Preferential Reasoning with Typicality in ASP over Weighted Argumentation Graphs in a Gradual Semantics

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Abstract

Recently some new gradual argumentation semantics have been proposed inspired by a fuzzy multi-preferential semantics for weighted conditional knowledge bases with typicality. In this paper we extend these semantics to the finitely-valued case, and develop an ASP approach for conditional reasoning over a weighted argumentation graph, through the verification of graded conditional implications over arguments and over boolean combination of arguments. The semantics defined in the paper is enforced via a custom propagator. The paper also develops a probabilistic semantics for gradual argumentation, which builds on the many-valued conditional semantics.

1. Introduction

The relationships between preferential, conditional approaches to non-monotonic reasoning and argumentation semantics are strong. In particular, for Dung-style argumentation semantics and Abstract Dialectical Frameworks (ADFs) the relationships with conditional reasoning have been deeply investigated [1, 2, 3, 4]. This is not the case for gradual argumentation, which has been studied through many different approaches and frameworks [5, 6, 7, 8, 9, 10, 11, 12, 13].

Some new gradual argumentation semantics [14, 15] have been recently proposed inspired by the multi-preferential semantics for weighted conditional knowledge bases with typicality in description logics [16, 17]. This suggests an approach to conditional reasoning over arguments in an argumentation graph, as well as a probabilistic interpretation for an argumentation graph with respect to a gradual semantics. The approach is a general one, and can be applied to different gradual argumentation semantics (under some conditions on the domain of argument valuation), but in this paper we will focus on some gradual argumentation semantics developed in [14, 15] for weighted argumentation graphs, and on their finitely-valued variants we are going to introduce.

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As a motivation of the work, it has been shown that a multilayer neural network can be mapped to a weighted conditional knowledge base [16, 17] and, hence, a weighted argumentation graph, with positive and negative weights, under the φ-coherent semantics [14, 15]. This is in agreement with previous work on the relationship between argumentation frameworks and neural networks [18, 13], see [15] for a comparison. This concrete application and the widespread interest in neural networks strongly motivates the development of proof methods for weighted argumentation graphs under the φ-coherent semantics.

The paper first introduces a finitely-valued variant of the φ-coherent semantics [14, 15] for weighted argumentation graphs, and develops an answer set programming (ASP) approach for conditional reasoning over weighted argumentation graphs, through the verification of graded (strict or defeasible) implications over arguments based on a *many-valued conditional logic with typicality*. We consider conditional implications of the form $\mathbf{T}(A_1) \rightarrow A_2$, meaning that "normally argument A_1 implies argument A_2", in the sense that "in the typical situations where A_1 holds, A_2 also holds", where $\mathbf{T}$ is also inspired by typicality operator in Propositional Typicality Logic (PTL) [19]. The truth degree of such implications can be determined with respect to the preferential interpretation built from a set of φ-coherent labellings of an argumentation graph.

More generally, we consider *graded conditionals* of the form $\mathbf{T}(\alpha) \rightarrow \beta \geq l$ ("normally argument α implies argument β with degree at least l"), where α and β are *boolean combination of arguments*. The satisfiability of such inclusions in a *multi-preferential* interpretation of an argumentation graph G, exploits the preference relations $<_{A_i}$ over φ-coherent labellings, which depend on arguments.

The preferential interpretation associated to an argumentation graph, based on the φ-coherent semantic, is further exploited to develop a probabilistic interpretation of the φ-coherent semantics. More precisely, we propose a probabilistic argumentation semantics, inspired by Zadeh's probability of fuzzy events [20], to extend the classical epistemic approach to probabilistic argumentation [21, 22] to the gradual case.

2. Many-valued Coherent, Faithful and φ-coherent Semantics for Weighted Argumentation Graphs

In this section, we generalize the gradual semantics for weighted argumentation graphs proposed in [14, 15], which have been introduced adopting the real unit interval $[0, 1]$ as the domain of argument valuation. We include the finitely-valued case. As a proof of concept, the φ-coherent semantics will be used in Section 4 for conditional reasoning over an argumentation graph.

More precisely, we let the *domain of argument valuation* $\mathcal{S}$ to be either the real unit interval $[0, 1]$ or the finite set $\mathcal{C}_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, for some integer $n \geq 1$. This allows to develop the notions of *many-valued coherent, faithful and φ-coherent labellings* for weighted argumentation graphs, which include both the infinitely and the finitely-valued case.

Let a *weighted argumentation graph* be a triple $G = \langle \mathcal{A}, \mathcal{R}, \pi \rangle$, where $\mathcal{A}$ is a set of arguments, $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ and $\pi : \mathcal{R} \rightarrow \mathbb{R}$. An example is in Figure 1.

This definition of weighted argumentation graph is similar to that of *weighted argument system* in [7], where only positive weights are allowed, representing the strength of attacks. Here, a pair $(B, A) \in \mathcal{R}$ is regarded as a *support* of argument B to argument A when the weight $\pi(B, A)$ is

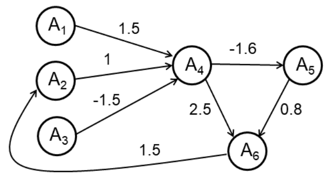

Figure 1: Example weighted argumentation graph G

positive and as an *attack* of argument B to argument A when $\pi(B, A)$ is negative. This leads to bipolar argumentation, which is well-studied in the literature [23, 10, 24, 13]. The argumentation semantics described below deals with positive and negative weights in a uniform way.

Given a weighted argumentation graph $G = \langle \mathcal{A}, \mathcal{R}, \pi \rangle$, a *many-valued labelling of* G is a function $\sigma : \mathcal{A} \rightarrow \mathcal{S}$ which assigns to each argument an *acceptability degree* in the domain of argument valuation $\mathcal{S}$.

Let $R^-(A) = \{B \mid (B, A) \in \mathcal{R}\}$. When $R^-(A) = \emptyset$, argument A has neither supports nor attacks. For $G = \langle \mathcal{A}, \mathcal{R}, \pi \rangle$ and a labelling σ, we introduce a *weight* W_σ^G *on* $\mathcal{A}$ as a partial function $W_\sigma^G : \mathcal{A} \rightarrow \mathbb{R}$, assigning a positive or negative support (relative to labelling σ) to all arguments $A_i \in \mathcal{A}$ such that $R^-(A_i) \neq \emptyset$, as follows:

$$W_\sigma^G(A_i) = \sum_{(A_j, A_i) \in \mathcal{R}} \pi(A_j, A_i)\, \sigma(A_j) \tag{1}$$

$W_\sigma^G(A_i)$ is left undefined when $R^-(A_i) = \emptyset$,

We exploit such a notion of weight of an argument with respect to a labelling to define some argumentation semantics for a graph G, which generalize the semantics in [14, 15].

Definition 1. *Given a weighted graph* $G = \langle \mathcal{A}, \mathcal{R}, \pi \rangle$ *and* $\sigma : \mathcal{A} \rightarrow \mathcal{S}$ *a labelling, we say that:*

- σ *is a* coherent labelling of G *if, for all arguments* $A, B \in \mathcal{A}$ *s.t.* $R^-(A) \neq \emptyset$ *and* $R^-(B) \neq \emptyset$*:* $\sigma(A) < \sigma(B) \iff W_\sigma^G(A) < W_\sigma^G(B)$;
- σ *is a* faithful labelling of G *if, for all arguments* $A, B \in \mathcal{A}$ *s.t.* $R^-(A) \neq \emptyset$ *and* $R^-(B) \neq \emptyset$*:* $\sigma(A) < \sigma(B) \Rightarrow W_\sigma^G(A) < W_\sigma^G(B)$;
- *given a function* $\varphi : \mathbb{R} \rightarrow \mathcal{S}$*,* σ *is a* φ-coherent labelling of G *if, for all arguments* $A \in \mathcal{A}$*, s.t.* $R^-(A) \neq \emptyset$*,*
$$\sigma(A) = \varphi(W_\sigma^G(A)) \tag{2}$$

Observe that the notion of φ-coherent labelling of G is defined through a set of equations, as in Gabbay's equational approach to argumentation networks [25]. The notions of coherent, faithful and φ-coherent labelling of a weighted argumentation graph G do not put constraints on the labelling of arguments without incoming edges, provided the constraints on the labellings of all other arguments can be satisfied, depending on the semantics considered. It can be proven that also for the finitely-valued case there are strong relationships between the three semantics, as in the fuzzy case [14].

Proposition 1. *Given $G = \langle \mathcal{A}, \mathcal{R}, \pi \rangle$ and $\varphi : \mathbb{R} \rightarrow \mathcal{S}$, with $\mathcal{S} = \mathcal{C}_n$, (1) a coherent labelling $\sigma : \mathcal{A} \rightarrow \mathcal{S}$ of G is a faithful labelling of G; (2) if φ is a* monotonically non-decreasing *function, a φ-coherent labelling σ of G is a faithful labelling of G; (3) if φ is* monotonically increasing*, a φ-coherent labelling σ of G is a coherent labelling of G.*

Restricting the domain of argument valuation to the finite number of values in $\mathcal{C}_n$, allows considering finitely-valued labellings which approximate the fuzzy semantics considered in [14, 15]. In the following, for $\mathcal{S} = [0, 1]$ we will assume function φ to be the logistic function $S(x) = 1/(1 + e^{-x})$ and, for $\mathcal{S} = \mathcal{C}_n$, its pointwise approximation (i.e., $S(x)$ is approximated to the closest value in $\mathcal{C}_n$).

Example 1. *For $\mathcal{S} = \mathcal{C}_n$ with $n = 5$, the graph G in Figure 1 has 36 φ-coherent labellings, while, for $n = 9$, G has 100 φ-coherent labellings. For instance, $\sigma = (0, 4/5, 3/5, 2/5, 2/5, 3/5)$ (meaning that $\sigma(A_1) = 0$, $\sigma(A_2) = 4/5$, and so on) is a labelling for $n = 5$. Consider, e.g., A_4; the φ-coherence condition indeed holds, i.e., $\sigma(A_4) = \varphi(W^G_\sigma(A_4))$:*
$W^G_\sigma(A_4) = 4/5 - 1.5 * 3/5 = -0.1$;
$S(W^G_\sigma(A_4)) = S(-0.1) = 0.475$, then $\varphi(W^G_\sigma(A_4)) = 2/5$.

In [14, 15] it has been shown that, for labellings with values in $[0, 1]$, the notion of φ-coherent labelling relates to the framework of gradual semantics studied by Amgoud and Doder [11], by considering a slight extension of their *gradual argumentation framework* so to deal with both positive and negative weights, to capture the strength of supports and attacks. The notion of bipolar argumentation has been widely studied in the literature [23, 5, 10, 12, 13]. In particular, an extension of the bipolar argumentation framework QBAFs by Baroni et al. [10] which includes the strength of attacks and supports has been developed and studied by Potyka [13].

As observed in [14], since MultiLayer Perceptrons (MLPs) can be mapped to weighted conditional knowledge bases, they can as well be seen as weighted argumentation graphs, with positive and negative weights, under the proposed semantics. In this view, φ-coherent labellings correspond to stationary states [26] of the network, where each unit in the network is associated to an argument, synaptic connections (with their weights) correspond to attacks/supports, and the activation of units to values of the corresponding arguments in a labelling. This is in agreement with previous work on the relationship between argumentation frameworks and neural networks investigated by d'Avila Garcez, Gabbay and Lamb [18] and recently by Potyka [13].

3. A Preferential Interpretation of Gradual Argumentation Semantics

The strong relations between the notions of coherent, faithful and φ-coherent labellings of a gradual argumentation graph and the corresponding semantics of weighted conditional knowledge bases have suggested an approach for defeasible reasoning over a weighted argumentation graph [14, 15], which builds on the semantics of the argumentation graph.

In the following, we first consider a propositional language to represent *boolean combinations of arguments* and a many-valued semantics for it on the domain of argument valuation. Then, we extend the language with a *typicality operator*, to introduce defeasible implications over boolean combinations of arguments and to define a *(multi-)preferential* interpretation associated with a weighted argumentation graph G, for the semantics introduced in Section 2). A similar extension with typicality of a propositional language has, for instance, been considered for the two-valued case in the Propositional Typicality Logic [19].

Let us consider a weighted argumentation graph $G = \langle \mathcal{A}, \mathcal{R}, \pi \rangle$. We introduce a propositional language $\mathcal{L}$, whose set of propositional variables $Prop$ is the set of arguments $\mathcal{A}$, and assume our language $\mathcal{L}$ contains the connectives $\wedge$, $\vee$, $\neg$ and $\rightarrow$, and that formulas are defined inductively, as usual. Formulas built from the propositional variables in $\mathcal{A}$ correspond to *boolean combination of arguments* considered, for instance, by Hunter, Polberg and Thimm [22]. Here we consider a many-valued semantics for boolean combination of arguments, using α, β, γ to denote them, e.g., $\alpha = (A_1 \wedge \neg A_2) \vee A_3$.

Let $\mathcal{S}$ be a *truth degree set*, equipped with a preorder relation $\leq$, a minimum and maximum element (denoted 0 and 1, resp.). Let $\otimes$, $\oplus$, $\rhd$ and $\ominus$ be the *truth degree functions* in $\mathcal{S}$ for the connectives $\wedge$, $\vee$, $\neg$ and $\rightarrow$ (respectively). When $\mathcal{S}$ is $[0, 1]$ or the finite set $\mathcal{C}_n$, as in our case of study, $\otimes$, $\oplus$, $\rhd$ and $\ominus$ can be chosen as a t-norm, s-norm, implication function, and negation function in some system of many-valued logic [27].

Let $\mathcal{S}$ be the domain of argument valuation for a gradual semantics S. A labelling $\sigma : \mathcal{A} \rightarrow \mathcal{S}$ of graph G assigns to each argument $A_i \in \mathcal{A}$ a truth degree in $\mathcal{S}$, that is, σ is a *many-valued valuation*. A valuation σ can be inductively extended to all propositional formulas of L as follows:

$$\sigma(\alpha \wedge \beta) = \sigma(\alpha) \otimes \sigma(\beta) \qquad \sigma(\alpha \vee \beta) = \sigma(\alpha) \oplus \sigma(\beta)$$
$$\sigma(\alpha \rightarrow \beta) = \sigma(\alpha) \rhd \sigma(\beta) \qquad \sigma(\neg \alpha) = \ominus \sigma(\alpha)$$

Based on the choice of the combination functions, a labelling σ uniquely assigns a truth degree to any boolean combination of arguments. We will assume that the false argument $\bot$ and the true argument $\top$ are formulas of L and that $\sigma(\bot) = 0$ and $\sigma(\top) = 1$, for all labellings σ.

Let Σ be a set of labellings of an argumentation graph $G = \langle \mathcal{A}, \mathcal{R}, \pi \rangle$, under some gradual semantics S in Section 2, with domain of argument valuation $\mathcal{S}$ (both $\mathcal{S} = [0, 1]$ and $\mathcal{S} = \mathcal{C}_n$ satisfy the conditions above and, actually, $<$ is a strict total order on $\mathcal{S}$).

Definition 2. *Given a set of labellings Σ, for each argument $A_i \in \mathcal{A}$, we define a* preference relation $<_{A_i}$ *on Σ, as follows: for $\sigma, \sigma' \in \Sigma$,*

$$\sigma <_{A_i} \sigma' \textit{ iff } \sigma'(A_i) < \sigma(A_i).$$

Labelling σ *is preferred to* σ' with respect to argument A_i (or σ is *more plausible* than σ' for argument A_i), when the degree of truth of A_i in σ is greater than the degree of truth of A_i in σ'. The preference relation $<_{A_i}$ is a strict modular partial order on Σ (where, modularity means that, for all $\sigma', \sigma'', \sigma''' \in \Sigma$, $\sigma' <_{A_i} \sigma''$ implies ($\sigma' <_{A_i} \sigma'''$ or $\sigma''' <_{A_i} \sigma''$)).

Example 2. *Referring to graph G in Figure 1, among the set Σ of 36 labellings of G for $n = 5$, there are σ and σ' such that $\sigma'(A_1) = 0 < \sigma(A_1) = 1/5$, and $\sigma'(A_3) = 1 > \sigma(A_3) = 4/5$. Hence $\sigma <_{A_1} \sigma'$ and $\sigma' <_{A_3} \sigma$. Labelling $\sigma = (1, 4/5, 0, 1, 1/5, 1)$ is preferred to all other ones with respect to $<_{A_6}$, being the only one with $\sigma(A_6) = 1$.*

The definition of preference wrt. arguments which is induced by a set of labellings Σ also extends to *boolean combination of arguments* α in the obvious way, based on the choice of combination functions. A set of labellings Σ *induces* a preference relation $<_\alpha$ on Σ, for each boolean combination of arguments α, as follows: $\sigma <_\alpha \sigma'$ iff $\sigma'(\alpha) < \sigma(\alpha)$.

When the set Σ is infinite, $<_{A_i}$ (or some $<_\alpha$) is not guaranteed to be *well-founded*, as there may be infinitely-descending chains of labellings $\sigma_2 <_{A_i} \sigma_1, \sigma_3 <_{A_i} \sigma_2, \ldots$. In the following, we restrict our consideration to *well-founded set of labellings* Σ, i.e., to Σ such that both $<_{A_i}$ and $<_{\neg A_i}$ are well-founded for all arguments A_i.

Example 3. *Under Gödel logic with standard involutive negation (i.e., with combination functions:* $a \otimes b = min\{a, b\}$*,* $a \oplus b = max\{a, b\}$*,* $a \rhd b = 1$ if $a \leq b$ and b otherwise*; and* $\ominus a = 1 - a$*) for the weighted graph G in Figure 1, with $n = 5$, the boolean combination of arguments $A_1 \wedge A_2 \wedge \neg A_3$ has 4 maximally preferred labellings, with $\sigma(A_1 \wedge A_2 \wedge \neg A_3) = 4/5$.*

We can now define a notion of preferential interpretation over many-valued valuations, which relates to preferential interpretations in KLM preferential logics [28]. Here preferences are defined over many-valued labellings, rather than over two-valued propositional valuations. A further difference with KLM approach is that the semantics exploits multiple preferences (one for each argument); it is a *multi-preferential* semantics. Other multi-preferential semantics for KLM conditional logics have been considered in [29, 30] and, for ranked and weighted knowledge bases in description logics, in [17, 16].

Definition 3. *Given a weighted argumentation graph $G = \langle \mathcal{A}, \mathcal{R}, \pi \rangle$, a pair $I_G^S = (\mathcal{S}, \Sigma)$ is a* preferential interpretation of *G in a gradual semantics S in Section 2, if Σ the set of labellings of G in S, and $\mathcal{S}$ is the domain of argument valuation.*

The preference relations $<_\alpha$ in I_G^S are left implicit, as they are induced by the labellings in Σ. Often, we will simply write I, rather than I_G^S.

We add a unary typicality operator $\mathbf{T}$ to the language L. The extended language is called $L^{\mathbf{T}}$, and the associated many-valued logic with typicality $\mathcal{L}^{\mathbf{T}}$. Intuitively, as in PTL, "a sentence of the form $\mathbf{T}(\alpha)$ is understood to refer to the *typical situations in which α holds*" [19]. The typicality operator allows the formulation of *conditional implications* (or *defeasible implications*) of the form $\mathbf{T}(\alpha) \rightarrow \beta$ whose meaning is that "normally, if α then β". In the two-valued case such implications correspond to conditional implications $\alpha \mathrel{|\!\sim} \beta$ of KLM preferential logics [28]. As usual [19], we do not allow nesting of the typicality operator. When α and β do not contain occurrences of the typicality operator, an implication $\alpha \rightarrow \beta$ is called *strict*. In the following, we do not restrict our consideration to strict or defeasible implications, but we allow in $L^{\mathbf{T}}$ *general implications* $\alpha \rightarrow \beta$, where α and β may contain occurrences of the typicality operator.

The interpretation of a typicality formula $\mathbf{T}(\alpha)$ is defined with respect to a preferential interpretation $I = (\mathcal{S}, \Sigma)$ with Σ well-founded.

Definition 4. *Given a preferential interpretation $I = (\mathcal{S}, \Sigma)$, and a labelling $\sigma \in \Sigma$, the valuation of a propositional formula $\mathbf{T}(\alpha)$ in σ is defined as follows:*

$$\sigma(\mathbf{T}(\alpha)) = \begin{cases} \sigma(\alpha) & \text{if } \sigma \in min_{<_\alpha}(\Sigma) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where $min_{<_\alpha}(\Sigma) = \{\sigma : \sigma \in \Sigma$ and $\nexists \sigma' \in \Sigma$ s.t. $\sigma' <_\alpha \sigma\}$.

When $(\mathbf{T}(A))^I(\sigma) > 0$, σ is a labelling assigning a maximal degree of acceptability to argument A in I, i.e., it maximizes the acceptability of argument A, among all the labellings in I.

Example 4. *Let us consider the interpretation $I = (\mathcal{C}_5, \Sigma)$, where Σ is the set of labellings of graph G in Figure 1 in the φ-coherent semantics with truth space $\mathcal{C}_5$. For the (unique) labelling σ preferred for A_6 (see Example 2), $\sigma(\mathbf{T}(A_6)) = 1$. For all other labellings $\sigma' \in \Sigma$, $\sigma'(\mathbf{T}(A_6)) = 0$. For the 4 labellings $\sigma_1, \ldots, \sigma_4$ that are the preferred ones for $A_1 \wedge A_2 \wedge \neg A_3$ (see example 3), $\sigma_i(\mathbf{T}(A_1 \wedge A_2 \wedge \neg A_3)) = 4/5$.*

Given a preferential interpretation $I = (\mathcal{S}, \Sigma)$, we can now define the satisfiability in I of a *graded implication*, having form $\alpha \rightarrow \beta \geq l$ or $\alpha \rightarrow \beta \leq u$, with l and u in $\mathcal{S}$ and α and β boolean combination of arguments. We first define the truth degree of an implication $\alpha \rightarrow \beta$ wrt a preferential interpretation I.

Definition 5. *Given a preferential interpretation $I = (\mathcal{S}, \Sigma)$ of an argumentation graph G, the* truth degree of an implication $\alpha \rightarrow \beta$ wrt. *I is defined as: $(\alpha \rightarrow \beta)^I = inf_{\sigma \in \Sigma}(\sigma(\alpha) \rhd \sigma(\beta))$.*

As a special case, for conditional implications, we have that: $(\mathbf{T}(\alpha) \rightarrow \beta)^I = inf_{\sigma \in \Sigma}(\sigma(\mathbf{T}(\alpha)) \rhd \sigma(\beta))$. Note that the interpretation of an implication (and of a conditional implication) is defined *globally* wrt a preferential interpretation I, as it is based on the whole set of labellings Σ in $I = (\mathcal{S}, \Sigma)$, in agreement with the interpretation of conditionals [31, 28].

We can now define the satisfiability of a *graded implication* in an interpretation $I = (\mathcal{S}, \Sigma)$.

Definition 6. *Given a preferential interpretation $I = (\mathcal{S}, \Sigma)$ of an argumentation graph G, I* satisfies a graded implication $\alpha \rightarrow \beta \geq l$ *(written $I \models \alpha \rightarrow \beta \geq l$) iff $(\alpha \rightarrow \beta)^I \geq l$; I satisfies a graded implication $\alpha \rightarrow \beta \leq u$ (written $I \models \alpha \rightarrow \beta \leq u$) iff $(\alpha \rightarrow \beta)^I \leq u$.*

Example 5. *As mentioned before, for the weighted argumentation graph in Figure 1, there are 36 labellings in case $n = 5$. The following graded conditionals are among the ones satisfied in the interpretation: $\mathbf{T}(A_1 \wedge A_2 \wedge \neg A_3) \rightarrow A_6 \geq 1$ (with 4 preferred labellings), $\mathbf{T}(A_1 \wedge A_2) \rightarrow A_6 \geq 4/5$ (12 preferred labellings), $\mathbf{T}(A_6) \rightarrow A_1 \wedge A_2 \geq 4/5$ (1 preferred labelling). On the other hand, for instance, the strict implication $A_6 \rightarrow A_1 \wedge A_2 \geq 1/5$ does not hold.*

Notice that the valuation of a graded implication (e.g., $\alpha \rightarrow \beta \geq l$) in a preferential interpretation I is two-valued, that is, either the graded implication is satisfied in I (i.e., $I \models \alpha \rightarrow \beta \geq l$) or it is not (i.e., $I \not\models \alpha \rightarrow \beta \geq l$). Hence, it is natural to consider *boolean combinations of graded implications*, such as $(\mathbf{T}(A_1) \rightarrow A_2 \wedge A_3 \leq 0.7) \wedge (\mathbf{T}(A_3) \rightarrow A_4) \geq 0.6) \rightarrow (\mathbf{T}(A_1) \rightarrow A_4) \geq 0.6)$, and define their satisfiability in an interpretation I in the obvious way, based on the semantics of classical propositional logic.

When the preferential interpretation I^S_G is finite (contains a finite set of labellings), the satisfiability of graded implications (or their boolean combinations) can be verified by *model checking* over the preferential interpretation I^S_G. In the next section, we develop an ASP approach for defeasible reasoning over an argumentation graph for the φ-coherent semantics.

It can be shown that one can reformulate the KLM properties of a *preferential consequence relation* in the many-valued setting, following the approach for weighted conditional DLs in [32].

It can be proven that (for the choice of combination functions as in Gödel logic) such properties are satisfied by the set of graded conditionals of the form $\mathbf{T}(\alpha) \rightarrow \beta \geq 1$, which hold in a given interpretation $I^S = (\mathcal{S}, \Sigma^S)$. A detailed description and the proof will be included in an extended version of the paper.

4. An ASP Approach for Conditional Reasoning in the Finitely-valued Case

In this section, we consider the φ-coherent finitely-valued semantics of a weighted argumentation graph G introduced in Section 2, with domain of argument valuation $\mathcal{C}_n$, for some integer $n \geq 1$, and we describe an ASP approach for reasoning about argumentation graphs. The idea is to represent a many-valued labelling in ASP as an answer set that encodes the assignment of a value in $\mathcal{C}_n$ to each argument A_i. The labelling is encoded by a set of atoms of the form $val(a, v)$, meaning that $\frac{v}{n} \in C_n$ is the acceptability degree of argument a in the labelling.

Answer set candidates are generated by the rule

$\mathit{1}\{val(A, V) : val(V)\}\mathit{1} \ \leftarrow arg(A).$

where facts $val(a, v)$ and $arg(a)$ are given for all v s.t. $\frac{v}{n} \in C_n$ and $a \in \mathcal{A}$.

Boolean combinations of interest are determined at grounding time and collected in predicate arg_comb by means of the rules

$arg\_comb(impl(Alpha, Beta)) \leftarrow query(typ(Alpha), Beta, L)$
$arg\_comb(A) \leftarrow arg\_comb(neg(A))$
$arg\_comb(A) \leftarrow arg\_comb(op_2(A, B))$

where op_2 stands for and, or, and $impl$, and the fact $query(typ(alpha), beta, l)$ is used to represent the query $\mathbf{T}(\alpha) \rightarrow \beta \geq \frac{l}{n}$, with $alpha$ and $beta$ given in the form of nested applications of neg, and, or, and $impl$.

The valuation of boolean combinations B of arguments is encoded as a predicate $eval(B, V)$. A rule is introduced for each connective to encode its semantics, based, e.g., on Gödel logic with standard involutive negation (see Example 3), but other choices of combination functions can as well be considered. Then we have:

$eval(A, V) \leftarrow arg(A), val(A, V).$
$eval(neg(A), n - V) \leftarrow arg\_comb(neg(A)), eval(A, V).$
$eval(and(A, B), @min(V1, V2)) \leftarrow arg\_comb(and(A, B)), eval(A, V1), eval(B, V2).$
$eval(or(A, B), @max(V1, V2)) \leftarrow arg\_comb(or(A, B)), eval(A, V1), eval(B, V2).$
$eval(impl(A, B), n) \leftarrow arg\_comb(impl(A, B)), eval(A, V1), eval(B, V2), V1 \leq V2.$
$eval(impl(A, B), V2) \leftarrow arg\_comb(impl(A, B)), eval(A, V1), eval(B, V2), V1 > V2.$

where $@min$ and $@max$ are @-terms respectively returning the minimum and maximum of their arguments.

To enforce the φ-coherent semantics we need to encode condition (2). Even if the computation of the weighted sum and of φ is expressible in terms of ASP rules, the evaluation of such rules would eventually materialize all combinations of truth degrees for all attacker arguments, which is practically feasible only if the maximum in-degree is bounded by a small number. Moreover, such ASP rules would require to represent weights on the graph in terms of integers, hence introducing an approximation. We opted for an alternative approach powered by the Propagator interface of the *clingo* API [33]. In a nutshell, we defined a custom propagator that enforces the condition

$$\bot \leftarrow arg(A), eval(A, V), V \neq \varphi(V/n).$$

for all arguments $A \in \mathcal{A}$ s.t. $R^{-}(A) \neq \emptyset$. The propagator takes as input the argument A, and is initialized after the grounding phase, when it can identify all attackers/supporters of A, the associated weights, and set watches for the boolean variables that *clingo* associates to the instances of predicate $eval$. After that, whenever a watched boolean variable is assigned true or unrolled to undefined, the propagator is notified and keeps track of the change. If all attackers/supporters of A have been assigned a truth degree, then the propagator can infer the truth degree of A, and provide an explanation to *clingo* for such an inference in terms of a clause. Similarly, if A has already a truth degree and all attackers/supporters of A have been assigned a truth degree which is incompatible with that of A, then the propagator can report a conflict, and provide an explanation to *clingo* for such a conflict in terms of a clause.

The preferential interpretation $I_G^S = (\mathcal{C}_n, \Sigma)$ which is built over the (finite) set Σ of all the φ-coherent labellings of the argumentation graph G in the finitely-valued semantics, is represented by all resulting answer sets.

Regarding the evaluation of the query $\mathbf{T}(\alpha) \rightarrow \beta \geq \frac{l}{n}$, it is satisfied over the preferential interpretation I_G^S if in all the labellings σ that maximize the value of $\sigma(\alpha)$ wrt all labellings in Σ, it holds that $\sigma(\mathbf{T}(\alpha)) \rhd \sigma(\beta) \geq \frac{l}{n}$. That is, one has to verify that in all the answer sets M that maximize the value of V such that $eval(\alpha, V) \in M$, if $eval(\alpha, v1), eval(\beta, v2) \in M$, then $v1 \leq v2$ or $v2 \geq l$ also hold in M. Equivalently, one can search for a counterexample, i.e. an answer set that maximizes the evaluation $v1$ of α, and such that, if $v2$ is the evaluation of β, $v1 > v2$ and $v2 < l$. The following weak constraints are used:

$$:\sim\ query(typ(Alpha), \_, \_), eval(Alpha, V). \quad [-1@V+2, Alpha, V]$$
$$:\sim\ query(typ(Alpha), Beta, L), eval(impl(Alpha, Beta), V), V < L.$$
$$[-1@1, Alpha, Beta, V, L]$$

The first weak constraint expresses a strong preference for answer sets where the evaluation of α is maximal, given that the presence of $eval(Alpha, V)$ has priority $V + 2$, then increasing with V. Among such answer sets, the second weak constraint expresses a preference for a witness that the query is not satisfied. Note that also in such answer sets the evaluation of $\mathbf{T}(\alpha) \rightarrow \beta$ coincides with the one for $\alpha \rightarrow \beta$ (because the evaluation of $\mathbf{T}(\alpha)$ and α coincide for typical α labellings), and the second weak constraint can therefore use $eval(impl(Alpha, Beta), V)$.

Our implementation is available at https://github.com/alviano/valphi, and it can be installed and used in a Python 3.10 environment by running the following commands:

```
pip install valphi
python -m valphi -t file.graph -v file.valphi solve
python -m valphi -t file.graph -v file.valphi query "query"
```

where *file.graph* and *file.valphi* are files encoding a weighted argumentation graph G and the activation thresholds for φ (where the value of φ changes from some $\frac{l}{n}$ to $\frac{l+1}{n}$), and $query$ is a string encoding a gradual implication. Graph files start with the preamble `#graph`, and then list the attack relation as space-separated triples of the form $attacker\ attacked\ weight$, where $attacker$ and $attacked$ are argument indices (starting by 1) and $weight$ is a real number. The graph shown in Figure 1 is encoded as

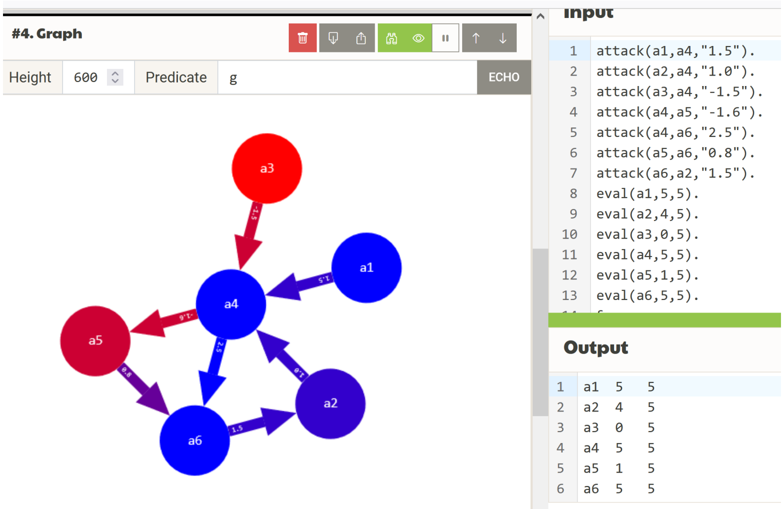

Figure 2: The weighted argumentation graph from Figure 1 and one of its φ-coherent labelling shown by the VALPHI system via a shareable ASP Chef link (https://github.com/alviano/asp-chef). Arguments are colored according to their value: blue if 1, red if 0, and intermediate shades for intermediate values. Attacks are shown in red, and supports in blue, with shades corresponding to magnitudes.

```
#graph
1 4 1.5
2 4 1
3 4 -1.5
4 5 -1.6
4 6 2.5
5 6 0.8
6 2 1.5
```

Thresholds files are sequences of real numbers, one per line. A query $\mathbf{T}(\alpha) \rightarrow \beta \geq l$ is encoded by the string $\alpha'\#\beta'\# >= \#l$, where α' and β' are the terms representing α and β. For example, $\mathbf{T}(A_1 \wedge A_2 \wedge \neg A_3) \rightarrow A_6 \geq 1$ is encoded as `and(a1,and(a2,neg(a3)))#a6#>=#1.0`. Finally, solutions can be shown in an interactive graph visualization by adding the command-line flag `--show-in-asp-chef`; an example is shown in Figure 2.

5. Towards a Probabilistic Semantics for Gradual Argumentation

When the domain of argument valuation is the interval $[0, 1]$, the definition of a preferential interpretation I^S associated to the gradual semantics S of a weighted argumentation graph also suggests a probabilistic interpretation of the weighted graph, inspired by Zadeh's *probability of fuzzy events* [20]. The approach has been previously considered in [34] for providing a probabilistic interpretation of Self-Organising Maps [35] after training, by exploiting a recent characterization of the continuous t-norms compatible with Zadeh's probability of fuzzy events

(P_Z-compatible t-norms) by Montes et al. [36]. In this section we explore this approach for weighted argumentation graphs, showing that it relates to the probabilistic semantics presented by Thimm [21]. We discuss some advantages and drawbacks of the approach.

Let Σ be the set of labellings of G in a gradual argumentation semantics S with domain of argument valuation in $[0, 1]$, and I_G^S an associated preferential interpretation. The probabilistic semantics we propose is inspired by Zadeh's probability of fuzzy events [20], as one can regard an argument $A \in \mathcal{A}$ as a *fuzzy event*, with *membership function* $\mu_A : \Sigma \rightarrow [0, 1]$, where $\mu_A(\sigma) = \sigma(A)$. Similarly, any boolean combination of arguments α can as well be regarded as a fuzzy event, with membership function $\mu_\alpha(\sigma) = \sigma(\alpha)$, where the extension of labellings to boolean combinations of arguments and to typicality formulas has been defined in Section 3.

We restrict ourselves to a P_Z-compatible t-norm $\otimes$ [36], with associated t-conorm $\oplus$ and the negation function $\ominus x = 1 - x$. For instance, one can take the minimum t-norm, product t-norm, or Lukasiewicz t-norm. Given $I_G^S = \langle \mathcal{S}, \Sigma \rangle$, we assume a discrete probability distribution $p : \Sigma \rightarrow [0, 1]$ over Σ, and define the *probability of a boolean combination of arguments* α as follows:

$$P(\alpha) = \sum_{\sigma \in \Sigma} \sigma(\alpha)\; p(\sigma) \tag{4}$$

For a single argument $A \in \mathcal{A}$, when labellings are two-valued (that is, $\sigma(A)$ is 0 or 1), the definition above becomes the following: $P(A) = \sum_{\sigma \in \Sigma \wedge \sigma(A)=1} p(\sigma)$, which relates to the probability of an argument in the probabilistic semantics by Thimm in [21]. Indeed, in [21] the probability of an argument A in Arg is "the degree of belief that A is in an extension", defined as the sum of the probabilities of all possible extensions e that contain argument A, i.e., $P(A) = \sum_{A \in e \subseteq Arg}\; p(e)$, where an extension $e \in 2^{Arg}$ is a set of arguments in Arg, and $p(e)$ is the probability of extension e. Here, on the other hand, we are considering many-valued labellings σ assigning an acceptability degree $\sigma(A)$ to each argument A, so it is not the case that an argument either belongs to an extension (a labelling) or it does not.

Following Smets [37], we let the *conditional probability of* α *given* β, where α and β are boolean combinations of arguments, to be defined as

$$P(\alpha|\beta) = P(\alpha \wedge \beta)/P(\beta)$$

(provided $P(\beta) > 0$). As observed by Dubois and Prade [38], this generalizes both conditional probability and the fuzzy inclusion index advocated by Kosko [39].

Let us extend the language $L^{\mathbf{T}}$ by introducing a new proposition $\{\sigma\}$, for each $\sigma \in \Sigma$, and the valuations σ to such propositions by letting: $\sigma(\{\sigma\}) = 1$ and $\sigma'(\{\sigma\}) = 0$, for any $\sigma' \in \Sigma$ such that $\sigma' \neq \sigma$. It can be proven (see [34]) that

$$P(A|\{\sigma\}) = \sigma(A).$$

The result holds when the t-norm is chosen as in Gödel, Łukasiewicz or Product logic. In such cases, $\sigma(A)$ can be interpreted as the conditional probability that argument A holds, given labelling σ, which can be regarded as a *subjective probability* (i.e., the degree of belief we put into A when we are in the state represented by labelling σ).

Under the assumption that the probability distribution p is *uniform* over the set Σ of labellings, it holds that $P(\alpha|\beta) = M(\alpha \wedge \beta)/M(\beta)$ (provided $M(\beta) > 0$), where $M(\alpha) = \sum_{x \in \Sigma} \sigma(A)$

is the *size* of the fuzzy event α. For a finite set of labellings $\Sigma^S = \{\sigma_1, \ldots, \sigma_m\}$ wrt. a given semantics S, assuming a uniform probability distribution, we have that $P(\alpha) = M(\alpha)/m = (\sigma_1(\alpha) + \ldots + \sigma_m(\alpha))/m$.

Example 6. *Reconsidering the example in Figure 1, for the φ-coherent semantics with truth degree set $\mathcal{C}_5$, assuming a uniform probability distribution, we get: $P(A_1) = 0.5$, $P(A_2) = 0.777$, $P(A_3) = 0.483$, $P(A_4) = 0.644$, $P(A_5) = 0.222$, $P(A_6) = 0.755$.*

Furthermore, $P(A_1 \wedge A_2|A_6) = 0.618$ and $P(A_1 \wedge A_2 \wedge \neg A_3|A_6) = 0.397$, while $P(A_1 \wedge A_2|\mathbf{T}(A_6)) = 0.8$ and $P(A_1 \wedge A_2 \wedge \neg A_3|\mathbf{T}(A_6)) = 0.8$.

While the notion of probability P defined by equation (4) satisfies Kolmogorov's axioms for any P_Z-compatible t-norm, with associated t-conorm, and the negation function $\ominus x = 1 - x$ [36], there are properties of classical probability which do not hold (depending on the choice of t-norm), as a consequence of the fact that not all classical logic equivalences hold in a fuzzy logic.

For instance, the truth degree of $A \wedge \neg A$ in a labelling σ may be different from 0 depending on the t-norm (e.g., with Gödel and Product t-norms). Hence, it may be the case that $P(A \wedge \neg A)$ is different from 0. Similarly, it may be the case that $P(A \vee \neg A)$ is different from 1 (e.g., with Göedel t-norm) and that $P(A|A) = P(A \wedge A)/P(A)$ is different from 1 (e.g., with Product t-norm). While $P(A) + P(\neg A) = 1$ holds (due to the choice of negation function), $P(A|B) + P(\neg A|B)$ may be different from 1.

In spite of the simplicity of this approach, on the negative side, some properties of classical probability are lost. Hence, we can consider the proposal in this section as a first step towards a probabilistic semantics for gradual argumentation.

6. Conclusions

In this paper we have developed an approach to define a many-valued preferential interpretation of a weighted argumentation graph, based on some gradual argumentation semantics. The approach allows for graded (strict or conditional) implications involving arguments and boolean combination of arguments (with typicality) to be evaluated in the preferential interpretation I^S of the argumentation graph, in a given gradual argumentation semantics S. It can be proven that the set of graded conditionals of the form $\mathbf{T}(\alpha) \rightarrow \beta \geq 1$, which are satisfied in I^S, satisfy the KLM postulates of a preferential consequence relation [31]. We have considered a finitely-valued version of the φ-coherent argumentation semantics in [14], for which an Answer set Programming approach can be used for the verification of graded conditionals. For the gradual semantics with domain of argument valuation in the unit real interval $[0, 1]$, the paper also proposes a probabilistic argumentation semantics, which builds on a gradual semantics S and on the preferential interpretation I^S_G.

Concerning the relationships between argumentation semantics and conditional reasoning, Weydert [1] has proposed one of the first approaches for combining abstract argumentation with a conditional semantics. He has studied "how to interpret abstract argumentation frameworks by instantiating the arguments and characterizing the attacks with suitable sets of conditionals describing constraints over ranking models". In doing this, he has exploited the JZ-evaluation semantics, which is based on system JZ [40]. Our approach aims to provide a preferential and conditional interpretation for some gradual argumentation semantics.

A correspondence between Abstract Dialectical Frameworks [2] and Nonmonotonic Conditional Logics has been studied in [3], with respect to the two-valued models, the stable, the preferred semantics and the grounded semantics of ADFs. Whether our approach can be extended to ADFs will be subject of future investigation.

In [4] Ordinal Conditional Functions (OCFs) are interpreted and formalized for Abstract Argumentation, by developing a framework that allows to rank sets of arguments wrt. their plausibility. An attack from argument a to argument b is interpreted as the conditional relationship, “if a is acceptable then b should not be acceptable". Based on this interpretation, an OCF inspired by System Z ranking function is defined. In this paper we focus on the gradual case, based on a many-valued logic.

In Section 5, we have proposed a probabilistic semantics for weighted argumentation graphs, which builds on the gradual argumentation semantics in Section 2 and on their preferential interpretation, and is inspired by Zadeh's *probability of fuzzy events* [20]. We have seen that the proposed approach relates to the probabilistic semantics by Thimm [21] and it allows the truth degree $\sigma(A)$ of an argument A in a labelling σ to be regarded as the conditional probability of A given σ. On the one hand, our approach does not require to introduce a notion of p-justifiable probability function, as we define the probability of an argument with respect to the set of labellings Σ of the weighted graph G in a given gradual semantics. On the other hand, as we have seen, some classical equivalences may not hold (depending on the choice of combination functions), and some properties of classical probability may be lost. This requests for further investigation. Alternative approaches for combining conditionals and probabilities, such as the one proposed recently by Flaminio et al. [41], might suggest alternative ways of defining a probabilistic semantics for gradual argumentation.

In the paper, we have adopted an epistemic approach to probabilistic argumentation, and we refer to [42] for a general survey on probabilistic argumentation. As a generalization of the epistemic approach to probabilistic argumentation, epistemic graphs [22] allow for epistemic constraints, that is, for boolean combinations of inequalities, involving statements about probabilities of formulae built out of arguments. While so far we only allow for combining graded conditionals, this is a possible direction to extend the probabilistic semantics in Section 5.

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