Bijective proofs for Eulerian numbers in types B and D

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Abstract

Let $\left\langle {n \atop k} \right\rangle$, $\left\langle {\mathsf{B}_n \atop k} \right\rangle$, and $\left\langle {\mathsf{D}_n \atop k} \right\rangle$ be the Eulerian numbers in the types A, B, and D, respectively—that is, number of permutations of n elements with k descents, the number of signed permutations (of n elements) with k type B descents, the number of even signed permutations (of n elements) with k type D descents. Let $S_n(t) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k$ and $B_n(t) = \sum_{k=0}^{n} \left\langle {\mathsf{B}_n \atop k} \right\rangle t^k$, and $D_n(t) = \sum_{k=0}^{n} \left\langle {\mathsf{D}_n \atop k} \right\rangle t^k$. We give bijective proofs of the identity

$$B_n(t^2) = (1+t)^{n+1} S_n(t) - 2^n t S_n(t^2)\,.$$

and of Stembridge's identity

$$D_n(t) = B_n(t) - n2^{n-1} t S_{n-1}(t)\,.$$

These bijective proofs rely on a representation of signed permutations as paths. Using the same representation we establish a bijective correspondence between even signed permutations and pairs (w, E) with $([n], E)$ a threshold graph and w a degree ordering of $([n], E)$.

1 Introduction

For $n \geq 0$, we use $[n]$ for the set $\{1, \ldots, n\}$ and S_n for the set of permutations of n-elements—that is, bijections of $[n]$. We write a permutation $w \in \mathsf{S}_n$ as a word $w_1 w_2 \ldots w_n$ with $w_i \in [n]$ and $w_i \neq w_j$ for $i \neq j$. A *descent* of $w \in \mathsf{S}_n$ is an index (or position) $i \in [n-1]$ such that $w_i > w_{i+1}$. The Eulerian number $\left\langle {n \atop k} \right\rangle$ counts the number of permutations $w \in \mathsf{S}_n$ that have k descent positions. This is, of course, one among the many interpretations that we can give to these numbers, see e.g. [14]. The given interpretation is closely related to order theory. Let us recall that the set S_n can be endowed with a lattice structure, see e.g. [11, 6]. The lattice S_n is known under the name of Permutohedron or weak (Bruhat) order. Exploiting the bijection between descent positions and lower covers of $w \in \mathsf{S}_n$, the Eulerian number $\left\langle {n \atop k} \right\rangle$ counts the number of permutations $w \in \mathsf{S}_n$ with k lower covers. In particular, $\left\langle {n \atop 1} \right\rangle = 2^n - n - 1$ is the number of join-irreducible elements in S_n. A subtler order-theoretic interpretation is given in [2]: since the S_n are (join-)semidistributive as lattices, every element can be written canonically as the join of join-irreducible elements [9]; the numbers $\left\langle {n \atop k} \right\rangle$ counts then the permutations $w \in \mathsf{S}_n$ that can be written canonically as the join of k join-irreducible elements.

The symmetric groups S_n are particular instances of Coxeter groups, see [4]. Under the usual classification of finite Coxeter groups, the symmetric group S_n yields a concrete model for the Coxeter group A_{n-1} in the family A. Similarly to the symmetric groups, notions of length, descent, and inversion, and a weak order as well, can be defined for elements of an arbitrary Coxeter group [3]. We move our focus to the families B and D of Coxeter groups. More precisely, this paper concerns the Eulerian numbers in the types B and D. The Eulerian number $\left\langle {\mathsf{B}_n \atop k} \right\rangle$ (resp., $\left\langle {\mathsf{D}_n \atop k} \right\rangle$) counts the number of elements in the group B_n (resp., D_n) with k descent positions. Order-theoretic interpretations of these numbers, analogous to the ones mentioned for the standard Eulerian numbers, are still valid. As the abstract group A_{n-1} has a concrete realization as the symmetric group S_n, the group B_n (resp., D_n) has a realization as the hyperoctahedral group of signed permutations (resp., the group of even signed permutations). Starting from these concrete representations of Coxeter groups in the types B and D, we pinpoint some new representations of signed permutations relying on which we provide bijective proofs of known formulas for Eulerian numbers in the types B and D. These formulas allow to compute the Eulerian numbers in the types B and D. from the better known Eulerian numbers in type A.

Let $S_n(t)$ and $B_n(t)$ be the Eulerian polynomials in the types A and B:

$$S_n(t) := \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k \,, \qquad\qquad B_n(t) := \sum_{k=0}^{n} \left\langle {\mathsf{B}_n \atop k} \right\rangle t^k \,. \tag{1}$$

In [14, §13, p. 215] the following polynomial identity is stated:

$$2B_n(t^2) = (1+t)^{n+1} S_n(t) + (1-t)^{n+1} S_n(-t)\,. \tag{2}$$

Considering that, for $f(t) = \sum_{k\geq 0} a_k t^k$,

$$f(t) + f(-t) = 2 \sum_{k\geq 0} a_{2k} t^{2k}\,,$$

the polynomial identity (2) amounts to the following identity among coefficients:

$$\left\langle {\mathsf{B}_n \atop k} \right\rangle = \sum_{i=0}^{2k} \left\langle {n \atop i} \right\rangle \binom{n+1}{2k-i}\,. \tag{3}$$

We present a bijective proof of (3) and also establish the identity

$$2^n \left\langle {n \atop k} \right\rangle = \sum_{i=0}^{2k+1} \left\langle {n \atop i} \right\rangle \binom{n+1}{2k+1-i}\,. \tag{4}$$

Considering that, for $f(t) = \sum_{k\geq 0} a_k t^k$,

$$f(t) - f(-t) = 2 \sum_{k\geq 0} a_{2k+1} t^{2k+1}\,,$$

the identity (4) yields the polynomial identity:

$$2^{n+1} t S_n(t^2) = (1+t)^{n+1} S_n(t) - (1-t)^{n+1} S_n(-t)\,.$$

More importantly, (3) and (4) jointly yield the polynomial identity

$$(1+t)^{n+1} S_n(t) = B_n(t^2) + 2^n t S_n(t^2)\,. \tag{5}$$

Let now $D_n(t)$ be the Eulerian polynomial in type D:

$$D_n(t) := \sum_{k=0}^{n} \left\langle {\mathsf{D}_n \atop k} \right\rangle t^k\,.$$

Investigating further the terms $2^n S_n(t)$, we also ended up finding a simple bijective proof, that we present here, of Stembridge's identity [22, Lemma 9.1]

$$D_n(t) = B_n(t) - n2^{n-1} t S_{n-1}(t)\,, \tag{6}$$

which, in terms of the Eulerian numbers in type D, amounts to

$$\left\langle {\mathsf{D}_n \atop k} \right\rangle = \left\langle {\mathsf{B}_n \atop k} \right\rangle - n2^{n-1} \left\langle {n-1 \atop k-1} \right\rangle\,.$$

Let us remark that the proofs presented here follow different paths from known proofs of the identities (2) and (6). As suggested in [14], the first identity may be derived by computing the f-vector of the type B Coxeter complex and then by applying the transform yielding h-vector from the f-vector. A similar method is used in [22] to prove the identity (6). On the other hand, our proofs directly rely on the combinatorial properties of signed/even signed permutations and on the representation of these objects that we call the path representation of a signed permutation and simply barred permutations.

2 Notation, elementary definitions, and facts

The notation used is chosen to be consistent with [14]. As mentioned before, we use $[n]$ for the set $\{1,\dots,n\}$ and S_n for the set of permutations of $[n]$. We use $[n]_0$ for the set $\{0,1,\dots,n\}$, $[-n]$ for $\{-n,\dots,-1\}$, and $[\pm n]$ for $\{-n,\dots,-1,1,\dots,n\}$. We write a permutation $w \in \mathsf{S}_n$ as a word $w = w_1 w_2 \dots w_n$, with $w_i \in [n]$. For $w \in \mathsf{S}_n$, its set of descents and its set of inversions[1] are defined follows:

$$\mathrm{Des}(w) := \{ i \in \{1,\dots,n-1\} \mid w_i > w_{i+1} \}, \quad \mathrm{Inv}(w) := \{ (i,j) \mid 1 \le i < j \le n, w^{-1}(i) > w^{-1}(j) \}.$$

Then, we let

$$\mathrm{des}(w) := |\mathrm{Des}(w)|.$$

The Eulerian number $\left\langle {n \atop k} \right\rangle$, counting the number of permutations of n elements with k descents, can be formally defined as follows:

$$\left\langle {n \atop k} \right\rangle := |\{ w \in \mathsf{S}_n \mid \mathrm{des}(w) = k \}|.$$

Let us define a *signed permutation of* $[n]$ as a permutation u of $[\pm n]$ such that, for each $i \in [\pm n]$, $u_{-i} = -u_i$. We use B_n for the set of signed permutations of $[n]$. When writing a signed permutation u as a word $u_{-n} \dots u_{-1} u_1 \dots u_n$, we prefer writing $u_i = \overline{x}$ in place of $-x$ if $u_i < 0$ and $|u_i| = x$. Also, we often write $u \in \mathsf{B}_n$ in *window notation*, that is, we only write the suffix $u_1 u_2 \dots u_n$; indeed, the prefix $u_{-n} u_{n-1} \dots u_{-1}$ is uniquely determined by the suffix $u_1 u_2 \dots u_n$, by mirroring it and by exchanging the signs. Obviously, the set B_n is a group and, as mentioned before, it is the standard model for a Coxeter group in the family B with n generators. Therefore, general notions from the theory of Coxeter groups (descent, inversion) apply to signed permutations. We present below as definitions the well-known explicit formulas for the descent and inversion sets of $u \in \mathsf{B}_n$. We let

$$\mathrm{Des}_{\mathsf{B}}(u) := \{ i \in \{0,\dots,n-1\} \mid u_i > u_{i+1} \}, \quad \mathrm{Inv}_{\mathsf{B}}(u) := \{ (i,j) \mid 1 \le |i| \le j \le n, u^{-1}(i) > u^{-1}(j) \},$$

where we set $u_0 := 0$, so 0 is a descent of u if and only if $u_1 < 0$,

$$\mathrm{des}_{\mathsf{B}}(u) = |\mathrm{Des}_{\mathsf{B}}(u)|, \qquad \left\langle {\mathsf{B}_n \atop k} \right\rangle := |\{ u \in \mathsf{B}_n \mid \mathrm{des}_{\mathsf{B}}(u) = k \}|.$$

Finally, and recalling the definition in (1) of the Eulerian polynomials in the types A and B, let us mention that the type A Eulerian polynomial is often (for example in [5]) defined as follows:

$$A_n(t) := \sum_{k=1}^{n} \left\langle {n \atop k-1} \right\rangle t^k = t S_n(t).$$

We shall exclusively manipulate the polynomials $S_n(t)$ and not the $A_n(t)$. Notice that $S_n(t)$ has degree $n-1$ and $B_n(t)$ has degree n.

We shall introduce later even signed permutations and their groups, as well as related notions arising from the fact that these groups are standard models for Coxeter groups in the family D.

For the time being, let us observe the following. For $u \in \mathsf{B}_n$ we let $\mathrm{Des}^+_{\mathsf{B}}(u) := \mathrm{Des}_{\mathsf{B}}(u) \setminus \{0\}$—thus $\mathrm{Des}^+_{\mathsf{B}}(u)$ is the set of strictly positive descents of u. We have then:

Lemma 2.1. $|\{ u \in \mathsf{B}_n \mid \mathrm{Des}^+_{\mathsf{B}}(u) = k \}| = 2^n \left\langle {n \atop k} \right\rangle$.

Proof. By considering its window notation, a signed permutation u yields a mapping $\tilde{u} : [n] \to [\pm n]$ with a unique decomposition of the form $\tilde{u} = \iota \circ w$ with $w \in \mathsf{S}_n$ and $\iota : [n] \to [\pm n]$ an order preserving injection such that $x \in \iota([n])$ iff $-x \notin \iota([n])$. The monotone injections with this property are uniquely determined by their positive image $\iota([n]) \cap [n]$, so there are 2^n such injections. Moreover, for $i = 1,\dots,n-1$, $w_i > w_{i+1}$ if and only if $u_i > u_{i+1}$, so $|\mathrm{Des}^+_{\mathsf{B}}(\iota \circ w)| = |\mathrm{Des}(w)|$. ∎

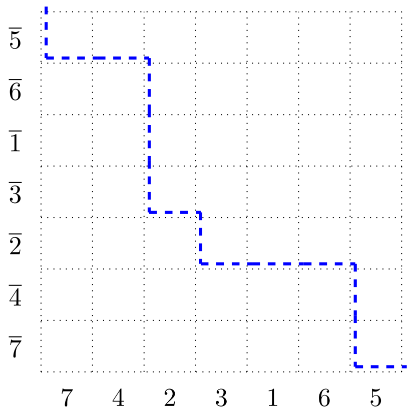
[1] It is also also possible to define $\mathrm{Inv}(w)$ as the set $\{ (i,j) \mid 1 \le i < j \le n, w_i > w_j \}$. We stick in this paper to the one given here.

3 Path representation of signed permutations, simply barred permutations

We present here our combinatorial tools to deal with signed permutations, the path representation and the simply barred permutations.

Definition 3.1. The *path representation* of $u \in \mathsf{B}_n$ is a triple $(\pi^u, \lambda_x^u, \lambda_y^u)$ where π^u is a discrete path, drawn on a grid $[n]_0 \times [n]_0$ and joining the point $(0, n)$ to the point $(n, 0)$, $\lambda_x^u : [n] \to [n]$, and $\lambda_y^u : [n] \to [-n]$. The triple $(\pi^u, \lambda_x^u, \lambda_y^u)$ is constructed from u according to the following algorithm: (i) u is written in full notation as a word and scanned from left to right: each positive letter yields an East step (a length 1 step along the x-axis towards the right), and each negative letter yields a South step (a length 1 step along the y-axis towards the bottom) ; (ii) the labelling $\lambda_x^u : [n] \to [n]$ is obtained by projecting each positive letter on the x-axis, (iii) the labelling $\lambda_y^u : [n] \to [-n]$ is obtained by projecting each negative letter on the y-axis.

Example 3.2. Consider the signed permutation $u := \overline{2}316\overline{4}75$, in window notation, that is, $\overline{5}74\overline{613}2\overline{2}316\overline{4}75$, in full notation. Applying the algorithm above, we draw the path π^u and the labellings λ_x^u, λ_y^u as follows:



Therefore, π^u is the dashed blue path, λ_x^u is the permutation 7423165, and λ_y^u is $\overline{7}\,\overline{4}\,\overline{2}\,\overline{3}\,\overline{1}\,\overline{6}\,\overline{5}$. ◇

It is easily seen that, for an arbitrary $u \in \mathsf{B}_n$, $(\pi^u, \lambda_x^u, \lambda_y^u)$ has the following properties:

(i) π^u is symmetric along the diagonal,

(ii) $\lambda_x^u \in \mathsf{S}_n$ and, moreover, it is the subword of u of positive letters,

(iii) for each $i \in [n]$, $\lambda_y(i) = \overline{\lambda_x(i)}$ and, moreover, λ_y^u is the mirror of the subword of u of negative letters.

In particular, we see that the data $(\pi^u, \lambda_x^u, \lambda_y^u)$ is redundant, since we could give away λ_y^u which is completely determined from λ_x^u.

Proposition 3.3. *The mapping $u \mapsto (\pi^u, \lambda_x^u)$ is a bijection from the set of signed permutations B_n to the set of pairs (π, w), where $w \in \mathsf{S}_n$ and π is a discrete path from $(0, n)$ to $(n, 0)$ with East and South steps which, moreover, is symmetric along the diagonal.*

We leave the reader convince himself of the above statement. Let us argue that the path representation of a signed permutation is, possibly, the more interesting combinatorial representation available to represent signed permutations. For example, the type B inversions of u can be identified with the type A inversions of λ_x^u and the unordered pairs or singletons $\{w_i, w_j\}$ with $1 \leq i \leq j \leq n$ such that the square (i, j) lies below π^u (we index squares from left to right and from bottom to top). Indeed, (i, j) lies below π^u if and only if (j, i) does, by symmetry of π^u.

With respect to Example 3.2, the set of type B inversions of $\overline{2}316\overline{47}5$ is the disjoint union of the set of type A inversions of 7423165 and the set

$$\{\,(-7,7),(-4,7),(-2,7),(-3,7),(-1,7),(-6,7),(-4,4),(-2,4),(-3,4),(-1,4),(-4,6),(-2,2)\,\}$$

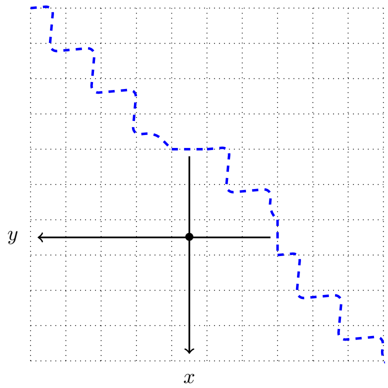
corresponding to the set of unordered pairs

$$\{\,\{7,7\},\{7,4\},\{7,2\},\{7,3\},\{7,1\},\{7,6\},\{4,4\},\{4,2\},\{4,3\},\{4,1\},\{4,6\},\{2,2\}\,\}\,.$$

Let us argue for this formally.

Proposition 3.4. *Let $u \in \mathsf{B}_n$. For each i, j with $1 \leq |i| \leq j \leq n$, $(i, j) \in \mathrm{Inv}_{\mathsf{B}}(u)$ if and only if either $1 \leq i < j \leq n$ and $(i, j) \in \mathrm{Inv}(\lambda^u_x)$ or $i < 0$ and $((\lambda^u_x)^{-1}(i), (\lambda^u_x)^{-1}(j))$ appears below π^u.*

Proof. Let in the following $w = \lambda^u_x$. If $i > 0$, then the statement that $(i, j) \in \mathrm{Inv}_{\mathsf{B}}(u)$ if and only if $(i, j) \in \mathrm{Inv}(w)$ follows since w is the subword of u (written in full notation) of positive integers. On the other hand, if $i < 0$, then recall that $(-i, j) \in \mathrm{Inv}_{\mathsf{B}}(u)$ if and only if $(-i, j) \in \mathrm{Inv}(u)$ if and only if $(-j, i) \in \mathrm{Inv}(u)$, where, in the expression $\mathrm{Inv}(u)$, u is considered as a permutation of the linear order $[\pm n]$. Moreover, since the π^u_x is symmetric along the diagonal, observe that (i, j) is below the π^u_x if and only (j, i) is below the π^u_x. Therefore, it will be enough to observe that for $y < 0$ and $x > 0$, (x, y) is below π^u_x if and only if x appears before y in u, as suggested in the picture on the right. □



We introduce next a second representation of signed permutations.

Definition 3.5. A *simply barred permutation* of $[n]$ is a pair (w, B) where $w \in \mathsf{S}_n$ and $B \subseteq \{1, \ldots, n\}$. We let $\mathtt{SBP}_n$ be the set of *simply barred permutations* of $[n]$.

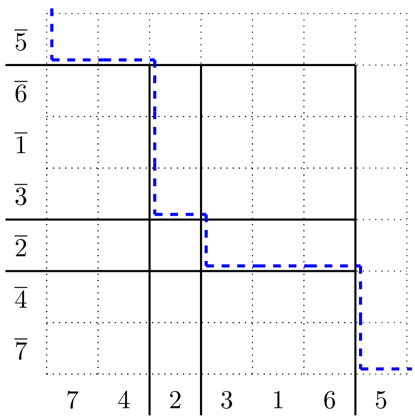
We think of B as a set of positions of w, the barred positions. We have added the adjective "simply" to "barred permutation" since we do not require that B is a superset of $\mathrm{Des}(w)$, as for example in [10].

Example 3.6. Consider (w, B) with $w = 7423165$ and $B = \{2, 4, 6\}$. We represent (w, B) as a permutation divided into blocks by the bars, placing a vertical bar after u_i for each $i \in B$, e.g. $74|23|16|5$. Notice that we allow a bar to appear in the last position, for example $34|1|265|7|$ stands for the simply barred permutation $(3412657, \{2, 3, 6, 7\})$. Thus, a bar appears in the last position if and only if the last block is empty. ♢

We describe next a bijection—that we call ψ—from the set $\mathtt{SBP}_n$ to B_n. Let us notice that, in order to establish equipotence of these two sets, other bijections are available and more straightforward.

Definition 3.7. For $(w, B) \in \mathtt{SBP}_n$, we define the signed permutation $\psi(w, B) \in \mathsf{B}_n$ according to the following algorithm: (i) draw the grid $[n]_0 \times [n]_0$; (ii) since $B \subseteq [n]$, $B \times B$ defines a subgrid of $[n]_0 \times [n]_0$, construct the upper anti-diagonal π of this subgrid; (iii) $\psi(w, B)$ is the signed permutation u whose path representation $(\pi^u, \lambda^u_x, \lambda^u_y)$ equals $(\pi, w, \overline{w})$.

Example 3.8. The required construction can be understood as raising the bars and transforming them into a grid. For example, for the simply barred permutation $74|2|316|5$ (that is (w, B) with $w = 7423165$ and $B = \{2, 3, 6\}$) the construction is as follows:



The dashed path is the anti-diagonal of the subgrid. The resulting signed permutation $\psi(w, B)$ is $\overline{2}316\overline{4}\overline{7}5$ as from Example 3.2. ♢

Notice that the inverse image of ψ has a possibly easier description, as it can be constructed according to the following algorithm: for $u \in \mathsf{B}_n$ (i) construct the path representation $(\pi^u, \lambda^u_x, \lambda^u_y)$ of u, (ii) insert a bar in w at each vertical step of π^u (and remove consecutive bars), (iii) remove a bar at position 0 if it exists. Said otherwise, $(w, B) = \psi^{-1}(u)$ is obtained from u by transforming each negative letter into a bar, by removing consecutive bars, and then by removing a bar at position 0 if needed.

In the following chapter we shall deal mostly with simply barred permutations. Even if we understand simply barred permutations just as shorthands for path representations of even signed permutations, some remarks are due:

Lemma 3.9. *If $u = \psi(w, B)$, then there is a bijection between the set B of bars and the set of xy-turns of π^u.*

4 Descents from simply barred permutations

We start investigating how the type B descent set can be recovered from a simply barred permutation.

Proposition 4.1. *For a simply barred permutation (w, B), we have*

$$\mathrm{des}_{\mathsf{B}}(\psi(w, B)) = |\mathrm{Des}(w) \setminus B| + \left\lceil \frac{|B|}{2} \right\rceil . \tag{7}$$

Proof. Write $u = \psi(w, B)$ in window notation and divide it in maximal blocks of consecutive letters having the same sign. If the first block has negative sign, add a zero positive block which is empty. Each change of sign $+-$ among consecutive blocks yields a descent. These changes of sign bijectively correspond to xy-turns of π^u that lie on or below the diagonal. By Lemma 3.9, each bar determines an xy-turn and, by symmetry of π^u along the diagonal, the number of xy-turns that are on or below the diagonal is $\left\lceil \frac{|B|}{2} \right\rceil$. Therefore this quantity counts the number of descents determined by a change of sign.

The other descents of $\psi(w, B)$ are either of the form $w_i w_{i+1}$ with $w_i > w_{i+1}$ and w_i, w_{i+1} belonging to the same positive block, or of the form $\overline{w_{i+1}}\,\overline{w_i}$ with $w_i > w_{i+1}$ and $\overline{w_i}, \overline{w_{i+1}}$ belonging to the same negative block. These descents are in bijection with the descent positions of w that do not belong to the set B. □

The following lemma might be immediately proved by considering that $0 \in \mathrm{Des}_{\mathsf{B}}(u)$ if and only if, in the path representation of $\psi(w, B)$, the first step of π^u is along the y-axis. In this case (and only in this case), π^u makes an xy-turn on the diagonal. This happens exactly when π^u has an odd number of xy-turns.

Lemma 4.2. *We have $0 \in \mathrm{Des}_{\mathsf{B}}(\psi(w, B))$ if and only if $|B|$ is odd.*

For each $k \in \{ 0, 1, \ldots, n \}$, in the following we let $\mathtt{SBP}_{n,k}$ be the set simply barred permutations $(w, B) \in \mathtt{SBP}_n$ such that $|D(w) \setminus B| + \left\lceil \frac{|B|}{2} \right\rceil = k$.

Corollary 4.3. *The set $\mathtt{SBP}_{n,k}$ is in bijection with the set of signed permutations of n with k descents.*

Definition 4.4. A *loosely barred permutation* of $[n]$ is a pair (w, B) where w is a permutation of $[n]$ and $B \subseteq \{ 0, \ldots, n \}$ is a set of positions (the bars). We let $\mathtt{LBP}_n$ be the set of *loosely barred permutation*s of $[n]$.

Loosely barred permutations are being introduced only as a tool to index simply barred permutations independently of the even/odd cardinalities of their set of bars. Namely, for a loosely barred permutation (w, B), let us define its simplification $\varsigma(w, B)$ by

$$\varsigma(w, B) := (w, B \setminus \{ 0 \}) .$$

Then $\varsigma(w, B)$ is a simply barred permutation whose set of bars has even (resp., odd) cardinality if either $0 \in B$ and $|B|$ is odd (resp., even), or $0 \notin B$ and $|B|$ is even (resp., odd). For a loosely barred permutations (w, B), we shall often need to evaluate the expression $\left\lceil \frac{|B \setminus \{ 0 \}|}{2} \right\rceil$. We record this value once for all in the lemma below.

Lemma 4.5. *For a loosely barred permutation (w, B) we have*

$$\left\lceil \frac{|B \setminus \{ 0 \}|}{2} \right\rceil = \begin{cases} \frac{|B|}{2} , & \text{if } |B| \text{ is even,} \\ \frac{|B|-1}{2} , & \text{if } |B| \text{ is odd and } 0 \in B , \\ \frac{|B|+1}{2} , & \text{if } |B| \text{ is odd and } 0 \notin B . \end{cases} \tag{8}$$

Next, we define an involution—that we name θ—from the set of loosely barred permutations to itself. For a loosely barred permutation (w, B), $\theta(w, B)$ is defined by:

$$\theta(w, B) := (w, \mathrm{Des}(w) \Delta B) , \tag{9}$$

where Δ stands for symmetric difference. Let us insist that this involution is defined for all loosely barred permutations, not just for the simply barred permutations.[2]

Lemma 4.6. *If $(u, C) = \theta(w, B)$, then*

$$|D(w)| + |B| = 2|D(u) \setminus C| + |C| \,. \tag{10}$$

Proof. Recall that $C = D(w)\Delta B$ and so $D(u) \setminus C = D(w) \cap B$. Equation (10) follows since $|D(w)| + |B| = |D(w)\Delta B| + 2|D(w) \cap B|$. □

More formally, we define a variant Θ_n of the correspondences θ defined in (9) as follows:

$$\Theta_n(w, B) := \varsigma(\theta(w, B)) = (w, \mathrm{Des}(w)\Delta B \setminus \{\,0\,\})\,.$$

Notice that, this time, $\Theta_n : \mathtt{LBP}_n \to \mathtt{SBP}_n$.

Definition 4.7. For each $n \geq 0$ and $k \in [2n]_0$, we let $\mathtt{LBP}_{n,k}$ be the set of loosely barred permutations (w, B) such that $|\mathrm{Des}(w)| + |B| = k$.

Proposition 4.8. *For each $n \geq 0$ and $k \in [n]_0$, the restriction of Θ_n to $\mathtt{LBP}_{n,2k}$ yields a bijection $\Theta_{n,k}$ from $\mathtt{LBP}_{n,2k}$ to $\mathtt{SBP}_{n,k}$.*

Proof. Let $(w, B) \in \mathtt{LBP}_{n,2k}$, so $|\mathrm{Des}(w)| + |B| = 2k$. Let also $(u, C) = \theta(w, B)$, so $\Theta_n(w, B) = (w, C \setminus \{\,0\,\})$. Then, by (10),

$$2k = 2|D(u) \setminus C| + |C|$$

and so $|C|$ is even. Therefore

$$|D(w) \setminus (C \setminus \{\,0\,\})| + \left\lceil \frac{|C \setminus \{\,0\,\}|}{2} \right\rceil = |D(u) \setminus C| + \left\lceil \frac{|C \setminus \{\,0\,\}|}{2} \right\rceil = |D(u) \setminus C| + \frac{|C|}{2} = k\,,$$

where in the last step we have used equation (8).

The transformation $\Theta_{n,k}$ is injective. If $\Theta_n(w, B) = \Theta_n(w', B')$, then $w = w'$ and $\mathrm{Des}(w) = \mathrm{Des}(w')$. Moreover, from $\mathrm{Des}(w)\Delta B \setminus \{\,0\,\} = \mathrm{Des}(w)\Delta B' \setminus \{\,0\,\}$ we deduce $B \setminus \{\,0\,\} = B' \setminus \{\,0\,\}$. If moreover $(w, B), (w', B') \in \mathtt{LBP}_{n,2k}$, then $|B| = 2k - |\mathrm{Des}(w)| = |B'|$ and $B \setminus \{\,0\,\} = B' \setminus \{\,0\,\}$ imply that $0 \in \mathrm{Des}(w)\Delta B$ if and only if $0 \in \mathrm{Des}(w)\Delta B'$. It follows that $B = B'$.

In order to show that the transformation $\Theta_{n,k}$ is surjective, let us fix $(u, C) \in \mathtt{SBP}_{n,k}$, so $|\mathrm{Des}(u) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil = k$. If $|C|$ is even, then $(w, B) := \theta(u, C)$ is such that $\Theta_n(w, B) = \theta(w, B) = (u, C)$ and, using equations (8) and (10), $(w, B) \in \mathtt{LBP}_{n,2k}$:

$$|\mathrm{Des}(w)| + |B| = 2|\mathrm{Des}(u) \setminus C| + |C| = 2(\,|\mathrm{Des}(u) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil\,) = 2k.$$

If $|C|$ is odd, then $(w, B) := \theta(u, C \cup \{\,0\,\})$ is such that $\Theta(w, B) = (u, C)$ and $(w, B) \in \mathtt{LBP}_{n,2k}$:

$$\begin{aligned}|\mathrm{Des}(w)| + |B| &= 2|\mathrm{Des}(u) \setminus (C \cup \{\,0\,\})| + |C \cup \{\,0\,\}| = 2|\mathrm{Des}(u) \setminus C| + 2\frac{|C| + 1}{2} \\ &= 2(\,|\mathrm{Des}(u) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil\,) = 2k\,.\end{aligned}$$

□

Definition 4.9. For each $k \in [n-1]_0$, we let $\mathtt{SBP}_n^k$ be the set of simply barred permutations $(w, B) \in \mathtt{SBP}_n$ such that $|\mathrm{Des}^+_{\mathsf{B}}(\psi(w, B))| = k$.

Notice that

$$|\mathrm{Des}^+_{\mathsf{B}}(\psi(w, B))| = k \quad \text{iff} \quad \begin{cases} 0 \notin \mathrm{Des}_{\mathsf{B}}(\psi(w, B)) \text{ and } \mathrm{des}_{\mathsf{B}}(\psi(w, B)) = k, \text{ or} \\ 0 \in \mathrm{Des}_{\mathsf{B}}(\psi(w, B)) \text{ and } \mathrm{des}_{\mathsf{B}}(\psi(w, B)) = k + 1\,. \end{cases}$$

Using Lemma 4.2 and equation (7), for $(w, B) \in \mathtt{SBP}_n$ we have

$$(w, B) \in \mathtt{SBP}_n^k \quad \text{iff} \quad \begin{cases} |B| \text{ is even and } |D(w) \setminus B| + \left\lceil \frac{|B|}{2} \right\rceil = k, \text{ or} \\ |B| \text{ is odd and } |D(w) \setminus B| + \left\lceil \frac{|B|}{2} \right\rceil = k + 1\,. \end{cases}$$

[2] At the moment of writing we cannot pinpoint any other usage of this involution apart, of course, from yielding our bijections and counting results.

Proposition 4.10. *For each $k \in [n-1]_0$, the restriction of Θ_n to $\mathtt{LBP}_{n,2k+1}$ yields a bijection Θ_n^k from $\mathtt{LBP}_{n,2k+1}$ to $\mathtt{SBP}_n^k$.*

Proof. Let $(w, B) \in \mathtt{LBP}_{n,2k+1}$ and $(u, C) = \theta(w, B)$, so $\Theta_n(w, B) = (w, C \setminus \{0\})$. Thus $|\mathrm{Des}(w)| + |B| = 2k+1$ and, by (10),

$$2k = 2|D(u) \setminus C| + |C| - 1\,,$$

so in particular $|C|$ is odd. Using equation (8), we obtain

$$|D(u) \setminus C| + \left\lceil \frac{|C \setminus \{0\}|}{2} \right\rceil = \begin{cases} |D(u) \setminus C| + \frac{|C|-1}{2} = k\,, & 0 \in C, \\ |D(u) \setminus C| + \frac{|C|+1}{2} = k+1\,, & 0 \notin C\,. \end{cases}$$

Considering that $|C|$ is odd, we have

$$|D(u) \setminus (C \setminus \{0\})| + \left\lceil \frac{|C \setminus \{0\}|}{2} \right\rceil = \begin{cases} k\,, & |C \setminus \{0\}| \text{ is even,} \\ k+1\,, & |C \setminus \{0\}| \text{ is odd,} \end{cases}$$

that is, $(u, C \setminus \{0\}) = \Theta_n(w, B) \in \mathtt{SBP}_n^k$.

The transformation Θ_n^k is injective, the reason being similar to the one for $\Theta_{n,k}$. If $\Theta_n(w, B) = \Theta_n(w', B')$ then $w = w'$ and $\mathrm{Des}(w) = \mathrm{Des}(w')$. Moreover, from $\mathrm{Des}(w)\Delta B \setminus \{0\} = \mathrm{Des}(w)\Delta B' \setminus \{0\}$ we deduce $B \setminus \{0\} = B' \setminus \{0\}$. Considering now that $|B| = 2k+1-|\mathrm{Des}(w)| = |B'|$ and $B \setminus \{0\} = B' \setminus \{0\}$, we derive $0 \in \mathrm{Des}(w)\Delta B$ if and only if $0 \in \mathrm{Des}(w)\Delta B'$, whence $B = B'$.

In order to show that the transformation Θ_n^k is surjective, let us fix $(u, C) \in \mathtt{SBP}_n^k$, so either (i) $|C|$ is even and $|D(u) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil = k$ or (ii) $|C|$ is odd and $|D(u) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil = k+1$.

Let us suppose (i). Then $(w, B) := \theta(u, C \cup \{0\})$ is such that $\Theta_n(w, B) = (u, C)$ and, using equations (8) and (10), $(w, B) \in \mathtt{LBP}_{n,2k+1}$:

$$\begin{aligned} |\mathrm{Des}(w)| + |B| &= 2|\mathrm{Des}(u) \setminus C| + |C \cup \{0\}| = 2|\mathrm{Des}(u) \setminus C| + 2\frac{|C|+1}{2} \\ &= 2(\,|\mathrm{Des}(u) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil + \frac{1}{2}\,) = 2k+1. \end{aligned}$$

Let us suppose (ii). Then $(w, B) := \theta(u, C)$ is such that $\Theta_n(w, B) = \theta(w, B) = (u, C)$ and, using equations (8) and (10), $(w, B) \in \mathtt{LBP}_{n,2k+1}$:

$$\begin{aligned} |\mathrm{Des}(w)| + |B| &= 2|\mathrm{Des}(u) \setminus C| + |C| = 2|\mathrm{Des}(u) \setminus C| + 2\frac{|C|}{2} \\ &= 2(\,|\mathrm{Des}(u) \setminus C| + \left\lceil \frac{|C|}{2} \right\rceil - \frac{1}{2}\,) = 2(k+1) - 1 = 2k+1. \end{aligned}$$ □

To end this section, we collect the consequences of the bijections established so far.

Theorem 4.11. *The following relations hold:*

$$\left\langle {\mathsf{B}_n \atop k} \right\rangle = \sum_{i=0}^{2k} \left\langle {n \atop i} \right\rangle \binom{n+1}{2k-i}\,, \tag{3}$$

$$2^n \left\langle {n \atop k} \right\rangle = \sum_{i=0}^{2k+1} \left\langle {n \atop i} \right\rangle \binom{n+1}{2k+1-i}\,. \tag{4}$$

Proof. We have seen that signed permutations $u \in \mathsf{B}_n$ such that $\mathrm{des}_{\mathsf{B}}(u) = k$ are in bijection (via the mapping ψ of Definition 3.7) with simply barred permutations in $\mathtt{SBP}_{n,k}$. Next, this set is in bijection (see Proposition (4.8)) with the set $\mathtt{LBP}_{n,2k}$ of loosely barred permutations $(w, B) \in \mathtt{LBP}_n$ such that $\mathrm{des}(w) + |B| = 2k$. The cardinality of $\mathtt{LBP}_{n,2k}$ is the right-hand side of equality (3).

The left-hand side of equality (4) is the cardinality of the set of signed permutations u such that $|\mathrm{Des}_{\mathsf{B}}^+(u)| = k$, see Lemma 2.1. This set is in bijection with the set $\mathtt{SBP}_n^k$ (via ψ defined in 3.7 and by the definition of $\mathtt{SBP}_n^k$) which, in turn, is in bijection (see Proposition (4.10)) with the set $\mathtt{LBP}_{n,2k+1}$ of loosely barred permutations $(w, B) \in \mathtt{LBP}_n$ such that $\mathrm{des}(w) + |B| = 2k+1$. The cardinality of this set is the right-hand side of equality (4). □

Theorem 4.12. *The following relation holds:*

$$B_n(t^2) = (1+t)^{n+1} S_n(t) - 2^n t S_n(t^2)\,. \tag{11}$$

Proof. By (3), $\left\langle {\mathsf{B}_n \atop k} \right\rangle$, which is the coefficient of t^{2k} in the polynomial $B_n(t^2)$, is also the coefficient of t^{2k} in $(1+t)^{n+1}S_n(t)$. By (4), $2^n \left\langle {n \atop k} \right\rangle$ is the coefficient of t^{2k+1} in the polynomials $2^n t S_n(t^2)$ and $(1+t)^{n+1}S_n(t)$. Therefore

$$B_n(t^2) + 2^n t S_n(t^2) = (1+t)^{n+1} S_n(t)\,, \tag{5}$$

whence equation (11). □

5 Stembridge's identity for Eulerian numbers in type D

Let us recall that a signed permutation $u \in \mathsf{B}_n$ is *even signed* if the number of negative letters in its window notation is even. The even signed permutations of B_n form a subgroup D_n of B_n and in fact the groups D_n are the standard models for abstract Coxeter groups of type D.

Definitions analogous to those given in Section 2 for the types A and B can be given for type D. Namely, for $u \in \mathsf{D}_n$, we set

$$\mathrm{Des}_{\mathsf{D}}(u) := \{\, i \in \{\, 0, 1, \ldots, n-1 \,\} \mid u_i > u_{i+1} \,\}\,, \tag{12}$$

where we have set $u_0 = -u_2$,

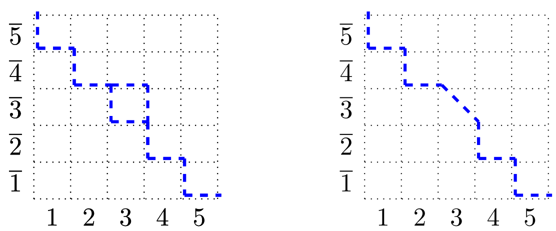
$$\mathrm{des}_{\mathsf{D}}(u) := |\mathrm{Des}_{\mathsf{D}}(u)|\,, \qquad \left\langle {\mathsf{D}_n \atop k} \right\rangle := |\{\, u \in \mathsf{D}_n \mid \mathrm{des}_{\mathsf{D}}(u) = k \,\}|\,, \qquad D_n(t) := \sum_{k=0}^{n} \left\langle {\mathsf{D}_n \atop k} \right\rangle t^k\,.$$

The formula in (12) is the standard one, see e.g. [4, §8.2] or [1]. The reader will have no difficulties verifying that, up to renaming 0 by -1, the type D descent set of u can also be defined using the following, see [14, §13]:

$$\mathrm{Des}_{\mathsf{D}}(u) := \{\, i \in \{\, -1, 1, \ldots, n-1 \,\} \mid u_i > u_{|i|+1} \,\}\,. \tag{13}$$

It is convenient to consider a more flexible representation of elements of D_n. If $u \in \mathsf{B}_n$, then its mate is the signed permutation $\underline{u} \in \mathsf{B}_n$ that differs from u only for the sign of the first letter. Notice that $\underline{\underline{u}} = u$. We define a *forked signed permutation* (see [14, §13]) as an unordered pair of the form $\{u, \underline{u}\}$ for some $u \in \mathsf{B}_n$. Clearly, just one of the mates is even signed and therefore forked signed permutations are combinatorial models of D_n.

The path representation of a forked signed permutation is insensitive of how the diagonal is crossed, either from the West, or from the North. The following are possible ways to draw a forked signed permutation on a grid:



Even if the formulas in (12) and (13) have been defined for even signed permutations, they still can be computed for all signed permutations. The formula in (13) is not invariant under taking mates, however the following lemma shows that this formula suffices to compute the number of type D descents of a forked signed permutation and therefore the Eulerian numbers $\left\langle {\mathsf{D}_n \atop k} \right\rangle$.

Lemma 5.1. *For each* $u \in \mathsf{B}_n$, $1 \in \mathrm{Des}_{\mathsf{D}}(u)$ *if and only if* $-1 \in \mathrm{Des}_{\mathsf{D}}(\underline{u})$. *Therefore* $\mathrm{des}_{\mathsf{D}}(u) = \mathrm{des}_{\mathsf{D}}(\underline{u})$.

Proof. Suppose $1 \in \mathrm{Des}_{\mathsf{D}}(u)$, that is $u_1 > u_2$. Then $\underline{u}_{-1} = -(-u_1) = u_1 > u_2$, and so $-1 \in \mathrm{Des}_{\mathsf{D}}(u)$. The opposite implication is proved similarly.

For the last statement, observe that $\mathrm{Des}_{\mathsf{D}}(u) = \Delta_u \cup \{\, i \in \{\, 2, \ldots, n-1 \,\} \mid u_i > u_{i+1} \,\}$ with $\Delta_u := \{\, i \in \{\, 1, -1 \,\} \mid u_i > u_{|i|+1} \,\}$ and, by what we have just remarked, we have $|\Delta_u| = |\Delta_{\underline{u}}|$. It follows that $|\mathrm{Des}_{\mathsf{D}}(u)| = |\mathrm{Des}_{\mathsf{D}}(\underline{u})|$. □

Our next aim is to derive Stembridge's identity

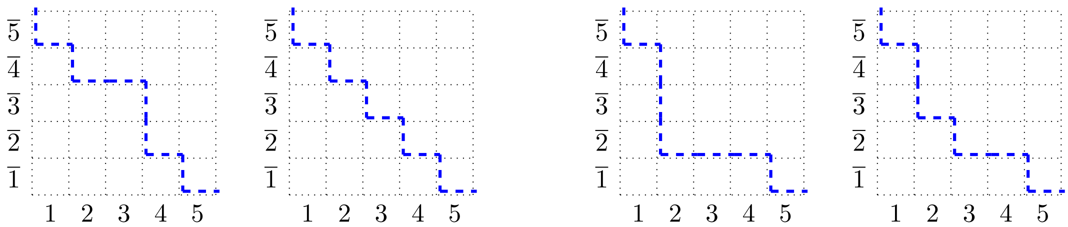
$$D_n(t) = B_n(t) - n2^{n-1} t S_{n-1}(t)\,, \tag{14}$$

see [22, Lemma 9.1], which, in term of the coefficients of these polynomials, amounts to

$$\left\langle \begin{matrix} \mathsf{D}_n \\ k \end{matrix} \right\rangle = \left\langle \begin{matrix} \mathsf{B}_n \\ k \end{matrix} \right\rangle - n2^{n-1} \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle . \tag{15}$$

Definition 5.2. A signed permutation u is *smooth* if u_1, u_2 have equal sign and, otherwise, it is non-smooth.

The reason for naming a signed permutation smooth arises again from the path representation of a signed permutation: the smooth signed permutation is, between the two mates, the one that minimizes the turns nearby the diagonal, as suggested below with two pairs of mates as examples:



Lemma 5.3. *If $u \in \mathsf{B}_n$ is smooth, then $-1 \in \mathrm{Des}_{\mathsf{D}}(u)$ if and only if $0 \in \mathrm{Des}_{\mathsf{B}}(u)$ and therefore $\mathrm{des}_{\mathsf{D}}(u) = \mathrm{des}_{\mathsf{B}}(u)$.*

Proof. Suppose $0 \in \mathrm{Des}_{\mathsf{B}}(u)$, so $u_1 < 0$ and $u_2 < 0$ as well, since u is smooth. Then $u_{-1} = -u_1 > 0 > u_2$, so $-1 \in \mathrm{Des}_{\mathsf{D}}(u)$. Conversely, suppose $-1 \in \mathrm{Des}_{\mathsf{D}}(u)$, that is, $u_{-1} > u_2$. If $u_1 > 0$, then $0 > -u_1 = u_{-1} > u_2$, so u_1, u_2 have different sign, a contradiction. Therefore $u_1 < 0$ and $0 \in \mathrm{Des}_{\mathsf{B}}(u)$. □

Next, we consider the correspondence—let us call it χ—sending a non-smooth signed permutation $u \in \mathsf{B}_n$ to the pair $(|u_1|, u')$, where u' is obtained from $u_2 \ldots u_n$ by normalising this sequence, so that it takes absolute values in the set $[n-1]$. For example $\chi(6\overline{12347}5) = (6, \overline{12346}5)$ and $\chi(\overline{2}316\overline{47}5) = (2, 215\overline{36}4)$, as suggested below:

$$6\overline{12347}5 \rightsquigarrow (6, \overline{12347}5) \rightsquigarrow (6, \overline{12346}5)\,, \qquad \overline{2}316\overline{47}5 \rightsquigarrow (2, 316\overline{47}5) \rightsquigarrow (2, 215\overline{36}4)\,.$$

The process of normalizing the sequence $u_2 \ldots u_n$ can be understood as applying to each letter of this sequence the unique order preserving bijection $N_{n,x} : [\pm n] \setminus \{ x, \overline{x} \} \to [\pm\, n-1]$ where, in general, $x \in [n]$ and, in this case, $x = |u_1|$.

Lemma 5.4. *Let $n \geq 2$. For each pair (x, v) with $x \in [n]$ and $v \in \mathsf{B}_{n-1}$, there exists a unique non-smooth $u \in \mathsf{B}_n$ such that $\chi(u) = (x, v)$.*

Proof. We construct u firstly by renaming v to v' so that none of $x, \overline{x}$ appears in v' (that is, we apply to each letter of v the inverse of $N_{n,x}$) and then by adding in front of v' either x or $\overline{x}$, according to the sign of the first letter of v'. □

Lemma 5.5. *The correspondence χ restricts to a bijection from the set of non-smooth signed permutations $u \in \mathsf{B}_n$ such that $\mathrm{des}_{\mathsf{B}}(u) = k$ to the set of pairs (x, v) where $x \in [n]$ and $v \in \mathsf{B}_{n-1}$ is such that $|\mathrm{Des}^+_{\mathsf{B}}(v)| = k - 1$.*

Proof. We have already argued that χ is a bijection from the set of non-smooth signed permutations u of $[n]$ to the set of pairs (x, v) with $x \in [n]$ and $v \in \mathsf{B}_{n-1}$. Therefore, we are left to argue that, for non-smooth u and v, if $\mathrm{des}_{\mathsf{B}}(u) = k$, then $|\mathrm{Des}^+_{\mathsf{B}}(v)| = k - 1$. Since u_1, u_2 have different sign, then $|\mathrm{Des}_{\mathsf{B}}(u) \cap \{ 0, 1 \}| = 1$. Clearly, if $\chi(u) = (x, v)$, then $\mathrm{Des}^+_{\mathsf{B}}(v) = \{ i - 1 \mid i \in \mathrm{Des}_{\mathsf{B}}(u) \cap \{ 2, \ldots, n-1 \} \}$, from which the statement of the lemma follows. □

Theorem 5.6. *The following relations hold:*

$$\left\langle \begin{matrix} \mathsf{B}_n \\ k \end{matrix} \right\rangle = \left\langle \begin{matrix} \mathsf{D}_n \\ k \end{matrix} \right\rangle + n2^{n-1} \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle , \qquad B_n(t) = D_n(t) + n2^{n-1} t S_{n-1}(t)\,.$$

Proof. Every signed permutation is either smooth or non-smooth. By Lemma 5.3, the smooth signed permutations with k type B descents are in bijection with the even signed permutations with k type D descents. By Lemma 5.5, the non-smooth signed permutations $u \in \mathsf{B}_n$ with k type B descents are in bijection with the pairs $(x, v) \in [n] \times \mathsf{B}_{n-1}$ such $|\mathrm{Des}^+_{\mathsf{B}}(v)| = k - 1$. Using Lemma 2.1, the number of these pairs is $n2^{n-1} \left\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rangle$. □

Example 5.7. We end this section exemplifying the use of formulas (3) and (15) by which computation of the Eulerian numbers in type B and D is reduced to computing Eulerian numbers in type A. Let us mention that our interest in Eulerian numbers originates from our lattice theoretic work on the lattice variety of Permutohedra [19] and its possible extensions to generalized forms of Permutohedra [15, 18, 17]. Among these generalizations, we count lattices of finite Coxeter groups in the types B and D [3]. While it is known that the lattices B_n span the same lattice variety of the permutohedra, see [6, Exercice 1.23], characterising the lattice variety spanned by the lattices D_n is an open problem. A first step towards solving this kind of problem is to characterize (and count) the join-irreducible elements of a class of lattices. In our case, this amounts to characterizing the elements u in B_n (resp., in D_n) such that $\mathrm{des}_{\mathsf{B}}(u) = 1$ (resp., such that $\mathrm{des}_{\mathsf{D}}(u) = 1$). The numbers $\left\langle {\mathsf{B}_n \atop 1} \right\rangle$ and $\left\langle {\mathsf{D}_n \atop 1} \right\rangle$ are known to be equal to $3^n - n - 1$ and $3^n - n - 1 - n2^{n-1}$ respectively, see [14, Propositions 13.3 and 13.4]. Let us see how to derive these identities using the formulas (3) and (15). To this end, we also need the alternating sum formula for Eulerian numbers, see e.g. [14, page 12]:

$$\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n \,. \tag{16}$$

For type B, we have

$$\begin{aligned}
\left\langle {\mathsf{B}_n \atop 1} \right\rangle &= \left\langle {n \atop 0} \right\rangle \binom{n+1}{2} + \left\langle {n \atop 1} \right\rangle \binom{n+1}{1} + \left\langle {n \atop 2} \right\rangle \binom{n+1}{0} \\
&= \binom{n+1}{2} + (2^n - n - 1)(n+1) + \left\langle {n \atop 2} \right\rangle \\
&= \binom{n+1}{2} + (2^n - n - 1)(n+1) + 3^n - 2^n(n+1) + \binom{n+1}{2}\,, && \text{by (16)} \\
&= 3^n - (n+1)^2 + 2\binom{n+1}{2} = 3^n - (n+1)(n+1-n) = 3^n - n - 1\,.
\end{aligned}$$

The computation in type D is then immediate from Stembridge's identity (15):

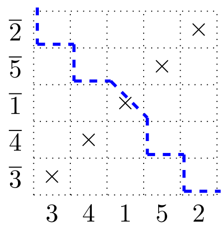
$$\left\langle {\mathsf{D}_n \atop 1} \right\rangle = \left\langle {\mathsf{B}_n \atop 1} \right\rangle - n2^{n-1} \left\langle {n-1 \atop 0} \right\rangle = 3^n - n - 1 - n2^{n-1}\,. \qquad \Diamond$$

6 Threshold graphs and their degree orderings

Besides presenting the bijective proofs, a goal of this paper is to exemplify the potential of the path representation of signed permutations. The attentive reader might object then that the path representation is not being used in Section 5. Indeed, after discovering this proof via this representation of signed permutations, we realized that the presentation of the proof could be simplified by avoiding mention of the path representation. It might be asked then whether the path representation yields something more, in particular with respect to the lattices of the Coxeter groups D_n. We answer. The type D set of inversions of an even signed permutation (or of a forked signed permutation) can be defined as follows:

$$\mathrm{Inv}_{\mathsf{D}}(u) := \mathrm{Inv}_{\mathsf{B}}(u) \setminus \{\, (-i, i) \mid i \in [n] \,\}\,,$$

which, graphically, amounts to ignoring boxes on the diagonal:



As mentioned in Proposition 3.4, we can identify the set of inversions of a signed permutation u with the disjoint union of $\mathrm{Inv}(\lambda^u_x)$ and a set of unordered pairs. For even signed permutations, this identification yields

$$\mathrm{Inv}_{\mathsf{D}}(u) = \mathrm{Inv}(\lambda^u_x) \cup E^u \quad \text{with} \quad E^u := \{\, \{i, j\} \mid i, j \in [n], i \neq j,\ ((\lambda^u_x)^{-1}(i), (\lambda^u_x)^{-1}(j)) \text{ lies below } \pi^u \,\}\,.$$

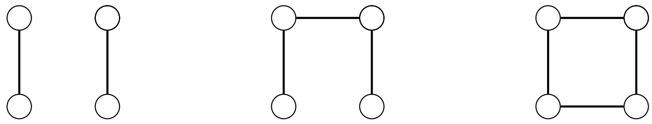


Figure 1: The (unlabelled) graphs $K_{2,2}$, P_3, and C_4

Thus, we may consider $([n], E^u)$ as a simple graph on the set of vertices $[n]$. Let us recall the following standard definitions that apply to an arbitrary simple graph (V, E) and to a vertex $v \in V$:

$$N_E(v) := \{ u \in V \mid \{v, u\} \in E \}, \qquad \deg_E(v) := |N_E(v)|, \qquad N_E[v] := N(v) \cup \{ v \}.$$

A linear ordering $v_1, \ldots, v_n$ of V is a *degree ordering* of (V, E) if $\deg_E(v_1) \geq \deg_E(v_2) \geq \ldots \geq \deg_E(v_n)$. The *vicinal preorder* of a graph (V, E), noted $\lhd_E$, is defined by saying that $v \lhd_E u$ iff $N_E(v) \subseteq N_E[u]$. That the vicinal preorder is indeed a preorder is well known, see e.g. [12]. Next, we take Theorem 1 in [7] as the definition of the class of threshold graphs and consider, among the possible characterisations of this class, the one that uses the vicinal preorder.

Definition 6.1. A graph (V, E) is *threshold* if it does not contain an induced subgraph isomorphic to one among $K_{2,2}$, P_3 and C_4 (these graphs are illustrated in Figure 1).

Proposition 6.2 (see e.g. [12, Theorem 1.2.4]). *A graph* (V, E) *is* threshold *if and only if the vicinal preorder is total.*

With these tools available, let us observe the following:

Theorem 6.3. *The mapping sending* u *to* (λ^u_x, E^u) *is a bijection from the set* D_n *to the set of pairs* (w, E) *such that* $([n], E)$ *is a threshold graph and* $w \in \mathsf{S}_n$ *is a degree ordering on this graph.*

Proof. We start arguing that, for $u \in \mathsf{D}_n$, $([n], E^u)$ is a threshold graph and that λ^u_x is a degree ordering on this graph. Notice that $E^u = E^{\underline{u}}$, so we can suppose that u is the mate such that $u_1 > 0$. Clearly, we can also suppose that λ^u_x is the identity permutation. Under these hypothesis, the paths π^u that are symmetric along the diagonal bijectively correspond to fixed-point free self-adjoint Galois connections, see e.g. [16] for the general correspondence between paths and sup-preserving functions. For a fixed-point free self-adjoint Galois connection we mean an antitone map $f : [n]_0 \to [n]_0$ such that, for each $x, y \in [n]_0$, $x \leq f(y)$ iff $y \leq f(x)$ and $x \neq f(x)$. Moreover, under these hypothesis, we have that $\{x, z\} \in E^u$ if and only if $z \neq x$ and $z \leq f(x)$, where f bijectively corresponds to π^u. Then, if $y < x$ and $z \in N_{E^u}(x)$, then $z \leq f(x) \leq f(y)$ and so either $z = y$ or $z \in N_{E^u}(y)$. We have argued that $x < y$ implies that $N_{E^u}(y) \subseteq N_{E^u}[x]$ which, in particular, implies that the identity permutation is a degree ordering on E^u and that the vicinal preorder is total, so $([n], E^u)$ is a threshold graph.

The mapping sending u to (λ^u_x, E^u) is clearly injective, so we are left to argue that every pair (w, E), with $([n], E)$ a threshold graph and $w \in \mathsf{S}_n$ a degree ordering on it, arises in this way. Again, we can assume that w is the identity permutation, so we need to find a fixed-point free self-adjoint Galois connection $f : [n]_0 \to [n]_0$ such that, for $x, z \in [n]$, $\{x, z\} \in E$ if and only if $x \neq z$ and $z \leq f(x)$.

Observe that if $\deg_E(y) \leq \deg_E(x)$, then necessarily we have $N_E(y) \subseteq N_E[x]$, otherwise $N_E(x) \subseteq N_E[y]$ and we get a contradiction. Thus, if $x < y$, then $N_E(y) \subseteq N_E[x]$, since the identity permutation is a degree ordering. Define then $f(x) = \max N_E(x)$, with the conventions that $\max \emptyset = 0$ and $N_E(0) = [n]_0$. For $x, z \in [n]$, if $\{x, z\} \in E$, then $z \in N_E(x)$ and then $z \leq f(x)$. Conversely, if $z \leq f(x)$, then $x \in N_E(f(x)) \subseteq N_E[z]$, so if $x \neq z$, then $x \in N_E(z)$ and $\{x, z\} \in E$. Finally f is a fixed-point free self-adjoint Galois connection: it is fixed-point free since $x \notin N_E(x)$, and it is easily seen that $y \leq f(x)$ if and only if $x \leq f(y)$, for each $x, y \in [n]_0$, property that also implies that f is antitone. □

Let us remark that Theorem 6.3 also yields a natural representation of the weak ordering on D_n as follows: under the bijection, $(w_1, E_1) \leq (w_2, E_2)$ holds if and only if $w_1 \leq w_2$ in the weak ordering of S_n and, moreover, $E_1 \subseteq E_2$. This poset (actually a lattice, since it is isomorphic to D_n) is built out from threshold graphs but is only loosely related to the lattice of threshold graphs of [13] where unlabeled (that is, up to isomorphism) threshold graphs are considered.

That threshold graphs are related to the families B and D in the theory of Coxeter groups has already been observed, see e.g. [8], [20, Exercise 5.25], and [21, Exercise 3.115]. As part of possible future research, it is tempting to investigate further the bijection presented in Theorem 6.3 (which can be further adapted to fit the type B) and try to understand if it plays any important role with respect to the problem, partly solved in [8], of characterizing free sub-arrangements of the Coxeter arrangements B_n.

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