Extending c-Representations and c-Inference for Reasoning with Infeasible Worlds

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Abstract
Inductive inference operators capture the process of completing a conditional belief base to an inference relation. One such operator is c-inference which is based on the c-representations of a belief base, c-representations being a special kind of ranking functions. c-Inference exhibits many desirable properties put forward for nonmonotonic reasoning; for instance, it fully complies with syntax splitting. A characterization of c-inference as a constraint satisfaction problem (CSP) yields a basis for implementing c-inference. However, the definitions of c-representations and of c-inference only take belief bases into account that satisfy a rather strong notion of consistency requiring every possible world to be at least somewhat plausible. In this paper, we extend the definition of c-representations to belief bases that need to satisfy only a weaker notion of consistency where some worlds may be completely infeasible. Based on these extended c-representations, we also extend the definition of c-inference correspondingly, thus covering all weakly consistent belief bases. Furthermore, we develop an adapted CSP characterizing the such extended c-inference that can be used as a basis for an implementation.

Keywords
c-inference, c-representations, inductive inference operator, infeasible worlds

1. Introduction

Ranking functions [1] are commonly used as models for conditional belief bases. The *c-representations* [2, 3] of a belief base Δ are a special kind of ranking functions modelling Δ. c-Representations define inductive inference operators that satisfy most advanced properties of nonmonotonic inference, particularly syntax splitting [4] and also conditional syntax splitting [5]. While initially introduced only for belief bases satisfying a rather strong notion of consistency, in this paper we define extended c-representations that also cover belief bases satisfying a weaker notion of consistency. In the such extended c-representations some possible worlds may be assigned a rank of ∞ indicating them to be completely infeasible according to Δ. This allows for realizing a kind of paraconsistent conditional reasoning based on the strong structural concept of c-representations.

The notion of c-inference was introduced in [6, 7] as nonmonotonic inference taking all c-representations into account. Therefore, the inductive inference operator c-inference inherits the restriction that it is only defined for strongly consistent belief bases. Using the extended c-representations we will introduce an extended version of c-inference that also covers weakly consistent belief bases.

The c-representations of a belief base Δ can be characterized by a constraint satisfaction problem (CSP), and in [6, 7] it is shown that c-inference can also be realized by a CSP. Here, we develop both a CSP that characterizes all extended c-representations and a simplified version of this CSP the solutions of which still cover all c-representations relevant for c-inference. Furthermore, we show how also extended c-inference can be realized by a CSP.

To summarize, the main contributions of this paper are:

- extension of c-representations for all weakly consistent belief bases;
- extension of c-inference to all weakly consistent belief bases;
- proof of some key properties of extended c-inference;
- construction of a CSP describing extended c-representations and then development of a simplified version of this CSP;
- development of a CSP realizing extended c-inference.

After recalling the background on conditional logic in Sec. 2 and inductive inference in Sec. 3 we present the different kinds of consistency in Sec. 4. We develop extended c-representations in Sec. 5 and extended c-inference in Sec. 6. Section 7 discusses the characterization and implementation of c-representations and c-inference by CSPs, before we conclude and point out future work in Sec. 8.

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2. Conditional Logic

A *(propositional) signature* is a finite set Σ of propositional variables. Assuming an underlying signature Σ, we denote the resulting propositional language by $\mathcal{L}_\Sigma$. Usually, we denote elements of signatures with lowercase letters $a, b, c, \ldots$ and formulas with uppercase letters $A, B, C, \ldots$. We may denote a conjunction $A \wedge B$ by AB and a negation $\neg A$ by $\overline{A}$ for brevity of notation. The set of interpretations over the underlying signature is denoted as Ω_Σ. Interpretations are also called *worlds* and Ω_Σ the *universe*. An interpretation $\omega \in \Omega_\Sigma$ is a *model* of a formula $A \in \mathcal{L}$ if A holds in ω, denoted as $\omega \models A$. The set of models of a formula (over a signature Σ) is denoted as $\mathit{Mod}_\Sigma(A) = \{\omega \in \Omega_\Sigma \mid \omega \models A\}$ or short as Ω_A. The Σ in Ω_Σ, $\mathcal{L}_\Sigma$ and $\mathit{Mod}_\Sigma(A)$ can be omitted if the signature is clear from the context or if the underlying signature is not relevant. A formula A *entails* a formula B, denoted by $A \models B$, if $\Omega_A \subseteq \Omega_B$. By slight abuse of notation we sometimes interpret worlds as the corresponding complete conjunction of all elements in the signature in either positive or negated form.

A *conditional* $(B|A)$ connects two formulas A, B and represents the rule "If A then usually B", where A is called the *antecedent* and B the *consequent* of the conditional. The conditional language is denoted as $(\mathcal{L}|\mathcal{L})_\Sigma = \{(B|A) \mid A, B \in \mathcal{L}_\Sigma\}$. A finite set of conditionals is called a *belief base*. We use a three-valued semantics of conditionals in this paper [8]. For a world ω a conditional $(B|A)$ is either *verified* by ω if $\omega \models AB$, *falsified* by ω if $\omega \models A\overline{B}$, or *not applicable* to ω if $\omega \models \overline{A}$. Popular models for belief bases are ranking functions (also called ordinal conditional functions, OCF) [1, 9] and total preorders (TPO) on Ω_Σ [10]. An OCF $\kappa : \Omega_\Sigma \to \mathbb{N}_0 \cup \{\infty\}$ maps worlds to a *rank* such that at least one world has rank 0, i.e., $\kappa^{-1}(0) \neq \emptyset$. The intuition is that worlds with lower ranks are more plausible than worlds with higher ranks; worlds with rank ∞ are considered infeasible. OCFs are lifted to formulas by mapping a formula A to the smallest rank of a model of A, or to ∞ if A has no models. An OCF κ is a model of a conditional $(B|A)$, denoted as $\kappa \models (B|A)$, if $\kappa(A) = \infty$ or if $\kappa(AB) < \kappa(A\overline{B})$; κ is a model of a belief base Δ, denoted as $\kappa \models \Delta$, if it is a model of every conditional in Δ.

Note that there are different definitions of consistency of a belief base in the literature. To distinguish two different notions of consistency that both occur in this paper we call one notion of consistency *strong consistency* and the other notion *weak consistency*, as suggested in [11].

Definition 1 ([11])**.** *A belief base Δ is called* strongly consistent *if there exists at least one ranking function κ with $\kappa \models \Delta$ and $\kappa^{-1}(\infty) = \emptyset$. A belief base Δ is* weakly consistent *if there is a ranking function κ with $\kappa \models \Delta$.*

Thus, Δ is strongly consistent if there is at least one ranking function modelling Δ that considers all worlds feasible. This notion of consistency is used in many approaches, e.g., [12]. The notion of weak consistency is equivalent to the more relaxed notion of consistency that is used in, e.g., [13, 14]. Trivially, strong consistency implies weak consistency.

3. Inductive Inference

The conditional beliefs of an agent are formally captured by a binary relation $\mid\!\sim$ on propositional formulas with $A \mid\!\sim B$ representing that A (defeasibly) entails B; this relation is called *inference* or *entailment relation*. Different sets of properties for inference relations have been suggested in literature, and often the set of postulates called *system P* is considered as minimal requirement for inference relations. Inference relations satisfying system P are called *preferential inference relations* [15, 16].

Every ranking function κ induces a preferential inference relation $\mid\!\sim_\kappa$ by

$$A \mid\!\sim_\kappa B \quad \text{iff} \quad \kappa(A) = \infty \text{ or } \kappa(AB) < \kappa(A\overline{B}). \quad (1)$$

Note that the condition $\kappa(A) = \infty$ in (1) ensures that system P's axiom (Reflexivity): $A \mid\!\sim_\kappa A$ is satisfied for $A \equiv \bot$.

Inductive inference is the process of completing a given belief base to an inference relation. To formally capture this we use the concept of inductive inference operators.

Definition 2 (inductive inference operator [4])**.** *An inductive inference operator is a mapping $C : \Delta \mapsto \mid\!\sim_\Delta$ that maps each belief base to an inference relation s.t. direct inference (DI) and trivial vacuity (TV) are fulfilled, i.e.,*

(DI) *if $(B|A) \in \Delta$ then $A \mid\!\sim_\Delta B$, and*

(TV) *if $\Delta = \emptyset$ and $A \mid\!\sim_\Delta B$ then $A \models B$.*

An inductive inference operator C is a *preferential inductive inference operator* if every inference relation $\mid\!\sim_\Delta$ in the image of C satisfies system P.

p-Entailment [15, 16] $C^p : \Delta \mapsto \mid\!\sim^p_\Delta$ is the most cautious preferential inductive inference operator. It is characterized by system P in the way that it only licenses inferences that can be obtained by iteratively applying the rules of system P to the belief base. Every other preferential inductive inference operator extends p-entailment. While extending p-entailment and adding some more inferences to the induced inference relations is usually desired, p-entailment can act as guidance for example for inferences of the form $A \mid\!\sim \bot$ which can be seen as representations of "strict" beliefs (i.e., A is completely unfeasible).

Postulate (Classic Preservation) (adapted from [14]). *An inductive inference operator $C : \Delta \mapsto {\mid\!\sim}_\Delta$ satisfies (Classic Preservation) if for all belief bases Δ and $A, B \in \mathcal{L}$ it holds that $A {\mid\!\sim}_\Delta \bot$ iff $A {\mid\!\sim}^p_\Delta \bot$.*

System Z is an inductive inference operator that is defined based on the Z-partition of a belief base [17]. Here we use an extended version of system Z that also covers weakly consistent belief bases and that was shown to be equivalent to *rational closure* [18] in [19].

Definition 3 ((extended) Z-partition)**.** *A conditional $(B|A)$ is* tolerated *by $\Delta = \{(B_i|A_i) \mid i = 1, \dots, n\}$ if there exists a world $\omega \in \Omega$ such that ω verifies $(B|A)$ and ω does not falsify any conditional in Δ, i.e., $\omega \models AB$ and $\omega \models \bigwedge_{i=1}^{n} (\overline{A_i} \vee B_i)$.*

The (extended) Z-partition *$EZP(\Delta) = (\Delta^0, \dots, \Delta^k, \Delta^\infty)$ of a belief base Δ is the ordered partition of Δ that is constructed by letting Δ^i be the inclusion maximal subset of $\bigcup_{j=i}^{n} \Delta^j$ that is tolerated by $\bigcup_{j=i}^{n} \Delta^j$ until $\Delta^{k+1} = \emptyset$. The set Δ^∞ is the remaining set of conditionals that contains no conditional which is tolerated by Δ^∞.*

Because the Δ^i are chosen inclusion-maximal, the Z-partition is unique [17].

Definition 4 ((extended) system Z)**.** *Let Δ be a belief base with $EZP(\Delta) = (\Delta^0, \dots, \Delta^k, \Delta^\infty)$. If Δ is not weakly consistent, then let $A {\mid\!\sim}^z_\Delta B$ for any $A, B \in \mathcal{L}$.*

Otherwise, the (extended) Z-ranking function κ^z_Δ is defined as follows: For $\omega \in \Omega$, if one of the conditionals in Δ^∞ is applicable to ω define $\kappa^z_\Delta(\omega) = \infty$. If not, let Δ^j be the last partition in $EZP(\Delta)$ that contains a conditional falsified by ω. Then let $\kappa^z_\Delta(\omega) = j + 1$. If ω does not falsify any conditional in Δ, then let $\kappa^z_\Delta(\omega) = 0$. (Extended) system Z *maps Δ to the inference relation ${\mid\!\sim}^z_\Delta$ induced by κ^z_Δ.*

For strongly consistent belief bases the extended system Z coincides with system Z as defined in [17, 12]. Note that for any belief base Δ the OCF κ^z_Δ is a model of Δ.

Lemma 1 ([11])**.** *For a weakly consistent belief base Δ and a formula A we have $\kappa^z_\Delta(A) = \infty$ iff $A {\mid\!\sim}^p_\Delta \bot$.*

Lemma 2 ([11])**.** *Let Δ be a belief base with $EZP(\Delta) = (\Delta^0, \dots, \Delta^k, \Delta^\infty)$. A world $\omega \in \Omega$ falsifies a conditional in Δ^∞ iff it is applicable for a conditional in Δ^∞.*

Proof. Direction ⇒: Assume that ω falsifies a conditional in Δ^∞. Then this conditional is applicable for ω.

Direction ⇐: Assume that ω is applicable for at least one conditional $(B|A) \in \Delta^\infty$. There are two possible cases: Either ω falsifies one of the other conditionals in Δ^∞ or not. In the first case the lemma holds. In the second case, towards a contradiction, we assume that ω does not falsify $(B|A)$. If ω is applicable and does not falsify $(B|A)$ then ω must verify $(B|A)$. That implies that $(B|A)$ is tolerated by Δ^∞ which contradicts the construction of $EZP(\Delta)$. □

4. Consistency of Belief Bases

Let us illustrate weak and strong consistency with an example.

Example 1. *Let $\Sigma = a, b, c, d$ be a signature. The belief bases $\Delta_1 = \{(\bot|\top)\}$ and $\Delta_2 = \{(\bot|a), (\overline{b}|\overline{a}), (b|\overline{a})\}$ are not weakly consistent and thus also not strongly consistent. The belief base $\Delta_3 = \{(\bot|a)\}$ is weakly consistent but not strongly consistent. The belief base $\Delta_4 = \{(b|a), (d|c)\}$ is strongly consistent and thus also weakly consistent.*

For every weakly consistent belief base Δ there is a world that does not falsify any conditional in Δ.

Lemma 3. *For every weakly consistent belief base Δ there is an $\omega \in \Omega$ s.t. ω does not falsify any conditional in Δ.*

Proof. Because Δ is weakly consistent, there is a ranking function κ with $\kappa \models \Delta$. Let $\omega \in \kappa^{-1}(0)$. Towards a contradiction, assume that there is a $(B|A) \in \Delta$ that is falsified by ω, i.e., $\omega \models A\overline{B}$. For κ to accept $(B|A)$ it must be either $\kappa(A) = \infty$ or $\kappa(AB) < \kappa(A\overline{B})$. Because $\omega \models A$ and $\kappa(\omega) = 0$ we have $\kappa(A) \neq \infty$. Because $\kappa(A\overline{B}) \leqslant 0$ and there are no ranks below 0 the condition $\kappa(AB) < \kappa(A\overline{B})$ does not hold. This is a contradiction; hence ω does not falsify any conditional in Δ. □

It is well-known that the construction of the extended Z-partition $EZP(\Delta)$ is successful with $\Delta^\infty = \emptyset$ iff Δ is strongly consistent. We can also use the extended Z-partition to check for weak consistency. The following proposition summarizes the relations between $EZP(\Delta)$ and the consistency of Δ.

Proposition 1. *Let $\Delta = \{(B_1|A_1), \dots, (B_n|A_n)\}$ be a belief base with $EZP(\Delta) = (\Delta^0, \dots, \Delta^k, \Delta^\infty)$.*

(1.) *Δ is not weakly consistent iff $\Delta^\infty = \Delta$ and $A_1 \vee \dots \vee A_n \equiv \top$.*

(2.) *Δ is weakly consistent iff $\Delta^\infty \neq \Delta$ or $A_1 \vee \dots \vee A_n \not\equiv \top$.*

(3.) *Δ is strongly consistent iff $\Delta^\infty = \emptyset$.*

Continuing Example 1, for the not weakly consistent Δ_2 we have $EZP(\Delta_2) = (\Delta_2^\infty)$ with $\Delta_2^\infty = \Delta$ and $a \vee \overline{a} \vee \overline{a} \equiv \top$. For the weakly consistent Δ_3 we have $EZP(\Delta_3) = (\Delta_3^\infty)$ with $\Delta_3^\infty = \Delta$ but $a \not\equiv \top$. For the strongly consistent Δ_3 we have $EZP(\Delta_4) = (\Delta_4^0)$ with $\Delta_4^0 = \Delta$ and $\Delta_4^\infty = \emptyset$.

5. Generalizing c-Representations

For strongly consistent belief bases, c-representations have been defined as follows.

Definition 5 (c-representation [2, 3]). *A c-representation of a belief base $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ over Σ is a ranking function $\kappa_{\vec{\eta}}$ constructed from integer impacts $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ with $\eta_i \in \mathbb{N}_0, i \in \{1, \ldots, n\}$ assigned to each conditional $(B_i|A_i)$ such that $\kappa_{\vec{\eta}}$ accepts Δ and is given by:*

$$\kappa_{\vec{\eta}}(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \eta_i. \tag{2}$$

We will denote the set of all c-representations of Δ by $Mod^{c}_{\Sigma}(\Delta)$.

A belief base Δ that is not strongly consistent will not have a c-representation: by Definition 5, a c-representation of Δ is a finite ranking function modelling Δ; if Δ is not strongly consistent, such a ranking function cannot exist.

To work with belief bases that are only weakly consistent, we need a more general definition of c-representations. A ranking function that is a model of a weakly but not strongly consistent belief base must assign rank ∞ to some worlds. To achieve this while keeping a construction of c-representations similar to the one given in (2), we extend the definition of c-representations to allow infinite impacts.

Definition 6 (extended c-representation). *An* extended c-representation *of a belief base $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ over Σ is a ranking function $\kappa_{\vec{\eta}}$ constructed from impacts $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ with $\eta_i \in \mathbb{N}_0 \cup \infty, i \in \{1, \ldots, n\}$ assigned to each conditional $(B_i|A_i)$ such that $\kappa_{\vec{\eta}}$ accepts Δ and is given by:*

$$\kappa_{\vec{\eta}}(\omega) = \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \eta_i \tag{3}$$

We will denote the set of all extended c-representations of Δ by $Mod^{ec}_{\Sigma}(\Delta)$.

Example 2. *Let $\Sigma = \{b, p, f\}$ and $\Delta = \{(b|p), (f|b), (\overline{b}|p)\}$. Note that Δ is weakly consistent but not strongly consistent. Then the ranking function $\kappa_{\vec{\eta}}$ displayed in Table 1 is an extended c-representation of Δ induced by the impacts $\vec{\eta} = (\infty, 1, \infty)$.*

Every c-representation of a strongly consistent belief base Δ is obviously an extended c-representation of Δ.

Proposition 2. *Let Δ be a strongly consistent belief base. Every c-representation $\kappa_{\vec{\eta}}$ of Δ is an extended c-representation of Δ.*

ω	$(b\|p)$	$(f\|b)$	$(\overline{b}\|p)$	impact on ω	$\kappa_{\vec{\eta}}(\omega)$
bpf	v	v	f	η_3	∞
$bp\overline{f}$	v	f	f	$\eta_2 + \eta_3$	∞
$b\overline{p}f$	–	v	–	0	0
$b\overline{p}\overline{f}$	–	f	–	η_2	1
$\overline{b}pf$	f	–	v	η_1	∞
$\overline{b}p\overline{f}$	f	–	v	η_1	∞
$\overline{b}\overline{p}f$	–	–	–	0	0
$\overline{b}\overline{p}\overline{f}$	–	–	–	0	0
impacts:	η_1	η_2	η_3		
$\vec{\eta}$	∞	1	∞		

Table 1
Verification (v) and falsification (f) of the conditionals in Δ from Example 2 and their corresponding impacts. The ranking function $\kappa_{\vec{\eta}}$ induced by the impacts $\vec{\eta} = (\eta_1, \eta_2, \eta_3) = (\infty, 1, \infty)$ is an extended c-representation for Δ.

Every weakly consistent belief base has at least one extended c-representation.

Proposition 3. *Let Δ be a weakly consistent belief base. Then $\kappa_{\vec{\eta}}$ with $\vec{\eta} = (\infty, \ldots, \infty)$ is an extended c-representation of Δ.*

Proof. Because Δ is weakly consistent, there is at least one world $\omega \in \Omega_{\Sigma}$ that does not falsify any of the conditionals (see Lemma 3). This implies $\kappa_{\vec{\eta}}(\omega) = 0$. Thus, $\kappa_{\vec{\eta}}$ is a ranking function.

For every $(B|A) \in \Delta$ it holds that $\kappa_{\vec{\eta}}(A\overline{B}) = \infty$ because every model of $A\overline{B}$ falsifies the conditional $(B|A)$ with impact ∞. For $\kappa_{\vec{\eta}}(AB)$ we have either (1.) $\kappa_{\vec{\eta}}(AB) = 0$ or (2.) $\kappa_{\vec{\eta}}(AB) = \infty$. In case (1.) we have $\kappa_{\vec{\eta}}(AB) = 0 < \infty = \kappa_{\vec{\eta}}(A\overline{B})$. In case (2.) we have $\kappa_{\vec{\eta}}(AB) = \infty$ and $\kappa_{\vec{\eta}}(A\overline{B}) = \infty$ and therefore $\kappa_{\vec{\eta}}(A) = \infty$ because $\kappa_{\vec{\eta}}(A) = \min\{\kappa_{\vec{\eta}}(AB), \kappa_{\vec{\eta}}(A\overline{B})\}$. In both cases $\kappa_{\vec{\eta}}$ accepts $(B|A)$. Thus, $\kappa_{\vec{\eta}} \models \Delta$. □

Proposition 3 also illustrates that in extended c-representations worlds may have rank infinity without the belief base requiring this. In an extended c-representation of Δ only those worlds need to have rank infinity that have rank infinity in the z-ranking κ^z_{Δ} of Δ.

Proposition 4. *Let Δ be a weakly consistent belief base. If $\kappa^z_{\Delta}(\omega) = \infty$ for a world ω, then $\kappa_{\vec{\eta}}(\omega) = \infty$ for all c-representations $\kappa_{\vec{\eta}}$ of Δ.*

Proof. Assume that $\kappa^z_{\Delta}(\omega) = \infty$. Let $EZP(\Delta) = \{\Delta^0, \ldots, \Delta^m, \Delta^\infty\}$ be the extended Z-partition of Δ. By definition of κ^z_{Δ} there exists a conditional $(B|A) \in \Delta^\infty$ s.t. $\omega \models A$. Because $(B|A) \in \Delta^\infty$ the conditional $(B|A)$ is not tolerated by Δ^∞, so there is a conditional

$(B'|A') \in \Delta^\infty$ that is falsified by ω (this can be $(B|A)$ again).

Towards a contradiction assume that there is a c-representation $\kappa_{\vec\eta}$ of Δ with $\kappa_{\vec\eta}(\omega) < \infty$. As $\kappa_{\vec\eta}$ models Δ and thus also $(B'|A')$ there must be a world ω^1 that verifies $(B'|A')$ and satisfies $\kappa_{\vec\eta}(\omega^1) < \kappa_{\vec\eta}(\omega)$. With the same argumentation there must be another conditional $(B^1|A^1) \in \Delta^\infty$ that is falsified by ω^1, and another world ω_2 that verifies $(B^1|A^1)$ and satisfies $\kappa_{\vec\eta}(\omega^2) < \kappa_{\vec\eta}(\omega^1)$. Repeating this argumentation we obtain an infinite chain of worlds $\omega_1, \omega_2, \ldots$ s.t. $\kappa_{\vec\eta}(\omega_1) > \kappa_{\vec\eta}(\omega_2) > \ldots$. But as there are only finitely many worlds (and also because there are only finitely many ranks below $\kappa_{\vec\eta}(\omega_1)$) such a chain cannot exist. Contradiction. □

Proposition 5. *Let Δ be a weakly consistent belief base. There is a c-representation $\kappa_{\vec\eta}$ of Δ with $\kappa_{\vec\eta}(\omega) < \infty$ for all worlds ω with $\kappa^z_\Delta(\omega) < \infty$.*

Proof. Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a weakly consistent belief base. Let $EZP(\Delta) = \{\Delta^0, \ldots, \Delta^m, \Delta^\infty\}$ be the extended Z-partition of Δ. Construct an impact vector $\vec\eta$ for Δ as follows. Let $\mu^0 = 1$ and $\mu^j = |\Delta^0 \cup \cdots \cup \Delta^{j-1}| \cdot \mu^{j-1} + 1$ for $j = 1, \ldots, m$. For $(B_i|A_i)$ with $(B_i|A_i) \in \Delta^j$ let $\eta_i = \mu^j$ for $j < \infty$ and $\eta_j = \infty$ for $j = \infty$. By construction, for worlds ω that do not falsify a conditional from $\Delta^j \cup \cdots \cup \Delta^m \cup \Delta^\infty$ we have $\kappa_{\vec\eta}(\omega) < \mu^j$.

$\kappa_{\vec\eta}$ is a c-representation of Δ: Let $(B_i|A_i)$ be any conditional in Δ. If $(B_i|A_i) \in \Delta^\infty$ then $\kappa^z_\Delta(A_i) = \infty$ by the definition of κ^z_Δ which implies with Proposition 4 that $\kappa_{\vec\eta}(A_i) = \infty$ and therefore $\kappa_{\vec\eta} \models (B_i|A_i)$. Otherwise, we have $(B_i|A_i) \in \Delta^j$ with $j < \infty$. Then for any world ω' falsifying $(B_i|A_i)$ we have $\kappa_{\vec\eta}(\omega) > \mu^j$; hence $\kappa_{\vec\eta}(A_i\overline{B_i}) \geqslant \mu_j$. Because $(B_i|A_i) \in \Delta^j$, there is a world ω' that verifies $(B_i|A_i)$ and does not falsify a conditional in $\Delta^j \cup \cdots \cup \Delta^m \cup \Delta^\infty$. Therefore, $\kappa_{\vec\eta}(A_iB_i) < \mu_j$. Thus, $\kappa_{\vec\eta}(A_iB_i) < \mu_j \leqslant \kappa_{\vec\eta}(A_i\overline{B_i})$ and $\kappa_{\vec\eta} \models (B_i|A_i)$.

Furthermore, it holds that $\kappa_{\vec\eta}(\omega) = \infty$ iff ω falsifies a conditional in Δ^∞. Therefore, $\kappa_{\vec\eta}(\omega) < \infty$ for all worlds ω with $\kappa^z_\Delta(\omega) < \infty$. □

Using Proposition 4 we can see that for all worlds ω the c-representation constructed in the proof of Proposition 5 satisfies that $\kappa_{\vec\eta}(\omega) < \infty$ iff $\kappa^z_\Delta(\omega) < \infty$. Using Lemma 1 we have $\kappa_{\vec\eta}(\omega) < \infty$ iff ω does not entail $\perp$ with p-entailment.

Lemma 4. *Let Δ be a weakly consistent belief base. There is an extended c-representation $\kappa_{\vec\eta}$ of Δ such that for all $\omega \in \Omega$ we have $\kappa_{\vec\eta}(\omega) < \infty$ iff $\omega \not\mid\!\sim^p_\Delta \perp$, where the world ω is considered as a formula on the right side of the "iff".*

Another consequence of Propositions 4 and 5 is the following.

Proposition 6. *Let Δ be a belief base with $EZP(\Delta) = \{\Delta^0, \ldots, \Delta^m, \Delta^\infty\}$, and let $\omega \in \Omega$. We have that $\kappa(\omega) = \infty$ for all $\kappa \in Mod^{ec}_\Delta$ iff $\omega \models A$ for some $(B|A) \in \Delta^\infty$.*

Proof. **Direction ⇒** If $\kappa(\omega) = \infty$ for all $\kappa \in Mod^{ec}_\Delta$ then there is no $\kappa_{\vec\eta} \in Mod^{ec}_\Delta$ with $\kappa_{\vec\eta}(\omega) < \infty$. With Proposition 5 this implies $\kappa^z_\Delta(\omega) = \infty$. By Definition 4 this is the case if a conditional in Δ^∞ is applicable for ω.

Direction ⇐ Assume $\omega \models A$ for some $(B|A) \in \Delta^\infty$. Then $\kappa^z_\Delta(\omega) = \infty$ and with Proposition 4 we have $\kappa(\omega) = \infty$ for all $\kappa \in Mod^{ec}_\Delta$. □

6. Extending c-Inference

c-Inference [6, 7] is an inference operator taking all c-representations of a belief base Δ into account. It was originally defined for strongly consistent belief bases.

Definition 7 (c-inference, $\mid\!\sim^c_\Delta$ [6]). *Let Δ be a strongly consistent belief base and let A, B be formulas. B is a* c-inference *from A in the context of Δ, denoted by $A \mid\!\sim^c_\Delta B$, iff $A \mid\!\sim_\kappa B$ holds for all c-representations κ of Δ.*

Now we use extended c-representations to extend c-inference for belief bases that may be only weakly consistent. Extended c-inference takes all extended c-representations of Δ into account.

Definition 8 (extended c-inference, $\mid\!\sim^{ec}_\Delta$). *Let Δ be a belief base and let $A, B \in \mathcal{L}$. Then B is an* extended c-inference *from A in the context of Δ, denoted by $A \mid\!\sim^{ec}_\Delta B$, iff $A \mid\!\sim_\kappa B$ holds for all extended c-representations κ of Δ.*

First, let us verify that extended c-inference is indeed a preferential inductive inference operator that coincides with c-inference for strongly consistent belief bases.

Proposition 7. *Extended c-inference is an inductive inference operator.*

Proof. We need to show that extended c-inference satisfies (DI) and (TV). (DI) is trivial: Every c-representation of Δ accepts the conditionals in Δ by definition. Therefore, $A \mid\!\sim^{ec}_\Delta B$ for every $(B|A) \in \Delta$. (TV) is also clear: For $\Delta = \emptyset$ the only c-representation is $\kappa = 0$. In this case κ accepts only conditionals $(B|A)$ with $A\overline{B} = \perp$, which are conditionals with $A \models B$. □

Proposition 8. *For strongly consistent belief bases, extended c-inference coincides with normal c-inference.*

Proof. Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a strongly consistent belief base and $C, D \in \mathcal{L}$. We need to show that $C \mid\!\sim^{ec}_\Delta D$ iff $C \mid\!\sim^c_\Delta D$.

Direction $\Rightarrow$: Let $C \mathrel{|\!\sim}^{ec}_{\Delta} D$, i.e., every extended c-representation models $(D|C)$. As every c-representation is an extended c-representation (Proposition 2), every c-representation models $(D|C)$. Thus, $D \mathrel{|\!\sim}^{c}_{kb} C$.

Direction $\Leftarrow$: Let $C \mathrel{|\!\sim}^{c}_{\Delta} D$, i.e., every c-representation models $(D|C)$. We need to show that any extended c-representation $\kappa_{\vec{\eta}}$ models $(D|C)$. If $\vec{\eta}$ contains only finite values it is a c-representation and thus models $(D|C)$ by assumption.

Assume that $\vec{\eta}$ contains infinite entries. Let $EZP(\Delta) = \{\Delta^0, \dots, \Delta^m, \Delta^\infty\}$ be the extended tolerance partition of Δ. Because Δ is strongly consistent, we have $\Delta^\infty = \emptyset$. Let $\mathit{fin}(\vec{\eta}) = \{\eta_i \mid i \in \{0, \dots, n\}, \eta_i < \infty\}$ be the set of finite values in impact vector $\vec{\eta}$ and $f_0 = 1 + |\mathit{fin}(\vec{\eta})| \cdot \max(\mathit{fin}(\vec{\eta}))$. Now construct $\vec{\eta}^f$ from $\vec{\eta}$ as follows. For $(B_i|A_i) \in \Delta^0$ with $\eta_i = \infty$ let $\eta_i^f = f_0$. Let $f_1 = (f_0 + 1) \cdot |\{(B_i|A_i) \in \Delta^0 \mid \eta_i = \infty\}|$. For $(B_i|A_i) \in \Delta^1$ with $\eta_i = \infty$ let $\eta_i^f = f_1$. Let $f_2 = (f_1 + 1) \cdot \left|\{(B_i|A_i) \in \Delta^1 \mid \eta_i = \infty\}\right|$. For $(B_i|A_i) \in \Delta^1$ with $\eta_i = \infty$ let $\eta_i^f = f_2$; and so on. By construction the sum of the impacts in $\mathit{fin}(\vec{\eta})$ is less than f_0 and the sum of the impacts of the conditionals in $\Delta^0 \cup \dots \cup \Delta^j$ is less than f_j for $j = 0, \dots, m$.

Let $\kappa^f = \kappa_{\vec{\eta}^f}$. Now verify that:

1. κ^f is a c-representation of Δ. For this we need to check that κ^f models all conditionals in Δ.
2. $\mathrel{|\!\sim}_{\kappa^f} \subseteq \mathrel{|\!\sim}_{\kappa_{\vec{\eta}}}$, i.e., every inference in $\mathrel{|\!\sim}_{\kappa^f}$ is also an inference in $\mathrel{|\!\sim}_{\kappa_{\vec{\eta}}}$.

From (1.) it follows that κ^f is a model of $(D|C)$, because $C \mathrel{|\!\sim}^{c}_{\Delta} D$. With (2.) it follows that $(D|C)$ is modelled by $\kappa_{\vec{\eta}}$.

Ad (1): Let $(B_i|A_i) \in \Delta$. We distinguish three cases.

<u>Case 1:</u> $\kappa_{\vec{\eta}}(A_iB_i) < \kappa_{\vec{\eta}}(A_i\overline{B_i}) < \infty$
In this case $\kappa^f(A_iB_i) < \kappa^f(A_i\overline{B_i}) < f_0$ and therefore $\kappa^f \models (B_i|A_i)$.

<u>Case 2:</u> $\kappa_{\vec{\eta}}(A_iB_i) < \infty$ and $\kappa_{\vec{\eta}}(A_i\overline{B_i}) = \infty$
In this case $\kappa^f(A_iB_i) < f_0 < \kappa^f(A_i\overline{B_i})$ and therefore $\kappa^f \models (B_i|A_i)$.

<u>Case 3:</u> $\kappa_{\vec{\eta}}(A_iB_i) = \infty$ and $\kappa_{\vec{\eta}}(A_i\overline{B_i}) = \infty$
Assume that $(B_i|A_i)$ is in Δ^j. Then there is a world ω s.t. $\omega \models A_iB_i$ and ω falsifies no conditional in $\Delta^0 \cup \dots \cup \Delta^j$. Therefore, $\kappa^f(\omega) < f_j$ and thus $\kappa^f(A_iB_i) < f_j$. Any model of $A_i\overline{B_i}$ falsifies $(B_i|A_i)$, therefore $\kappa^f(A_i\overline{B_i}) > f_j$. Thus, we have $\kappa^f(A_iB_i) < f_j < \kappa^f(A_i\overline{B_i})$ and therefore $\kappa^f \models (B_i|A_i)$.

Ad (2.): Assume that $X \mathrel{|\!\sim}_{\kappa^f} Y$. There are two cases.

<u>Case 1:</u> $\kappa^f(X\overline{Y}) < f_0$
In this case $\kappa^f(XY) < \kappa^f(X\overline{Y}) < f_0$ and therefore $\kappa_{\vec{\eta}}(XY) < \kappa_{\vec{\eta}}(X\overline{Y}) < \infty$. Hence, $X \mathrel{|\!\sim}_{\kappa_{\vec{\eta}}} Y$.

<u>Case 2:</u> $\kappa^f(X\overline{Y}) \geqslant f_0$
In this case $\kappa_{\vec{\eta}}(X\overline{Y}) = \infty$ and therefore $X \mathrel{|\!\sim}_{\kappa_{\vec{\eta}}} Y$. ☐

Let us continue by showing some further properties of extended c-inference.

Proposition 9. *Extended c-inference is preferential, i.e., it satisfies system P.*

Proof. Every ranking function, and thus every extended c-representation, induces a preferential inference relation. The intersection of two preferential inference relations is again preferential. As extended c-inference is the intersection of the inference relations induced by each extended c-representation, extended c-inference is preferential. ☐

Proposition 9 implies that extended c-inference captures p-entailment, i.e., if $A \mathrel{|\!\sim}^{p}_{\Delta} B$ then $A \mathrel{|\!\sim}^{ec}_{\Delta} B$. Furthermore, exteded c-inference coincides with p-entailment on entailments of the form $A \mathrel{|\!\sim} \bot$ which can be seen as representations of "strict" beliefs (i.e., A is completely unfeasible).

Proposition 10. *Extended c-inference satisfies (Classic Preservation).*

Proof. We need to show that $A \mathrel{|\!\sim}^{ec}_{\Delta} \bot$ iff $A \mathrel{|\!\sim}^{p}_{\Delta} \bot$. Using Lemma 1 it is sufficient to show that $A \mathrel{|\!\sim}^{ec}_{\Delta} \bot$ iff $\kappa^z(A) = \infty$.

Direction $\Leftarrow$: Let $\kappa^z_\Delta(A) = \infty$. Then Proposition 4 states that $\kappa_{\vec{\eta}}(A) = \infty$ for every c-representation $\kappa_{\vec{\eta}}(A)$ of Δ. Thus, $A \mathrel{|\!\sim}^{c}_{\Delta} \bot$.

Direction $\Rightarrow$: Let $A \mathrel{|\!\sim}^{c}_{\Delta} \bot$, i.e., there is no c-representation $\kappa_{\vec{\eta}}$ of Δ s.t. $\kappa_{\vec{\eta}}(A) < \infty$. By Proposition 5 we have $\kappa^z_\Delta(A) = \infty$. ☐

Extended c-inference does not satisfy *Rational Monotony (RM)* as c-inference already violates (RM).

7. CSPs for Extended c-Representations

In this section, we investigate constraint satisfaction problems (CSPs) dealing with extended c-representations. In Section 7.1, after presenting a constraint system describing all extended c-representations of a belief base, we develop a simplification of this constraint system that takes the effects of conditionals in Δ^∞ into account right from the beginning. In Section 7.2 we show how extended c-inference can be realized by a CSP.

7.1. Describing Extended c-Representations by CSPs

The c-representations of a belief base Δ can conveniently be characterized by the solutions of a constraint satisfaction problem. In [7], the following modelling of c-representations as solutions of a CSP is introduced. For a belief base $\Delta = \{(B_1|A_1), \dots, (B_n|A_n)\}$ over Σ the constraint satisfaction problem for c-representations of Δ, denoted by $CR_\Sigma(\Delta)$, on the constraint variables $\{\eta_1, \dots, \eta_n\}$ ranging over $\mathbb{N}_0$ is given by the constraints cr_i^Δ, for all $i \in \{1, \dots, n\}$:

$$(cr_i^\Delta)$$

$$\eta_i > \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_iB_i}} \sum_{\substack{j\neq i \\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_i\overline{B}_i}} \sum_{\substack{j\neq i \\ \omega\models A_j\overline{B_j}}} \eta_j.$$

The constraint cr_i^Δ is the constraint corresponding to the conditional $(B_i|A_i)$. The sum terms are induced by the worlds verifying and falsifying $(B_i|A_i)$, respectively. A solution of $CR_\Sigma(\Delta)$ is an n-tuple $(\eta_1, \dots, \eta_n) \in \mathbb{N}_0^n$. For a constraint satisfaction problem CSP, the set of solutions is denoted by $Sol(CSP)$. Thus, with $Sol(CR_\Sigma(\Delta))$ we denote the set of all solutions of $CR_\Sigma(\Delta)$. The solutions of $CR_\Sigma(\Delta)$ correspond to the c-representations of Δ.

Proposition 11 (soundness and completeness of $CR_\Sigma(\Delta)$ [7]). *Let $\Delta = \{(B_1|A_1), \dots, (B_n|A_n)\}$ be a belief base over Σ. Then we have:*

$$Mod^c_\Sigma(\Delta) = \{\kappa_{\vec\eta} \mid \vec\eta \in Sol(CR_\Sigma(\Delta))\} \tag{4}$$

If we want to construct a similar CSP for extended c-representations, we have to take worlds and formulas with infinite rank into account.

Definition 9 ($CR^{ex}_\Sigma(\Delta)$). *Let $\Delta = \{(B_1|A_1), \dots, (B_n|A_n)\}$ be a belief base over Σ. The constraint satisfaction problem for extended c-representations of Δ, denoted by $CR^{ex}_\Sigma(\Delta)$, on the constraint variables $\{\eta_1, \dots, \eta_n\}$ ranging over $\mathbb{N}_0 \cup \{\infty\}$ is given by the constraints $cr_i^{ex\,\Delta}$, for all $i \in \{1, \dots, n\}$:*

$$(cr_i^{ex\,\Delta}) \quad \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_i}} \sum_{\substack{1\leqslant j\leqslant n \\ \omega\models A_j\overline{B_j}}} \eta_j = \infty \qquad \textit{or}$$

$$\eta_i > \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_iB_i}} \sum_{\substack{j\neq i \\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_i\overline{B}_i}} \sum_{\substack{j\neq i \\ \omega\models A_j\overline{B_j}}} \eta_j$$

Again, each constraint $cr_i^{ex\,\Delta}$ corresponds to the conditional $(B_i|A_i) \in \Delta$.

Proposition 12 (soundness and completeness of $CR^{ex}_\Sigma(\Delta)$). *Let $\Delta = \{(B_1|A_1), \dots, (B_n|A_n)\}$ be a weakly consistent belief base over Σ. Then we have:*

$$Mod^{ec}_\Sigma(\Delta) = \{\kappa_{\vec\eta} \mid \vec\eta \in Sol(CR^{ex}_\Sigma(\Delta))\} \tag{5}$$

Proof. **Soundness:** Let $\vec\eta$ be an impact vector in $Sol(CR^{ex}_\Sigma(\Delta))$. Because Δ is weakly consistent, there is a world ω that does not falsify any conditional in Δ; therefore $\kappa_{\vec\eta}(\omega) = 0$ and $\kappa_{\vec\eta}$ is a ranking function. It is left to show that $\kappa_{\vec\eta}$ satisfies all conditionals in Δ.

Let $(B_i|A_i) \in \Delta$. There are three cases.

Case 1: $\kappa_{\vec\eta}(A_i\overline{B_i}) = \infty$ and $\kappa_{\vec\eta}(A_iB_i) = \infty$
In this case $\kappa_{\vec\eta}(A_i) = \infty$ and therefore $\kappa_{\vec\eta} \models (B_i|A_i)$.

Case 2: $\kappa_{\vec\eta}(A_i\overline{B_i}) = \infty$ and $\kappa_{\vec\eta}(A_iB_i) < \infty$
In this case $\kappa_{\vec\eta}(A_i\overline{B_i} > \kappa_{\vec\eta}(A_iB_i) < \infty$ and therefore $\kappa_{\vec\eta} \models (B_i|A_i)$.

Case 3: $\kappa_{\vec\eta}(A_i\overline{B_i}) < \infty$
In this case $\min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_i}} \sum_{\substack{1\leqslant j\leqslant n \\ \omega\models A_j\overline{B_j}}} \eta_j = \kappa_{\vec\eta}(A_i\overline{B_i}) < \infty$; hence the condition in $(cr_i^{ex\,\Delta})$ before the *or* is not satisfied.

Because $\vec\eta \in Sol(CR^{ex}_\Sigma(\Delta))$ it must satisfy all constraints in $CR^{ex}_\Sigma(\Delta)$ including $(cr_i^{ex\,\Delta})$. Because the condition before the *or* is violated, it must hold that

$$\eta_i > \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_iB_i}} \sum_{\substack{j\neq i \\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_i\overline{B}_i}} \sum_{\substack{j\neq i \\ \omega\models A_j\overline{B_j}}} \eta_j$$

$$\Leftrightarrow \quad \eta_i + \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_i\overline{B}_i}} \sum_{\substack{j\neq i \\ \omega\models A_j\overline{B_j}}} \eta_j > \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_iB_i}} \sum_{\substack{j\neq i \\ \omega\models A_j\overline{B_j}}} \eta_j$$

$$\Leftrightarrow \quad \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_i\overline{B}_i}} \sum_{\substack{1\leqslant j\leqslant n \\ \omega\models A_j\overline{B_j}}} \eta_j > \min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_iB_i}} \sum_{\substack{1\leqslant j\leqslant n \\ \omega\models A_j\overline{B_j}}} \eta_j$$

$$\Leftrightarrow \quad \kappa_{\vec\eta}(A_i\overline{B_i}) > \kappa_{\vec\eta}(A_iB_i)$$

and therefore $\kappa_{\vec\eta} \models (B_i|A_i)$.

Completeness: Let $\kappa_{\vec\eta}$ be an extended c-representation of Δ with impact vector $\vec\eta$. We need to show that $\vec\eta \in Sol(CR^{ex}_\Sigma(\Delta))$, i.e., that $\vec\eta$ satisfies every constraint $(cr_i^{ex\,\Delta})$ in $CR^{ex}_\Sigma(\Delta)$. Because $\kappa_{\vec\eta}$ is an extended c-representation of Δ, we have $\kappa_{\vec\eta} \models (B_i|A_i)$. This requires either (1.) $\kappa_{\vec\eta}(A_i) = \infty$ or (2.) $\kappa_{\vec\eta}(A_i\overline{B_i}) > \kappa_{\vec\eta}(A_iB_i)$. In case (1.) it is $\min_{\substack{\omega\in\Omega_\Sigma \\ \omega\models A_i}} \sum_{\substack{1\leqslant j\leqslant n \\ \omega\models A_j\overline{B_j}}} \eta_j = \kappa_{\vec\eta}(A_i\overline{B_i}) = \infty$ and the condition before the *or* in $(cr_i^{ex\,\Delta})$ is satisfied. In case (2.) we can see with the equivelence transformations in the *Soundness* part of this proof that the condition behind the *or* is satisfied. In both cases $\vec\eta$ satisfies $(cr_i^{ex\,\Delta})$. ☐

The requirement for weak consistency in Proposition 12 is necessary because for a belief base Δ that is not weakly consistent it holds that $CMod^{ec}_\Delta = \emptyset$ but $Sol(CR^{ex}_\Sigma(\Delta)) = (\infty, \dots, \infty)$. If we rule out the solution $(\infty, \dots, \infty)$ by adding a constraint, Proposition 12 also holds for not weakly consistent belief bases.

The resulting CSP $CR^{ex}_\Sigma(\Delta)$ is not a conjunction of inequalities any more, but it now contains disjunctions and is thus more complex. However, for the computation of extended c-inference we can construct a simplified CSP $CRS^{ex}_\Sigma(\Delta)$ that still yields all extended c-representations necessary for c-inference. This is possible, because from Propositions 4 and 5 we already know which worlds must have rank infinity and which worlds may have finite rank in the extended c-representations of Δ. The simplified CSP not only uses fewer constraint variables but also fewer constraints than $CR^{ex}_{\Sigma}(\Delta)$ for weakly but not strongly consistent belief bases.

Before stating $CRS^{ex}_{\Sigma}(\Delta)$, we show some proposition we will use for proving the correctness of $CRS^{ex}_{\Sigma}(\Delta)$.

We can assume the impacts of conditionals in Δ^{∞} to be infinity.

Proposition 13. *Let Δ be a weakly consistent belief base with extended Z-partition $EZP(\Delta) = \{\Delta^0, \ldots, \Delta^m, \Delta^{\infty}\}$. Let $\vec{\eta}$ be impacts such that $\kappa_{\vec{\eta}}$ is an extended c-representation of Δ. Let $\vec{\eta}'$ be the impact vector defined by $\eta'_i = \infty$ if $(B_i|A_i) \in \Delta^{\infty}$ and $\eta'_i = \eta_i$ otherwise. Then $\kappa_{\vec{\eta}} = \kappa_{\vec{\eta}'}$.*

Proof. Let ω be a world. There are two cases.

Case 1: There is a conditional $(B_i|A_i) \in \Delta^{\infty}$ that is falsified by ω. Then $\kappa^z_{\Delta}(\omega) = \infty$ and therefore $\kappa_{\vec{\eta}}(\omega) = \infty$ by Proposition 4. Because $\eta'_i = \infty$ we have $\kappa_{\vec{\eta}'}(\omega) = \sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j \overline{B}_j}} \eta'_j = \infty = \kappa_{\vec{\eta}}(\omega)$.

Case 2: There is no conditional in Δ^{∞} that is falsified by ω. Because $\eta_i = \eta'_i$ for all i with $\omega \models A_j \overline{B}_j$ we have $\kappa_{\vec{\eta}'}(\omega) = \sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j \overline{B}_j}} \eta'_j = \sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j \overline{B}_j}} \eta_j = \kappa_{\vec{\eta}}(\omega)$. □

For c-inference, it is sufficient to take only a subset of all c-representations of a belief base into account.

Definition 10. *Let Δ be a belief base. Then $CMod^{ec}_{\Delta}$ is the set of c-representations $\kappa_{\vec{\eta}}$ of Δ with $\kappa_{\vec{\eta}}(\omega) < \infty$ for all worlds ω with $\kappa^z_{\Delta}(\omega) < \infty$.*

Proposition 14. *Let Δ be a belief base. Then $A \mid\!\sim_{\kappa} B$ holds for all c-representations κ in $CMod^{ec}_{\Delta}$ iff $A \mid\!\sim_{\kappa} B$ holds for all c-representations κ in Mod^{ec}_{Δ}.*

Proof. Direction $\Leftarrow$: Observe that $CMod^{ec}_{\Delta} \subseteq Mod^{ec}_{\Delta}$. Therefore, if $A \mid\!\sim_{\kappa} B$ holds for all c-representations κ in Mod^{ec}_{Δ}, then $A \mid\!\sim_{\kappa} B$ holds for all c-representations κ in $CMod^{ec}_{\Delta}$.

Direction $\Rightarrow$: Show this by contraposition. Assume that $\kappa \in CMod^{ec}_{\Delta}$ with $A \not\mid\!\sim_{\kappa} B$. Using the construction of κ^f in the proof of Proposition 8 we can find a $\kappa' = \kappa^f$ that is a c-inference of Δ and satisfies $\mid\!\sim_{\kappa'} \subseteq \mid\!\sim_{\kappa}$. Therefore, if $A \not\mid\!\sim_{\kappa} B$ then $A \not\mid\!\sim_{\kappa'} B$. Hence, there is a c-representation κ' with $A \not\mid\!\sim_{\kappa'} B$. □

As already indicated above, the c-representations in $CMod^{ec}_{\Delta}$ can then be represented by a simplified CSP.

Definition 11 ($CRS^{ex}_{\Sigma}(\Delta)$)**.** *Let $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ be a belief base over Σ with the extended tolerance partition $EZP(\Delta) = \{\Delta^0, \ldots, \Delta^m, \Delta^{\infty}\}$. Let*

$$J_{\Delta} = \{j \mid (B_j|A_j) \in \Delta \setminus \Delta^{\infty} \text{ s.t. } A_j\overline{B_j} \wedge \Big(\bigwedge_{(D|C) \in \Delta^{\infty}} (\overline{C} \vee D) \Big) \not\equiv \bot \}.$$

The simplified constraint satisfaction problem for extended c-inference of Δ, denoted by $CRS^{ex}_{\Sigma}(\Delta)$, on the constraint variables $\{\eta_{j_1}, \ldots, \eta_{j_l}\}$, $j_k \in J_{\Delta}$ ranging over $\mathbb{N}_0$ is given by the constraints $crs^{ex\,\Delta}_j$, for all $j \in J_{\Delta}$:

$$(crs^{ex\,\Delta}_j)$$

$$\eta_i > \min_{\substack{\omega \in \Omega_{\Sigma} \\ \omega \models A_i B_i}} \sum_{\substack{j \in J_{\Delta} \\ j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j - \min_{\substack{\omega \in \Omega_{\Sigma} \\ \omega \models A_i \overline{B}_i}} \sum_{\substack{j \in J_{\Delta} \\ j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j .$$

The condition $A_j\overline{B_j} \wedge \big(\bigwedge_{(D|C) \in \Delta^{\infty}} (\overline{C} \vee D) \big) \not\equiv \bot$ in the definition of J_{Δ} is equivalent to there being a world $\omega \in \Omega_{A_j\overline{B_j}}$ that does not falsify a conditional in Δ^{∞}.

Definition 12. *Let Δ be a belief base, $n = |\Delta|$, and let J_{Δ} be defined as above. For every $\vec{\eta}^J \in Sol(CRS^{ex}_{\Sigma}(\Delta))$ let $\vec{\eta}^{J+\infty} \in (\mathbb{N}_0 \cup \{\infty\})^n$ be the impact vector with*

$$\eta^{J+\infty}_i = \begin{cases} \eta_i & \text{for } i \in J_{\Delta} \\ \infty & \text{otherwise.} \end{cases}$$

Then $Sol^{J+\infty}_{\Delta} := \{\vec{\eta}^{J+\infty} \mid \vec{\eta}^J \in Sol(CRS^{ex}_{\Sigma}(\Delta))\}$.

Proposition 15 (soundness and completeness of $CRS^{ex}_{\Sigma}(\Delta)$)**.** *Let Δ be a weakly consistent belief base over Σ. Then*

$$CMod^{ec}_{\Sigma}(\Delta) = \{\kappa_{\vec{\eta}} \mid \vec{\eta} \in Sol^{J+\infty}_{\Delta}\}. \tag{6}$$

Proof. Let $EZP(\Delta) = (\Delta^0, \ldots, \Delta^k, \Delta^{\infty})$, and let J_{Δ} be defined as in Definition 11.

Soundness: Let $\vec{\eta} \in Sol^{J+\infty}_{\Delta}$. By definition, there is a vector $\vec{\eta}^J \in Sol(CRS^{ex}_{\Sigma}(\Delta))$ such that $\vec{\eta} = \vec{\eta}^{J+\infty}$. Because $\eta_i = \infty$ for every $(B_i|A_i) \in \Delta^{\infty}$ and due to Lemma 2, all worlds ω for which one of the conditionals in Δ^{∞} is applicable have rank $\kappa_{\vec{\eta}}(\omega) = \infty$. Therefore, all conditionals in Δ^{∞} are accepted by $\kappa_{\vec{\eta}}$.

For any conditional $(B_i|A_i) \in \Delta \setminus \Delta^{\infty}$ there is at least one world ω that verifies $(B_i|A_i)$ without falsifying a conditional in Δ^{∞} (otherwise $(B_i|A_i)$ would not be tolerated by Δ^{∞}). Because every world that falsifies a conditional $(B_j|A_j)$ with $j \notin J_{\Delta}$ also falsifies a conditional in Δ^{∞}, the world ω does not falsify any such conditional $(B_j|A_j)$ with impact ∞. Therefore, $\kappa_{\vec{\eta}}(A_iB_i) < \infty$. If $\kappa_{\vec{\eta}}(A_i\overline{B_i}) = \infty$ then $\kappa_{\vec{\eta}} \models (B_i|A_i)$. Otherwise, for $\kappa_{\vec{\eta}}(A_i\overline{B_i}) < \infty$, there is a world that falsifies $(B|A)$ without falsifying a conditional in Δ^{∞}. In this case it is $i \in J_{\Delta}$ and the CSP $CRS^{ex}_{\Sigma}(\Delta)$ contains the constraint $(crs^{ex\,\Delta}_j)$ which must hold for $\vec{\eta}^J$:

$$\eta_i > \min_{\substack{\omega \in \Omega_{\Sigma} \\ \omega \models A_i B_i}} \sum_{\substack{j \in J_{\Delta} \\ j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j - \min_{\substack{\omega \in \Omega_{\Sigma} \\ \omega \models A_i \overline{B}_i}} \sum_{\substack{j \in J_{\Delta} \\ j \neq i \\ \omega \models A_j \overline{B_j}}} \eta_j$$

$$\Leftrightarrow \quad \eta_i + \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i\overline{B}_i}} \sum_{\substack{j\in J_\Delta\\ j\neq i\\ \omega\models A_j\overline{B_j}}} \eta_j > \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_iB_i}} \sum_{\substack{j\in J_\Delta\\ j\neq i\\ \omega\models A_j\overline{B_j}}} \eta_j$$

$$\Leftrightarrow \quad \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i\overline{B}_i}} \sum_{\substack{j\in J_\Delta\\ \omega\models A_j\overline{B_j}}} \eta_j > \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_iB_i}} \sum_{\substack{j\in J_\Delta\\ \omega\models A_j\overline{B_j}}} \eta_j$$

$$\overset{(*)}{\Leftrightarrow} \quad \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i\overline{B}_i}} \sum_{\substack{1\leqslant j\leqslant n\\ \omega\models A_j\overline{B_j}}} \eta_j > \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_iB_i}} \sum_{\substack{1\leqslant j\leqslant n\\ \omega\models A_j\overline{B_j}}} \eta_j$$

$$\Leftrightarrow \quad \kappa_{\vec\eta}(A_i\overline{B_i}) > \kappa_{\vec\eta}(A_iB_i).$$

Therefore, $\kappa_{\vec\eta} \models (B_i|A_i)$.

The equivalence $(*)$ holds, because there is a model for $A_i\dot{B}_i$ that does not falsify a conditional in Δ^∞, we have $\eta_j = \infty$ for all $(B_j|A_j)$ with $j \notin J_\Delta$, and therefore

$$\min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i\dot{B}_i}} \sum_{\substack{j\in J_\Delta\\ \omega\models A_j\overline{B_j}}} \eta_j = \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models A_i\dot{B}_i}} \sum_{\substack{1\leqslant j\leqslant n\\ \omega\models A_j\overline{B_j}}} \eta_j.$$

For any world ω with $\kappa^z_\Delta(\omega) < \infty$ it holds that all conditionals in Δ^∞ are not applicable in ω. Therefore, $\kappa_{\vec\eta}(\omega)$ is the sum of some of the impacts in $\vec\eta^J$; and because $\vec\eta^J \in \mathbb{N}_0^l$ we have $\kappa_{\vec\eta}(\omega) < \infty$.

In summary, $\kappa_{\vec\eta} \in \mathit{CMod}^{ec}_\Delta$.

Completeness: Let $\kappa \in \mathit{CMod}^{ec}_\Delta$ be an extended c-representation. Let $\vec\eta \in (\mathbb{N}_0 \cup \infty)^n$ be an impact vector such that $\kappa = \kappa_{\vec\eta}$. Because of Proposition 13, w.l.o.g. we can assume $\eta_i = \infty$ for all $(B_i|A_i) \in \Delta^\infty$. Furthermore, w.l.o.g, we can assume $\eta_i = \infty$ for all conditionals $(B_i|A_i) \in \Delta \setminus \Delta^\infty$ which are falsified only by worlds ω that also falsify a conditional in Δ^∞ – all worlds for which these impacts apply already have rank ∞ because of the impacts for Δ^∞.

The vector $\vec\eta$ is a combination of a vector $\vec\eta^J$ of impacts η_j for $j \in J_\Delta$, and a vector $(\infty, \dots, \infty)$ of size $n - |J_\Delta|$ of impacts for conditionals $(B_j|A_j)$ with $j \notin J_\Delta$.

For every $i \in J_\Delta$, by construction of J_Δ there is at least one world ω falsifying $(B_i|A_i)$ without falsifying a conditional in Δ^∞. Then, $\kappa^z_\Delta(\omega) < \infty$ because ω falsifies no conditionals in Δ^∞ and due to Lemma 2; therefore $\eta_i < \kappa_{\vec\eta}(\omega) < \infty$ because $\kappa_{\vec\eta} \in \mathit{CMod}^{ec}_\Delta$. Hence, $\vec\eta^J \in \mathbb{N}_0$.

It is left to show that $\vec\eta^J$ is a solution of $CRS^{ex}_\Sigma(\Delta)$, i.e., that for every $j \in J_\Delta$ it satisfies the constraint $(crs^{ex\,\Delta}_j)$. As $\kappa_{\vec\eta}$ is a model of Δ, it satisfies the conditional $(B_j|A_j) \in \Delta$. By construction of J_Δ, there is at least one world ω falsifying $(B_j|A_j)$ without falsifying a conditional in Δ^∞. As established above, the rank of such a world in $\kappa_{\vec\eta}$ is finite, and thus $\kappa_{\vec\eta}(A)$ is finite. To satisfy$(B_j|A_j)$ it is necessary that $\kappa_{\vec\eta}(A_i\overline{B_i}) > \kappa_{\vec\eta}(A_iB_i)$. Using the equivalence transformation in the *Soundness* part of this proof, we obtain that $(crs^{ex\,\Delta}_j)$ holds for η_j. □

Propositions 14 and 15 imply the following result.

Proposition 16. *Let Δ be a weakly consistent belief base. Then $A \mathrel{|\!\sim}^{ec}_\Delta B$ iff $A \mathrel{|\!\sim}_{\kappa_{\vec\eta}} B$ for every $\vec\eta \in \mathit{Sol}^{J+\infty}_\Delta$.*

The following example illustrates how $CRS^{ex}_\Sigma(\Delta)$ is simpler than $CR^{ex}_\Sigma(\Delta)$.

Example 3. *Let $\Sigma = \{a, b, c\}$ and $\Delta = \{(\bot|a), (\overline{a}|b), (b|c)\}$. The CSP $CR^{ex}_\Sigma(\Delta)$ over $\eta_1, \eta_1, \eta_3 \in \mathbb{N}_0 \cup \infty$ contains the constraints*

$$(cr^{ex\,\Delta}_1) \quad \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models a}} \sum_{\substack{1\leqslant j\leqslant n\\ \omega\models A_j\overline{B_j}}} \eta_j = \infty \quad \textit{or}$$

$$\eta_1 > \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models a\wedge\bot}} \sum_{\substack{j\neq 1\\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models a\wedge\top}} \sum_{\substack{j\neq 1\\ \omega\models A_j\overline{B_j}}} \eta_j,$$

$$(cr^{ex\,\Delta}_2) \quad \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models b}} \sum_{\substack{1\leqslant j\leqslant n\\ \omega\models A_j\overline{B_j}}} \eta_j = \infty \quad \textit{or}$$

$$\eta_i > \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models b\overline{a}}} \sum_{\substack{j\neq 2\\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models ba}} \sum_{\substack{j\neq 2\\ \omega\models A_j\overline{B_j}}} \eta_j,$$

$$(cr^{ex\,\Delta}_3) \quad \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models c}} \sum_{\substack{1\leqslant j\leqslant n\\ \omega\models A_j\overline{B_j}}} \eta_j = \infty \quad \textit{or}$$

$$\eta_i > \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models cb}} \sum_{\substack{j\neq 3\\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models c\overline{b}}} \sum_{\substack{j\neq 3\\ \omega\models A_j\overline{B_j}}} \eta_j.$$

The extended Z-partition of Δ is $EZP(\Delta) = (\Delta^0, \Delta^\infty)$ with $\Delta^0 = \{(\overline{a}|b), (b|c)\}$ and $\Delta^\infty = \{(\bot|a)\}$. The conditional $(\overline{a}|b)$ cannot be falsified without also falsifying $(\bot|a) \in \Delta^\infty$. Therefore, $J_\Delta = \{3\}$ and the CSP $CRS^{ex}_\Sigma(\Delta)$ over $\eta_3 \in \mathbb{N}_0$ contains only the constraint

$$(crs^{ex\,\Delta}_3)$$

$$\eta_3 > \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models bc}} \sum_{\substack{j\in J_\Delta\\ j\neq 3\\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models b\overline{c}}} \sum_{\substack{j\in J_\Delta\\ j\neq 3\\ \omega\models A_j\overline{B_j}}} \eta_j$$

which simplifies to $\eta_3 > 0$. For $\vec\eta \in \mathit{Sol}^{J+\infty}_\Delta$ it holds that $\eta_1 = \eta_2 = \infty$ and $\eta_3 \in \mathit{Sol}(CRS^{ex}_\Sigma(\Delta))$.

7.2. Check for Extended c-Entailment by Testing a CSP for Solvability

In [7] a method is developed that realizes c-inference as a CSP. The idea of this approach is that in order to check wheter $A \mathrel{|\!\sim}^{c}_\Delta B$ holds, a constraint encoding that $A \mathrel{|\!\sim}_{\kappa_{\vec\eta}} B$ does not hold is added to $CR_\Sigma(\Delta)$. If the resulting CSP is unsovable, $A \mathrel{|\!\sim}_{\kappa_{\vec\eta}} B$ holds for all solutions $\vec{\eta}$ of $CR_\Sigma(\Delta)$. Based on this idea, we develop a CSP that allows doing something similar for extended c-inference.

Definition 13. *Let $\Delta = \{(B_1|A_1), \dots, (B_n|A_n)\}$ be a belief base and let J_Δ be as defined in Definition 11. The constraint $\neg CR_\Delta(B|A)$ is given by*

$$\min_{\omega \models AB} \sum_{\substack{i \in J_\Delta \\ \omega \models A_i \overline{B_i}}} \eta_i \geqslant \min_{\omega \models A\overline{B}} \sum_{\substack{i \in J_\Delta \\ \omega \models A_i \overline{B_i}}} \eta_i. \quad (7)$$

Proposition 17. *Let Δ be a weakly consistent belief base. Then $A \mid\!\sim^{ec}_\Delta B$ iff either $\kappa^z_\Delta(A\overline{B}) = \infty$ or $\big(\kappa^z_\Delta(AB) < \infty$ and $CRS^{ex}_\Sigma(\Delta) \cup \neg CR_\Delta(B|A)$ is unsolvable$\big)$.*

Proof. **Direction ⇒:** Assume that $A \mid\!\sim^{ec}_\Delta B$ and that $\kappa^z_\Delta(A\overline{B}) < \infty$. Then $\kappa(A\overline{B}) < \infty$ for all $\kappa \in CMod^{ec}_\Delta$ by the definition of $CMod^{ec}_\Delta$. Therefore, $\kappa(A) < \infty$ for all $\kappa \in CMod^{ec}_\Delta$. Furthermore, $A \mid\!\sim^{ec}_\Delta B$ implies that for every $\kappa \in CMod^{ec}_\Delta$, we have $A \mid\!\sim_\kappa B$. Therefore, $\kappa(AB) < \kappa(A\overline{B})$ for every $\kappa \in CMod^{ec}_\Delta$, and because of Proposition 15 $\kappa_{\vec{\eta}}(AB) < \kappa_{\vec{\eta}}(A\overline{B})$ for every $\vec{\eta} \in Sol^{J+\infty}_\Delta$. We have

$$\kappa_{\vec{\eta}}(AB) < \kappa_{\vec{\eta}}(A\overline{B})$$

$$\Leftrightarrow \quad \min_{\omega \models AB} \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \eta_i < \min_{\omega \models A\overline{B}} \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \eta_i$$

$$\overset{(*)}{\Leftrightarrow} \quad \min_{\omega \models AB} \sum_{\substack{i \in J_\Delta \\ \omega \models A_i \overline{B_i}}} \eta_i < \min_{\omega \models A\overline{B}} \sum_{\substack{i \in J_\Delta \\ \omega \models A_i \overline{B_i}}} \eta_i.$$

Equivalence $(*)$ holds because the ranks of the minimal models of AB and $A\overline{B}$ are finite and therefore do not violate a conditional $(B_i|A_i)$ with $i \notin J_\Delta$.

Therefore, $\neg CR_\Delta(B|A)$ does not hold for any solution of $CRS^{ex}_\Sigma(\Delta)$, implying that $CRS^{ex}_\Sigma(\Delta) \cup \neg CR_\Delta(B|A)$ is unsolvable.

Direction ⇐: Assume that either $\kappa^z_\Delta(A\overline{B}) = \infty$ or $\big(\kappa^z_\Delta(AB) < \infty$ and $CRS^{ex}_\Sigma(\Delta) \cup \neg CR_\Delta(B|A)$ is unsolvable$\big)$. There are three cases.

Case 1: $\kappa^z_\Delta(AB) = \infty$ and $\kappa^z_\Delta(A\overline{B}) = \infty$
Then $\kappa^z_\Delta(A) = \infty$ and, by Proposition 4, $\kappa(A) = \infty$ for every $\kappa \in Mod^{ec}_\Delta$. Therefore $A \mid\!\sim^{ec}_\Delta B$.

Case 2: $\kappa^z_\Delta(AB) < \infty$ and $\kappa^z_\Delta(A\overline{B}) = \infty$
Then, by the definition of $CMod^{ec}_\Delta$, we have $\kappa(AB) < \infty$ and, by Proposition 4, $\kappa(A\overline{B}) = \infty$ for every $\kappa \in CMod^{ec}_\Delta$. Therefore, $\kappa(AB) < \kappa(A\overline{B})$ for every $\kappa \in CMod^{ec}_\Delta$ and hence $A \mid\!\sim^{ec}_\Delta B$ by Proposition 14.

Case 3: $\kappa^z_\Delta(A\overline{B}) < \infty$
Then, by assumption, $CRS^{ex}_\Sigma(\Delta) \cup \neg CR_\Delta(B|A)$ is unsolvable and $\kappa^z_\Delta(AB) < \infty$. This implies that $\neg CR_\Delta(B|A)$ is false for every $\vec{\eta}^J \in Sol(CRS^{ex}_\Sigma(\Delta))$. In this case, using the equivalence transformations in the part of the proof for *Direction ⇒*, we have $\kappa_{\vec{\eta}}(AB) < \kappa_{\vec{\eta}}(A\overline{B})$ for every $\vec{\eta} \in Sol^{J+\infty}_\Delta$. With Proposition 16 it follows that $A \mid\!\sim^{ec}_\Delta B$. □

8. Conclusions and Future Work

In this paper, we introduced extended c-representations as a generalization of c-representations for also weakly consistent belief bases. Based on extended c-representations we developed extended c-inference which is an extension of c-inference. We investigated the basic properties of extended c-representations and extended c-inference. Additionally, we developed a CSP that characterizes extended c-representations. We introduced a simplified version of this CSP that still describes all extended c-representations relevant for c-inference, and we showed how extended c-inference can be realized by a CSP. Note that our concept of extended c-representations can be used not only to define extended c-inference; analogously, it yields extended versions of credulous and weakly skeptical c-inference [20, 21] covering also weakly consistent belief bases.

Nonmonotonic inference is closely connected to belief revision [22]. The idea that some formulas are completely infeasible, that is used for inference here, also occurs in credibility limited revision [23]. In [24], a single "inconsistent world" is used for the representation of inconsistent belief states in the context of belief expansion. Drawing the connection between inductive inference from weakly consistent belief bases to these belief change approaches remains for future work.

Future work also includes to further investigate the properties of extended c-inference. For instance, we will investigate whether extended c-inference also satisfies syntax splitting and conditional syntax splitting [4, 5], and we will broaden the map of relations among inductive inference operators developed in [25] to extended c-inference and to other inductive inference operators taking also weakly consistent belief bases into account. Similarly as it has been done for c-inference [26, 27], we plan to realize extended c-inference as a SAT and as an SMT problem and to implement it in the InfOCF platform [28, 29].

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