The Counting Problem of Slot Mereology

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Abstract

Bennett proposed in 2013 a new mereological theory based on the decomposition of the parthood relation into two relations: having a slot and occupying that slot. Slot mereology can be used to represent the mereological structure of a variety of entities that can have the same part multiple times, including (but not only) structural universals. We show here that this theory is not compatible with a counting criterion that would enable us to count appropriately how many times a whole has a part. We propose news axioms to fix those flaws.

Keywords

ontology, mereology, slot mereology, structural universal, counting

1. Introduction

Bennett proposed in 2013 [1] a new mereological theory that makes it possible for a whole to have the same part multiple times. In this theory, the parthood relation is analysed in terms of two relations: *having a slot*, and *filling a slot*. More precisely, x is a part of y iff x fills a slot of y. Thus, y can have the same part x several times, by x filling several slots of y. Bennett motivated her theory by a problem, exposed by Lewis in 1986 [2], that appears with *structural universals*, namely universals that are composed of other universals. Indeed, a structural universal can have the same universal as part multiple times. Bennett illustrates her theory by using the example of the methane molecule universal, first introduced by Lewis [2]: in her theory, the methane molecule universal CH_4 can have the hydrogen atom universal H as part four times, reflecting the structure of methane particulars. Structural universals were debated by Armstrong [3] and Bigelow [4] in 1986, and by Fisher (2018) [5], Masolo and Vieu (2018) [6] and Garbacz (2020) [7] in articles discussing mereological theories for structural universals, including Bennett's slot mereology. The slot mereology is in particular analysed by Fisher (2013) [8] and Garbacz (2016) [9]. Note however that Bennett's mereology is not restricted to structural universals and can be applied to other entities, such as informational entities (see [10] and [11]). This theory is inspired

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by a role-based analysis in which the slots would be identified with roles and the fillers with the role-players, although Bennett does not propose a full account of roles in terms of slot mereology and does not exploit any well-developed formal ontology of roles (such as [12] or [13]).

In spite of its aim to account for having a part multiple times, as we are going to show, Bennett's proposition has counterintuitive implications when counting parts is at stake. For example, according to some counting criterion and some model of the METHANE universal, the METHANE universal would have hydrogen as part not four times, but five. According to another counting criterion, the METHANE universal would have the ELECTRON universal as part not ten times, as expected (six times from the carbon atom and four times from the four hydrogen atoms), but seven times. We will thus propose a modification of Bennett's theory that solves such problems. We will illustrate our theory by using the METHANE example, occasionally completed by other examples of structural universals and informational entities when needed.

2. The Slot Mereology

The slot mereology separates the parthood relation into two primitive relations: *being the parthood slot of* and *occupying a parthood slot*, respectively named *slot_of* and *fills*.[1] This theory was motivated by the possibility for a whole to have the same part multiple times. For example, a structural universal can have the same universal as part multiple times (e.g. CH_4 having H as part four times), or an informational entity particular can have the same informational entity particular as part multiple times (e.g. 'aa' having 'a' as part twice). Thus, one could expect this theory to enable a correct count of the number of appearances of each part (even if Bennett does not discuss countability). However, the system fails in that respect, as we are going to see. Let us first present the system.

2.1. Presentation of the System

The theory is based on the two primitive relations *slot_of* and *fills*. Five relations are then defined on this basis, as presented in Table 1. The definitions, axioms and theorems from Bennett's paper [1] are identified by, respectively, "BD", "BA" and "BT", followed by a number.

Table 1
Slot Mereology Definitions

Number	Name	Definition
BD1	Parthood	$part\_of(x,y) \triangleq \exists z(slot\_of(z,y) \wedge fills(x,z))$
BD2	Proper Parthood	$proper\_part\_of(x,y) \triangleq part\_of(x,y) \wedge \neg part\_of(y,x)$
BD3	Overlap	$overlap(x,y) \triangleq \exists z(part\_of(z,x) \wedge part\_of(z,y))$
BD4	Slot-overlap	$slot\_overlap(x,y) \triangleq \exists z(slot\_of(z,x) \wedge slot\_of(z,y))$
BD5	Proper Parthood Slot	$proper\_slot\_of(x,y) \triangleq slot\_of(x,y) \wedge \neg fills(y,x)$

[1]The names have been changed. The original names in Bennett's work were P_s and F. Note also that we do not generally conceive of slots in spatial terms, although the connection between slots and spatial locations might be worth investigating in the future.

Eight axioms constrain the system, presented in Table 2. Axiom (BA5) makes slots inheritable.

Table 2
Slot Mereology Axioms

Number	Description	Axiom
BA1	Only Slots are Filled	$fills(x,y) \rightarrow \exists z(slot\_of(y,z))$
BA2	Slots Cannot Fill	$fills(x,y) \rightarrow \neg\exists z(slot\_of(x,z))$
BA3	Slots Don't Have Slots	$slot\_of(x,y) \rightarrow \neg\exists z(slot\_of(z,x))$
BA4	Improper Parthood Slots	$\exists y(slot\_of(y,x)) \rightarrow \exists z(slot\_of(z,x) \wedge fills(x,z))$
BA5	Slot Inheritance	$[slot\_of(z_1,y) \wedge fills(x,z_1) \wedge slot\_of(z_2,x)] \rightarrow slot\_of(z_2,y)$
BA6	Mutual Occupancy is Identity	$(slot\_of(z_1,y) \wedge fills(x,z_1)) \wedge$ $(slot\_of(z_2,x) \wedge fills(y,z_2)) \rightarrow x = y$
BA7	Single Occupancy[2]	$slot\_of(x,y) \rightarrow \exists! z(fills(z,x))$
BA8	Slot Strong Supplementation	$\exists z(slot\_of(z,x)) \wedge \exists z(slot\_of(z,y)) \rightarrow$ $[\neg(\exists z(slot\_of(z,x) \wedge fills(y,z))) \rightarrow$ $\exists z(slot\_of(z,y) \wedge \neg slot\_of(z,x))]$

If s is a slot of a, we say that s is a "direct slot" of a iff there is no proper part b of a such that s is also a slot of b (see (D1)[3]). If s is a slot of a that is not a direct slot, we call it a "non-direct slot". Axiom (BA5) implies the existence of non-direct slots in non-trivial models. Note that in absence of any discreteness axiom, the existence of direct slots is not guaranteed.

Definition D1. $direct\_slot\_of(s,a) \triangleq slot\_of(s,a) \wedge \neg\exists b(proper\_part\_of(b,a) \wedge slot\_of(s,b))$

We will call "filler" an entity that fills a slot. (BA4) implies that anything that has at least one slot is also a filler. At this point, we can make some remarks:

- Being a proper slot is not an intrinsic property of slots, but a relational property: the same slot can be a proper slot of a filler and an improper slot of another filler.
- Even though every whole that has a proper part also has an improper slot (axiom (BA4)), the converse is not true: a filler can have an improper slot without having any proper parts.
- Although Bennett's paper does not mention this possibility, nothing in her system prevents a filler from having several improper slots.

Table 3 presents Bennett's theorems that are used in this paper.

2.2. Representing the Methane Molecule

Any methane molecule particular is composed of five atoms: one carbon and four hydrogens, each bound to the carbon atom. Moreover, any carbon atom particular has six electrons, and each hydrogen atom particular has one electron.[4] Using the slot mereology, the universal METHANE is

[2] A typographical mistake in Bennett's paper has been corrected, following Garbacz [9].

[3] Our definitions, axioms and theorems are denoted using only the first letter of the word, respectively D, A and T. Therefore, they can be distinguished from Bennett's.

[4] Note that atoms also have other parts, such as nuclei, but for the sake of simplicity, we only represent electrons here. The reasoning we will develop for electrons also applies to other parts such as nuclei.

Table 3
Slot Mereology Theorems

Number	Description	Theorem
BT7	Transitivity	$part\_of(x,y) \wedge part\_of(y,z) \rightarrow part\_of(x,z)$
BT9	Conditional Reflexivity	$\exists z(slot\_of(z,x)) \rightarrow part\_of(x,x)$
BT13	Slot Weak Supplementation	$proper\_part\_of(x,y) \rightarrow \exists z(slot\_of(z,y) \wedge \neg slot\_of(z,x))$

described as having five proper slots, one for the CARBON universal and four for the HYDROGEN universal. The current structure is described by Facts (1) and pictured in Figure 1, where, as well as in the following ones, $F \xleftarrow{S_i} \square$ represents $slot\_of(S_i, F)$, $\overset{S_i}{\boxed{F}}$ represents $fills(F, S_i)$ and $\overset{S_i}{\boxed{F}}\circlearrowleft$ represents $slot\_of(S_i, F) \wedge fills(F, S_i)$. Note that axiom (BA4) entails that S_0 exists and that it is an improper slot of METHANE. For now and in the remainder, different constant symbols are supposed to be interpreted as different fillers and different slots.

$$
\begin{array}{rl}
slot\_of(S_i, \text{METHANE}) & 0 \leq i \leq 5 \text{ [5]} \\
fills(\text{METHANE}, S_0) & \\
fills(\text{CARBON}, S_1) & \\
fills(\text{HYDROGEN}, S_i) & 2 \leq i \leq 5
\end{array} \tag{1}
$$

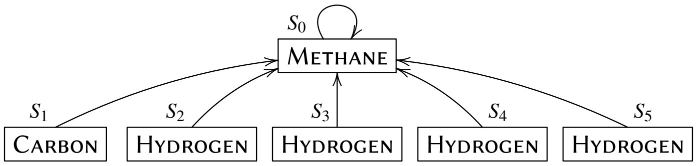


Figure 1: Representation of METHANE and its parts, CARBON and HYDROGEN.

Let's represent the electrons of each atom. Each carbon atom has six electrons, each hydrogen atom has one. So, in our case, the universal CARBON owns six slots filled with ELECTRON, whereas the universal HYDROGEN owns only one, as described by Facts (2).

$$
\begin{array}{rl}
slot\_of(S_i, \text{CARBON}) & 6 \leq i \leq 11 \\
slot\_of(S_{12}, \text{HYDROGEN}) & \\
fills(\text{ELECTRON}, S_i) & 6 \leq i \leq 12
\end{array} \tag{2}
$$

One thing to keep in mind is the fact that even if there are four slots of METHANE filled by HYDROGEN, there is only one universal of HYDROGEN, which has only one slot filled by ELECTRON. Bennett's axiom (BA5) states that slots are inherited by wholes. In our case, this means that METHANE inherits from CARBON and HYDROGEN their slots filled by ELECTRON. METHANE inherits six slots from CARBON and only one slot from HYDROGEN. Considering that METHANE has no other slots filled by ELECTRON, METHANE has in total seven slots filled by ELECTRON.

[5] This is a compact notation for six formulas. This notation is used in the remainder of the paper.

According to axioms (BA4) and (BA5), there are two additional slots, called S_{13} and S_{14}, that are improper slots of CARBON and HYDROGEN, respectively (see Facts (3)).

$$\begin{array}{ll} \mathit{slot\_of}(S_{13}, \text{CARBON}) & \mathit{slot\_of}(S_{14}, \text{HYDROGEN}) \\ \mathit{fills}(\text{CARBON}, S_{13}) & \mathit{fills}(\text{HYDROGEN}, S_{14}) \end{array} \tag{3}$$

Regarding improper slots, there are two possibilities: either those two slots are different from the ones previously mentioned, or some of them are identical to some of the previously mentioned slots. In this example, we chose the first possibility: S_{13} and S_{14} are different from all the other slots. All the slots of CARBON and HYDROGEN are inherited by METHANE, due to axiom (BA5): $\mathit{slot\_of}(S_i, \text{METHANE})$ $(6 \leq i \leq 14)$. Note that it is also the case for the improper slots.

3. Counting the Parts

3.1. Counting Criteria

"Counting how many times filler A has filler B as part" means counting the number of appearances of B in A. But what counts as a genuine appearance? As a matter of fact, because Bennett's theory includes improper slots, we can define two counting criteria. The first counting criterion, **C1**, is to count the number of different slots owned by A that are filled by B, whether they are also owned by B or not. The second counting criterion, **C2**, is to count the number of different slots owned by A that are filled by B and that are not owned by B (that is, that are not improper slots of B).

The results obtained with the two criteria will be compared to the result obtained when analysing a methane molecule particular, which has one carbon atom particular, four hydrogen atom particulars and ten electron particulars. Thus, we expect from our mereological theory and counting criterion to lead to the methane molecule universal METHANE having as parts the carbon atom universal CARBON once, the hydrogen atom universal HYDROGEN four times and the electron universal ELECTRON ten times.

3.2. Counting Problems

Bennett's theory raises two issues concerning the countability of parts: a first one stems from the existence of improper slots, and a second one from slots of parts. Ultimately, as we will see, both are caused by the slot inheritance axiom (BA5).

3.2.1. Improper Slot Problem

Let's say we want to count the number of times HYDROGEN is part of METHANE. METHANE has five different slots filled with HYDROGEN: four direct slots (S_2 to S_5) and one inherited slot (S_{14}), the latter being the improper slot of HYDROGEN itself. HYDROGEN is part of METHANE five times according to **C1** and four times according to **C2**.[6] By comparing those results with the

[6] Note that different results from the ones presented are possible if a different representation of the slot structure of the methane molecule is used.

expected result when we count how many hydrogen particulars belong to a particular of methane molecule, we can state that criterion **C2** leads to a correct result, whereas criterion **C1** leads to an incorrect result.

Among the possible models of the slot mereology, Figure 2 illustrates three models worthy of interest. In these models, *A* and *B* are different and *B* is part of *A*. Here is an informal description of the three models:

(a) *A* has only one slot S_1 that is filled by *B* and this slot is not owned by *B*;
(b) *A* has only one slot S_1 that is filled by *B* and this slot is also owned by *B* (and thus, is an improper slot of *B*);
(c) *A* has exactly two different slots S_1 and S_2 that are filled by *B*. One of these slots (say S_2) is also owned by *B*.

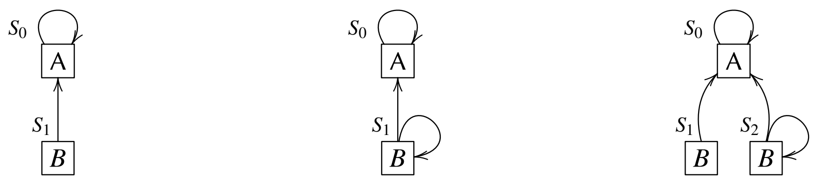


Figure 2: Three possible models of the slot mereology.

From the facts represented in Figure 2, we can deduce the following facts:

- in the model (a), according to both **C1** and **C2**, *B* is part of *A* once;
- in model (b), by **C1**, *B* is part of *A* once. However, by **C2**, *B* is part of *A*, but zero times ;
- in model (c), by **C1**, *B* is part of *A* twice. However, by **C2**, *B* is part of *A* once.[7]

According to the counting criteria, we get different results for models (b) and (c). Since it is obviously absurd for *B* to be part of *A*, but zero times, **C2** is inappropriate. Thus, we have shown that neither **C1** (by the METHANE example) nor **C2** (by examples of Figure 2) can be compatible with Bennett's theory. Therefore, we need to change Bennett's theory.

3.2.2. Parts of Parts Problem

The second problem stems from the parts of the parts. Let's say we want to count how many times ELECTRON is part of METHANE. If we do so, we will find six slots inherited from CARBON (namely S_6, S_7, S_8, S_9, S_{10}, S_{11}) and one slot inherited from HYDROGEN (namely S_{12}), that is, seven slots in total. **C1** would thus lead to METHANE having ELECTRON as part seven times, whereas **C2** would lead to it having ELECTRON as part at most seven times (depending on whether some of those slots also are improper slots of ELECTRON). This result is different from the expected result of ELECTRON being part of METHANE ten times.

[7] S_1 and S_2 are filled by the very same universal. In contrast to Bennett's figures in which slots can be drawn one inside another, we chose to separate them, even if it implies to repeat the filler.

3.2.3. Conclusion on the Two Counting Problems

These two problems are in fact caused by the same axiom of slot inheritance (BA5), which i) makes improper slots inheritable and ii) does not make the subparts inheritable multiple times. Since this axiom was presumably introduced by Bennett to allow parthood transitivity, we will need to replace it by alternative axioms that do not lead to the same problems, while still ensuring parthood transitivity.

4. Fixing Bennett's Theory

Our analysis above suggests that the theory lacks axioms that could ensure correct counting results according to our counting criteria. We suggest a few solutions in the following sections.

4.1. Constraining Improper Slots Further

Bennett does not state explicitly why she admitted in her theory the possibility for fillers to have improper slots. However, we can assume that they were introduced to satisfy conditional reflexivity (BT9) as they are not used for any other purpose in Bennett's paper. Hence, we could impose that an improper slot should not be owned by anything else than the filler it is an improper slot of, with axiom (A1).

Axiom A1. *Improper Slots are only owned by their Filler*

$$slot\_of(s,x) \wedge fills(x,s) \rightarrow \forall y(slot\_of(s,y) \rightarrow x = y)$$

Using axiom (A1) and definitions (BD1), (BD2), we can deduce the following theorem, which states that a proper part of A is a part of A by filling a slot it does not own itself:

Theorem T1. $\forall x,y,s(proper\_part\_of(y,x) \rightarrow \exists s(slot\_of(s,x) \wedge fills(y,s) \wedge \neg slot\_of(s,y)))$

Proof. Let x and y be two fillers such that y is a proper part of x. By definition of proper parthood (BD2), we know that $part\_of(y,x) \wedge \neg part\_of(x,y)$, which leads to $\exists s(slot\_of(s,x) \wedge fills(y,s))$.

To complete the proof, we need to show that s is not a slot of y. Suppose that s is a slot of y. We have $slot\_of(s,y) \wedge fills(y,s)$, which, according to (1), leads to $x = y$. However, we know that $\neg part\_of(x,y)$: contradiction. Therefore, we have $\neg slot\_of(s,y)$. □

This would make sure that every proper part of a filler fills a slot of this filler which it does not own. Therefore, models like model (b) in Figure 2 are excluded.

(A1) and Bennett's slot inheritance axiom (BA5) and definition (BD1) lead together to the following theorem (T2), stating that every part which has an improper slot is identical to its whole, which is way too restrictive, and leads to trivial models only. For this reason, the slot inheritance axiom should be revised.

Theorem T2. $\forall x,y,s(part\_of(y,x) \wedge slot\_of(s,y) \wedge fills(y,s) \rightarrow x = y)$

Proof. Let x and y be two fillers such that y is part of x. Let s be an improper slot of y. According to axiom (BA5), s is also a slot of x. Therefore, according to axiom (A1), $x = y$. □

We revise axiom (BA5) by accepting instead that if x is a part of y and s is a proper slot of x, then s is also a slot of y. That is, we restrict slot inheritance to proper slots, by the following axiom (A2).

Axiom A2. *Proper Slot Inheritance*

$$part\_of(x,y) \wedge slot\_of(s,x) \wedge \neg fills(x,s) \rightarrow slot\_of(s,y)$$

Transitivity of parthood (BT7) still holds when replacing axiom (BA5) by axiom (A2):

Proof. Let x, y and z be three fillers such that x is a part of y and y is a part of z.
If $x = y$ then $part\_of(x,z)$. Let's suppose now that $x \neq y$.
By definition of *part_of* (BD1), there is a slot s such that $slot\_of(s,y) \wedge fills(x,s)$.
By unicity of the filler and $x \neq y$, y does not fill s. Then, since y is a part of z, by (A2) s is a slot of z. Since x fills s, x is a part of z by definition of parthood (BD1).
Thus, in all cases, x is a part of z. □

There are still two things to discuss: the generalisation of improper slots to all fillers, and the possibility for a filler to have multiple improper slots.

Bennett's axiom (BA4) ensures that anything that possesses a slot has an improper slot. Bennett justifies the conditional reflexivity by stating that "the reflexivity of parthood is restricted to things that have parthood slots. That's because [(BA3)] and the definition of parthood entail that parthood slots cannot have parts at all" [1, p. 94]. This certainly justifies why slots are not part of themselves, but it does not justify why fillers without slots are not part of themselves. We found no other justification for this. Garbacz [9] made the same observation. We therefore generalize Bennett's axiom (BA4) by adding that every filler has an improper slot, in line with what Garbacz proposed in his axiom (GA9). We thus add (A3) to the theory.

Axiom A3. *Additional Improper Parthood Slots*

$$fills(x,s) \rightarrow \exists t(slot\_of(t,x) \wedge fills(x,t))$$

We can broaden the theorem of conditional reflexivity (BT9) as theorem (T3).

Theorem T3. *General Conditional Reflexivity*

$$\exists s(slot\_of(s,x) \vee fills(x,s)) \rightarrow part\_of(x,x)$$

Proof. This is a trivial consequence of (BD1), (BA4) and (A3). □

The last point to discuss about improper slots is that in Bennett's theory, an entity can have several improper slots. Remember that we determine the number of slots of a universal by considering the number of parts of a particular that would instantiate this universal. For example, a particular of METHANE has arguably itself as part only once. From this viewpoint, METHANE should have a unique improper slot. More generally, we add the following axiom (A4) asserting that a thing has only one improper slot.

Axiom A4. *Only One Improper Slot per Filler*

$$slot\_of(s,x) \land fills(x,s) \land slot\_of(t,x) \land fills(x,t) \rightarrow s = t$$

With these new axioms, we can reconsider the METHANE universal. The mereological structure of METHANE is described by Facts (4) for the proper slots as well as Facts (5) for the improper slots.

$$\begin{array}{rl|rl} slot\_of(S_i, \text{METHANE}) & 1 \leq i \leq 12 & slot\_of(S_i, \text{CARBON}) & 6 \leq i \leq 11 \\ slot\_of(S_{12}, \text{HYDROGEN}) & & fills(\text{CARBON}, S_1) & \\ fills(\text{HYDROGEN}, S_i) & 2 \leq i \leq 5 & fills(\text{ELECTRON}, S_i) & 6 \leq i \leq 12 \end{array} \quad (4)$$

$$\begin{array}{ll} slot\_of(S_0, \text{METHANE}) & fills(\text{METHANE}, S_0) \\ slot\_of(S_{13}, \text{CARBON}) & fills(\text{CARBON}, S_{13}) \\ slot\_of(S_{14}, \text{HYDROGEN}) & fills(\text{HYDROGEN}, S_{14}) \\ slot\_of(S_{15}, \text{ELECTRON}) & fills(\text{ELECTRON}, S_{15}) \end{array} \quad (5)$$

If we count how many times HYDROGEN is part of METHANE, the result is four times, for both counting criteria **C1** and **C2**. In the remainder, we will no longer refer to counting criteria **C1** and **C2** for proper parts, as they are equivalent in the new theory, since improper slots are not inheritable anymore. Also, as every filler has now a unique improper slot, improper slots will no longer be represented on figures in the remainder of the paper.

4.2. Parts of Parts

4.2.1. Pre-Formal Idea

The second problem comes from the inheritance by the whole of the proper slots owned by its parts. As shown with the slots filled by ELECTRON in section 3.2.2, these slots are not inherited the correct number of times. To solve this problem, we will propose a different system. This system will rest on the pre-formal intuition that slots should not be inherited but *copied*.

To implement this idea, we drop the slot inheritance axioms (BA5) and (A2) altogether and "simulate" a (controlled) slot inheritance using slots that we will call "copy-slots". If a whole is a part of a bigger whole, its entire structure is copied using copy-slots. These copy-slots are slots that have the same filler as the slot they are copied from. Improper slots are not copied.

We can represent from which slot a copy-slot is copied from with the relation *copied_from*. We can also represent through which HYDROGEN-filled slots the copy-slots (filled by ELECTRON) are copied thanks to another relation: *copied_through* between a copy-slot and a slot. Those two relations are represented in Figure 3, where $\overset{S_i}{\square} \leftarrow - \overset{S_j}{\square}$ represents *copied_through*(S_j, S_i) and $\overset{S_i}{\square} \leftarrow \cdots \overset{S_j}{\square}$ represents *copied_from*(S_j, S_i). On this figure, S_3 is a copy-slot owned by A which is copied from S_2 through S_1.

Let's see what would happen on METHANE by using this pre-formal idea. CARBON fills one slot of METHANE. Therefore, its structure is present only once: METHANE has six copy-slots filled by ELECTRON. HYDROGEN fills four slots of METHANE. Hence, its structure is repeated four times: METHANE has four additional copy-slots filled by ELECTRON. With this structure

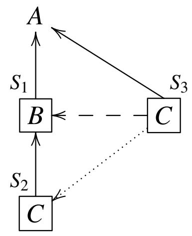


Figure 3: Example of application of *copied_from* and *copied_through*.

repetition, METHANE has now ten copy-slots filled by ELECTRON, which is the correct number. That is, among the ten copy-slots of METHANE filled by ELECTRON, six of them are copied from the six electron-filled slots of CARBON and four of them are the result of copying four times from the one electron-filled slot of HYDROGEN.

4.2.2. Axiomatizing Copy-Slots

According to our pre-formal idea, the first thing to do is to get rid of the revised slot inheritance axiom (A2) (as well as the original (BA5)) and endorse an axiom of anti-inheritance (A5).[8]

Axiom A5. *Anti-inheritance.* $[slot\_of(s,y) \wedge fills(x,s) \wedge slot\_of(t,x)] \rightarrow \neg slot\_of(t,y)$

Since (A2) was previously used to prove parthood transitivity (BT7), we need to accept new axioms involving copy-slots that would enable to prove (BT7). To illustrate how copy-slots work, let's use a simpler example: the HELIUMDIMER universal, whose mereological structure is pictured in Figure 4a.

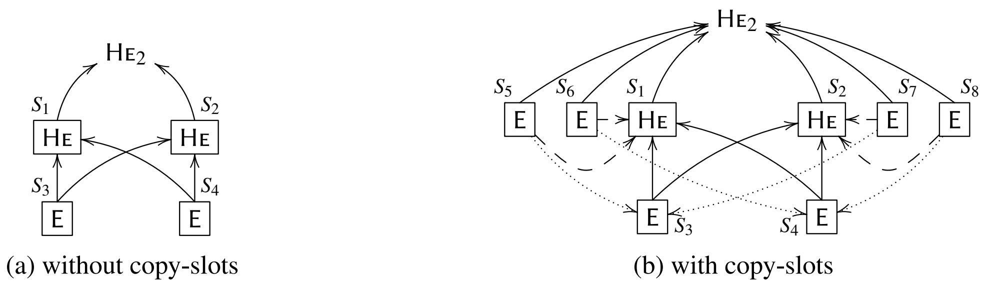


Figure 4: The mereological structure of HELIUMDIMER without and with copy-slots. The chemical symbols are used: He$_2$ is HELIUMDIMER, He is HELIUM and E is ELECTRON.

Note that since a particular of HELIUMDIMER has four particulars of ELECTRON as parts, the universal HELIUMDIMER should have four slots filled by ELECTRON. And this is indeed the case in our theory. As a matter of fact, the HELIUM universal has two slots filled by ELECTRON, which are each copied twice, through each of the two slots of HELIUMDIMER filled by HELIUM.

More generally, we can say that there are as many copy-slots as there are possible pairs of slots (s,t) such that the first element is the slot through which the copy-slot copies, called "path-slot",

[8] This implies that parthood is discrete [14], which is not an issue for the examples considered here.

and the second element, called "source-slot", is the slot from which the copy-slot is copied. In the case of the HELIUMDIMER, those pairs are (S_1, S_3), (S_1, S_4), (S_2, S_3) and (S_2, S_4). This is what axiom (A6) describes. Axiom (A7) imposes that the copy-slot has the same filler as its source-slot. Figure 4b pictures how copy-slots work with the HELIUMDIMER universal.

Axiom A6. *Existence of a Unique Copy-Slot for each Whole and Path-Slot, Source-Slot Pair.*

$$proper\_slot\_of(s,x) \land fills(y,s) \land proper\_slot\_of(t,y) \rightarrow$$
$$\exists! u(slot\_of(u,x) \land copied\_through(u,s) \land copied\_from(u,t))$$

Axiom A7. *Copied Slot has the Same Filler as its Source.*

$$copied\_from(t,s) \rightarrow \exists x(fills(x,s) \land fills(x,t))$$

Axioms (A6) and (A7) and definition (BD1) are sufficient to prove the theorem of transitivity (BT7):

Proof. Let x, y and z be three fillers such that x is a part of y and y is a part of z.

If $x = y$ then x is a part of z. So let's suppose that $x \neq y$.

By definition of *part_of* (BD1), there are two slots s and t such that $slot\_of(s,y) \land fills(x,s) \land slot\_of(t,z) \land fills(y,t)$.

According to axiom (A6), there is a slot u of z copied from s through t.

By (A7), since x fill s, x also fills u. Therefore, x is a part of z. □

As pictured in Figure 3, any copy-slot is owned by the same filler as the path-slot it copies through is. Axiom (A8) ensures that both slots are owned by the same filler. Also, any pair of path-slot and source-slot are related by a filler: the path-slot is filled by it, and the source-slot is owned by it. Axiom (A9) ensures that the path-slot and the source-slot are related.

Axiom A8. $proper\_slot\_of(t,x) \land copied\_through(t,s) \rightarrow proper\_slot\_of(s,x)$

Axiom A9. $copied\_through(u,s) \land copied\_from(u,t) \rightarrow \exists x(fills(x,s) \land proper\_slot\_of(t,x))$

Finally, both relations *copied_from* and *copied_through* are constrained to be functional, by axioms (A10) and (A11).[9] This ensures that a copy-slot is only related to one pair. Otherwise, we cannot be sure that counting yields a proper result; if the same copy-slot is used for multiple pairs of slots filled with CARBON and ELECTRON, METHANE will not have the right number of ELECTRON parts.

Axiom A10. $copied\_from(s,t) \land copied\_from(s,u) \rightarrow t = u$

Axiom A11. $copied\_through(s,t) \land copied\_through(s,u) \rightarrow t = u$

The resulting theory, with axioms (BA1)-(BA4), (BA6)-(BA8), (A1), (A3)-(A11), along with definitions (BD1)-(BD5) and (D1), is sufficient to prove that the problem of counting inherited slots is solved. The proof below focuses on the representative case of a filler a that has b as part m times, where b itself has c as part n times. It shows that a has c as part $m \times n$ times.

Proof. Let a, b and c be three different fillers, let $s_1, \ldots, s_m$ be different slots of a filled by b and let $z_1, \ldots, z_n$ be different slots of b filled by c.

[9] These axioms are discussed in the context of overlap in section 5.1.

We want to make sure that a has exactly one slot filled by c for each pair (s_i, z_j) of slots.

Let's first prove that a has at least one slot filled by c for each pair. Let (s_i, z_j) and (s_k, z_l) be two different pairs. Since $\mathit{proper\_slot\_of}(s_i, a) \wedge \mathit{fills}(b, s_i) \wedge \mathit{proper\_slot\_of}(z_j, b)$, according to axiom (A6), there is a copy-slot v such that $\mathit{slot\_of}(v, a)$, $\mathit{copied\_through}(v, s_i)$ and $\mathit{copied\_from}(v, z_j)$. Assume that v is also the copy-slot for the pair (s_k, z_l), i.e., $\mathit{copied\_from}(v, z_l)$ and $\mathit{copied\_through}(v, s_k)$. According to axioms (A10) and (A11), both relations *copied_from* and *copied_through* are functional. Therefore, we deduce that $s_i = s_k$ and $z_j = z_l$, making the two pairs the same: contradiction. Hence, there is a different slot of a filled by c for each pair. Due to the unicity in axiom (A6), there is at most one slot for each pair.

We can conclude that a, having at least and at most one slot for each pair, has exactly the right number of slots filled by c. □

Let's illustrate this on the METHANE universal. Facts (6) describe the mereological structure without improper slots nor copy-slots; note that Facts (5) still hold in addition. Facts (7) describe the copy-slots. With copy-slots, METHANE has exactly ten slots (S_{16} to S_{25}) filled by ELECTRON, which is the expected result.

$$\begin{array}{rl|rl} \mathit{slot\_of}(S_i, \text{METHANE}) & 1 \leq i \leq 5 & \mathit{slot\_of}(S_i, \text{CARBON}) & 6 \leq i \leq 11 \\ \mathit{slot\_of}(S_{12}, \text{HYDROGEN}) & & \mathit{fills}(\text{CARBON}, S_1) & \\ \mathit{fills}(\text{HYDROGEN}, S_i) & 2 \leq i \leq 5 & \mathit{fills}(\text{ELECTRON}, S_i) & 6 \leq i \leq 12 \end{array} \quad (6)$$

$$\begin{array}{rl|rl} \mathit{slot\_of}(S_i, \text{METHANE}) & 16 \leq i \leq 25 & \mathit{fills}(\text{ELECTRON}, S_i) & 16 \leq i \leq 25 \\ \mathit{copied\_through}(S_i, S_1) & 16 \leq i \leq 21 & \mathit{copied\_through}(S_i, S_{i-20}) & 22 \leq i \leq 25 \\ \mathit{copied\_from}(S_i, S_{i-10}) & 16 \leq i \leq 21 & \mathit{copied\_from}(S_i, S_{12}) & 22 \leq i \leq 25 \end{array} \quad (7)$$

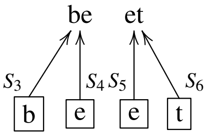
5. Discussion

5.1. Overlap

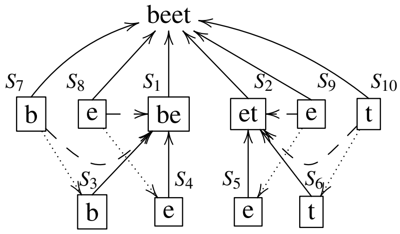
Bennett's system has two relations related to overlap: classical *overlap* (BD3) and *slot_overlap* (BD4). However, these relations are of little interest for structural universals. Let's consider our standard examples, universals of molecules including METHANE. First, any molecule universal that has CARBON as part overlaps all other molecules having CARBON as part. Furthermore, if CARBON has proper parts, it also has an improper slot. This improper slot is inherited by every molecule CARBON is part of. Therefore, all those molecules are also slot-overlapping in Bennett's theory.

Nevertheless, provided we ignore slot-overlapping on improper slots, slot-overlapping is an interesting feature. As argued by Bennett herself, in the case of structural universals, what best mimics Classical Mereology's overlap in standard domains is *slot_overlap* rather than *overlap*.

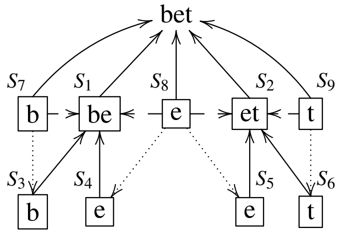
The system proposed in this paper disables improper slot inheritance. Therefore, wholes can only slot-overlap on proper slots. Yet, the current copy-slots theory to handle slot inheritance makes overlap nearly impossible. Figure 5a pictures the mereological structures of two strings “be” and “et”, respectively composed of letters “b” and “e”, and “e” and “t”. These two strings can be composed to create (at least) two other strings: “beet” and “bet”. The theoretical extension of slot mereology exposed in this paper enables representing the mereological structure of “beet”,



(a) “be” and “et”



(b) “beet”



(c) “bet”

Figure 5: Mereological structures of some strings.

as pictured in Figure 5b.[10] However, the mereological structure of “bet” is not representable. Indeed, we would like a structure like the one pictured on Figure 5c. In this figure, “bet” has only one slot filled by “e” (namely S_8), otherwise, the counting criterion would not be satisfied. As “e” is part of “be” and “et” (respectively by slots S_4 and S_5), S_8 should be copied from both S_4 and S_5, and through S_1 and S_2. But axioms (A10) and (A11) prevent this. Therefore, in our system, “bet”, as composed of “be” and “et”, is not representable. If we want to be able to handle slot-overlap, the first step is to remove (A10) and (A11). The resulting theoretical issue—making sure that copy-slots are not reused when overlap is not at stake, but may be reused when overlap is involved—will be addressed in future work.

5.2. Constraining the Slot Supplementation Further

Garbacz [9] analysed Bennett’s slot mereology inadequacies, in particular slot supplementation. He noted that Bennett’s theorem of weak slot supplementation (BT13) is arguably rather superficial.[11] In Classical Mereology, weak supplementation is meant to forbid the decomposition of a whole into a single proper part [15]. In Bennett’s theory, theorem (BT13), although presented as a reformulated version of weak supplementation, misses the whole point. Figure 6a illustrates a model challenging Bennett’s weak slot supplementation: theorem (BT13) is satisfied while X is a single proper part of Y. (BT13) stipulates that there is a slot z owned by Y but not by X. In 6a, two slots are in this situation. The first one is the improper slot of Y, S_0. To reject this model, the weak slot supplementation could be changed so as to only consider proper slots. However, and quite surprisingly, in 6a a second slot plays the role of z in the theorem, namely, the slot that makes X a proper part of Y, S_1 itself. To fix this problem, one could think to change weak slot supplementation into stating that the slot z should not be filled by x.

We do not implement such changes here, because this constraint would hinder models such as the one pictured in Figure 6b, which is a similar model as HeliumDimer presented above. More work, beyond Garbacz’s considerations on the proof of weak supplementation in Bennett’s theory, is needed to obtain an adequate version of weak supplementation in slot mereology.

[10]Here, only the mereological structure of strings is considered, order is ignored. Therefore, “beet” and “bete” are actually the same individual.

[11]Theorem (BT13) still holds in our theory, since, as proved by Garbacz [9], axiom (BA4), which we keep, suffices to prove it.

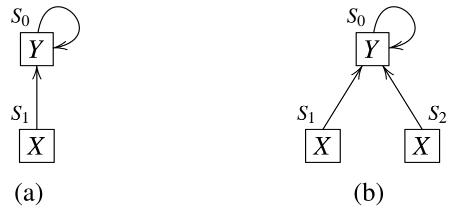


Figure 6: Models challenging Weak Slot Supplementation.

5.3. Using the Revised Slot Mereology with Particulars

As mentioned earlier, Bennett's slot mereology can be applied not only to structural universals, but also to informational entity particulars, as they also can have a part several times. However, it does not seem to bring any benefit for the analysis of material entity particulars such as a bike, a human body or a table (although we could still apply a degenerated slot mereology in which each part could appear only once). Indeed, a canonical human body would have two arms, but not twice the same arm. Therefore, a classical mereology theory should probably be used in complement of the revised slot mereology we proposed here. This will imply analyzing how these two theories interact and, in particular, how a slot-mereological relation between universals is reflected by a classical mereological relation among their instances. Such articulation might also enable to represent "silent change" [16] in which an assertion remains true at the universal level while the involved particulars may change (such as "Every car has an engine as part", even though one engine instance might be replaced by another one in the same car).

6. Conclusion

In this paper we discussed Bennett's slot mereology [1], mainly illustrated on standard examples of structural universals such as METHANE. In section 3, we exposed two counting criteria **C1** and **C2** and showed how Bennett's system, despite having been introduced with the motivation to account for having a part several times over, fails on counting parts with both counting criteria. More precisely, we first showed that Bennett's axiom of slot inheritance (BA5) was problematic for counting. As shown in section 3.2.1, improper slots are inherited in Bennett's system, and therefore are taken into account by counting criterion **C1**—although they arguably should not. Therefore, in section 4.1, we proposed to constrain improper slots. Those constraints make improper slots non-inheritable and only relevant for ensuring parthood reflexivity—but not for counting. Second, in section 3.2.2, we showed that neither slot inheritance (BA5) nor the proposed revised slot inheritance (A2) yield a theory with models satisfying any of the two counting criteria when parts of parts are involved. Therefore, we proposed in section 4.2 to replace slot inheritance by slot copy. The new system was shown to yield correct results for both counting criteria with non-overlapping mereological structures. We are confident that this revision can serve as a basis for further extensions with potential for multiple applications beyond structural universals, for instance, for informational entities.

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