Navigating the $\mathcal{EL}$ Subsumption Hierarchy *

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Abstract. The $\mathcal{EL}$ subsumption hierarchy consists of all $\mathcal{EL}$ concept descriptions and is partially ordered by subsumption. In order to navigate within this hierarchy, one can go up to subsumers and down to subsumees. We analyze how smallest steps can be made. Specifically, we show how all upper neighbors as well as all lower neighbors of a given $\mathcal{EL}$ concept description can be efficiently computed, where two concepts are neighbors if one subsumes the other and there is no third concept in between. We further show that the hierarchy contains very long chains: there is a sequence of concepts C_n with size linear in n such that each chain of neighbors from $\top$ to C_n has at least n-fold exponential length. As applications, we provide a template for determining upper complexity bounds for deciding whether a concept is maximally general or maximally specific w.r.t. a property, we construct a metric on the set of all $\mathcal{EL}$ concept descriptions, we introduce a similarity measure that fulfills the triangle inequality, and we conclude that an uninformed search for a target concept by subsequently computing neighbors or, equivalently, along an ideal refinement operator is not feasible in practical applications.

Keywords: Description logic · Subsumption · Upper neighbor · Lower neighbor · Distance measure · Metric · Refinement operator

1 Introduction

The $\mathcal{EL}$ subsumption hierarchy consists of all $\mathcal{EL}$ concept descriptions and is partially ordered by subsumption. In order to navigate within this hierarchy, one can go up to subsumers and down to subsumees. We analyze how smallest steps can be made. Specifically, we show how all upper neighbors as well as all lower neighbors of a given $\mathcal{EL}$ concept description can be efficiently computed, where two concepts are neighbors if one subsumes the other and there is no third concept in between. The complexity for both computation tasks is determined: while all upper neighbors can be computed in polynomial time, computation of all lower neighbors needs exponential time.

Secondly, we employ a standard construction from Lattice Theory [6] for constructing a distance measure (metric) on the set of all $\mathcal{EL}$ concept descriptions. Specifically, the distance between two concepts C and D coincides with the

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length of a shortest path in the undirected graph the vertex set of which consists of all $\mathcal{EL}$ concept descriptions and in which neighboring concept descriptions are connected by an edge. An asymptotic analysis reveals that the distance between $\top$ and $\exists r_1. \cdots \exists r_n.(A_1 \sqcap \cdots \sqcap A_k)$ where $k \geq 3$ is asymptotically bounded from below by the n-fold iterated exponential

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^{2^k}}}}}}}_{n \text{ times}}.$$

It follows that the subsumption hierarchy contains very long chains: each chain of neighbors from $\top$ to $\exists r_1. \cdots \exists r_n.(A_1 \sqcap \cdots \sqcap A_k)$ has n-fold exponential length.

Thirdly, we devise some consequences and applications. We provide a template for determining upper complexity bounds for deciding whether a concept description is maximally specific or maximally general w.r.t. a given property. We construct a similarity measure on $\mathcal{EL}$ concept descriptions that fulfills the triangle inequality. We conclude that an uninformed search for a target concept by subsequently computing neighbors or, equivalently, along an ideal refinement operator is not feasible in practical applications. The latter applies, in particular, to an approach that tries to learn a target concept [20] as well as to the problem of computing a maximally strong weakening w.r.t. $\succ^{\text{sub}}$ (but only for this particular weakening relation!) in the context of computing a gentle repair [3].

In addition to citing and rephrasing existing results on the enumeration of upper neighbors as well as a construction of the distance measure, this article provides as new contributions an efficient approach to computing lower neighbors and an asymptotic analysis of the distance measure. Specifically, a contained result is a previously unpublished, new contribution if and only it is followed only by a reference to [16]. Proofs can be found in the mentioned references.

2 Preliminaries

The $\mathcal{EL}$ Subsumption Hierarchy. We assume basic knowledge of the description logic $\mathcal{EL}$. For a *signature* $\Sigma \coloneqq \Sigma_\mathsf{C} \cup \Sigma_\mathsf{R}$ consisting of concept names and role names, *$\mathcal{EL}$ concept descriptions* are built from Σ by means of the constructors $\top$, $\sqcap$, and $\exists$, and the set of all concept descriptions is denoted as $\mathcal{EL}(\Sigma)$. We treat (nested) conjunctions like sets, i.e., nestings, order, and repetitions are not relevant. Concept names and existential restrictions are also called *atoms*. Each concept C is a conjunction of atoms, the *top-level conjuncts* of C, and we denote the set of all these atoms as $\mathsf{Conj}(C)$. $\top$ is the empty conjunction. The *role depth* of a concept is the maximal number of nestings of existential restrictions.

Given two concept descriptions C and D, we say that *C is subsumed by D*, written $C \sqsubseteq_\emptyset D$, if $C^\mathcal{I} \subseteq D^\mathcal{I}$ for each interpretation $\mathcal{I}$. We use the subscript $\emptyset$ to indicate that we consider subsumption w.r.t. the empty TBox. It is well-known that the *subsumption relation* $\sqsubseteq_\emptyset$ is a partial order — a reflexive, transitive binary relation — on the set $\mathcal{EL}(\Sigma)$. Consequently, the quotient of $\mathcal{EL}(\Sigma)$ w.r.t. the induced equivalence relation $\equiv_\emptyset$ is a partially ordered set. In what follows, we will not distinguish between the equivalence classes and their representatives, with the consequence that $\sqsubseteq_\emptyset$ becomes a partial order — a reflexive, transitive, antisymmetric binary relation. As representatives we usually take the reduced concepts, where a concept is *reduced* if none of its conjunctions contains atoms that are comparable w.r.t. $\sqsubseteq_\emptyset$. According to [17], each concept has a unique reduced form to which it is equivalent. The $\mathcal{EL}$ *subsumption hierarchy* is the partially ordered set (poset) $(\mathcal{EL}(\Sigma), \sqsubseteq_\emptyset)$.

Upper and Lower Neighbors. For a partially ordered set $(P, \leq)$, its *neighborhood relation*[1] consists of all pairs (p, q) where $p < q$ and there exists no x such that $p < x < q$. Usually, if $p \prec q$, then one calls p a *lower neighbor* of q and, vice versa, q is called an *upper neighbor* of p.[2] We say that $\leq$ is *neighborhood generated* if the transitive closure $\prec^+$ equals the irreflexive part $<$. Obviously, this is the case if and only if, for each pair $p \leq q$, there is a finite chain of neighbors from p to q, i.e., there are elements $x_0, \ldots, x_n$ such that $p = x_0 \prec x_1 \prec \cdots \prec x_n = q$.

Clearly, if the underlying set P is finite, then each partial order on P is neighborhood generated. However, there are infinite posets where this does not hold true; even worse, there are cases where $\prec$ and thus also $\prec^+$ are both empty. For instance, consider the set $\mathbb{R}$ of real numbers with their usual ordering $\leq$. It is well known that $\mathbb{R}$ is dense in itself, i.e., for each pair $x < y$, there is another real number z in between, i.e., such that $x < z < y$ — thus, there are no neighboring real numbers. Another example of a partially ordered set that is not neighborhood generated is the set $\mathbb{N}$ of all natural numbers augmented with a greatest element ∞, which then does not have any neighbors.

As an alternative formulation, a partial order $\leq$ is *not* neighborhood generated if and only if there exists a pair $p < q$ such that every finite chain[3] $p = x_0 < x_1 < \cdots < x_n = q$ can be refined, i.e., there is some index i and an element y such that $x_i < y < x_{i+1}$. It follows that $\leq$ is neighborhood generated if $\leq$ is *bounded*, i.e., for each element p, there is a finite upper bound on the lengths of chains starting with p. Although boundedness is sufficient for neighborhood-generatedness, it is not necessary: the following result immediately implies that each unbounded poset can be order-embedded[4] into a neighborhood generated poset, which must be unbounded as well.

Proposition 1. [12], [16, Page 114]. *Each poset can be order-embedded into some neighborhood generated poset.*

Proof sketch. Let $(P, \leq)$ be a poset. Firstly, we define the poset $(Q, \sqsubseteq)$ by adding a fresh element between each two $\leq$-comparable elements of P. It is easy to see that $(Q, \sqsubseteq)$ must be neighborhood generated. Secondly, we can map each element of P to itself in Q in order to obtain an order-embedding. □

[1] While in [13] the name "neighborhood relation" is used, it is called "covers relation" in [6] and "one-step relation" in [3]. A further alternative name is "transitive reduction."

[2] Alternative denotations are "lower cover" and "upper cover."

[3] Formally, a *chain* in a poset $(P, \leq)$ is a subset $C \subseteq P$ such that $p \leq q$ or $q \leq p$ for each two elements $p, q \in C$. It *starts with* p if $p \leq q$ for each $q \in C$.

[4] An *order-embedding* h from $(P, \leq)$ into $(Q, \sqsubseteq)$ is an injective mapping $h\colon P \to Q$ such that $p \leq q$ if and only if $h(p) \sqsubseteq h(q)$ for all $p, q \in P$.

3 Neighbors of $\mathcal{EL}$ Concept Descriptions

This section is concerned with the questions whether the subsumption relation is neighborhood generated and how upper and lower neighbors can be enumerated.

Definition 2. *Let C and D be concept descriptions. We call C a* lower neighbor *of D and D an* upper neighbor[5] *of C, written $C \prec_\emptyset D$ and $D \succ_\emptyset C$, respectively, if*[6] *$C \sqsubset_\emptyset D$ and there does not exist a concept E such that $C \sqsubset_\emptyset E \sqsubset_\emptyset D$.*

Here, we only consider the empty TBox — other types of TBoxes (cycle-restricted, acyclic, general) as well as two extensions of $\mathcal{EL}$ (with $\bot$, with greatest fixed-points) are considered in [16] but left out here due to a lack of space. As short summary, the subsumption relation is neighborhood generated if the TBox is cycle-restricted or acylic, but not neighborhood generated for general TBoxes or if $\bot$ or greatest fixed-points are added.

In the proof of Proposition 3.5 in [5] it was shown that the subsumption relation $\sqsubseteq_\emptyset$ is bounded and so we can immediately draw the following conclusion.

Proposition 3. [15, Proposition 2], [16, Proposition 5.1.2]. *The subsumption relation $\sqsubseteq_\emptyset$ on $\mathcal{EL}(\Sigma)$ is neighborhood generated.*

Next, we show how all upper and all lower neighbors of a given $\mathcal{EL}$ concept description C can be computed. Recall that there is the following recursive characterization of the subsumption relation [5]: $C \sqsubseteq_\emptyset D$ if and only if $A \in \mathsf{Conj}(D)$ implies $A \in \mathsf{Conj}(C)$ for each concept name A, and for each existential restriction $\exists r.F \in \mathsf{Conj}(D)$, there is $\exists r.E \in \mathsf{Conj}(C)$ such that $E \sqsubseteq_\emptyset F$. It follows that, given a concept description C, we obtain an upper neighbor of C by removing one concept name from $\mathsf{Conj}(C)$ and, vice versa, we get a lower neighbor of C by adding a concept name to $\mathsf{Conj}(C)$ that is not already there.

Dealing with the existential restrictions is more involved. To construct an upper neighbor, we cannot simply remove an existential restriction $\exists r.D$ from $\mathsf{Conj}(C)$, since the resulting concept might be too general — according to the above characterization of $\sqsubseteq_\emptyset$ it holds true that $\exists r.D \sqsubset_\emptyset \bigsqcap\{\, \exists r.E \mid D \prec_\emptyset E \,\}$ and so we infer that[7] the concept $(C \setminus \exists r.D) \sqcap \bigsqcap\{\, \exists r.E \mid D \prec_\emptyset E \,\}$ is between C and $C \setminus \exists r.D$. In fact, we can prove that the concept in the middle is always an upper neighbor of C and further that each upper neighbor is either of this form or of the form $C \setminus A$ for some concept name $A \in \mathsf{Conj}(C)$.

Definition 4. *For each reduced $\mathcal{EL}$ concept description C, we recursively define the set $\mathsf{Upper}(C) := \{\, C^{\uparrow G} \mid G \in \mathsf{Conj}(C) \,\}$ where $C^{\uparrow A} := C \setminus A$ and $C^{\uparrow \exists r.D} := (C \setminus \exists r.D) \sqcap \bigsqcap\{\, \exists r.E \mid E \in \mathsf{Upper}(D) \,\}$.*

Proposition 5. [15, Proposition 4], [3, Lemma 22], [16, Proposition 5.1.5]. *For each reduced $\mathcal{EL}$ concept description C, the set $\mathsf{Upper}(C)$ consists of all upper neighbors of C (modulo equivalence), i.e., a concept description D is an upper neighbor of C if and only if there is some $D' \in \mathsf{Upper}(C)$ such that $D \equiv_\emptyset D'$.*

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[5] Alternative denotations could be "direct subsumee" and "direct subsumer".
[6] We write $C \sqsubset_\emptyset D$ to indicate that $C \sqsubseteq_\emptyset D$ and $D \not\sqsubseteq_\emptyset C$.
[7] For $D \in \mathsf{Conj}(C)$, we abbreviate by $C \setminus D$ the concept description $\bigsqcap(\mathsf{Conj}(C) \setminus \{D\})$.

The mapping $G \mapsto C^{\uparrow G}$ is a bijection from $\mathsf{Conj}(C)$ to $\mathsf{Upper}(C)$, cf. Proposition 5.1.6 in [16]. Thus, each top-level conjunct of C induces a unique upper neighbor of C.

Proposition 6. [16, Proposition 5.1.16]. $\mathsf{Upper}(C)$ *can be computed in quadratic time (w.r.t. C), each neighbor in* $\mathsf{Upper}(C)$ *has quadratic size (w.r.t. C),* $\mathsf{Upper}(C)$ *consists of linearly many neighbors (w.r.t. C), and each two different neighbors in* $\mathsf{Upper}(C)$ *are not equivalent.*

An immediate consequence of Proposition 6 is the following result.

Proposition 7. [16, Theorem 5.1.17]. *The neighborhood relation $\prec_\emptyset$ can be decided in polynomial time.*

Example 8. Consider the concept $C \coloneqq A \sqcap \exists r.(B_1 \sqcap B_2 \sqcap B_3)$. The upper neighbor induced by the first top-level conjunct A is $\exists r.(B_1 \sqcap B_2 \sqcap B_3)$. In order to compute the upper neighbor induced by the second top-level conjunct $\exists r.(B_1 \sqcap B_2 \sqcap B_3)$, we remove that atom and in its stead add the existential restrictions $\exists r.D$ where D ranges over all upper neighbors of $B_1 \sqcap B_2 \sqcap B_3$ — we obtain $A \sqcap \exists r.(B_1 \sqcap B_2) \sqcap \exists r.(B_1 \sqcap B_3) \sqcap \exists r.(B_2 \sqcap B_3)$ as the second upper neighbor of C.

It remains to investigate how lower neighbors can be obtained by adding an existential restriction. The above characterization of upper neighbors provides a first hint: a lower neighbor of a concept C can be obtained by adding an existential restriction $\exists r.D$ for which all $\exists r.E$ for all upper neighbors E of D are already present in C (in a certain sense). More specifically, we observe that $C \sqcap \exists r.D$ is a lower neighbor of C if and only if (L1) $C \not\sqsubseteq_\emptyset \exists r.D$ and (L2) $C \sqsubseteq_\emptyset \exists r.E$ for each upper neighbor E of D, since the concept $C \sqcap \exists r.D$ is a lower neighbor of $C \sqcap \bigsqcap\{\, \exists r.E \mid E \in \mathsf{Upper}(D) \,\}$, cf. Proposition 5.

According to the recursive characterization of $\sqsubseteq_\emptyset$ and using the notation $\mathsf{Succ}(C, r) \coloneqq \{\, F \mid \exists r.F \in \mathsf{Conj}(C) \,\}$, the above Condition (L2) is satisfied if and only if there is a function $\psi\colon \mathsf{Upper}(D) \to \mathsf{Succ}(C, r)$ such that $\psi(E) \sqsubseteq_\emptyset E$ for each upper neighbor E of D. This function ψ is injective. (Assume there would be two different upper neighbors U and V of D such that $\psi(U) = \psi(V)$. We would obtain $\psi(U) \sqsubseteq_\emptyset U \sqcap V$. Since $U \sqcap V$ is equivalent to D, we would obtain a contradiction to the above condition (L1).)

By means of the above bijection between $\mathsf{Conj}(D)$ and $\mathsf{Upper}(D)$ we can transform ψ to an injective function $\phi\colon \mathsf{Conj}(D) \to \mathsf{Succ}(C, r)$ where $\phi(G) \sqsubseteq_\emptyset D^{\uparrow G}$ for each $G \in \mathsf{Conj}(D)$. Since ϕ is injective, we can invert it and so obtain a surjective partial mapping[8] $\chi\colon \mathsf{Succ}(C, r) \nrightarrow \mathsf{Conj}(D)$ such that $F \sqsubseteq_\emptyset D^{\uparrow \chi(F)}$ for each $F \in \mathsf{Dom}(\chi)$. Since χ is surjective, we conclude that D must be the conjunction of all atoms in $\mathsf{Ran}(\chi)$. Furthermore, it is not hard to prove that $F \sqcap \chi(F)$ is a lower neighbor of F[9] for each $F \in \mathsf{Dom}(\chi)$ and further that $F' \sqsubseteq_\emptyset \chi(F)$ for each two different $F, F' \in \mathsf{Dom}(\chi)$. We call χ a lowering function

[8] For each partial function $f\colon A \nrightarrow B$, its *domain* is $\mathsf{Dom}(f) \coloneqq \{\, a \mid f(a) \text{ is defined} \,\}$ and its *range* is $\mathsf{Ran}(f) \coloneqq \{\, f(a) \mid a \in \mathsf{Dom}(f) \,\}$.

[9] That's why $\chi(F)$ could be called a *lowering atom* of F as in [2].

and, to sum up, we have seen that there exists such a lowering function χ if $C \sqcap \exists r.\, D$ is a lower neighbor of C. Actually, the converse direction holds true as well and we obtain the following formal characterization.

Definition 9. *Let C be a reduced $\mathcal{EL}$ concept description. We define the following by mutual recursion on the role depth of C. Firstly, define the set*

$$\begin{aligned}\mathsf{Lower}(C) &\coloneqq \{\, C \sqcap A \mid A \in \Sigma_{\mathsf{C}} \text{ and } A \notin \mathsf{Conj}(C) \,\} \\ &\cup \{\, C \sqcap \exists r.\, \textstyle\bigsqcap \mathsf{Ran}(\chi) \mid r \in \Sigma_{\mathsf{R}} \text{ and } \chi \text{ is a lowering function for } r \,\}\,.\end{aligned}$$

Secondly, a lowering function *for a role name r is a partial mapping $\chi\colon \mathsf{Succ}(C, r) \nrightarrow \mathcal{EL}(\Sigma)$ such that (1) $F \sqcap \chi(F) \in \mathsf{Lower}(F)$ for each $F \in \mathsf{Dom}(\chi)$, (2) $F' \sqsubseteq_\emptyset \chi(F)$ for each two different $F, F' \in \mathsf{Dom}(\chi)$, and (3) $C \not\sqsubseteq_\emptyset \exists r.\, \bigsqcap \mathsf{Ran}(\chi)$.*

Proposition 10. [16, Corollary 5.1.13]. *For each reduced $\mathcal{EL}$ concept C, the set $\mathsf{Lower}(C)$ consists of all lower neighbors of C (modulo equivalence), i.e., a concept D is a lower neighbor of C if and only if $D \equiv_\emptyset D'$ for some $D' \in \mathsf{Lower}(C)$.*

Note that in [2] an efficient non-deterministic algorithm is described that produces all lower neighbors on its successful computation paths.

Proposition 11. [16, Propositions 5.1.19 and 5.1.21]. *$\mathsf{Lower}(C)$ can be computed in exponential time (w.r.t. C and Σ), each neighbor in $\mathsf{Lower}(C)$ has quadratic size (w.r.t. C), $\mathsf{Lower}(C)$ consists of at most exponentially many neighbors (w.r.t. C and Σ), and each two different neighbors in $\mathsf{Lower}(C)$ are not equivalent.*

The following example shows that the exponential blow-up cannot be avoided.

Example 12. Fix the signature Σ consisting only of two role names r and s. Define the concept $C_n \coloneqq \bigsqcap\{\, \exists r.\, D_n^i \mid i \in \{1, \ldots, n\} \,\}$ for each $n \geq 2$ where $D_n^i \coloneqq \bigsqcap\{\, E_j, F_j \mid j \in \{1, \ldots, n\} \setminus \{i\} \,\} \sqcap \exists r^n.\, \top \sqcap \exists s^n.\, \top$, and $E_j \coloneqq \exists r^j.\, \exists s.\, \top$, and $F_j \coloneqq \exists s^j.\, \exists r.\, \top$. There are exponentially many lowering functions for r, namely the partial functions $\chi\colon \mathsf{Succ}(C_n, r) \nrightarrow \mathcal{EL}(\Sigma)$ where $\mathsf{Dom}(\chi) \coloneqq \mathsf{Succ}(C_n, r)$ and $\chi(D_n^i) \in \{E_i, F_i\}$ for each index $i \in \{1, \ldots, n\}$, and these induce the lower neighbors $C_n \sqcap \exists r.\, (G_1 \sqcap \cdots \sqcap G_n)$ where each G_i is either E_i or F_i.

4 Distances between $\mathcal{EL}$ Concept Descriptions

Using the results from the previous section on neighbors of $\mathcal{EL}$ concept descriptions, we are now able to introduce a *natural* distance measure (metric) on the set $\mathcal{EL}(\Sigma)$, namely where the distance between two $\mathcal{EL}$ concepts C and D is the length of a shortest path between C and D in the undirected graph with vertex set $\mathcal{EL}(\Sigma)$ and in which two concepts are connected by an edge if they are neighbors of each other.[10] This distance measure fulfills all conditions of a *metric*, i.e., the distance between two concepts is never negative, two concepts have distance 0 if and only if they are equivalent, the distance from C to D equals the distance from D to C, and the distance between two concepts never exceeds the sum of the distances with a third concept in the middle (*triangle inequality*).

In what follows, we will first explore some lattice-theoretic properties of the partially ordered set $(\mathcal{EL}(\Sigma), \sqsubseteq_\emptyset)$ and then utilize a well-known construction from Lattice Theory [6] to formally introduce the above mentioned metric. At the end of this section, an asymptotic analysis of this metric reveals that it can reach huge values, i.e., computing it is highly infeasible.

First of all, the binary conjunction $\sqcap$ is the *infimum* operation in the poset $(\mathcal{EL}(\Sigma), \sqsubseteq_\emptyset)$, i.e., for each two concepts C and D, it holds true that $C \sqcap D \sqsubseteq_\emptyset C$ and $C \sqcap D \sqsubseteq_\emptyset D$, and further $E \sqsubseteq_\emptyset C$ and $E \sqsubseteq_\emptyset D$ implies $E \sqsubseteq_\emptyset C \sqcap D$ for each concept E. The dual notion is the *supremum* operation; for each two concepts C and D, the supremum (or *least common subsumer*) $C \vee D$ satisfies $C \sqsubseteq_\emptyset C \vee D$ as well as $D \sqsubseteq_\emptyset C \vee D$, and further $C \sqsubseteq_\emptyset E$ and $D \sqsubseteq_\emptyset E$ implies $C \vee D \sqsubseteq_\emptyset E$ for each concept E. According to [4], least common subsumers can be computed by means of products of graphs/trees, yielding the following recursive formula:

$$\begin{aligned} C \vee D \equiv_\emptyset {} & \textstyle\bigsqcap\{\, A \mid A \in \mathsf{Conj}(C) \text{ and } A \in \mathsf{Conj}(D) \,\} \\ & \sqcap \textstyle\bigsqcap\{\, \exists r.\,(E \vee F) \mid \exists r.\,E \in \mathsf{Conj}(C) \text{ and } \exists r.\,F \in \mathsf{Conj}(D) \,\}. \end{aligned}$$

The existence of the infimum and the supremum operation turns the poset $(\mathcal{EL}(\Sigma), \sqsubseteq_\emptyset)$ into a lattice. In [16, Proposition 5.2.1] it is shown that this lattice is *distributive*, i.e., the equivalences $C \sqcap (D \vee E) \equiv_\emptyset (C \sqcap D) \vee (C \sqcap E)$ and $C \vee (D \sqcap E) \equiv_\emptyset (C \vee D) \sqcap (C \vee E)$ hold true for each three concepts C, D, E.

Recall that, for each concept C, there is a finite upper bound on the lengths of chains starting with C [5, proof of Proposition 3.5]. We conclude that, for each two concepts C and D, every chain from C to D has finite length. According to [6], it follows that the lattice $(\mathcal{EL}(\Sigma), \sqsubseteq_\emptyset)$ satisfies the *Jordan-Dedekind chain condition*, i.e., for each two concepts C and D with $C \sqsubseteq_\emptyset D$, all maximal chains[11] from C to D have the same length. It is now straightforward to define the *distance* between two concepts C and D with $C \sqsubseteq_\emptyset D$ as the length of *some* maximal chain from C to D. If the condition $C \sqsubseteq_\emptyset D$ is not satisfied, we instead employ the length of *some* maximal chain from the infimum $C \sqcap D$ to the supremum $C \vee D$.

Proposition 13. [15, Proposition 16], [16, Proposition 5.4.3]. *Consider the mapping* $\mathsf{d}\colon \mathcal{EL}(\Sigma) \times \mathcal{EL}(\Sigma) \to \mathbb{N}$ *where* $\mathsf{d}(C, D) \coloneqq n$ *if there exist concepts* $E_0, \ldots, E_n$ *such that* $C \sqcap D = E_0 \prec_\emptyset E_1 \prec_\emptyset \cdots \prec_\emptyset E_n = C \vee D$. *It is well-defined and it is a metric, i.e., the following properties are fulfilled for all concepts* C, D, E*: (1)* $\mathsf{d}(C, D) = 0$ *if and only if* $C \equiv_\emptyset D$*, (2)* $\mathsf{d}(C, D) = \mathsf{d}(D, C)$*, and (3)* $\mathsf{d}(C, E) \leq \mathsf{d}(C, D) + \mathsf{d}(D, E)$ *(*triangle inequality*).*

[10] Note that, in graph theory, it is well-known that the *shortest path distance* yields a metric on each undirected graph.

[11] A maximal chain is one that cannot be refined, i.e., which is not a strict subset of another chain.

As shown in [16, Proposition 5.4.4], the distance $\mathsf{d}(C, D)$ coincides with the length of a shortest path from C to D in the undirected graph (V, E) with vertex set $V := \mathcal{EL}(\Sigma)$ and with edge set $E := \{ \{C, D\} \mid C \prec_\emptyset D \}$. Specifically, it follows that the distance $\mathsf{d}(C, D)$ equals the sum of the lengths of a chain of neighbors from C to $C \sqcap D$ and of another from D to $C \sqcap D$.

We close this section with an asymptotic analysis. As a simple example, consider the concept description $\exists r.(A_1 \sqcap \cdots \sqcap A_k)$. Let $X_1, \ldots, X_\ell$ be an enumeration of the exponentially many subsets of $\{A_1, \ldots, A_k\}$ with $\lfloor \frac{k}{2} \rfloor$ elements, and further let $D_m := \bigsqcap\{ \exists r. \bigsqcap X_i \mid i \in \{m, \ldots, \ell\} \}$ for each index $m \in \{1, \ldots, \ell\}$. It follows that $\exists r.(A_1 \sqcap \cdots \sqcap A_k) \sqsubset_\emptyset D_1 \sqsubset_\emptyset D_2 \sqsubset_\emptyset \cdots \sqsubset_\emptyset D_\ell \sqsubset_\emptyset \top$ is an exponentially long chain of strict subsumptions, which implies that the distance from $\exists r.(A_1 \sqcap \cdots \sqcap A_k)$ to $\top$ must be at least exponential. Even worse, the following proposition shows that there exists a sequence of concept descriptions C_n such that C_n has role depth n (and size linear in n) and the distance from C_n to $\top$ is n-fold exponential.

Proposition 14. [16, Corollary 5.6.9]. *Fix a natural number $k \geq 3$ and let $A_1, \ldots, A_k$ be different concept names from Σ. Further let r_i be a role name in Σ for each natural number i. The distance from $\exists r_1. \cdots \exists r_n.(A_1 \sqcap \cdots \sqcap A_k)$ to $\top$, where n ranges over the natural numbers, is asymptotically bounded from below by*[12] *the iterated exponential*

$$\underbrace{2^{2^{\cdot^{\cdot^{2^{2^k}}}}}}_{n\ \textit{times}}.$$

The precondition $k \geq 3$ is crucial for the multi-exponential growth. Specifically, the above distance is only linear if $k < 3$, cf. Proposition 5.6.12 in [16]. The below table shows some values of the distance $\mathsf{d}(\top, \exists r_1. \cdots \exists r_n.(A_1 \sqcap \cdots \sqcap A_k))$.

$k \backslash n$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	7
2	2	4	6	8	10	12	14
3	3	$8 \geq \binom{3}{1}$	$20 \geq 4$	$84 \geq \binom{4}{2}$	$8573 \geq \binom{6}{3}$	$? \geq \binom{20}{10}$	$? \geq \binom{184756}{92378}$
4	4	$16 \geq \binom{4}{2}$	$168 \geq \binom{6}{3}$	$? \geq \binom{20}{10}$	$? \geq \binom{184756}{92378}$	$? \gtrapprox \binom{2.33 \cdot 10^{55614}}{1.16 \cdot 10^{55614}}$	?
5	5	$32 \geq \binom{5}{2}$	$7581 \geq \binom{10}{5}$	$? \geq \binom{252}{126}$	$? \gtrapprox \binom{3.63 \cdot 10^{74}}{1.82 \cdot 10^{74}}$	?	?
6	6	$64 \geq \binom{6}{3}$	$? \geq \binom{20}{10}$	$? \geq \binom{184756}{92378}$	$? \gtrapprox \binom{2.33 \cdot 10^{55614}}{1.16 \cdot 10^{55614}}$	?	?
7	7	$128 \geq \binom{7}{3}$	$? \geq \binom{35}{17}$	$? \gtrapprox \binom{4.54 \cdot 10^{9}}{2.27 \cdot 10^{9}}$	?	?	?
8	8	$256 \geq \binom{8}{4}$	$? \geq \binom{70}{35}$	$? \gtrapprox \binom{1.12 \cdot 10^{20}}{5.61 \cdot 10^{19}}$	?	?	?
9	9	$512 \geq \binom{9}{4}$	$? \geq \binom{126}{63}$	$? \gtrapprox \binom{6.03 \cdot 10^{36}}{3.02 \cdot 10^{36}}$	?	?	?
10	10	$1024 \geq \binom{10}{5}$	$? \geq \binom{252}{126}$	$? \gtrapprox \binom{3.63 \cdot 10^{74}}{1.82 \cdot 10^{74}}$	?	?	?

[12] A function $f \colon \mathbb{N} \to \mathbb{R}$ is *asymptotically bounded from below by* a function $g \colon \mathbb{N} \to \mathbb{R}$ if there is a constant $c \in \mathbb{R}_+$ and a number $n_0 \in \mathbb{N}$ such that $c \cdot g(n) \leq f(n)$ for each $n \geq n_0$.

5 Applications and Consequences

Deciding Minimality and Maximality for a Property. [2, Section 5], [16, Section 5.1.2]. Let $\mathcal{P} \subseteq \mathcal{EL}(\Sigma)$ be a decision problem that is closed under subsumees, i.e., $C \in \mathcal{P}$ and $C \sqsupseteq_\emptyset D$ implies $D \in \mathcal{P}$. Further consider the problem $\mathsf{Max}(\mathcal{P})$ that consists of all maximally general concepts in $\mathcal{P}$, i.e., $C \in \mathsf{Max}(\mathcal{P})$ if and only if $C \in \mathcal{P}$ and there is no D such that $C \sqsubset_\emptyset D$ and $D \in \mathcal{P}$.

A deterministic, polynomial time procedure that decides $\mathsf{Max}(\mathcal{P})$ is as follows: (1) On input C, check whether $C \in \mathcal{P}$. If not, then reject C. (2) Compute the set $\mathsf{Upper}(C)$ from Proposition 5. (3) If there exists an upper neighbor $D \in \mathsf{Upper}(C)$ such that $D \in \mathcal{P}$, then reject C; otherwise accept C.

We conclude that $\mathcal{P} \in \mathsf{C}$ implies $\mathsf{Max}(\mathcal{P}) \in \mathsf{P}^{\mathsf{C}}$ for each complexity class C, which specifically yields that $\mathcal{P} \in \mathsf{P}$ implies $\mathsf{Max}(\mathcal{P}) \in \mathsf{P}$ and that $\mathcal{P} \in \Sigma_n^{\mathsf{P}}$ implies $\mathsf{Max}(\mathcal{P}) \in \Delta_{n+1}^{\mathsf{P}}$ for each number $n \in \mathbb{N}$. Furthermore, $\mathcal{P} \in \mathsf{PSpace}$ implies $\mathsf{Max}(\mathcal{P}) \in \mathsf{PSpace}$, since $\mathsf{P}^{\mathsf{PSpace}} \subseteq \mathsf{PSpace}$ [24]. More generally, $\mathcal{P} \in \mathsf{C}$ implies $\mathsf{Max}(\mathcal{P}) \in \mathsf{C}$ for each complexity class C where $\mathsf{PSpace} \subseteq \mathsf{C}$.

Dually, let $\mathcal{P} \subseteq \mathcal{EL}(\Sigma)$ be a decision problem that is closed under subsumers, i.e., $C \in \mathcal{P}$ and $C \sqsubseteq_\emptyset D$ implies $D \in \mathcal{P}$, and consider the problem $\mathsf{Min}(\mathcal{P})$ that consists of all maximally specific concepts in $\mathcal{P}$, i.e., $C \in \mathsf{Min}(\mathcal{P})$ if and only if $C \in \mathcal{P}$ and there is no D such that $C \sqsupset_\emptyset D$ and $D \in \mathcal{P}$.

A non-deterministic, polynomial time procedure that decides the complement of $\mathsf{Min}(\mathcal{P})$ is the following: (1) On input C, check if $C \in \mathcal{P}$. If not, then accept C. (2) Guess a concept D the size of which is quadratic in C. (3) Check if C is an upper neighbor of D. If not, then accept C. (4) Check if $D \in \mathcal{P}$. If yes, then accept C; otherwise reject C.

It follows that $\mathcal{P} \in \mathsf{C}$ implies $\mathsf{Min}(\mathcal{P}) \in \mathsf{co}(\mathsf{NP}^{\mathsf{C}})$ for each complexity class C. In particular, we infer that $\mathcal{P} \in \mathsf{P}$ implies $\mathsf{Min}(\mathcal{P}) \in \mathsf{coNP}$, and that $\mathcal{P} \in \Sigma_n^{\mathsf{P}}$ implies $\mathsf{Min}(\mathcal{P}) \in \Pi_{n+1}^{\mathsf{P}}$ for each $n \in \mathbb{N}$. As $\mathsf{co}(\mathsf{NP}^{\mathsf{PSpace}}) \subseteq \mathsf{PSpace}$ [24], we further conclude that $\mathcal{P} \in \mathsf{PSpace}$ implies $\mathsf{Min}(\mathcal{P}) \in \mathsf{PSpace}$ and, more generally, that $\mathcal{P} \in \mathsf{C}$ implies $\mathsf{Min}(\mathcal{P}) \in \mathsf{coC}$ for each complexity class C with $\mathsf{PSpace} \subseteq \mathsf{C}$.

A Similarity Measure that Fulfills the Triangle Inequality. [16, Section 5.5]. *Similarity measures* quantify the similarity of two objects. In the context of DLs, usually either concepts or individuals are compared. In an early article on similarity measures on DL concepts [7], three approaches to defining such measures are proposed: based on features, based on semantic networks, and based on information content. However, no concrete measure is devised. Later, [9] compares existing (concrete) DL similarity measures, divides them into extensional-based and intentional-based ones, and proposes a new such measure. A framework for constructing similarity measures between concepts in the mild extension of $\mathcal{EL}$ with simple role inclusions $r \sqsubseteq s$ is developed in [22]. Additionally, an overview on other existing similarity measures and their properties is provided. Notably, none satisfies the *triangle inequality.* In [10] a general framework for constructing similarity measures based on concept relaxation operators (which are similar to upward refinement operators) is introduced. Each such similarity measure satisfies the triangle inequality.

Different kinds of similarity measures have also been employed in the development of novel DLs. For instance, [23] extends an expressive DL by means to specify degrees of similarity between individuals, but that logic turns out to be undecidable. So-called *threshold DLs* are investigated in [1], where (values of) graded membership functions replace the Boolean membership values 0 and 1. Decidability and computational complexity is determined and it is further explained how a graded membership function can be obtained from a concept similarity measure.

In the following, we explain how the distance measure d in Proposition 13 can be transformed into a similarity measure. Firstly, we convert d into a metric with range $[0, 1)$ and, secondly, we twist the range (i.e., value x becomes $1 - x$). Formally, if $f\colon [0, \infty) \to [0, 1)$ is an order-preserving, sub-additive function, then the mapping s defined by $\mathsf{s}(C, D) \coloneqq 1 - f(\mathsf{d}(C, D))$ is a similarity measure. For instance, suitable functions are $f(x) \coloneqq (\frac{x}{1+x})^y$ for each $y > 0$ and $f(x) \coloneqq 1 - y^x$ for each $y \in (0, 1)$. Such a similarity measure s fulfills all properties listed in [22] including the triangle inequality, but not the property of structural dependence.

Potential Uselessness of Ideal Downward Refinement Operators. [16, Section 5.1.9]. A *downward refinement operator* takes an object as input and returns a set of objects that are more specific w.r.t. a given partial order. Such operators have been employed in *inductive logic programming* in order to learn theories from facts [27]. The analysis in [18] reveals that refinement operators with certain desirable properties cannot exist. In the context of DLs, such an operator usually maps a concept C to a set of concepts subsumed by C. In [21], an *ideal* downward refinement operator in a slight extension of $\mathcal{EL}$ is introduced and, in [20], it is proposed to use this operator to learn $\mathcal{EL}$ concepts. A similar downward refinement operator is described in [25], where it further serves as a means to defining a similarity measure (by counting refinement steps). Expanding on the latter results, a downward refinement operator on conjunctive queries (with a fixed set of answer variables) is introduced in [26], which allows for the definition of a similarity measure in a similar manner.

In [20, Section 2.2] it is stated that "refinement operators are used to structure a search process for concepts." While this argument holds true in theory, we show in the following that utilizing the ideal downward refinement operator, as introduced in [21], for learning a target concept is not feasible in applications.

Given a TBox $\mathcal{T}$, a *downward refinement operator* is a function ρ that maps each concept description to a set of concept descriptions such that $D \in \rho(C)$ implies $D \sqsubseteq_{\mathcal{T}} C$.[13] Furthermore, ρ is *ideal* if it additionally fulfills the following conditions: (1) ρ is *finite*, i.e., $\rho(C)$ is finite for each C, (2) ρ is *proper*, i.e., $D \in \rho(C)$ implies $D \not\equiv_{\mathcal{T}} C$, and (3) ρ is *complete*, i.e., $D \sqsubset_{\mathcal{T}} C$ implies the existence of concept descriptions $E_1, \ldots, E_n$ such that $E_1 \in \rho(C)$, $E_2 \in \rho(E_1)$, $\ldots$, $E_n \in \rho(E_{n-1})$, and $E_n \equiv_{\mathcal{T}} D$.

There is a connection between the notion of neighbors of concept descrip-

[13] By $C \sqsubseteq_{\mathcal{T}} D$ we indicate that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for each model $\mathcal{I}$ of $\mathcal{T}$.

tions and ideal downward refinement operators. In what follows, we consider the particular downward refinement operator ρ^* in [21, Theorem 1]. Firstly, ρ^* is defined with respect to TBoxes that may only contain axioms of the following forms: concept inclusions $A \sqsubseteq B$, concept disjointness axioms $A \sqcap B \equiv \bot$, role inclusions $r \sqsubseteq s$, domain restrictions $\mathsf{Dom}(r) \sqsubseteq A$, and range restrictions $\mathsf{Ran}(r) \sqsubseteq A$, where A and B are concept names and where r and s are role names. Secondly, ρ^* is defined by means of another refinement operator ρ, which constructs refinements by applying one of four operations to the syntax tree of the concept description to be refined: (1) add a concept name as label to some node, (2) refine a concept name that labels a node (i.e., if $A \sqsubseteq B$ is in $\mathcal{T}$, then a label B can be replaced by A), (3) refine a role name that labels an edge, and (4) attach a new subtree to some node (see [21] for the concrete definition). ρ is only *weakly complete*, i.e., the above condition for completeness is only guaranteed for $C = \top$. The refinements of some concept description C w.r.t. ρ^* are then defined as the w.r.t. $\mathcal{T}$ maximally general concept descriptions D satisfying the following conditions: (1) there is a sequence of ρ-refinements starting with $\top$ and ending with D, (2) $D \sqsubset_{\mathcal{T}} C$, and (3) the role depth of D exceeds the role depth of C by at most 1. Note that, since ρ is weakly complete, the first condition is redundant. It immediately follows from the required maximal generality and from Condition (2) that D must be a lower neighbor of C w.r.t. $\mathcal{T}$.[14] Furthermore, since ρ^* is complete, each set $\rho^*(C)$ must contain all lower neighbors of C w.r.t. $\mathcal{T}$ modulo equivalence. We conclude that ρ^* is the set of all lower neighbors of C w.r.t. $\mathcal{T}$ modulo equivalence.

For the case where the TBox $\mathcal{T}$ is empty, it follows from Proposition 14 that the search for a particular $\mathcal{EL}$ concept description may need a non-elementary number of consecutive refinement steps with ρ^*. More specifically, constructing the target concept $\exists r_1. \cdots \exists r_n.(A_1 \sqcap \cdots \sqcap A_k)$ starting from $\top$ by means of ρ^* needs a number of steps that is asymptotically bounded from below by

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^{2^k}}}}}}}_{n \text{ times}}.$$

For the same reason, utilizing the refinement operators in [25, 26] for an uninformed search is not feasible in practical settings as well.

Computing Maximally Strong Weakenings for Gentle Repairs. In order to resolve inconsistency or to remove an unwanted consequence, the classical approach to repairing an ontology is deleting enough axioms from it. More fine-grained repairs can be obtained by *weakening axioms* instead of removing them completely [3, 8, 11, 14, 19, 28].

A general framework for computing *gentle repairs* by axiom weakening is developed in [3], where conditions on the weakening relations are formulated that guarantee termination. Furthermore, an instantiation of the framework for $\mathcal{EL}$ is provided specifically for the case of repairing TBoxes. One proposed weakening

[14] C is a *lower neighbor* of D if $C \sqsubset_{\mathcal{T}} D$ and there is no E such that $C \sqsubset_{\mathcal{T}} E \sqsubset_{\mathcal{T}} D$.

relation, namely $\succ^{\text{sub}}$, takes a concept inclusion $C \sqsubseteq D$ and weakens it to a concept inclusion $C \sqsubseteq E$ where $D \sqsubset_\emptyset E$ and such that $C \sqsubseteq E$ does not entail $C \sqsubseteq D$. A so-called *maximally strong weakening* is, put simply, a weakening that loses a minimal amount of information, see Definition 17 in [3] for details.

According to Proposition 18 in [3], maximally strong weakenings of a concept inclusion $C \sqsubseteq D$ w.r.t. $\succ^{\text{sub}}$ can be effectively computed by a breadth-first search, namely by traversing the $\mathcal{EL}$ subsumption hierarchy above D (all subsumers of D) by means of the neighborhood relation $\prec_\emptyset$. As already pointed out after Corollary 25 in [3], the search tree has $(n+1)$-fold exponential size where n is the role depth of D. Now Proposition 14 yields that this search tree can have n-fold exponential depth — more specifically, if $C \sqsubseteq \top$ is the maximally strong weakening of $C \sqsubseteq \exists r_1. \cdots \exists r_n. (A_1 \sqcap \cdots \sqcap A_k)$, then computing it by means of the procedure in Proposition 18 in [3] needs n-fold exponential time. Thus, in order to utilize the weakening relation $\succ^{\text{sub}}$ in implementations, more sophisticated methods for computing the maximally strong weakenings must be developed.

6 Conclusion

We have analyzed the $\mathcal{EL}$ subsumption hierarchy and, in particular, we have showed how smallest steps can be made when navigating within this hierarchy. Such a smallest step is either going up to an upper neighbor or going down to a lower neighbor. We have explained how all neighbors of a given concept can be computed and we have determined the computational complexity: all upper neighbors can be obtained in polynomial time, while enumerating all lower neighbors needs exponential time. By employing a standard construction from Lattice Theory, we have devised a distance measure (metric) on the set of all $\mathcal{EL}$ concepts. We have further seen that the $\mathcal{EL}$ subsumption hierarchy contains very long chains, since the distance between supposedly simple concepts can easily be multi-exponential — specifically, the distance between $\top$ and $\exists r_1. \cdots \exists r_n. (A_1 \sqcap \cdots \sqcap A_k)$ is n-fold exponential if $k \geq 3$ — which directly prohibits its practical usage. Additionally, we have given some applications and consequences of the aforementioned results.

In this article, we have looked at subsumption w.r.t. the empty TBox only. Other types of TBoxes (cycle-restricted, acyclic, general) as well as two extensions of $\mathcal{EL}$ (with $\bot$, with greatest fixed-points) are considered in [16]. Future research could investigate more expressive DLs. First investigations show that similar characterizations of upper and lower neighbors can be developed for $\mathcal{FLE}$. Since within $\mathcal{FLE}$ each $\mathcal{EL}$ concept has only $\mathcal{EL}$ concepts as subsumers, we immediately obtain that also the $\mathcal{FLE}$ subsumption hierarchy must contain chains of multi-exponential length.

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References

1. Baader, F., Gil, O.F.: Decidability and Complexity of Threshold Description Logics induced by Concept Similarity Measures. In: Seffah, A., Penzenstadler, B., Alves, C., Peng, X. (eds.) Proceedings of the Symposium on Applied Computing, SAC 2017, Marrakech, Morocco, April 3-7, 2017. pp. 983–988. ACM (2017)
2. Baader, F., Kriegel, F., Nuradiansyah, A.: Privacy-Preserving Ontology Publishing for $\mathcal{EL}$ Instance Stores. In: Calimeri, F., Leone, N., Manna, M. (eds.) 16th European Conference on Logics in Artificial Intelligence, JELIA 2019, Rende, Italy, May 7-11, 2019, Proceedings. Lecture Notes in Computer Science, vol. 11468, pp. 323–338. Springer (2019)
3. Baader, F., Kriegel, F., Nuradiansyah, A., Peñaloza, R.: Making Repairs in Description Logics More Gentle. In: Thielscher, M., Toni, F., Wolter, F. (eds.) Principles of Knowledge Representation and Reasoning: Proceedings of the Sixteenth International Conference, KR 2018, Tempe, Arizona, 30 October - 2 November 2018. pp. 319–328. AAAI Press (2018)
4. Baader, F., Küsters, R., Molitor, R.: Computing Least Common Subsumers in Description Logics with Existential Restrictions. In: Dean, T. (ed.) Proceedings of the Sixteenth International Joint Conference on Artificial Intelligence, IJCAI 99, Stockholm, Sweden, July 31 - August 6, 1999. 2 Volumes, 1450 pages. pp. 96–103. Morgan Kaufmann (1999)
5. Baader, F., Morawska, B.: Unification in the Description Logic $\mathcal{EL}$. Log. Methods Comput. Sci. **6**(3) (2010)
6. Birkhoff, G.: Lattice Theory, Colloquium Publications, vol. 25. American Mathematical Society, Providence, RI, USA (1940)
7. Borgida, A., Walsh, T.J., Hirsh, H.: Towards Measuring Similarity in Description Logics. In: Horrocks, I., Sattler, U., Wolter, F. (eds.) Proceedings of the 2005 International Workshop on Description Logics (DL2005), Edinburgh, Scotland, UK, July 26-28, 2005. CEUR Workshop Proceedings, vol. 147. CEUR-WS.org (2005)
8. Confalonieri, R., Galliani, P., Kutz, O., Porello, D., Righetti, G., Troquard, N.: Towards Even More Irresistible Axiom Weakening. In: Borgwardt, S., Meyer, T. (eds.) Proceedings of the 33rd International Workshop on Description Logics (DL 2020) co-located with the 17th International Conference on Principles of Knowledge Representation and Reasoning (KR 2020), Online Event [Rhodes, Greece], September 12th to 14th, 2020. CEUR Workshop Proceedings, vol. 2663. CEUR-WS.org (2020)
9. d'Amato, C., Staab, S., Fanizzi, N.: On the Influence of Description Logics Ontologies on Conceptual Similarity. In: Gangemi, A., Euzenat, J. (eds.) Knowledge Engineering: Practice and Patterns, 16th International Conference, EKAW 2008, Acitrezza, Italy, September 29 - October 2, 2008. Proceedings. Lecture Notes in Computer Science, vol. 5268, pp. 48–63. Springer (2008)
10. Distel, F., Atif, J., Bloch, I.: Concept Dissimilarity with Triangle Inequality. In: Baral, C., Giacomo, G.D., Eiter, T. (eds.) Principles of Knowledge Representation and Reasoning: Proceedings of the Fourteenth International Conference, KR 2014, Vienna, Austria, July 20-24, 2014. AAAI Press (2014)
11. Du, J., Qi, G., Fu, X.: A Practical Fine-grained Approach to Resolving Incoherent OWL 2 DL Terminologies. In: Li, J., Wang, X.S., Garofalakis, M.N., Soboroff, I., Suel, T., Wang, M. (eds.) Proceedings of the 23rd ACM International Conference on Conference on Information and Knowledge Management, CIKM 2014, Shanghai, China, November 3-7, 2014. pp. 919–928. ACM (2014)

12. Ganter, B.: Personal communication (April 2018)
13. Ganter, B., Wille, R.: Formal Concept Analysis - Mathematical Foundations. Springer (1999)
14. Horridge, M., Parsia, B., Sattler, U.: Laconic and precise justifications in OWL. In: Sheth, A.P., Staab, S., Dean, M., Paolucci, M., Maynard, D., Finin, T.W., Thirunarayan, K. (eds.) The Semantic Web - ISWC 2008, 7th International Semantic Web Conference, ISWC 2008, Karlsruhe, Germany, October 26-30, 2008. Proceedings. Lecture Notes in Computer Science, vol. 5318, pp. 323–338. Springer (2008)
15. Kriegel, F.: The Distributive, Graded Lattice of $\mathcal{EL}$ Concept Descriptions and its Neighborhood Relation. In: Ignatov, D.I., Nourine, L. (eds.) Proceedings of the Fourteenth International Conference on Concept Lattices and Their Applications, CLA 2018, Olomouc, Czech Republic, June 12-14, 2018. CEUR Workshop Proceedings, vol. 2123, pp. 267–278. CEUR-WS.org (2018)
16. Kriegel, F.: Constructing and Extending Description Logic Ontologies using Methods of Formal Concept Analysis. Doctoral thesis, Technische Universität Dresden, Dresden, Germany (2019)
17. Küsters, R.: Non-Standard Inferences in Description Logics, Lecture Notes in Computer Science, vol. 2100. Springer (2001)
18. van der Laag, P.R.J., Nienhuys-Cheng, S.: Existence and Nonexistence of Complete Refinement Operators. In: Bergadano, F., Raedt, L.D. (eds.) Machine Learning: ECML-94, European Conference on Machine Learning, Catania, Italy, April 6-8, 1994, Proceedings. Lecture Notes in Computer Science, vol. 784, pp. 307–322. Springer (1994)
19. Lam, J.S.C., Sleeman, D.H., Pan, J.Z., Vasconcelos, W.W.: A fine-grained approach to resolving unsatisfiable ontologies. J. Data Semant. **10**, 62–95 (2008)
20. Lehmann, J.: Learning OWL Class Expressions, Studies on the Semantic Web, vol. 6. IOS Press (2010)
21. Lehmann, J., Haase, C.: Ideal Downward Refinement in the $\mathcal{EL}$ Description Logic. In: Raedt, L.D. (ed.) Inductive Logic Programming, 19th International Conference, ILP 2009, Leuven, Belgium, July 02-04, 2009. Revised Papers. Lecture Notes in Computer Science, vol. 5989, pp. 73–87. Springer (2009)
22. Lehmann, K., Turhan, A.: A Framework for Semantic-Based Similarity Measures for $\mathcal{ELH}$-Concepts. In: del Cerro, L.F., Herzig, A., Mengin, J. (eds.) Logics in Artificial Intelligence - 13th European Conference, JELIA 2012, Toulouse, France, September 26-28, 2012. Proceedings. Lecture Notes in Computer Science, vol. 7519, pp. 307–319. Springer (2012)
23. Lutz, C., Wolter, F., Zakharyaschev, M.: Reasoning about Concepts and Similarity. In: Calvanese, D., Giacomo, G.D., Franconi, E. (eds.) Proceedings of the 2003 International Workshop on Description Logics (DL2003), Rome, Italy September 5-7, 2003. CEUR Workshop Proceedings, vol. 81. CEUR-WS.org (2003)
24. Papadimitriou, C.H.: Computational Complexity. Addison-Wesley (1994)
25. Sánchez-Ruiz-Granados, A.A., Ontañón, S., González-Calero, P.A., Plaza, E.: Measuring Similarity in Description Logics Using Refinement Operators. In: Ram, A., Wiratunga, N. (eds.) Case-Based Reasoning Research and Development - 19th International Conference on Case-Based Reasoning, ICCBR 2011, London, UK, September 12-15, 2011. Proceedings. Lecture Notes in Computer Science, vol. 6880, pp. 289–303. Springer (2011)
26. Sánchez-Ruiz-Granados, A.A., Ontañón, S., González-Calero, P.A., Plaza, E.: Refinement-Based Similarity Measure over DL Conjunctive Queries. In: Delany,

S.J., Ontañón, S. (eds.) Case-Based Reasoning Research and Development - 21st International Conference, ICCBR 2013, Saratoga Springs, NY, USA, July 8-11, 2013. Proceedings. Lecture Notes in Computer Science, vol. 7969, pp. 270–284. Springer (2013)

27. Shapiro, E.Y.: An Algorithm that Infers Theories from Facts. In: Hayes, P.J. (ed.) Proceedings of the 7th International Joint Conference on Artificial Intelligence, IJCAI '81, Vancouver, BC, Canada, August 24-28, 1981. pp. 446–451. William Kaufmann (1981)
28. Troquard, N., Confalonieri, R., Galliani, P., Peñaloza, R., Porello, D., Kutz, O.: Repairing Ontologies via Axiom Weakening. In: McIlraith, S.A., Weinberger, K.Q. (eds.) Proceedings of the Thirty-Second AAAI Conference on Artificial Intelligence, (AAAI-18), the 30th innovative Applications of Artificial Intelligence (IAAI-18), and the 8th AAAI Symposium on Educational Advances in Artificial Intelligence (EAAI-18), New Orleans, Louisiana, USA, February 2-7, 2018. pp. 1981–1988. AAAI Press (2018)