrt{2}}\right)\psi_2\left(\frac{x-y}{\sqrt{2}}\right) \qquad a.e. \tag{7}$$

Then $f, \phi_1, \phi_2, \psi_1$, *and* ψ_2 *are Gaussian functions. Moreover* ϕ_1 *and* ϕ_2 *have the same covariance, and are otherwise unrestricted.*

Proof. First suppose that each of the functions in (7) is real and non-negaive. Let P_t denote the heat semigroup on $L^p(R^{2n})$ and let Q_t denote the heat semigroup on $L^p(R^n)$. Let $u = (x+y)/\sqrt{2}$ and $v = (x-y)/\sqrt{2}$.

Then on account of the Euclidean invariance of the heat semigroup and *its* product structure,

$$P_t f(x, y) = Q_t\phi_1(x)\, Q_t\phi_2(y) = Q_t\psi_1(u)\, Q_t\psi_2(v) \tag{8}$$

for all $t > 0$. All of the functions in (8) are continuously differentiable as often as we like, and sine the heat semigroup is positivity improving, they are all strictly positive. Thus we may logarithmically differentiate $P_t f$:

$$\frac{\partial^2}{\partial x_j \partial y_k} \log P_t f(x, y) = \frac{\partial^2}{\partial x_j \partial y_k} (\log Q_t\phi_1(x) + \log Q_t\phi_2(y)) = 0.$$

But

$$\frac{\partial^2}{\partial x_j \partial y_k} = \frac{1}{2}\left(\frac{\partial^2}{\partial u_j \partial u_k} - \frac{\partial^2}{\partial v_j \partial v_k} + \frac{\partial^2}{\partial u_k \partial v_j} - \frac{\partial^2}{\partial u_j \partial v_k}\right)$$

so rewriting the above equation in terms of ψ_1 and ψ_2 we have

$$\text{Hess}(\log Q_t\psi_1)(u) = \text{Hess}(\log Q_t\psi_2)(v),$$

where Hess denotes the Hessian. Since u and v are independent variables, both Hessians are equal and constant, and so $Q_t\psi_1$ and $Q_t\psi_2$ are Gaussian functions with the same covariance. Hence also $P_t f$ is Gaussian.

Next, $\iint f(x, y)\, d^nx d^ny = \iint P_t f(x, y)\, d^nx d^ny < \infty$ since the second integrand is Gaussian, and the heat kernel preserves integrals. Therefore f is integrable, and so are both ϕ_1 and ϕ_2. But the heat semigroup is strongly continuous on $L^1(R^n, d^nx)$ so that $f = \lim_{t \to 0} P_t f$ in norm, and the strong limit of Gaussians is Gaussian. In fact, by a theorem of Cramér [10] which we will use a bit later, the convolution of two non-negative integrable functions is Gaussian just when both functions are themselves Gaussian, so we need only use that $P_1 f$ is Gaussian. Thus the theorem is proved when f is non-negative.

To remove the assumption that f is non-negative, first apply the theorem to $|P_t f| = |Q_t \phi_1| \, |Q_t \phi_2| = |Q_t \psi_1| \, |Q_t \psi_2|$ and conclude that each of these functions is Gaussian, and in particular the functions inside the absolute value signs are nowhere vanishing, and of course they are smooth. Then the logarithmic derivatives of the functions inside the absolute value signs are well defined as ratios, and taking two logarithmic derivatives as above we get two constant diagonalizable matrices. Diagonalizing these yields n independent ordinary differential equations uniquely solved by Gaussians. In this way, we see that the regularized functions are Gaussian with the same covariances. We take the limit as t vanishes in the same way.

Remarks. There is nothing special about coordinates rotated through $\pi/4$; any angle which is not a multiple of $\pi/2$ will do. Also, it is not really required that $f \in L^p$; one could permit arbitrary exponential growth at infinity for example. Then $P_t f$, etc., is well defined, and the argument proceeds as above.

Related results when f is a non-negative integrale function go back to Bernstein [4] and Kac [17], who both moreover required moment conditions on f. Skitovich [22] treated the problem—in fact a more general problem—without moment conditions; but his method against works only for non-negative integrable functions. This limitation is inherent in existing methods which rely on either analytic properties of the Fourier transform as in [4, 22], or on the central limit theorem as in [14]. Since the restriction to non-negative integrable f is crucial in Cramér's characterization of Gaussian functions [10], it is interesting that it can be dispensed with here. Moreover, we shall need to apply our result in the case where f is only assumed to be square integrable.

It is also interesting to note that the proof makes crucial use of the fact that Gaussian functions have a certain property to prove that this property actually characterizes them. The application of the mixed partial derivatives to the logarithm of f is inspired here by Brascamp and Lieb's proof [6] that only certain Gaussian functions saturate the sharp Young's inequality in cases where the constant is less than unity. That proof is an argument based on analytic properties of the Laplace transform, and the result does not subsume Theorem 1. The logarithmic derivative argument in this analytic function setting goes back at least to Paul Lévy; see [18, p. 338].

We will be able to apply the above result on account of the condition for equality in the following inequality, for which we now introduce some notation.

Let $L^p(R^m, d^m x) \otimes W^{1,p}(R^n)$ be equipped with the norm

$$|||f||| = \left(\int |\nabla_y f(x, y)|^p \, d^m x d^n y \right)^{1/p} + \|f\|_{L^p},$$

where throughout the following ∇ denotes the distributional gradient, and ∇_y denotes the partial distributional gradient in the y variables. Given $f \in L^p(R^m \times R^n, d^m x d^n y)$, define

$$G(y) = \left(\int |f(x, y)|^p \, d^m x \right)^{1/p}.$$

Clearly $G \in L^p(R^n, d^n y)$, and by Minkowski's inequality the *marginal map* M defined by

$$M : f \mapsto G$$

is continuous from $L^p(R^m \times R^n, d^m x d^n y)$ to $L^p(R^n, d^n y)$ for all $p \geqslant 1$.

THEOREM 2. *For all $p \geqslant 1$, the marginal map M is continuous from $L^p(R^m, d^m x) \otimes W^{1,p}(R^n)$ to $W^{1,p}(R^n)$, and if $G = Mf$ for any function $f(x, y)$ in $L^p(R^m, d^m x) \otimes W^{1,p}(R^n)$,*

$$\nabla G(y) = G(y)^{1-p} \int |f(x, y)|^{p-2} \operatorname{Re}(f^* \nabla_y f(x, y)) \, d^m x.$$

Moreover,

$$\int \left| \nabla_y \left(\int |f(x, y)|^p \, d^m x \right)^{1/p} \right|^p d^n y \leqslant \iint |\nabla_y f(x, y)|^p \, d^m x d^n y \tag{9}$$

and whenever there is equality $|f(x, y)| = |f_1(x)| \, |f_2(y)|$.

Remark. Inequality (9) closely resembles Minkowski's inequality in both its form and its proof.

Proof. Let $p' = p/(p-1)$, and let $\mathbf{h}$ be an R^n valued $L^{p'}$ function with $\|\mathbf{h}\|_{p'} = |||\mathbf{h}|||_{p'}$. Then

$$\nabla G_j(y) = G_j(y)^{1-p} \int |f_j(x, y)|^{p-2} \operatorname{Re}(f_j^* \nabla_y f_j(x, y)) \, d^m x$$

holds for any smooth function f_j in, say, Schwartz space on $R^m \times R^n$. To establish this formula for a general $f \in L^p(R^m, d^m x) \otimes W^{1,p}(R^n)$, consider a sequence $\{f_j\}$ of Schwartz space functions with $\lim_{j\to\infty} |||f-f_j|||=0$. Set $G_j = Mf_j$ and $G = Mf$. Finally set

$$Y(y)=G(y)^{1-p}\int |f(x,y)|^{p-2}\,\mathrm{Re}(f^*\nabla_y f(x,y))\,d^m x.$$

Then $\|\nabla G_j - Y\|_p \leqslant$

$$\sup\left\{\iint G_j^{1-p}\,|f_j|^{p-2}\,\mathrm{Re}(f_j^*(\nabla_y f_j-\nabla_y f)\cdot \mathbf{h} d^m x d^n y\,\Big|\,\|\mathbf{h}\|_{p'}=1\right\}$$

$$+\sup\left\{\iint (\nabla_y f\cdot \mathbf{h})(u_j-u)\,a d^m x d^n y\,\Big|\,\|\mathbf{h}\|=1\right\},$$

where $u_j(x,y)=G_j^{1-p}(y)\,|f_j(x,y)|^{p-2}f_1^*(x,y)$ and similarly for $u(x,y)$.

Repeatedly applying Hölder's inequality we obtain $\|\nabla G_j - Y\|_p \leqslant$

$$\left(\iint |\nabla_y(f_j(x,y)-f(x,y))|^p\,d^m x d^n y\right)^{1/p}$$

$$+\left(\iint |\nabla_y f(x,y)|^p\,d^m x d^n y\right)^{1/p}$$

$$\times\left(\int |\mathbf{h}(y)|^{p'}\left(\int |u_j(x,y)-u(x,y)|^{p'}\,d^m x\right)d^n y\right)^{1/p'}$$

$$\leqslant |||f-f_j||| + |||f|||\left(\int g_j(y)\,d^n y\right)^{1/p'},$$

where we have set

$$g_j(y)=|\mathbf{h}(y)|^{p'}\left(\int |u_j(x,y)-u(x,y)|^{p'}\,d^m x\right).$$

We will now show that $\lim_{j\to\infty}\int g_j(y)\,d^n y=0$. First note that $u_j(\cdot,y)$ and $u(\cdot,y)$ are unit vectors for almost every y, so that $(\int |u_j(x,y)-u(x,y)|^{p'}\,d^m x)^{1/p'}\leqslant 2$ almost everywhere in y so that

$$g_j \leqslant 2^{p'}\,|\mathbf{h}|^{p'}$$

almost everywhere for all j.

Suppose $\int g_j(y)\,d^n y$ fails to converge to zero; then there is a subsequence along which the integral always exceeds some $\varepsilon>0$. But since $|||f-f_j|||$ tends to zero, so that $\|G-G_j\|_p$ also tends to zero, we can select further subsequence along which u_j tend to u almost everywhere. Then along this subsequence g_j tends to zero almost everywhere, and thus along this subsequence $\int g_j(y)\,d^ny$ tends to zero by dominated convergence. This contradiction establishes that $\lim_{j\to\infty}\int g_j(y)\,d^ny=0$, and thus that $\lim_{j\to\infty}\|\nabla G_j - Y\|_p=0$. Since $\lim_{j\to\infty}\|G_j-G\|_p=0$, it follows from the fact that the gradient is closed that $Y=\nabla G$, which is what we set out to prove.

Having established the asserted formula for ∇G, the estimates above may be applied to arbitrary convergent n $L^p(R^m, d^mx)\otimes W^{1,p}(R^n)$, and we see that the marginal map is continuous.

We now prove (9).

$$\left(\int |\nabla G(y)|^p\,d^ny\right)^{1/p} = \sup\left\{\int \mathbf{h}(y)\cdot\nabla G(y)\,d^ny \,\middle|\, \|\mathbf{h}\|_{p'}=1\right\}$$

and for any h in $W^{1,p'}$,

$$\begin{aligned}\int \mathbf{h}(y)\cdot\nabla G(y)\,d^ny &= \iint \frac{1}{G(y)^{p-1}}\,|f(x,y)|^{p-2}\\ &\quad\times \mathrm{Re}(f(x,y)^*\,\nabla_y f(x,y))\cdot\mathbf{h}(y)\,d^mx d^ny\\ &\leqslant \iint \frac{1}{G(y)^{p-1}}\,|f(x,y)|^{p-1}\,|\nabla_y f(x,y)|\,|\mathbf{h}(y)|\,d^mx d^ny\\ &\leqslant \int \frac{1}{G(y)^{p-1}}\left(\int |f(x,y)|^p\,d^mx\right)^{1/p'}\\ &\quad\times\left(\int |\nabla_y f(x,y)|^p\,d^mx\right)^{1/p}|\mathbf{h}(y)|\,d^ny\\ &=\int\left(\int |\nabla_y f(x,y)|^p\,d^mx\right)^{1/p}|\mathbf{h}(y)|\,d^ny\\ &\leqslant\left(\iint |\nabla_y f(x,y)|^p\,d^mx d^ny\right)^{1/p}\left(\int |\mathbf{h}(y)|^{p'}\,d^ny\right)^{1/p'}.\end{aligned}$$

There is equality in the last inequality just when $|\nabla_y f(x,y)|^p$ is proportional to $|\mathbf{h}(y)|^{p'}$ for almost every x, which implies that $|\nabla_y f(x,y)|$ is a product function. But then for equality to hold in the second inequality, we must have that $|f(x,y)|$ is itself a product function.

Now let ρ be a probability density on R^l; that is, ρ is non-negative and $\int \rho(x)\,d^lx=1$. For $\rho^{1/2}\in W^{1,2}(R^l)$ we define the Fisher information of ρ, $I(\rho)$ by

$$I(\rho)=4\int |\nabla\rho^{1/2}(x)|^2\,d^lx=\int |\nabla\log\rho(x)|^2\,\rho(x)\,d^lx. \tag{10}$$

For all other ρ, we set $I(\rho)=\infty$. This quantity was introduced by R. A. Fisher [11] in his theory of sufficient statistics. The superadditivity property we discuss below seems not to be in the literature on this subject; probably it has no use there.

Corresponding to any orthogonal decomposition $R^t = R^r \oplus R^s$, $t = r + s$, we have the marginal densities

$$\rho_1(x)=\int_{R^s}\rho(x, y)\,d^s y, \qquad \rho_2(y)=\int_{R^r}\rho(x, y)\,d^r x.$$

An immediate consequence of the previous theorem is the strict superadditivity of the information:

THEOREM 3. *With ρ, ρ_1, and ρ_2 related as above,*

$$I(\rho)\geqslant I(\rho_1)+I(\rho_2) \tag{11}$$

with equality just when $\rho(x, y)=\rho_1(x)\,\rho_2(y)$ almost everywhere.

Proof. Let $f(x, y)=\rho^{1/2}(x, y)$. Then

$$\begin{aligned}
I(\rho) &= 4\iint |\nabla_x f(x, y)|^2\,d^r x\,d^s y + 4\iint |\nabla_y f(x, y)|^2\,d^s y\,d^r x\\
&\geqslant 4\int \left|\nabla\left(\int f^2(x, y)\,d^s y\right)^{1/2}\right|^2 d^r x\\
&\quad + 4\int \left|\nabla\left(\int f^2(x, y)\,d^r x\right)^{1/2}\right|^2 d^s y\\
&= I(\rho_1)+I(\rho_2).
\end{aligned}$$

The inequality results from two applications of the previous theorem with $p=2$; by the conditions for equality there, we must have $\rho(x, y)=\rho_1(x)\,\rho_2(y)$ to obtain equality here.

Remark. This theorem is a direct analog of the well known theorem asserting strict subadditivity of the entropy. It is weaker than Theorem 2, so that it will be convenient to refer to the $p=2$ case of Theorem 2 as "superadditivity of Fisher's information."

We are now ready to settle the cases of equality in the logarithmic Sobolev inequalities.

THEOREM 4. *Equality holds in (4) exactly when g is one of the Gaussian functions*

$$g_a(x)=t^{-n/4}e^{-\pi|x-a|^2/2t}, \qquad a\in R^n.$$

Proof. Suppose g is a function saturating (4). Suppose first that $g \geqslant 0$; we will remove this restriction below. Define

$$f(x, y) = g\left(\frac{x+y}{\sqrt{2}}\right) g\left(\frac{x-y}{\sqrt{2}}\right)$$

on R^{2n}. Clearly, by rotation symmetry, f saturates the $2n$ dimensional version of (4). On the other hand, making successive applications of the n dimensional version of (4),

$$\begin{aligned} &\int \left(\int f^2(x, y) \log f^2(x, y)\, d^n x \right) d^n y \\ &\quad \leqslant \int \left(\frac{1}{\pi} \int |\nabla_x f(x, y)|^2\, d^n x - n \int f^2(x, y)\, d^n x \right) \\ &\qquad + \left(\int f^2(x, y)\, d^n x \right) \log \left(\int f^2(x, y)\, d^n x \right) d^n y. \end{aligned}$$

But by (4) again and then Theorem 2,

$$\begin{aligned} &\int \left(\int f^2(x, y)\, d^n x \right) \log \left(\int f^2(x, y)\, d^n x \right) \\ &\quad \leqslant \frac{1}{\pi} \int \left| \nabla \left(\int f^2(x, y)\, d^n x \right)^{1/2} \right|^2 d^n y - n \\ &\quad \leqslant \frac{1}{\pi} \int |\nabla_y f(x, y)|^2\, d^n x\, d^n y - n. \end{aligned}$$

Altogether we then have

$$\begin{aligned} &\iint f^2(x, y) \log f^2(x, y)\, d^n y \\ &\quad \leqslant \frac{1}{\pi} \iint (|\nabla_x f(x, y)|^2 + |\nabla_y f(x, y)|^2)\, d^n x\, d^n y - 2n \end{aligned}$$

and we have already observed that we have equality here. Thus all the inequalities above are actually equalities. In particular, since the inequality of Theorem 2 is saturated, we must have

$$f^2(x, y) = \left(\int f^2(x, y)\, d^n x \right) \left(\int f^2(x, y)\, d^n y \right)$$

almost everywhere; and so by Theorem 1, f and hence g must be Gaussian.

The inequality (4) is translation invariant, so nothing fixes the center, and a simple calculation determines the covariance to be that specified above.

Now consider a general saturating function g. Clearly $|g|$ is also a saturating function, and so it must be one of the Gaussians specified above. But then $|g|$ is strictly bounded below on every ball in $\mathbf{R}^n$. For such functions, $\int |\nabla g|^2 \, d^n x = \int |\nabla |g||^2 \, d^n x$ implies that $g = \alpha |g|$ for some $\alpha \in \mathbf{C}$.

Remark. An attempt to prove Theorem 4 has been published in Appendix B of [16]. These authors attempt to restrict the search for saturating functions to translates of radial decreasing functions by means of a strict rearrangement inequality for the Dirichlet integral. The statement they make about cases of equality in this rearrangement inequality is incorrect. Counterexamples are however known and are discussed by Brothers and Zeimer in [7] where optimal conditions for equality in $\int |\nabla f^*(x)|^2 \, d^n x \leqslant \int |\nabla f(x)|^2 \, d^n x$ are obtained; here f^* is the symmetric decreasing rearrangement of f. With some further argument though, it may be possible to use the deep result of [7] to patch the proof in [16]. We have not pursued this since we already possessed the above clean proof before encountering [16].

THEOREM 5. *Equality holds in* (1) *exactly when f is one of the exponential functions*

$$f_a(x) = e^{\pi t(a \cdot x - |a|^2/2)}, \qquad a \in R_n.$$

Proof. Suppose f saturates (1); then define $g(x) = e^{-\pi t |x|^2/2} f(x)$. It is well known that finiteness of $\int |\nabla f|^2 \, dm$ means that f is in the form domain of the operator $(1/\pi t)(-\Delta + 2\pi t x \cdot \nabla)$ on $L^2(R^n, dm)$. This is unitarily equivalent, under multiplication by $e^{-\pi t |x|^2/2}$, to the operator $((1/\pi t)(-\Delta + |\pi t x|^2) - n))$ on $L^2(R^n, d^n x)$; the harmonic oscillator Hamiltonian. Thus g is in the form domain of this operator; and in particular, $\int |x|^2 g^2(x) \, d^n x$ is finite. On this account, the cancellation of second moments in (3) is valid, and g saturates (4), and then by the previous theorem, $g(x) = t^{-n/2} e^{-\pi |x-a|^2/2t}$ for some $a \in R^n$.

We will now prove a stronger result: we will find a non-negative "remainder term" which can be added to the left side of (1), and which vanishes only on the exponential functions f_a.

Our starting point is the Beckner–Hirschman inequality [3, 14] concerning entropy and the Fourier transform. Let the Fourier transform $\mathscr{F}$ be given by

$$\mathscr{F} f(x) = \int e^{-2\pi i x \cdot y} f(y) \, d^n y \tag{12}$$

for integrable functions f; $\mathscr{F}$ is defined on $L^2(R^n, d^nx)$ in the usual way. If $\int |f(x)|^2 \, d^nx = 1$, then of course $|f(x)|^2$ and $|\mathscr{F}f(x)|^2$ are both probability densities. For any probability density ρ, we define its entropy $S(\rho)$ by

$$S(\rho) = -\int \rho(x) \log \rho(x) \, d^nx \tag{13}$$

provided at least one of $\rho \log \rho_-$ or $\rho \log \rho_+$ is integrable. Of course, this admits both $+\infty$ and $-\infty$ as possible values. If the integrability condition is not satisfied, the entropy is undefined. In this respect, the entropy is a little more delicate than the Fisher information which is always defined.

Let h be a normalised square integrable function. Then the Beckner–Hirschman inequality states that

$$S(|h|^2) + S(|\mathscr{F}h|^2) \geqslant n(1 - \log 2) \tag{14}$$

provided only that the left side is well defined. In particular, if both entropies are defined, and one is $-\infty$, the other must be $+\infty$.

Remark. Suppose h belongs to the form domain of the harmonic oscillator Hamiltonian; then $(1/2\pi) \int (|\nabla h(x)|^2 + |2\pi x h(x)|^2) \, d^nx < \infty$. Then both $|h|^2$ and $|\mathscr{F}h|^2$ have finite second moments. By a well known equality, their entropies are then bounded above by the entropies of the Gaussians of the same variances. The entropies are bounded below by (3). So for this dense set of functions, the Beckner–Hirschman inequality holds with all terms finite.

The Beckner–Hirschman inequality has the following history. Hirschman [14] proved that by differentiating the Hausdorf-Young inequality in p at $p = 2$ where it is an equality, one obtains (14) with the constant 0 on the right side. The differentiation is unusually delicate for reasons alluded to above. The sharp constants in the Hausdorf–Young inequality were not then known, but they were conjectured. Hirschman noted that with the conjectured sharp constants, one would obtain (14). When Beckner [3] proved the outstanding conjecture on the sharp constants in the Hausdorff–Young inequality, he cited Hirschman and noted that together the results in [3, 14] proved (14). Bialnicki-Birula and Mycielski gave this same proof in [15] citing Beckner for his sharp constants in the Hausdorf–Young inequality, but were unaware Beckner or Hirschman on (14).

Let $\mathscr{Q}$ denote the form domain of the harmonic oscillator Hamiltonian as in the remark above; this will arise often in the following considerations.

To state the next result, we need to recall a few facts concerning $\mathscr{F}$ and its Gauss measure relative, the Wiener transform $\mathscr{W}$ [21]. Suppose f has the expansion

$$h(x)=\sum_{\alpha} a_{\alpha} H_{\alpha}(x)\, e^{-\pi|x|^2},$$

where the H_α are the normalised Hermite polynomials generated by $dm_{1/2}$. It is well known that

$$\mathscr{F}h(x)=\sum_{\alpha} i^{|\alpha|} a_{\alpha} H_{\alpha}(x)\, e^{-\pi|x|^2}.$$

Let $U\colon L^2(R^n, d^nx)\mapsto L^2(R^n, dm_{1/2})$ be given by $Uh(x)=2^{-n/4}e^{\pi|x|^2}h(x)$. Clearly U is unitary. The Wiener transform on $L^2(R^n, dm_{1/2})$ is defined by

$$\mathscr{W}\left(\sum_{\alpha} a_{\alpha} H_{\alpha}\right)=\sum_{\alpha} i^{|\alpha|} a_{\alpha} H_{\alpha}. \tag{15}$$

Clearly $\mathscr{W}=U\mathscr{F}U^*$.

THEOREM 6. *For arbitrarily normalised* $f\in L^2(R^n, dm_{1/2})$

$$\int |f|^2 \log |f|^2\, dm_{1/2}+\int |\mathscr{W}f|^2 \log |\mathscr{W}f|^2\, dm_{1/2}\leqslant\frac{1}{2\pi}\int |\nabla f|^2\, dm_{1/2}. \tag{16}$$

Moreover when $f\geqslant 0$ *and* $\int |\mathscr{W}f|^2 \log|\mathscr{W}f|^2\, dm_2=0$, *then* f *is one of the functions*

$$f_a=e^{2\pi(a\cdot x-|a|^2/2)}, \qquad a\in R^n.$$

Proof. Choose any f for which the right side of (16) is finite. Define $g=U^*f$; then g belongs to $\mathscr{Q}$ by the remark above, and so the Beckner–Hirschman inequality applies with all terms finite. Simply computing in terms of the definitions, $S(|g|^2)=-\int |f|^2 (\log|f|^2-2\pi\,|x|^2+(n/2)\log 2)\, dm_{1/2}$. A similar computation for $S(|\mathscr{F}g|^2)$ yields

$$\begin{aligned} &S(|g|^2)+S(|\mathscr{F}g|^2)\\ &\quad=-\int |f|^2\log|f|^2\, dm_{1/2}-\int |\mathscr{W}f|^2\log|\mathscr{W}f|^2\, dm_{1/2}\\ &\qquad+2\pi\int |x|^2\,(|g(x)|^2+|\mathscr{F}g|^2)\, d^nx-n\log 2. \end{aligned} \tag{17}$$

But

$$2\pi \int |x|^2 (|g(x)|^2 + |\mathscr{F}g|^2)\, d^n x - n$$

$$= \frac{1}{2\pi} \int (|\nabla g(x)|^2 + |2\pi x g(x)|^2)\, d^n x - n$$

$$= \frac{1}{2\pi} \int |\nabla f|^2\, dm_{1/2}. \tag{18}$$

Combining (17) and (18) and the Beckner–Hirschman inequality yields (16) whenever the right side is finite.

Now suppose $f \geqslant 0$ and $\int |\mathscr{W}f|^2 \log |\mathscr{W}f|^2\, dm_{1/2} = 0$. Then $g = U^* f \geqslant 0$ also, and $|\mathscr{W}f(x)|^2 = 1$ a.e. which means $|\mathscr{F}g(x)|^2 = e^{n/2} e^{-2\pi |x|^2}$ a.e. But then $\mathscr{F}g(x)\, \mathscr{F}g(-x) = 2^{n/2} e^{-2\pi |x|^2}$ a.e., and since everything in sight is integrable, the Fourier inversion theorem yields

$$\int g(x)\, g(x+y)\, d^n y = 2^{-n/2} e^{-\pi |x|^2/2}. \tag{19}$$

Since $g \geqslant 0$, and the right side is integrable, g itself is integrable. We can now appeal to Cramér's theorem [10] which asserts that whenever the convolution of two non-negative integrable functions is Gaussian, each of the factors is Gaussian. Therefore g, and hence $\mathscr{F}g$ is Gaussian. Since $|U\mathscr{F}g| = 1$, $\mathscr{W}f(x) = e^{ia \cdot x}$, and this determines f to be of the asserted form.

Remarks. This result independently implies Theorem 5. To see this, consider any function saturating the $dm_{1/2}$ version of (2) with the right side finite. Then f must be real, and $|f|$ also saturates (2). By Theorem (6), $|f|$ must be an exponential function. Since such a function never vanishes, $f = |f|$ or $f = -|f|$.

A weaker relation—which does not provide the extra term on the left in (16)—between (14) and (4) has been found by Bialnicki-Birula and Mycielski [15].

Next, the proof shows that (16) is equivalent to the Beckner–Hirschman inequality restricted to functions in $\mathscr{Q}$. Elliott Lieb has shown me his proof that only Gaussian functions saturate (14) (they all do; the inequality is invariant under all manner of dilations, translations, etc.) provided certain formal cancellations can be justified a priori for saturating functions. The method is similar to that which he used to show that only Gaussians saturate Beckner's sharp form of the Haussdorf–Young–Titchmarsh inequality: subadditivity of the entropy replaces the use of Minkowski's inequality, but things are complicated by the fact that in general the entropy can take on both the values plus and minus infinity.

Using Theorem 2 and a remark above, one can control this problem, and show that the formal cancellations are legitimate whenever $f \in \mathcal{Q}$. But again, the right side of (16) is finite just when $U^*f \in \mathcal{Q}$. Thus the saturating functions for (16) are precisely the functions Ug, g any normalised Gaussian.

Finally, as a byproduct of our argument, we have proved that when $g \geqslant 0$ and $|\mathscr{F}g|$ is Gaussian, then g itself is Gaussian. We proved this for a particular Gaussian, but the conditions are invariant under translation and changes of scale. This result is an interesting complement to a result of Hardy [13] which implies the same if f satisfies a decay condition instead of our positivity condition.

We close this paper by showing how an interesting inequality due to Blachman and Stamm follows immediately from Theorem 3, and how it in turn implies both the inequality (1) and Theorem 5. We first establish some notation which will be convenient for this purpose.

Let X be any R^n valued random variable on some probability space $(\Omega, \mathscr{B}, Pr)$. If X has a density ρ_X, i.e., if $Pr\{X \in A\} = \int_A \rho_x(x)\, d^nx$ for all Borel sets A, we define $I(X) = I(\rho_X)$. Otherwise we put $I(X) = \infty$. In the same manner, we define the entropy $S(X)$ of X in terms of ρ_X. Also let E denote the expectation, let $M_n(X)$ denote the nth moment $E\,|X|^n$; and let $\operatorname{cov}(X)$ denote the covariance matrix of X, $\operatorname{cov}_{j,k}(X) = E(X_j X_k) - E(X_j)\, E(X_k)$.

THEOREM 7 (Blachman [5] and Stam [23]). *Let X and Y be independent random variables with values in R^n. Let $0 < a < 1$. Then*

$$I(\sqrt{a}\, X + \sqrt{1-a}\, Y) \leqslant aI(X) + (1-a)\, I(Y) \tag{20}$$

and equality holds just when X and Y have Gaussian densities with

$$\operatorname{cov}(X) = \operatorname{cov}(Y). \tag{21}$$

Proof. Define

$$\rho(x, y) = \rho_X(\sqrt{a}\, x - \sqrt{1-a}\, y)\, \rho_Y(\sqrt{1-a}\, x + \sqrt{a}\, y)$$

on R^{2n}. Then clearly the marginal $\int \rho(x, y)\, d^nx$ is the density of $\sqrt{a}\, X + \sqrt{1-a}\, Y$. Also clearly

$$\iint |\nabla_y \rho^{1/2}(x, y)|^2\, d^nx d^ny = aI(X) + (1-a)\, I(Y).$$

But by Theorem 2,

$$\iint |\nabla_y \rho^{1/2}(x, y)|^2\, d^nx d^ny \geqslant \int \left|\nabla_y \left(\int \rho(x, y)\, d^nx\right)^{1/2}\right|^2 d^ny$$

with equality implying that ρ is a product of densities in x and y. Thus by Theorem 1 (and the first remark following it) equality in (20) holds just when X and Y are Gaussian with the same covariance.

Remark. The proofs in [5] and [23] involve formal manipulations with distributional derivatives; this might be serious as far as the determination of cases of equality is concerned.

The present proof is easily extends to more that two random variables.

Finally, let us show how the Blachman–Stam inequality implies (1) and Theorem 5. We will only sketch the argumant, but the differentiations and limits can all be easily justified. In fact, our original proof of Theorem 4 proceeded by first proving Theorem 5 this way, and then reversing the argument in the proof that Theorem 4 implies Theorem 5.

Proof of Theorem 5 via Theorem 7. Let ρ be any probability density on R^n with $\rho^{1/2} \in \mathcal{Q}$. Then ρ has a finite second moment, and $I(\rho) < \infty$. Let X be any random variable with this density, and let Y be any independent random variable with the density $\rho_Y(x) = e^{-\pi |x|^2}$. For every $t > 0$ define

$$Z_t = e^{-t} X + (1 - e^{-2t})^{1/2}\, Y.$$

Then Z_t has a density ρ_t at each t, and ρ_t is evolved from ρ under the action of the adjoint Ornstein–Uhlenbeck semigroup. Therefore ρ_t satisfies

$$\frac{\partial}{\partial t}\rho_t = \frac{1}{2\pi} \nabla \cdot (\nabla + 2\pi x)\, \rho_t$$

which can of course be checked directly from the definition.

Next, the relative entropy of Z_t with respect to Y, $S(Z_t \mid Y)$ is defined by

$$S(Z_t \mid Y) = -\int (\rho_t/\rho_Y) \log(\rho_t/\rho_Y)\, \rho_Y d^n x.$$

It is easy to see [2] that $\lim_{t \to \infty} S(Z_t \mid Y) = 0$, and that

$$-S(X \mid Y) = \int_0^\infty \frac{d}{dt} S(Z_t \mid Y)\, dt.$$

But

$$\frac{d}{dt} S(Z_t \mid Y) = -\int \left(\frac{1}{2\pi} \nabla \cdot (\nabla + 2\pi x)\, \rho_t \right) \log \rho_t d^n x - \pi \frac{d}{dt} M_2(Z_t).$$

After integrations by parts on the first term on the right, it becomes

$$\frac{1}{2\pi} I(e^{-t} X + (1 - e^{-2t})^{1/2}\, Y) - n. \tag{22}$$

Now we can apply the last theorem and an explicit calculation of $I(Y)$ to dominate this by

$$\frac{1}{2\pi} e^{-2t} I(X) - e^{-2t} n. \tag{23}$$

Integration in t now yields

$$-S(X \mid Y) \leqslant \frac{1}{4\pi} I(X) + \pi E\,|X|^2 - n \tag{24}$$

which is equivalent to Gross's inequality as explained at the beginning of the paper.

Equality holds just when equality holds in the passage from (22) to (23) for each t. This means that X has a Gaussian density with the same covariance as Y, and so $\rho^{1/2}$ has the form claimed in Theorem 5.

Remark. Blachman [5] made a very similar application of Theorem 7, using the heat semigroup where we have used the Ornstein–Uhlenbeck semigroup, to prove Shannon's entropy power inequality. See also the paper of Barron [2].

This proof of Gross's inequality is closely related to a proof of Bakry and Emery [1]; however, they use a different method for obtaining the exponential decrease of $I(Z_t)$ which does not yield the cases of equality.

The inequality (24) can of course be rewritten as

$$-S(X) \leqslant \frac{1}{4\pi} I(X) - n. \tag{25}$$

Then since for all $a > 0$, $S(aX) = S(X) + n \ln a$ and $I(aX) = a^{-2} I(X)$, we can insert aX into (25) and optimize over a. This yields

$$e^{-2S(X)/n} \leqslant \frac{1}{2\pi n e} I(X). \tag{26}$$

Clearly the argument can be reversed so that (26) is equivalent to (25) and in turn to Gross's inequality—in finite dimension of course.

The quantity on the left in (26) is called Shannon's entropy power. Stam [23] proved (26) by displacing it as a differentiated form of Shannon's entropy power inequality which Stam has just proved. This proof yields no information on cases of equality.

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