An Extension of Regularity Conditions for Complex Role Inclusion Axioms

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1 Introduction

The description logic (DL) $\mathcal{SROIQ}$ [1] provides a logical foundation for the new version of the web ontology language OWL 2.[1] In comparison to the DL $\mathcal{SHOIN}$ which underpins the first version of OWL,[2] $\mathcal{SROIQ}$ provides several new constructors for classes and axioms. One of the new powerful features of $\mathcal{SROIQ}$ are so-called complex role inclusion axioms (RIAs) which allow for expressing implications between role chains and roles $R_1 \cdots R_n \sqsubseteq R$. It is well-known that unrestricted usage of such axioms can easily lead to undecidability for modal and description logics [2–5], with a notable exception of the DL $\mathcal{EL}^{++}$ [6]. Therefore certain syntactic restrictions are imposed on RIAs in $\mathcal{SROIQ}$ to regain decidability. Specifically, the restrictions ensure that RIAs $R_1 \cdots R_n \sqsubseteq R$ when viewed as production rules for context-free grammars $R \to R_1 \ldots R_n$ induce a regular language. In fact, the tableau-based reasoning procedure for $\mathcal{SROIQ}$ [1] does not use the syntactic restrictions directly, but takes as an input the resulting non-deterministic finite automata for RIAs. This means that it is possible to use exactly the same procedure for any set of RIAs for which the corresponding regular automata can be constructed.

Unfortunately, the syntactic restrictions on RIAs in $\mathcal{SROIQ}$ are not satisfied in all cases when the language induced by the RIAs is regular. In fact, it is not possible to come up with such a most general syntactic restriction since, given a context-free grammar, it is in general not possible to determine whether it defines a regular language (see, e.g., [7]). In this paper we analyse several common use cases of RIAs which correspond to regular languages but cannot be expressed within $\mathcal{SROIQ}$. We then propose an extension of the syntactic restrictions for RIAs, which can capture such cases. Our restrictions have several nice properties, which could allow for their seamless integration into future revisions of OWL:

1. Our restrictions are conservative over the restrictions in $\mathcal{SROIQ}$. That is, every set of RIAs that satisfies the restriction in $\mathcal{SROIQ}$ will automatically satisfy our restrictions.
2. Our restrictions can be verified in polynomial time in the size of RIAs.
3. Our restrictions are constructive, which means that there is a procedure that builds the corresponding regular automaton for every set of RIAs that satisfy our restrictions.

[1] `http://www.w3.org/TR/owl2-syntax/`
[2] `http://www.w3.org/TR/owl-ref/`

Name	Syntax	Semantics
Concepts		
atomic concept	A	$A^{\mathcal{I}}$ (given)
nominal	$\{a\}$	$\{a^{I}\}$
top concept	$\top$	$\Delta^{\mathcal{I}}$
negation	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
existential restriction	$\exists R.C$	$\{x \mid R^{\mathcal{I}}(x, C^{\mathcal{I}}) \neq \emptyset\}$
min cardinality	$\geqslant nS.C$	$\{x \mid \|S^{\mathcal{I}}(x, C^{\mathcal{I}})\| \geq n\}$
exists self	$\exists S.\mathsf{Self}$	$\{x \mid \langle x, x\rangle \in S^{\mathcal{I}}\}$
Axioms		
complex role inclusion	$\rho \sqsubseteq R$	$\rho^{\mathcal{I}} \subseteq R_2^{\mathcal{I}}$
disjoint roles	$\mathsf{Disj}(S_1, S_2)$	$S_1^{\mathcal{I}} \cap S_2^{\mathcal{I}} = \emptyset$
concept inclusion	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
concept assertion	$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
role assertion	$R(a, b)$	$\langle a, b\rangle \in R^{\mathcal{I}}$

Table 1. The syntax and semantics of $\mathcal{SROIQ}$

4. Finally, for any set of RIAs that induce a regular language, there exists a set of RIAs (containing possibly new roles) that satisfies our syntactic restrictions and preserves the semantics of the old roles. This means that unlike the restrictions in $\mathcal{SROIQ}$, our syntactic restrictions allow to express any regular complex role inclusion properties.

2 Preliminaries

In this section we introduce syntax and semantics of the DL $\mathcal{SROIQ}$ [1]. A $\mathcal{SROIQ}$ *vocabulary* consists of countably infinite sets N_C of *atomic concepts*, N_R of *atomic roles*, and N_I of *individuals*. A $\mathcal{SROIQ}$ *role* is either $r \in \mathsf{N}_R$, an *inverse role* r^- with $r \in \mathsf{N}_R$, or the *universal role* U. A *role chain* is a sequence of roles $\rho = R_1 \cdots R_n$, $n \geq 0$, where $R_i \neq U$, $1 \leq i \leq n$; when $n = 0$, ρ is called the *empty role chain* and is denoted by ϵ. With $\rho_1\rho_2$ we denote the *concatenation* of role chains ρ_1 and ρ_2, and with ρR ($R\rho$) we denote the role chain obtained by appending (prepending) R to ρ. We denote by $\mathsf{Inv}(R)$ the *inverse of a role* R defined by $\mathsf{Inv}(R) := r^-$ when $R = r$, $\mathsf{Inv}(R) = r$ when $R = r^-$, and $\mathsf{Inv}(R) = U$ when $R = U$. The *inverse of a role chain* $\rho = R_1 \cdots R_n$ is a role chain $\mathsf{Inv}(\rho) := \mathsf{Inv}(R_n) \cdots \mathsf{Inv}(R_1)$.

The syntax and semantics of $\mathcal{SROIQ}$ is summarised in Table 1. The set of $\mathcal{SROIQ}$ concepts is recursively defined using the constructors in the upper part of the table, where $A \in \mathsf{N}_C$, C, D are concepts, R, S roles, a an individual, and n a positive integer. A $\mathcal{SROIQ}$ *ontology* is a set $\mathcal{O}$ of axioms listed in the lower part of Table 1, where ρ is a role chain, $R_{(i)}$ and $S_{(i)}$ are roles, C, D concepts, and a, b individuals.

A *regular order on roles* is an irreflexive transitive binary relation $\prec$ on roles such that $R_1 \prec R_2$ iff $\mathsf{Inv}(R_1) \prec R_2$. A (complex) role inclusion axiom (RIA) $R_1 \cdots R_n \sqsubseteq R$ is said to be *$\prec$-regular*, if either: (*i*) $n = 2$ and $R_1 = R_2 = R$, or (*ii*) $n = 1$ and $R_1 = \mathsf{Inv}(R)$, or (*iii*) $R_i \prec R$ for $1 \leq i \leq n$, or (*iv*) $R_1 = R$ and $R_i \prec R$ for $1 < i \leq n$, or (*v*) $R_n = R$ and $R_i \prec R$ for $1 \leq i < n$. It is assumed that all RIAs in $\mathcal{O}$ are regular for some order $\prec$.

Given an ontology $\mathcal{O}$, let $\bar{\mathcal{O}}$ be the extension of $\mathcal{O}$ with RIAs $\mathsf{Inv}(\rho) \sqsubseteq \mathsf{Inv}(R)$ for every $\rho \sqsubseteq R \in \mathcal{O}$. Let $\rho \sqsubseteq_{\mathcal{O}} R$ be the smallest relation between role chains and roles such that: (*i*) $R \sqsubseteq_{\mathcal{O}} R$ for every role R, and (*ii*) $\rho \sqsubseteq_{\mathcal{O}} R$ and $\rho_1 R \rho_2 \sqsubseteq R' \in \bar{\mathcal{O}}$ implies $\rho_1 \rho \rho_2 \sqsubseteq_{\mathcal{O}} R'$.

Lemma 1. *Given an ontology $\mathcal{O}$, role chain ρ, and role R, it is possible to decide in polynomial time whether $\rho \sqsubseteq_{\mathcal{O}} R$.*

Proof. We define a context-free grammar with terminal symbols s_R and non-terminal symbols A_R for every role R, and production rules $A_R \to s_R$ for every role R and $A_R \to A_{R_1} \ldots A_{R_n}$ for every RIA $R_1 \cdots R_n \sqsubseteq R \in \bar{\mathcal{O}}$. It is easy to show that $A_R \to s_{R_1} \ldots s_{R_n}$ w.r.t. this grammar iff $R_1 \cdots R_n \sqsubseteq_{\mathcal{O}} R$. Since the word problem (membership in the language) for context-free grammars is decidable in polynomial time (see, e.g. [8]), then so is the property $\rho \sqsubseteq_{\mathcal{O}} R$. □

A role S is *simple* (w.r.t. $\mathcal{O}$) if $\rho \sqsubseteq_{\mathcal{O}} S$ implies $\rho = R$ for some role R. It is required that all roles $S_{(i)}$ in Table 1 are simple w.r.t. $\mathcal{O}$. Other constructors and axioms of $\mathcal{SROIQ}$ such as *disjunction*, *universal restriction*, *role (ir)reflexivity*, and *role (a)symetry* can be expressed using the given ones.

The semantics of $\mathcal{SROIQ}$ is defined using interpretations. An *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty set called *the domain of the interpretation* and $\cdot^{\mathcal{I}}$ is the *interpretation function*, which assigns to every $A \in \mathsf{N}_C$ a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, to every $r \in \mathsf{N}_R$ a relation $r^{\mathcal{I}} \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and to every $a \in \mathsf{N}_I$ an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. The interpretation is extended to roles by $U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and $(r^-)^{\mathcal{I}} = \{\langle x, y\rangle \mid \langle y, x\rangle \in r^{\mathcal{I}}\}$, and to role chains by $(R_1 \cdots R_n)^{\mathcal{I}} = R_1^{\mathcal{I}} \circ \cdots \circ R_n^{\mathcal{I}}$ where $\circ$ is the composition of binary relations. The empty role chain ϵ is interpreted by $\epsilon^{\mathcal{I}} = \{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\}$.

The interpretation of concepts is defined according to the right column of the upper part of Table 1, where $\delta(x, V)$ for $\delta \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $V \subseteq \Delta^{\mathcal{I}}$, and $x \in \Delta^{\mathcal{I}}$ denotes the set $\{y \mid \langle x, y\rangle \in \delta \wedge y \in V\}$, and $\|V\|$ denotes the cardinality of a set $V \subseteq \Delta^{\mathcal{I}}$. An interpretation $\mathcal{I}$ *satisfies an axiom* α (written $\mathcal{I} \models \alpha$) if the respective condition to the right of the axiom in Table 1 holds; $\mathcal{I}$ is a *model of an ontology* $\mathcal{O}$ (written $\mathcal{I} \models \mathcal{O}$) if $\mathcal{I}$ satisfies every axiom in $\mathcal{O}$. We say that α is a *(logical) consequence* of $\mathcal{O}$ or *is entailed* by $\mathcal{O}$ (written $\mathcal{O} \models \alpha$) if every model of $\mathcal{O}$ satisfies α.

3 Regularity of Role Inclusion Axioms

Given an ontology $\mathcal{O}$, for every role R, we define a language $L_{\mathcal{O}}(R)$ consisting of role chains (viewed as finite words over roles) as follows:

$$L_{\mathcal{O}}(R) := \{\rho \mid \rho \sqsubseteq_{\mathcal{O}} R\} \tag{1}$$

We say that the set of RIAs of $\mathcal{O}$ is *regular* if the language $L_\mathcal{O}(R)$ is regular for every role R. It has been shown in [1] that $\prec$-regularity for RIAs implies regularity. The converse of this property, however, does not always hold, as we demonstrate in the following example.

Example 1. Consider an ontology $\mathcal{O}$ consisting of the RIAs (2)–(4) below:

$$\mathsf{isProperPartOf} \sqsubseteq \mathsf{isPartOf} \tag{2}$$

$$\mathsf{isPartOf} \cdot \mathsf{isPartOf} \sqsubseteq \mathsf{isPartOf} \tag{3}$$

$$\mathsf{isPartOf} \cdot \mathsf{isProperPartOf} \sqsubseteq \mathsf{isProperPartOf} \tag{4}$$

RIAs (2)–(4) express properties of parthood relations isPartOf and isProperPartOf: axiom (2) says that isProperPartOf is a sub-relation of isPartOf; axiom (3) says that isPartOf is transitive; finally, axiom (4) says that if x is a part of y which is a proper part of z, then x is a proper part of z. Since any role chain consisting of isPartOf and isProperPartOf can be rewritten to isPartOf by applying axioms (2) and (3), it is easy to see that:

$$L_\mathcal{O}(\mathsf{isPartOf}) = (\mathsf{isPartOf} \mid \mathsf{isProperPartOf})^+ \tag{5}$$

Since isProperPartOf is implied only by axiom (4), it is easy to see that:

$$L_\mathcal{O}(\mathsf{isProperPartOf}) = (\mathsf{isPartOf} \mid \mathsf{isProperPartOf})^* \cdot \mathsf{isProperPartOf} \tag{6}$$

Thus, the languages $L_\mathcal{O}(\mathsf{isPartOf})$ and $L_\mathcal{O}(\mathsf{isProperPartOf})$ induced by RIAs (2)–(4) are regular. However, there is no ordering $\prec$ for which axioms (2)–(4) are $\prec$-regular. Indeed, by conditions (i)–(v) of $\prec$-regularity, it follows from (2) that $\mathsf{isProperPartOf} \prec \mathsf{isPartOf}$, and from (4) that $\mathsf{isPartOf} \prec \mathsf{isProperPartOf}$, which contradicts the fact that $\prec$ is a transitive irreflexive relation.

In fact, there is no $\mathcal{SROIQ}$ ontology $\mathcal{O}$ that could express properties (2)–(4) using $\prec$-regular RIAs. It is easy to show by induction over the definition of $\sqsubseteq_\mathcal{O}$ that if the RIAs of $\mathcal{O}$ are $\prec$-regular, then $R_1 \cdots R_n \sqsubseteq_\mathcal{O} R$ implies that for every i with $1 \leq i \leq n$, we have either $R_i = R$, or $R_i = \mathsf{Inv}(R)$, or $R_i \prec R$. This means that for every role R, the language $L_\mathcal{O}(R)$ can contain only words over R, $\mathsf{Inv}(R)$, or R' with $R' \prec R$. Clearly, this is not possible if $L_\mathcal{O}(\mathsf{isPartOf})$ and $L_\mathcal{O}(\mathsf{isProperPartOf})$ contain the languages defined in (5) and (6).

Axioms such as (2)–(4) in Example 1 naturally appear in ontologies describing parthood relations, such as those between anatomical parts of the human body. For example, release 7 of the GRAIL version of the Galen ontology[3] contains the following axioms that are analogous to (2)–(4):

$$\mathsf{isNonPartitivelyContainedIn} \sqsubseteq \mathsf{isContainedIn} \tag{7}$$

$$\mathsf{isContainedIn} \cdot \mathsf{isContainedIn} \sqsubseteq \mathsf{isContainedIn} \tag{8}$$

$$\mathsf{isNonPartitivelyContainedIn} \cdot \mathsf{isContainedIn} \sqsubseteq \mathsf{isNonPartitivelyContainedIn} \tag{9}$$

[3] http://www.opengalen.org/

Complex RIAs such as (7)–(9) are used in Galen to propagate properties over chains of various parthood relations. For example, the next axiom taken from Galen expresses that every instance of body structure contained in spinal canal is a structural component of nervous system:

$$\begin{aligned}\mathsf{BodyStructure} \sqcap \exists \mathsf{isContainedIn}.\mathsf{SpinalCanal}&\\ \sqsubseteq \exists \mathsf{isStructuralComponentOf}.\mathsf{NervousSystem}&\end{aligned} \tag{10}$$

Recently, complex RIAs over parthood relations have been proposed as an alternative to SEP-triplet encoding [9]. The SEP-triplet encoding was introduced [10] as a technique to enable the propagation of some properties over parthood relations, while ensuring that other properties are not propagated. For example, if a finger is defined as part of a hand, then an injury to a finger should be classified as an injury to the hand, however, the amputation of a finger should not be classified as an amputation of the hand. The SEP-triplet encoding does not use the parthood relations explicitly, but simulates them via inclusion axioms between special triplets of classes. Every primary class U gives rise to a triplet of classes consisting of the *structure class* U_S describing all parts of U including U itself, the *entity class* U_E that is equivalent to U, and the *part class* U_P describing the proper parts of U. Thus the axioms $\mathsf{Finger}_E \sqsubseteq \mathsf{Finger}_S$, $\mathsf{Finger}_P \sqsubseteq \mathsf{Finger}_S$, as well as $\mathsf{Hand}_E \sqsubseteq \mathsf{Hand}_S$, $\mathsf{Hand}_P \sqsubseteq \mathsf{Hand}_S$ describe the relations between the classes within the triples, and one can use the axiom $\mathsf{Finger}_S \sqsubseteq \mathsf{Hand}_P$ to express that finger is a proper part of hand. Several drawbacks of this encoding was mentioned and it had been argued that the explicit usage of the parthood relations can reduce the complexity of the ontology and at the same time eliminate the potential problems [9]. Thus, the axiom stating that finger is a proper part of hand can be written in this setting directly as follows:

$$\mathsf{Finger} \sqsubseteq \exists \mathsf{isProperPartOf}.\mathsf{Hand} \tag{11}$$

The explicit usage of parthood relation requires, however, complex RIAs such as (2)–(3), which do not satisfy $\prec$-regularity conditions of $\mathcal{SROIQ}$. This would not be a problem for ontologies expressible within the tractable DL $\mathcal{EL}^{++}$ [6] such as SNoMed CT,[4] since $\mathcal{EL}^{++}$ does not require regularity for RIAs. But it could be a problem when an expressivity beyond $\mathcal{EL}^{++}$ is required, such as for translating the Galen ontology into OWL 2. In this paper we propose an extension of regularity conditions, which, in particular, can handle axioms such as (2)–(3).

Another situation where $\prec$-regularity is too restrictive, is when "sibling relations" between elements having common parents are to be expressed. Such relations appear, for example, when speaking about parts that belong to the same vehicle. The sibling relations can be expressed using the following RIAs:

$$\mathsf{isChildOf} \cdot \mathsf{isChildOf}^- \sqsubseteq \mathsf{isSiblingOf} \tag{12}$$

$$\mathsf{isSiblingOf} \cdot \mathsf{isSiblingOf} \sqsubseteq \mathsf{isSiblingOf} \tag{13}$$

$$\mathsf{isSiblingOf} \cdot \mathsf{isChildOf} \sqsubseteq \mathsf{isChildOf} \tag{14}$$

[4] `http://www.ihtsdo.org/`

It can easily be shown that properties (12)–(14) could not be expressed using $\prec$-regular RIAs since this would require that $\mathsf{isChildOf} \prec \mathsf{isSiblingOf}$ and $\mathsf{isSiblingOf} \prec \mathsf{isChildOf}$. On the other hand, they induce regular languages:

$$L_{\mathcal{O}}(\mathsf{isSiblingOf}) = ((\mathsf{isChildOf} \cdot \mathsf{isChildOf}^{-}) \mid \mathsf{isSiblingOf})^{+} \tag{15}$$

$$L_{\mathcal{O}}(\mathsf{isChildOf}) = ((\mathsf{isChildOf} \cdot \mathsf{isChildOf}^{-}) \mid \mathsf{isSiblingOf})^{*} \cdot \mathsf{isChildOf} \tag{16}$$

In the next section, we demonstrate how our extended regularity conditions can capture such kind of axioms as well.

4 Stratified Role Inclusion Axioms

Definition 1. *A preorder (a transitive reflexive relation) $\precsim$ on roles is said to be* admissible *for an ontology $\mathcal{O}$ if: (i) $\rho_1 R \rho_2 \sqsubseteq R' \in \mathcal{O}$ implies $R \precsim R'$, and (ii) $R \precsim R'$ implies $\mathsf{Inv}(R) \precsim \mathsf{Inv}(R')$. We write $R_1 \approx R_2$ if $R_1 \precsim R_2$ and $R_2 \precsim R_1$, and $R_1 \prec R_2$ if $R_1 \precsim R_2$ and not $R_2 \precsim R_1$.*

Definition 2. *Let $\mathcal{O}$ be an ontology and $\precsim$ and admissible preorder for $\mathcal{O}$. We say that RIA $\rho \sqsubseteq R'$ is* stratified w.r.t. $\mathcal{O}$ and $\precsim$, *if for every $R \approx R'$ such that $\rho = \rho_1 R \rho_2$, there exist R_1 and R_2 such that $\rho_1 R \sqsubseteq_{\mathcal{O}} R_1$, $R_1 \rho_2 \sqsubseteq_{\mathcal{O}} R'$, $R \rho_2 \sqsubseteq_{\mathcal{O}} R_2$, and $\rho_1 R_2 \sqsubseteq_{\mathcal{O}} R'$. We say that $\mathcal{O}$ is* stratified w.r.t $\precsim$ *if every RIA $\rho \sqsubseteq R$ such that $\rho \sqsubseteq_{\mathcal{O}} R$ is stratified w.r.t. $\mathcal{O}$ and $\precsim$. We often omit "w.r.t. $\mathcal{O}$" and "w.r.t. $\precsim$" if $\mathcal{O}$ and $\precsim$ are clear from the context.*

Definition 3. *We say that a RIA $\rho_1 R \rho_2 \sqsubseteq R'$ is an* overlap *of two RIAs $\rho_1 R_1 \sqsubseteq R_1'$ and $R_2 \rho_2 \sqsubseteq R_2'$ (w.r.t. $\mathcal{O}$) if either (i) $R = R_1$, $R_1' \sqsubseteq_{\mathcal{O}} R_2$, and $R_2' = R'$, or (ii) $R = R_2$, $R_2' \sqsubseteq_{\mathcal{O}} R_1$, and $R_1' = R'$. In cases (i) and (ii) we also say that the RIAs $\rho_1 R_1 \sqsubseteq R_1'$ and $R_2 \rho_2 \sqsubseteq R_2'$* overlap *(w.r.t $\mathcal{O}$).*

In the next lemma recall that $\bar{\mathcal{O}}$ is the extension of $\mathcal{O}$ with RIAs $\mathsf{Inv}(\rho) \sqsubseteq \mathsf{Inv}(R)$ such that $\rho \sqsubseteq R \in \mathcal{O}$.

Lemma 2. *Let $\mathcal{O}$ be an ontology and $\precsim$ an admissible preorder for $\mathcal{O}$. Then $\mathcal{O}$ is stratified w.r.t. $\precsim$ if and only if:*

1. *Every RIA in $\mathcal{O}$ (and hence in $\bar{\mathcal{O}}$) is stratified w.r.t. $\mathcal{O}$ and $\precsim$;*
2. *Every overlap of two RIAs in $\bar{\mathcal{O}}$ is stratified w.r.t $\mathcal{O}$ and $\precsim$.*

Proof. The "only if" direction of the lemma follows immediately from the definition of a stratified ontology.

For proving the "if" direction of the lemma, without loss of generality we may assume that whenever $\rho_1 R \rho_2 \sqsubseteq R' \in \mathcal{O}$ with $R \approx R'$, then either ρ_1 or ρ_2 is empty. Indeed, by Condition 1 of the lemma, there exist R_1 such that $\rho_1 R \sqsubseteq_{\mathcal{O}} R_1$ and $R_1 \rho_2 \sqsubseteq_{\mathcal{O}} R'$. Hence by removing $\rho_1 R \rho_2 \sqsubseteq R'$ from $\mathcal{O}$ we preserve the relation $\sqsubseteq_{\mathcal{O}}$ and the conditions of the lemma.

We prove that $\rho' \sqsubseteq_{\mathcal{O}} R''$ implies that $\rho' \sqsubseteq R''$ is stratified w.r.t. $\mathcal{O}$ and $\precsim$. The proof is by two-fold induction: the outermost induction is over the length of ρ', the innermost induction is over the proof of $\rho' \sqsubseteq_{\mathcal{O}} R''$ according to the definition of $\sqsubseteq_{\mathcal{O}}$. Pick an arbitrary R in ρ' such that $R \approx R''$. The following cases match all possible ways of deriving $\rho' \sqsubseteq_{\mathcal{O}} R''$ using the definition of $\sqsubseteq_{\mathcal{O}}$:

– $\rho' = R'' = R$. In this case $R'' \sqsubseteq R''$ is trivially stratified.
– $\rho' = \rho'_1\rho_1\rho'_2R\rho'_3$ such that $\rho_1 \sqsubseteq_{\mathcal{O}} R_1$ and $\rho'_1R_1\rho'_2R\rho'_3 \sqsubseteq R'' \in \bar{\mathcal{O}}$.
In this case, by Condition 1 of the lemma, $\rho'_1R_1\rho'_2R\rho'_3 \sqsubseteq R''$ is stratified. Thus, there exist R'_1 and R'_2 such that $\rho'_1R_1\rho'_2R \sqsubseteq_{\mathcal{O}} R'_1$, $R'_1\rho'_3 \sqsubseteq_{\mathcal{O}} R''$, $R\rho'_3 \sqsubseteq_{\mathcal{O}} R'_2$, and $\rho'_1R_1\rho'_2R'_2 \sqsubseteq_{\mathcal{O}} R''$. Since $\rho_1 \sqsubseteq_{\mathcal{O}} R_1$, we have found R'_1 and R'_2 such that $\rho'_1\rho_1\rho'_2R \sqsubseteq_{\mathcal{O}} R'_1$, $R'_1\rho'_3 \sqsubseteq_{\mathcal{O}} R''$, $R\rho'_3 \sqsubseteq_{\mathcal{O}} R'_2$, and $\rho'_1\rho_1\rho'_2R'_2 \sqsubseteq R''$.
– $\rho' = \rho'_1R\rho'_2\rho_2\rho'_3$ such that $\rho_2 \sqsubseteq_{\mathcal{O}} R_2$ and $\rho'_1R\rho'_2R_2\rho'_3 \sqsubseteq_{\mathcal{O}} R''$. This case is analogous to the previous case.
– $\rho' = \rho'_1\rho_1R\rho_2$ such that $\rho_1R\rho_2 \sqsubseteq_{\mathcal{O}} R'$, $\rho'_1R' \sqsubseteq R'' \in \bar{\mathcal{O}}$, and ρ_1 is not empty. Since $\rho_1R\rho_2 \sqsubseteq_{\mathcal{O}} R'$ is a subproof of $\rho' \sqsubseteq R''$, by the induction hypothesis, $\rho_1R\rho_2 \sqsubseteq R'$ is stratified. Since $R \approx R' \approx R''$, there exist R_1 and R_2 such that $\rho_1R \sqsubseteq_{\mathcal{O}} R_1$, $R_1\rho_2 \sqsubseteq_{\mathcal{O}} R'$, $R\rho_2 \sqsubseteq_{\mathcal{O}} R_2$, and $\rho_1R_2 \sqsubseteq_{\mathcal{O}} R'$. In particular, since $R_1\rho_2 \sqsubseteq_{\mathcal{O}} R'$ and $\rho'_1R' \sqsubseteq R'' \in \bar{\mathcal{O}}$, we have $\rho'_1R_1\rho_2 \sqsubseteq_{\mathcal{O}} R''$. Since ρ_1 is not empty, $\rho'_1R_1\rho_2$ is shorter than $\rho' = \rho'_1\rho_1R\rho_2$. Hence, by the induction hypothesis, $\rho'_1R_1\rho_2 \sqsubseteq R''$ is stratified. Since $R \approx R'_1 \approx R''$, there exists R'_1 such that $\rho'_1R_1 \sqsubseteq_{\mathcal{O}} R'_1$ and $R'_1\rho_2 \sqsubseteq_{\mathcal{O}} R''$. Thus, we have found R'_1 and R_2 such that $\rho'_1\rho_1R \sqsubseteq_{\mathcal{O}} R'_1$, $R\rho_2 \sqsubseteq_{\mathcal{O}} R_2$, $R'_1\rho_2 \sqsubseteq_{\mathcal{O}} R''$, and $\rho'_1\rho_1R_2 \sqsubseteq_{\mathcal{O}} R''$.
– $\rho' = \rho_1R\rho_2\rho'_2$ such that $\rho_1R\rho_2 \sqsubseteq_{\mathcal{O}} R'$, $R'\rho'_2 \sqsubseteq R'' \in \bar{\mathcal{O}}$, and ρ'_2 is not empty. This case is analogous to the previous case.
– $\rho' = \rho'_1R\rho_2\rho'_2$ such that $R\rho_2 \sqsubseteq_{\mathcal{O}} R_2$, $R_2\rho'_2 \sqsubseteq R'_2 \in \bar{\mathcal{O}}$, ρ'_2 is not empty, $R'_2 \sqsubseteq_{\mathcal{O}} R'$, and $\rho'_1R' \sqsubseteq R'' \in \bar{\mathcal{O}}$.
In this case RIAs $\rho'_1R_2\rho'_2 \sqsubseteq R''$ is an overlap of two RIAs $R_2\rho'_2 \sqsubseteq R'_2$ and $\rho'_1R' \sqsubseteq R''$ in $\bar{\mathcal{O}}$. Hence, by Condition 2 of the lemma, $\rho'_1R_2\rho'_2 \sqsubseteq R''$ is stratified. Since $R \approx R_2 \approx R''$, there exists R_1 such that $\rho'_1R_2 \sqsubseteq_{\mathcal{O}} R_1$ and $R_1\rho'_2 \sqsubseteq_{\mathcal{O}} R''$. In particular, since $R\rho_2 \sqsubseteq_{\mathcal{O}} R_2$ and $\rho'_1R_2 \sqsubseteq_{\mathcal{O}} R_1$, we have $\rho'_1R\rho_2 \sqsubseteq_{\mathcal{O}} R_1$. Since ρ'_2 is not empty, $\rho'_1R\rho_2$ is shorter than $\rho' = \rho'_1R\rho_2\rho'_2$. Hence, by the induction hypothesis, $\rho'_1R\rho_2 \sqsubseteq_{\mathcal{O}} R_1$ is stratified. Since $R \approx R_1 \approx R''$, there exists R'_1 such that $\rho'_1R \sqsubseteq_{\mathcal{O}} R'_1$ and $R'_1\rho_2 \sqsubseteq_{\mathcal{O}} R_1$. Thus we have found R'_1 and R' such that $\rho'_1R \sqsubseteq_{\mathcal{O}} R'_1$, $R'_1\rho_2\rho'_2 \sqsubseteq_{\mathcal{O}} R''$, $R\rho_2\rho'_2 \sqsubseteq_{\mathcal{O}} R'$, and $\rho'_1R' \sqsubseteq_{\mathcal{O}} R''$.
– $\rho' = \rho'_1\rho_1R\rho'_2$ such that $\rho_1R \sqsubseteq_{\mathcal{O}} R_1$, $\rho'_1R_1 \sqsubseteq R'_1 \in \bar{\mathcal{O}}$, ρ'_1 is not empty, $R'_1 \sqsubseteq_{\mathcal{O}} R'$, and $R'\rho'_2 \sqsubseteq R'' \in \bar{\mathcal{O}}$. This case is analogous to the previous case.
– $\rho' = \rho'_1R$ or $\rho' = R\rho'_2$. This case is trivial. □

Lemma 3. *Given an ontology $\mathcal{O}$ it is possible to decide in polynomial time in the size of $\mathcal{O}$ whether $\mathcal{O}$ is stratified.*

Proof. By Lemma 2, it is sufficient to show that every RIA in $\mathcal{O}$ is stratified and every overlap of two RIAs is stratified. Hence there are only polynomially many RIAs to test. In order to test whether $\rho_1R\rho_2 \sqsubseteq R'$ is stratified for R, we enumerate all possible roles R'_1 and R'_2 and check whether $\rho_1R \sqsubseteq_{\mathcal{O}} R'_1$, $R'_1\rho_2 \sqsubseteq R'$, $R\rho_2 \sqsubseteq_{\mathcal{O}} R'_2$, and $\rho_1R'_2 \sqsubseteq_{\mathcal{O}} R'$. By Lemma 1, each of these conditions can be checked in polynomial time. □

Now, as we have an algorithm for deciding whether an ontology is stratified, it is easy to see that every ontology that satisfies the original $\prec$-regularity conditions for RIAs, is automatically stratified for the ordering $\precsim$ defined by $R_1 \precsim R_2$ if either $R_1 = R_2$, or $R_1 = \mathsf{Inv}(R_2)$, or $R_1 \prec R_2$. Indeed, the only overlap between $\prec$-regular RIAs can occur between axioms $\rho_1 R_1 \sqsubseteq R_1$ of type (v) and axioms $R_2\rho_2 \sqsubseteq R_2$ of type (iv) when $R_1 = R_2$ or $R_1 = \mathsf{Inv}(R_2)$, which can easily be shown to be stratified since $R_1 \sqsubseteq_{\mathcal{O}} R_2$ and $R_2 \sqsubseteq_{\mathcal{O}} R_1$ in these cases.

Our next goal is to show that the set of RIAs in stratified ontologies is always regular. Fix an ontology $\mathcal{O}$ and an admissible preorder $\precsim$ for $\mathcal{O}$ such that $\mathcal{O}$ is stratified w.r.t. $\mathcal{O}$. Define the *level* of a role R w.r.t. $\precsim$ as follows:

- If there is no R' such that $R' \prec R$, then the level of R is 0;
- Otherwise the level of R is the maximum over the levels of $R' \prec R$ plus 1.

The level of a RIA $\rho \sqsubseteq R$ is the level of R. We write $\rho \prec_{\mathcal{O}} R'$ if $R \prec_{\mathcal{O}} R'$ for every R in ρ. As in the proof of Lemma 2, without loss of generality, we can assume that for every RIA $\rho_1 R \rho_2 \sqsubseteq R' \in \mathcal{O}$ with $R \approx R'$, either ρ_1 or ρ_2 is empty. Hence there are four types of RIAs in $\bar{\mathcal{O}}$ possible:

- Type 0: $\rho \sqsubseteq R'$, where $\rho \prec R'$;
- Type 1: $R_1\rho \sqsubseteq R'$, where $R_1 \approx R'$ and $\rho \prec R'$;
- Type 2: $\rho R_2 \sqsubseteq R'$, where $R_2 \approx R'$.

Note that RIAs of the form $R \sqsubseteq R'$ with $R \approx R'$ are of both Types 1 and 2.

One nice property of stratified ontologies is that for proving $\rho \sqsubseteq_{\mathcal{O}} R$ one can apply the RIAs in $\bar{\mathcal{O}}$ in some particular order, namely: (i) apply RIAs in the non-decreasing order of their levels, (ii) for the same level, apply the RIAs in the non-decreasing order of their types. To formalize this strategy, we introduce a notion of *stratified proof*:

Definition 4. *Given an ontology $\mathcal{O}$, we define the relations $\sqsubseteq^i_{n,\mathcal{O}}$, $0 \leq i \leq 2$, $n \geq 0$ between role chains and roles by induction on $n \geq 0$ as follows:*

1. *If $\rho \sqsubseteq R' \in \bar{\mathcal{O}}$ has Type 0 and level n, then $\rho \sqsubseteq^0_{n,\mathcal{O}} R'$;*
2. *If role R has level n, then $R \sqsubseteq^i_{n,\mathcal{O}} R$, $i = 1, 2$;*
3. *If $\rho_1 \sqsubseteq^1_{n,\mathcal{O}} R_1$ and $R_1\rho \sqsubseteq R' \in \bar{\mathcal{O}}$ has Type 1 and level n, then $\rho_1\rho \sqsubseteq^1_{n,\mathcal{O}} R'$;*
4. *If $\rho_2 \sqsubseteq^2_{n,\mathcal{O}} R_2$ and $\rho R_2 \sqsubseteq R' \in \bar{\mathcal{O}}$ has Type 2 and level n, then $\rho\rho_2 \sqsubseteq^2_{n,\mathcal{O}} R'$;*
5. *If $\rho \sqsubseteq^i_{n,\mathcal{O}} R$ and $\rho_1 R \rho_2 \sqsubseteq^j_{m,\mathcal{O}} R'$ with $(n,i) <_{lex} (m,j)$ then $\rho_1\rho\rho_2 \sqsubseteq^j_{m,\mathcal{O}} R'$;*

where $(n,i) <_{lex} (m,j)$ if either $n < m$, or $n = m$ and $i < j$. We say that a RIA $\rho \sqsubseteq R'$ has a stratified proof *in $\mathcal{O}$ if $\rho \sqsubseteq^i_{n,\mathcal{O}} R'$, for some n and i $(0 \leq i \leq 2)$.*

Lemma 4. *For every stratified ontology $\mathcal{O}$, if $\rho' \sqsubseteq_{\mathcal{O}} R''$ then $\rho' \sqsubseteq R''$ has a stratified proof in $\mathcal{O}$.*

Proof (Sketch). We repeatedly apply the following transformation to the proof of $\rho' \sqsubseteq_{\mathcal{O}} R''$, which tries to swap the RIAs in $\bar{\mathcal{O}}$ applied in the wrong order:

For every overlap $\rho_1 R \rho_2 \sqsubseteq R'$ of RIAs $\rho_1 R \sqsubseteq R_1$ and $R_2\rho_2 \sqsubseteq R'$ of Types 2 and 1 in the proof of $\rho' \sqsubseteq_{\mathcal{O}} R''$, where $R_1 \sqsubseteq_{\mathcal{O}} R_2$, and ρ_1 and ρ_2 are non-empty, we take R_2' such that $R_1\rho_2 \sqsubseteq_{\mathcal{O}} R_2'$ and $\rho_1 R_2' \sqsubseteq_{\mathcal{O}} R'$, which exists since $\mathcal{O}$ is stratified, and replace the sub-proof of $\rho_1 R \rho_2 \sqsubseteq R'$ in $\rho' \sqsubseteq_{\mathcal{O}} R''$ with the proof using $R_1 \rho_2 \sqsubseteq_{\mathcal{O}} R'_2$ and $\rho_1 R'_2 \sqsubseteq_{\mathcal{O}} R'$;

This transformation always terminates. Indeed, after each transformation step, either the number of axioms $R_1 \cdots R_n \sqsubseteq R'$ with $n \geq 2$ used in the proof increases (when the proof of the overlap is replaced with a longer proof), or, otherwise, the number of such axioms remains the same, but the number of pairs $(\rho_1 R \sqsubseteq R_1, R_2 \rho_2 \sqsubseteq R')$ of RIAs respectively of Types 2 and 1 such that ρ_1 and ρ_2 are non-empty and $\rho_1 R \sqsubseteq R_1$ is used in the proof before $R_2 \rho_2 \sqsubseteq R'$, decreases.

After the transformation terminates, it is easy to see that the proof becomes stratified. □

Theorem 1. *For every stratified ontology $\mathcal{O}$ and role R one can construct an automaton for $L_{\mathcal{O}}(R)$ which is at most exponential in the level of R in $\mathcal{O}$.*

Proof (Sketch). By Lemma 4 for every $\rho \in L_{\mathcal{O}}(R)$, the RIA $\rho \sqsubseteq R$ has a stratified proof. It is easy to show that the language generated by RIAs of the same level n and the same type is regular. This is a consequence of the fact that the RIAs of Types 1 and 2 of the same level correspond to left-linear and right-linear grammars respectively, which generate regular languages. The RIAs $\rho \sqsubseteq R'$ of Type 0 correspond to finite languages since the level of the roles in ρ is smaller than the level of R', and therefore, only one step rewritings are possible. Now the fact that $L_{\mathcal{O}}(R)$ generated by RIAs of all levels is regular, follows from the fact that regular languages are closed under substitution. The size of the automaton for every level is at most polynomial in the size of $\mathcal{O}$; thus the size of the automaton for $L_{\mathcal{O}}(R)$ is at most exponential in the level of R. □

Returning to Example 1, we can show that the set of RIAs (2)–(4) is stratified. Indeed, it can be shown that RIAs (3)–(4) can result in the following overlaps:

$$\mathsf{isPartOf} \cdot \mathsf{isPartOf} \cdot \mathsf{isPartOf} \sqsubseteq \mathsf{isPartOf} \tag{17}$$

$$\mathsf{isPartOf} \cdot \mathsf{isProperPartOf} \cdot \mathsf{isPartOf} \sqsubseteq \mathsf{isPartOf} \tag{18}$$

$$\mathsf{isPartOf} \cdot \mathsf{isPartOf} \cdot \mathsf{isProperPartOf} \sqsubseteq \mathsf{isPartOf} \tag{19}$$

$$\mathsf{isPartOf} \cdot \mathsf{isPartOf} \cdot \mathsf{isProperPartOf} \sqsubseteq \mathsf{isProperPartOf} \tag{20}$$

All of the above RIAs can be stratified using (2)–(4).

On the other hand, RIAs (12)–(14) are not stratified. Indeed there are the following overlaps between RIAs (12) and (13) and between RIAs (12) and (14):

$$\mathsf{isChildOf} \cdot \mathsf{isChildOf}^- \cdot \mathsf{isSiblingOf} \sqsubseteq \mathsf{isSiblingOf} \tag{21}$$

$$\mathsf{isChildOf} \cdot \mathsf{isChildOf}^- \cdot \mathsf{isChildOf} \sqsubseteq \mathsf{isChildOf} \tag{22}$$

The problem with (21) is that there is no R such that $\mathsf{isChildOf}^- {\cdot} \mathsf{isSiblingOf} \sqsubseteq_{\mathcal{O}} R$. Intuitively, this property should hold for $R = \mathsf{isChildOf}^-$, but RIAs (12)–(14) are not sufficient to derive this property. Fortunately the problem can be fixed by declaring the role $\mathsf{isSiblingOf}$ to be symmetric:

$$\mathsf{isSiblingOf}^- \sqsubseteq \mathsf{isSiblingOf}, \tag{23}$$

which implies: $\mathsf{isChildOf}^- \cdot \mathsf{isSiblingOf} \sqsubseteq \mathsf{isChildOf}^- \cdot \mathsf{isSiblingOf}^- \sqsubseteq (\mathsf{isSiblingOf} \cdot \mathsf{isChildOf})^- \sqsubseteq \mathsf{isChildOf}^-$ using (14). Now (12) implies that (21) is stratified.

The problem with (22) is more involved, since $\mathsf{isChildOf}^- \cdot \mathsf{isChildOf} \sqsubseteq_{\mathcal{O}} R$ does not seem to hold for any role R introduced so far. Intuitively, by going to a child and then going back to a parent, we should come to the "partner"—an individual who has the same children as the initial individual. Hence we can introduce a fresh role $\mathsf{isPartnerOf}$, and add properties similar to those of $\mathsf{isSiblingOf}$:

$$\mathsf{isChildOf}^- \cdot \mathsf{isChildOf} \sqsubseteq \mathsf{isPartnerOf} \tag{24}$$

$$\mathsf{isPartnerOf} \cdot \mathsf{isPartnerOf} \sqsubseteq \mathsf{isPartnerOf} \tag{25}$$

$$\mathsf{isChildOf} \cdot \mathsf{isPartnerOf} \sqsubseteq \mathsf{isChildOf} \tag{26}$$

$$\mathsf{isPartnerOf}^- \sqsubseteq \mathsf{isPartnerOf} \tag{27}$$

It is now possible to show that RIAs (12)–(14) and (23)–(27) are stratified.

It is a natural question, whether any set of RIAs that induce regular languages, can be extended, as in the example above, to a set of RIAs that is stratified. The following theorem gives a positive answer to this question.

Theorem 2. *Let $\mathcal{O}$ be an ontology such that for every role R the language $L_{\mathcal{O}}(R)$ is regular. Then there exists a conservative extension $\mathcal{O}'$ of $\mathcal{O}$ which is stratified for every preorder $\precsim$ that is admissible for $\mathcal{O}$.*

Proof. For every role R and a role chain ρ we introduce a fresh role R^ρ. We use R^ϵ as a synonym for R. For every R_1, R_2, ρ_1, ρ_2, and $\rho \sqsubseteq_{\mathcal{O}} R$ we add axioms:

$$R_1^{R_2\rho_2} \cdot R_2^{\rho_1} \sqsubseteq R_1^{\rho_1\rho_2} \tag{28}$$

$$\epsilon \sqsubseteq R^\rho \tag{29}$$

Let $\mathcal{O}'$ be the extension of $\mathcal{O}$ with the above axioms. It is easy to show that $\mathcal{O}'$ is a conservative extension of $\mathcal{O}$: for every model $\mathcal{I}$ of $\mathcal{O}$ one can interpret $(R^\rho)^\mathcal{I} := \bigcup\{(\rho')^\mathcal{I} \mid \rho'\rho \sqsubseteq_{\mathcal{O}} R\}$ so that the axioms (28) and (29) are satisfied. Moreover, it can be shown that the resulting ontology is stratified. First, the original RIAs follow from (28) and (29): if $\rho \sqsubseteq_{\mathcal{O}} R$ for $\rho = R_1 R_2 \cdots R_n$, then $R_1^\epsilon \cdots R_n^\epsilon \sqsubseteq \epsilon R_1^\epsilon \cdots R_n^\epsilon \sqsubseteq R^\rho R_1^\epsilon \cdots R_n^\epsilon = R^{R_1 \cdots R_n} R_1^\epsilon \cdots R_n^\epsilon \sqsubseteq R^{R_2 \cdots R_n} R_2^\epsilon \cdots R_n^\epsilon \sqsubseteq \cdots \sqsubseteq R^\epsilon$. Hence by removing the original RIAs, the relation $\sqsubseteq_{\mathcal{O}'}$ does not change. The remaining axioms of the form (28) and (29) are stratified, since every overlap of axioms (28) $R_1^{R_3\rho_3} \cdot R_3^{R_2\rho_2} \cdot R_2^{\rho_1} \sqsubseteq R_1^{\rho_1\rho_2\rho_3}$ is provable by $R_1^{R_3\rho_3} \cdot R_3^{R_2\rho_2} \sqsubseteq R_1^{R_2\rho_2\rho_3}$, $R_1^{R_2\rho_2\rho_3} \cdot R_2^{\rho_1} \sqsubseteq R_1^{\rho_1\rho_2\rho_3}$, and by $R_3^{R_2\rho_2} \cdot R_2^{\rho_1} \sqsubseteq R_3^{\rho_1\rho_2}$, $R_1^{R_3\rho_3} \cdot R_3^{\rho_1\rho_2} \sqsubseteq R_1^{\rho_1\rho_2\rho_3}$.

The only problem with $\mathcal{O}'$ is that it requires infinitely many axioms (28) and (29), since the number of new roles R^ρ is not bounded. To bound the number of roles, we use the regularity property for languages $L_{\mathcal{O}}(R)$. Define $L_{\mathcal{O}}^\rho(R) := \{\rho' \mid \rho'\rho \in L_{\mathcal{O}}(R)\}$, and set $R_1^{\rho_1} \sim_{\mathcal{O}} R_2^{\rho_2}$ if and only if $L_{\mathcal{O}}^{\rho_1}(R_1) = L_{\mathcal{O}}^{\rho_2}(R_2)$. By Myhill-Nerode theorem (see, e.g., [7]), since $L_{\mathcal{O}}(R)$ is regular for each R, there are at most finitely many equivalence classes induced by $\sim_{\mathcal{O}}$. Since $\sim_{\mathcal{O}}$-equivalent roles have the same interpretations, we can identify those roles, and thus obtain only finitely many axioms of form (28) and (29). $\square$

5 Related Works and Conclusions

In this paper we have introduced a notion of *stratified role inclusion axioms* which provides a syntactically-checkable sufficient condition for regularity of RIAs—a condition that ensures decidability of $\mathcal{SROIQ}$. We have demonstrated that for every stratified $\mathcal{SROIQ}$ ontology, one can construct a regular automaton representing the RIAs, which is worst case exponential in the size of the ontology. The result in [11] then implies that the complexity of reasoning with stratified $\mathcal{SROIQ}$ ontologies remains the same as the complexity of original $\mathcal{SROIQ}$, namely N2ExpTime-complete. Moreover, we have demonstrated that our conditions for regularity are in a sense maximal—every ontology $\mathcal{O}$ with regular RIAs can be conservatively extended to an ontology with stratified RIAs.

Complex RIAs are closely related to *interaction axioms* in *grammar modal logics* $\Box_{i_1} \cdots \Box_{i_n} X \to \Box_{i_1} \cdots \Box_{i_n} X$ [2, 12, 3]. Such axioms often cause undecidability, however in [12, 3] a decidable class called the *regular grammar modal logics* has been described. In [4] a decision procedure for this class is given by a translation into the two-variable guarded fragment. Because it is in general undecidable if the given interaction axioms are regular, the decision procedure assumes that a regular automaton for them is also provided. When applying these techniques to ontologies and complex RIAs, such a restriction poses a serious practical problem: the users are unlikely to provide such automata, and even if they do, it is in general not possible to verify if the automaton really corresponds to the given axioms. A solution to this problem, proposed in [13, 1], is to use a sufficient syntactical condition for regularity called $\prec$-regularity. Another sufficient condition proposed in [14] requires *associativity* of RIAs, which means that if $R_1 R_2 \sqsubseteq R'_1$ and $R'_1 R_3 \sqsubseteq R'$ then there exists R'_2 such that $R_2 R_3 \sqsubseteq R'_2$ and $R_1 R'_2 \sqsubseteq R'$. It is easy to see that associativity is a partial case of our sufficient conditions, when $\precsim$ is a total relation on roles. Note that Theorem 2 holds for any preorder $\precsim$, and in particular, when $\precsim$ is total.

In this paper we have mainly addressed theoretical properties of ontologies with stratified RIAs, and have argued that they can be used to model properties which otherwise are not possible to model within $\mathcal{SROIQ}$. One problem that has not been addressed in this paper, is how to ensure that an ontology modeler produces stratified RIAs in practice. We made a small experiment to check how many of complex RIAs in release 7 of Galen are stratified. We found that the total of 385 complex role inclusions in Galen produce 3604 non-stratified overlaps, for which additional axioms are required to fix the problems. Clearly, the process of finding the missing axioms can be very time consuming. The conditions of stratified overlaps in such cases could provide valuable hits for finding the missing axioms. Methods for finding stratified axioms could be one of the topics for our future works.

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