Abduction in Description Logics using Formal Concept Analysis and Mathematical Morphology: Application to Image Interpretation

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Abstract. We propose an original way of enriching Description Logics with abduction reasoning services by computing the best explanations of an observation through mathematical morphology (using erosions) over the Concept Lattice of a background theory. The intended application is scene understanding and spatial reasoning.
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1 Introduction and notations

Scene interpretation can benefit from prior knowledge expressed as ontologies and from description logics (DL) endowed with spatial reasoning tools as illustrated in our previous work [5, 6]. The challenge in this work was to derive reasoning tools that are able to handle in a unified way quantitative information supplied by the image domain and qualitative pieces of knowledge supplied by the ontology level. Object recognition and interpretation are seen as the satisfiability of a current situation (spatial configuration) encoded in the ABox of the DL and its TBox part. However, when the expert knowledge is not crisply consistent with the observations, which is common in image interpretation, then this approach does not apply or leads to inconsistent results. Adapting DL reasoning tools to such situations can be performed using abduction. Our aim is thus to compute the "best explanation" to the observed phenomena in such situations. Formally, given a background theory $\mathcal{K}$ representing the expert knowledge and a formula C representing an observation on the problem domain, abductive reasoning searches for an explanation formula D such that D is satisfiable w.r.t. $\mathcal{K}$ and it holds that $\mathcal{K} \models D \rightarrow C$ ($\mathcal{K} \cup D \models C$). We propose to add abductive reasoning tools to DL by associating ingredients from mathematical morphology, DL and Formal Concept Analysis (FCA), and by computing the best explanations of an observation through algebraic erosion over the concept lattice of a background theory which is efficiently constructed using tools from FCA. We show that the defined operators satisfy important rationality postulates of abductive reasoning.

Based on the TBox $\mathcal{T}$ and the ABox $\mathcal{A}$ parts of a knowledge base $\mathcal{K}$, we consider ABox abduction [3]: if for every $a \in A$ it holds that $\mathcal{K} \not\models \neg a$, an ABox Abduction Problem, denoted as $\langle \mathcal{K}, A \rangle$, consists in finding a set of assertions γ such that $\mathcal{K} \cup \gamma \models A$.

The set γ (consistent with $\mathcal{K}$) is said to be an explanation of A. Explanatory reasoning is concerned with preferred explanations rather than just plain explanations. So, explaining an observation requires that some formulas must be "selected" as preferred explanations.

We also rely on classical notions of (FCA), and denote a formal context by $\mathbb{K} = (G, M, I)$, where G is the set of objects, M the set of attributes and $I \subseteq G \times M$ a relation between the objects and attributes. For $X \subseteq G$ and $Y \subseteq M$, the derivation operators are denoted by α and β, with $\alpha(X) = \{m \in M \mid \forall g \in X, (g, m) \in I\}$, and $\beta(Y) = \{g \in G \mid \forall m \in Y, (g, m) \in I\}$. The concept lattice is defined from the classical partial ordering $(X_1, Y_1) \leq (X_2, Y_2) \Leftrightarrow X_1 \subseteq X_2$ ($\Leftrightarrow Y_2 \subseteq Y_1$).

Links between FCA and DL can be formalized via the notion of semantic context $\mathbb{K}_{\mathcal{T}} := (G, M, I)$ defined as [1]: $G := \{(\mathcal{I}, d) \mid \mathcal{I}$ is a model of $\mathcal{T}$ and $d \in \Delta^{\mathcal{I}}\}$, $M := \{m_1, \ldots, m_n\}$, and $I := \{((\mathcal{I}, d), m) \mid d \in m^{\mathcal{I}}\}$, where $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ denotes an interpretation. The lattice can be constructed using the distributive concept exploration algorithm [9].

2 Abduction Operators from Mathematical Morphology on Complete Lattices

Let $(L, \preceq)$ and $(L', \preceq')$ be two complete lattices (which do not need to be equal). An operator $\delta : L \to L'$ is a dilation if it commutes with the supremum. An operator $\varepsilon : L' \to L$ is an erosion if it commutes with the infimum. Classical properties of mathematical morphology operators on complete lattices can be found in [4, 8].

Here, with the aim of performing ABox abduction, we would like to reason on subsets of G in order to find their best explanations (in G). Hence we consider the complete lattice $(\mathcal{P}(G), \subseteq)$ and operations from $\mathcal{P}(G)$ into $\mathcal{P}(G)$, where $\mathcal{P}(G)$ is the set of subsets of G. Since the ordering on G is equivalent to the one of M, reasoning on G will directly lead to results on M. In order to define explicit operations on $\mathcal{P}(G)$, we will make use of particular erosions and dilations, called morphological ones [8], which involve the notion of structuring element, i.e. a binary relation b between elements of G. For $g \in G$, we denote by $b(g)$ the set of elements of G in relation with g. It can be typically derived from a distance d: $b(g) = \{g' \in G \mid \exists X \in \mathcal{P}(G), g' \in X, d(\{g\}, X) \leq 1\}$. The morphological erosion of X is then expressed as $\varepsilon_b(X) = \{g \in G \mid b(g) \subseteq X\}$. Defining b from a distance is particularly interesting in the context of abduction, where the "most central" parts of models will have to be defined. Erosion is then expressed as $\varepsilon^n(X) = \{g \in G \mid d(g, X^C) > n\}$, where X^C denotes the complement of X in G. Here G is a discrete finite space, and therefore only integer values of n are considered. All classical properties of mathematical morphology hold in this framework.

Last Non-empty Erosion. As shown in [2] in the framework of propositional logic, erosions can be used to find explanations. In this context, the idea was to find the *most central part* of a formula as the best explanation. This approach was shown to have good properties with respect to rationality postulates of abductive reasoning [7]. In this paper, we propose similar ideas, but adapted to the context of concept lattices, using erosions as defined above. For any $X \subseteq G$, we define its last erosion as $\varepsilon_\ell(X) = \varepsilon^n(X) \Leftrightarrow$

$\varepsilon^n(X) \neq \emptyset,$ and $\forall m > n,\ \varepsilon^m(X) = \emptyset$. This last non-empty erosion defines the subset of models in G that are the furthest ones from the complement of X (according to the distance d), i.e. the most central in X.

Definition 1 *Let A be a set of ABox assertions. A preferred explanation γ of A is defined from the last non-empty erosion as $A \rhd^{\ell ne} \gamma \overset{def}{\Leftrightarrow} \gamma^{\mathcal{I}} \subseteq \varepsilon_\ell(A^{\mathcal{I}})$. In this equation, $A^{\mathcal{I}}$ should be understood as the extent of the semantic concept associated with the DL concept A. When a constraint (e.g. a set of hypotheses belonging to the backgroud theory) $\mathcal{H}$ has to be introduced, then this definition is modified as $A \rhd^{\ell ne} \gamma \overset{def}{\Leftrightarrow} \gamma^{\mathcal{I}} \subseteq \varepsilon_\ell(\mathcal{H}^{\mathcal{I}} \cap A^{\mathcal{I}})$.*

Starting from the subset to be explained, performing successive erosions amounts to "go down" in the lattice as much as possible, in order to find a non-empty set of interpretations.

Last Consistent Erosion. Another idea to introduce the constraint $\mathcal{H}$ is to erode it, as soon as it remains consistent with A. This leads to a second explanatory relation.

Definition 2 *A preferred explanation γ of A is defined from the last consistent erosion as: $A \rhd^{\ell c} \gamma \overset{def}{\Leftrightarrow} \gamma^{\mathcal{I}} \subseteq \varepsilon_{\ell c}(\mathcal{H}^{\mathcal{I}}, A^{\mathcal{I}}) \cap A^{\mathcal{I}}$, where $A^{\mathcal{I}}$ corresponds to the extent of the semantic context and $\varepsilon_{\ell c}$ is the last consistent erosion defined as $\varepsilon_{\ell c}(\mathcal{H}^{\mathcal{I}}, A^{\mathcal{I}}) = \varepsilon^n(\mathcal{H}^{\mathcal{I}})$ where $n = \max\{k \mid \varepsilon^k(\mathcal{H}^{\mathcal{I}}) \cap A^{\mathcal{I}} \neq \emptyset\}$.*

Here we consider erosion of $\mathcal{H}$ (i.e. $\mathcal{H}^{\mathcal{I}}$) alone, which means that we are looking at the subsets (submodels) of the models of A while being the most in the constraint.

Properties and interpretations. A first important property is that reasoning on G actually amounts to reason on the whole formal context. Here, explanations where defined from ABox reasoning, leading to erosions of subsets of G (models). Let (X, Y) be a formal concept, with $X \subseteq G$ and $Y \subseteq M$. From the definitions of explanations of X, we can derive directly the corresponding concepts for Y, using the derivation operator, i.e. $\alpha(\gamma) = \{m \in M \mid \forall g \in \gamma, (g, m) \in I\}$. Note that eroding X amounts to dilate Y, which is in accordance with the correspondence between the Galois connection property between derivation operators and the adjunction properties of dilation and erosion. Let us now consider the rationality postulates introduced in [7] for explanation relations. It has been proved that most of them hold for explanations derived from last non-empty erosion and from last consistent erosion [2]. These results extend to the DL context as follows:

- Both $\rhd^{\ell ne}$ and $\rhd^{\ell c}$ are independent of the syntax (since they are computed on models).
- Definitions are consistent in the sense that $\mathcal{K} \not\models \neg A$ iff $\exists \gamma, A \rhd \gamma$.
- A reflexivity property holds for both definitions: if $A \rhd \gamma$, then $\gamma \rhd \gamma$.
- Disjunctions of explanations: if $A \rhd \gamma$ and $A \rhd \delta$, then $A \rhd (\gamma \sqcup \delta)$, for both definitions. This means that if there are several possible explanations, their disjunction is an explanation as well, which is an expected result.

- Disjunction on the left: if $C \rhd^{\ell c} \gamma$ and $D \rhd^{\ell c} \gamma$, then $(C \sqcup D) \rhd^{\ell c} \gamma$ (since the erosion is always performed on $\mathcal{H}^{\mathcal{I}}$). However this property does not hold for $\rhd^{\ell ne}$ since erosion does not commute with the supremum.
- For the same reasons, we have the following property for $\rhd^{\ell c}$: if $C \rhd^{\ell c} \gamma$ and $D \rhd^{\ell c} \delta$, then $(C \sqcup D) \rhd^{\ell c} \gamma$ or $(C \sqcup D) \rhd^{\ell c} \delta$, but it does not hold for $\rhd^{\ell ne}$.
- For conjunctions, we have a monotony property for $\rhd^{\ell c}$: if $C \rhd^{\ell c} \gamma$ and $\gamma^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ (i.e. $D \models \gamma$), then $(C \sqcap D) \rhd^{\ell c} \gamma$. For $\rhd^{\ell ne}$, only a weaker form holds: if $C \rhd^{\ell ne} \gamma$ and $D \rhd^{\ell ne} \gamma$, then $(C \sqcap D) \rhd^{\ell ne} \gamma$. Note that this weaker form is also very natural and interesting.

Since both $\rhd^{\ell ne}$ and $\rhd^{\ell c}$ operators perform erosion in the interpretation set $\Delta^{\mathcal{I}}$, any solution belongs then to this set and $\mathcal{K}$ is a model of the obtained solution. Hence we have the following theorems:
- Soundness: If $\exists \gamma \mid A \rhd \gamma$ then $\mathcal{K} \models \gamma$.
- Completeness: $\mathcal{K} \models \gamma \Rightarrow \exists A \mid \mathcal{K} \models A : A \rhd \gamma$.

3 Conclusion

With the aim of image interpretation, we have proposed abductive inference services in DL based on mathematical morphology over concept lattices, whose construction is based on exploiting the advances of using FCA in DL. The properties and interpretations of the introduced explanatory operators were analyzed, and the rational postulates of abductive reasoning were stated and extended to our context. Future work will concern the complexity analysis of these operators and associated algorithms, and a deeper investigation of their applications to image interpretation.

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