Iterated fixpoint well-founded semantics for hybrid knowledge bases

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Abstract. MKNF-based Hybrid Knowledge Bases (HKBs) integrate Logic Programming (LP) and Description Logics (DLs) offering the combined expressiveness of the two formalisms. In particular, HKB allow to make different closure assumptions for different predicates. HKBs have been given a well-founded semantics in terms of an alternate fixpoint. In this paper we provide an alternative definition of the semantics using an iterated fixpoint. In this way the computation of the well-founded model proceeds uniformly bottom-up, making the semantics easier to understand, to reason with and to automate. We also present slightly different but equivalent versions of our definition. We then discuss the relationships of HKBs with other formalisms. The results show that overall HKBs seem to be those that more tightly integrate LP and DL, even if there exist incomparable languages such as the recent FO(ID) formalism.

Keywords: Hybrid Knowledge Bases, MKNF, Well-foudned semantics, Description Logics

1 Introduction

Logic Programming (LP) languages and Description Logic (DL) languages are successful tools for modeling complex domains. They are both based on first order logic but differ for the domain closure assumption: LP makes the closed-world assumption while DLs make the open-world assumption.

In some domains, such as in legal reasoning [1], there is information requiring open world assumption and information requiring closed world assumption, so there is a considerable interest in combining LP and DL.

Among the several integration approaches proposed in the last decades, one of the most effective is Hybrid Knowledge Bases (HKBs) [13], composed of a logic program and a DL Knowledge Base (KB), with a semantics based on the logic of Minimal Knowledge with Negation as Failure (MKNF) [12]. As argued by [13], this approach is faithful, tight, flexible and decidable; the other approaches lack one or more of those properties.

Recently, HKBs have been given a well-founded semantics (WFS) [11] which is an extension of the WFS [18] for LP. The WFS assigns a three-valued model to a normal logic program, i.e., it identifies a consistent three-valued interpretation as the meaning of the program. The WFS interprets negation in LP in a way that has become a standard, together with the alternative interpretation offered by stable models [7]. The two semantics, while closely related, serve different purposes and differ for their computational cost: while computing the well-founded model of a propositional program has polynomial complexity, the stable model semantics is more expensive, for example, determining if a propositional program has a stable model is NP-complete.

The WFS for logic programs was given in [18] in terms of the least fixpoint of an operator that is composed by two sub-operators, one computing consequences and the other computing unfounded sets. The definition of second sub-operator is not constructive so a different, fully constructive definition was given in [6] that alternates between the computation of the fixpoint of two operators constructively defined. Another constructive definition was given in [14], where the model is built bottom-up by iterating an inner fixpoint computation until an outer fixpoint is achieved. This definition is more intuitive because it allows a fully bottom-up computation of the model, without the need to alternate operators.

The WFS for HKBs of [11] is given in terms of an alternating fixpoint. In this paper we aim at providing at iterating fixpoint definition of the well-founded semantics of HKB that is equivalent to that of [11]. In this way, we aim at giving a more intuitive definition of the semantics together with a more straightforward way of building the model bottom-up. This sheds some light on the properties of the semantics and provides new tools for reasoning on HKBs. We also provide slightly different but equivalent formulations of the iterative definition that can be useful in specific contexts.

We also discuss the relationships between HKBs and other formalisms for combining DL and LP, concentrating on FO(ID) [19] which integrates inductive definitions into FOL and that was not compared to HKBs before. We show that, while the two formalisms are related, there is no obvious mapping between them and thus they can be viewed as complementary tools.

The paper is structured as follows: in Section 2 we provide some background notions. In Section 3 we introduce our iterated fixpoint definition of the semantics of HKB. Section 4 presents alternative but equivalent formulations. Section 5 discusses related work and Section 6 concludes the paper.

2 Background and notation

This section is devoted to introducing the background notions. We first discuss fixpoints and LP in Sections 2.1 and 2.2 following [15]. Then we present DLs in Section 2.3 and HKBs in Section 2.4.

2.1 Partial orders, complete lattices, fixpoints

A relation on a set S is a *partial order* if it is reflexive, antisymmetric and transitive. In the following, let S be a set with a partial order $\leq$. $a \in S$ is an *upper bound* of a subset X of S if $x \leq a$ for all $x \in X$. Similarly, $b \in S$ is a *lower bound* of X if $b \leq x$ for all $x \in X$.

An element $a \in S$ is the *least upper bound* of a subset X of S if a is an upper bound of X and, for all upper bounds a' of X, we have $a \leq a'$. Similarly, $b \in S$ is the *greatest lower bound* of a subset X of S if b is a lower bound of X and, for all lower bounds b' of X, we have $b' \leq b$. The least upper bound of X is unique, if it exists, and is denoted by $lub(X)$. Similarly, the greatest lower bound of X is unique, if it exists, and is denoted by $glb(X)$.

A partially ordered set L is a *complete lattice* if $lub(X)$ and $glb(X)$ exist for every subset X of L. We let $\top$ denote the top element $lub(L)$ and $\bot$ denote the bottom element $glb(L)$ of the complete lattice L.

Let L be a complete lattice and $T : L \to L$ be a mapping. We say T is *monotonic* if $T(x) \leq T(y)$, whenever $x \leq y$. We say that $a \in L$ is a *fixpoint* of T if $T(a) = a$. We say that $a \in L$ is the *least fixpoint* of T if a is a fixpoint and, for all fixpoints b of T, we have $a \leq b$. Similarly, we define *greatest fixpoint.*

Let L be a complete lattice and $T : L \to L$ be monotonic. Then we define $T \uparrow 0 = \bot$; $T \uparrow \alpha = T(T \uparrow (\alpha - 1))$, if α is a successor ordinal; $T \uparrow \alpha = lub(\{T \uparrow \beta | \beta < \alpha\})$, if α is a limit ordinal; $T \downarrow 0 = \top$; $T \downarrow \alpha = T(T \downarrow (\alpha - 1))$, if α is a successor ordinal; $T \downarrow \alpha = glb(\{T \downarrow \beta | \beta < \alpha\})$, if α is a limit ordinal.

Proposition 1. *Let L be a complete lattice and $T : L \to L$ be monotonic. Then T has a least fixpoint $lfp(T)$ and a greatest fixpoint $gfp(T)$.*

2.2 Logic programming

A *normal program* P is a set of normal rules. A normal rule has the form

$$r = h \leftarrow b_1, \ldots, b_n, \neg c_1, \ldots, \neg c_m \tag{1}$$

where $h, b_1, \ldots, b_n, c_1, \ldots, c_m$ are atoms.

The set of ground atoms that can be built with the symbols of a program P is called the *Herbrand base* and is denoted as $\mathcal{B}_P$.

A *two-valued interpretation* I is a subset of $\mathcal{B}_P$. I is the set of true atoms, so a is true in I if $a \in I$ and is false if $a \notin I$. The set Int_2 of two-valued interpretations for a program P forms a complete lattice where the partial order $\leq$ is given by the subset relation $\subseteq$. The least upper bound and greatest lower bound are defined as $lub(X) = \bigcup_{I \in X} I$ and $glb(X) = \bigcap_{I \in X} I$. The bottom and top element are respectively $\emptyset$ and $\mathcal{B}_P$.

A *three-valued interpretation* $\mathcal{I}$ is a pair $\langle I_T; I_F \rangle$ where I_T and I_F are subsets of $\mathcal{B}_P$ and represent respectively the set of true and false atoms. So a is true in $\mathcal{I}$ if $a \in I_T$ and is false in $\mathcal{I}$ if $a \in I_F$. A *consistent* three-valued interpretation $\mathcal{I} = \langle I_T; I_F \rangle$ is such that $I_T \cap I_F = \emptyset$. The union of two three-valued interpretations $\langle I_T, I_F \rangle$ and $\langle J_T, J_F \rangle$ is defined as $\langle I_T, I_F \rangle \cup \langle J_T, J_F \rangle = \langle I_T \cup J_T, I_F \cup J_F \rangle$.

The intersection of two three-valued interpretations $\langle I_T, I_F\rangle$ and $\langle J_T, J_F\rangle$ is defined as $\langle I_T, I_F\rangle \cap \langle J_T, J_F\rangle = \langle I_T \cap J_T, I_F \cap J_F\rangle$. The set Int_3 of three-valued interpretations for a program P forms a complete lattice where the partial order $\leq$ is defined as $\langle I_T, I_F\rangle \leq \langle J_T, J_F\rangle$ if $I_T \subseteq J_T$ and $I_F \subseteq J_F$. The least upper bound and greatest lower bound are defined as $lub(X) = \bigcup_{I\in X} I$ and $glb(X) = \bigcap_{I\in X} I$. The bottom and top element are respectively $\langle\emptyset,\emptyset\rangle$ and $\langle\mathcal{B}_P,\mathcal{B}_P\rangle$.

The WFS assigns a three-valued model to a program, i.e., it identifies a consistent three-valued interpretation as the meaning of the program. The WFS was given in [18] in terms of the least fixpoint of an operator that is composed by two sub-operators, one computing consequences and the other computing unfounded sets. We give here the alternative definition of the WFS of [14] that is based on a different iterated fixpoint.

Definition 1. *For a normal program P, sets Tr and Fa of ground atoms, and a 3-valued interpretation $\mathcal{I}$ we define the operators $OpTrue^P_{\mathcal{I}} : Int_2 \to Int_2$ and $OpFalse^P_{\mathcal{I}} : Int_2 \to Int_2$ as*

$OpTrue^P_{\mathcal{I}}(Tr) = \{a|a$ *is not true in* $\mathcal{I}$*; and there is a clause* $b \leftarrow l_1, ..., l_n$ *in* P*, a grounding substitution* θ *such that* $a = b\theta$ *and for every* $1 \leq i \leq n$ *either* $l_i\theta$ *is true in* $\mathcal{I}$*, or* $l_i\theta \in Tr\}$*;*
$OpFalse^P_{\mathcal{I}}(Fa) = \{a|a$ *is not false in* $\mathcal{I}$*; and for every clause* $b \leftarrow l_1, ..., l_n$ *in* P *and grounding substitution* θ *such that* $a = b\theta$ *there is some* i $(1 \leq i \leq n)$ *such that* $l_i\theta$ *is false in* $\mathcal{I}$ *or* $l_i\theta \in Fa\}$.

In words, the operator $OpTrue^P_{\mathcal{I}}(Tr)$ extends the interpretation $\mathcal{I}$ to add the new true atoms that can be derived from P knowing $\mathcal{I}$ and true atoms Tr, while $OpFalse^P_{\mathcal{I}}(Fa)$ computes new false atoms in P by knowing $\mathcal{I}$ and false atoms Fa. $OpTrue^P_{\mathcal{I}}$ and $OpFalse^P_{\mathcal{I}}$ are both monotonic [14], so they both have least and greatest fixpoints. An iterated fixpoint operator builds up *dynamic strata* by constructing successive three-valued interpretations as follows.

Definition 2 (Iterated Fixed Point). *For a normal program P, let $IFP^P : Int_3 \to Int_3$ be defined as $IFP^P(\mathcal{I}) = \mathcal{I} \cup \langle lfp(OpTrue^P_{\mathcal{I}}), gfp(OpFalse^P_{\mathcal{I}})\rangle$.*

IFP^P is monotonic [14] and thus has a least fixed point $lfp(IFP^P)$. Moreover, the well-founded model $WFM(P)$ of P is in fact $lfp(IFP^P)$. Let δ be the smallest ordinal such that $WFM(P) = IFP^P \uparrow \delta$. We refer to δ as the *depth* of P. The *stratum* of atom a is the least ordinal β such that $a \in IFP^P \uparrow \beta$ (where a may be either in the true or false component of $IFP^P \uparrow \beta$). Undefined atoms of the well-founded model do not belong to any stratum – i.e. they are not added to $IFP^P \uparrow \delta$ for any ordinal δ.

2.3 Description logics

Description Logics (DLs) are a fragment of First Order Logic (FOL) used to model ontologies [2]. Usually, their syntax is based on concepts and roles, corresponding respectively to sets of individuals and sets of pairs of individuals of the domain. In the following we briefly recall the DL $\mathcal{ALC}$, one of the most used and a basis for many other DLs.

Let us consider a set $\mathbf{C}$ of *atomic concepts*, a set $\mathbf{R}$ of *atomic roles* and a set $\mathbf{I}$ of individuals. A *concept* C is defined by the syntax rule:

$$C ::= C_1 | \bot | \top | (C \sqcap C) | (C \sqcup C) | \neg C | \exists R.C | \forall R.C$$

where $C_1 \in \mathbf{C}$, $R \in \mathbf{R}$.

A *TBox* $\mathcal{T}$ is a finite set of *concept inclusion axioms* $C \sqsubseteq D$, where C and D are concepts. An *ABox* $\mathcal{A}$ is a finite set of *concept membership axioms* $a : C$ and *role membership axioms* $(a, b) : R$, where $C \in \mathbf{C}$, $R \in \mathbf{R}$ and $a, b \in \mathbf{I}$.

A $\mathcal{ALC}$ KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ consists of a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$. It is usually assigned a semantics in terms of interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty *domain* and $\cdot^{\mathcal{I}}$ is the *interpretation function*. This function assigns an element in $\Delta^{\mathcal{I}}$ to each $a \in \mathbf{I}$, a subset of $\Delta^{\mathcal{I}}$ to each $C \in \mathbf{C}$ and a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to each $R \in \mathbf{R}$.

DLs can be translated into FOL by defining a function π, which maps axioms to first order formulas.

2.4 MKNF hybrid knowledge bases

The logic of MKNF was introduced in [12]. The syntax of MKNF is the syntax of first order logic augmented with modal operators $\mathbf{K}$ and $\mathbf{not}$. In the following, Δ is the universe of the signature at hand.

The original MKNF semantics is two-valued; we now recap the three-valued semantics for MKNF formulas proposed in [11], which is the base for the well-founded semantics of HKBs, introduced later in this section.

A three-valued MKNF structure $(I, \mathcal{M}, \mathcal{N})$ consists of a first-order interpretation I and two pairs $\mathcal{M} = (M, M_1)$ and $\mathcal{N} = (N, N_1)$ of sets of first-order interpretations where $M_1 \subseteq M$ and $N_1 \subseteq N$. Satisfaction of a closed formula by a three-valued MKNF structure is defined as follows (where p is an atom and ψ is a formula and the values *true*, *undefined* and *false* follow the order *false* < *undefined* < *true*):

$$\begin{array}{ll}
(I, \mathcal{M}, \mathcal{N})(p) & \mathit{true} \;\Leftrightarrow\; p \in I \\
 & \mathit{false} \;\Leftrightarrow p \notin I \\
(I, \mathcal{M}, \mathcal{N})(\neg\psi) & \mathit{true} \;\Leftrightarrow\; (I, \mathcal{M}, \mathcal{N})\psi = \mathit{false}, \\
 & \mathit{undefined} \;\Leftrightarrow\; (I, \mathcal{M}, \mathcal{N})\psi = \mathit{undefined}, \\
 & \mathit{false} \;\Leftrightarrow\; (I, \mathcal{M}, \mathcal{N})\psi = \mathit{true} \\
(I, \mathcal{M}, \mathcal{N})(\psi_1 \wedge \psi_2) & \mathit{min}\{(I, \mathcal{M}, \mathcal{N})\psi_1, (I, \mathcal{M}, \mathcal{N})\psi_2\} \\
(I, \mathcal{M}, \mathcal{N})(\psi_1 \supset \psi_2) & \mathit{true} \;\Leftrightarrow (I, \mathcal{M}, \mathcal{N})\psi_1 < (I, \mathcal{M}, \mathcal{N})\psi_2, \\
 & \mathit{false} \;\; \text{otherwise} \\
(I, \mathcal{M}, \mathcal{N})(\exists x : \psi) & \mathit{max}\{(I, \mathcal{M}, \mathcal{N})\psi[\alpha/x]\} \text{ of the } \alpha \in \Delta
\end{array}$$

$$
\begin{array}{lll}
(I,\mathcal{M},\mathcal{N})(\mathbf{K}\,\psi) & \mathit{true} & \Leftrightarrow (J,(M,M_1),\mathcal{N})\psi = \mathit{true} \text{ for all } J \in M, \\
& \mathit{false} & \Leftrightarrow (J,(M,M_1),\mathcal{N})\psi = \mathit{false} \text{ for some } J \in M_1, \\
& \mathit{undefined} & \text{otherwise} \\
(I,\mathcal{M},\mathcal{N})(\mathbf{not}\,\psi) & \mathit{true} & \Leftrightarrow (J,\mathcal{M},(N,N_1))\psi = \mathit{false} \text{ for some } J \in N_1, \\
& \mathit{false} & \Leftrightarrow (J,\mathcal{M},(N,N_1))\psi = \mathit{true} \text{ for all } J \in N, \\
& \mathit{undefined} & \text{otherwise}
\end{array}
$$

An MKNF interpretation pair (M, N) consists of two MKNF interpretations M, N with $\emptyset \subset N \subseteq M$. An MKNF interpretation pair satisfies a closed MKNF formula ψ iff $(I, (M, N), (M, N))(\psi) = \mathit{true}$ for each $I \in M$. If $M = N$, then the MKNF interpretation pair (M, N) is called total. If there exists an MKNF interpretation pair satisfying ψ, then ψ is consistent. An MKNF interpretation pair (M, N) is a three-valued MKNF model for a given closed MKNF formula ψ if

- (M, N) satisfies ψ, and
- for each MKNF interpretation pair (M', N') with $M \subseteq M'$ and $N \subseteq N'$, where at least one of the inclusions is proper and $M' = N'$ if $M = N$, there is an $I' \in M'$ such that $(I', (M', N'), (M, N))(\psi) \neq \mathit{true}$.

An MKNF HKB [13] is a pair $\mathcal{K} = \langle \mathcal{O}, \mathcal{P} \rangle$ where $\mathcal{O}$ is a DL knowledge base (see Section 2.3) and $\mathcal{P}$ is a set of LP rules of the form $h \leftarrow a_1, \ldots, a_n, \sim b_1, \ldots, \sim b_m$, where a_i and b_i are atoms; $\sim$ represents default negation. A HKB is *positive* if no negative literals, i.e., no default negated atoms, occur in it. Note that we simplify the definition in [13] by disallowing disjunctions in LP rule heads for ease of presentation.

Given a HKB $\mathcal{K} = \langle \mathcal{O}, \mathcal{P} \rangle$, an atom in $\mathcal{P}$ is a *DL-atom* if its predicate occurs in $\mathcal{O}$, a non-DL-atom otherwise. An LP rule is *DL-safe* if each of its variables occurs in at least one positive non-DL-atom in the body; a HKB is *DL-safe* if all its LP-rules are DL-safe.

In the following, we recall the well-founded semantics for hybrid MKNF KBs presented in [11]. DL-safe HKBs have the same well-founded model of their grounding over the constants appearing in the HKBs [11], so their semantics can be given considering such grounding. Throughout the paper, we assume HKBs are DL-safe.

Let $\mathcal{K} = \langle \mathcal{O}, \mathcal{P} \rangle$ be a ground HKB. The set of *known atoms* of $\mathcal{K}$, $\mathsf{KA}(\mathcal{K})$, is the smallest set that contains all positive literals occurring in $\mathcal{P}$, and a positive literal ξ for each literal $\sim\xi$ occurring in $\mathcal{P}$. Given $S \subseteq \mathsf{KA}(\mathcal{K})$, the *objective knowledge of* $\mathcal{K}$ *with respect to* S is the set $\mathsf{OB}_{\mathcal{K},S} = \{\pi(\mathcal{O})\} \cup S$. The operators $R_\mathcal{K}$, $D_\mathcal{K}$ and $T_\mathcal{K}$ derive atoms that are consequences of a positive HKB $\mathcal{K}$ and a set S of atoms. $R_\mathcal{K}(S)$ is the set of consequences due to rules, i.e., the heads of rules in $\mathcal{P}$ whose bodies are composed of atoms that are a subset of S; $D_\mathcal{K}(S)$ is the set of consequences due to axioms, i.e., the atoms from $\mathsf{KA}(\mathcal{K})$ entailed by $\mathsf{OB}_{\mathcal{K},S}$; and $T_\mathcal{K}(S) = R_\mathcal{K}(S) \cup D_\mathcal{K}(S)$. Given a HKB $\mathcal{K}$ and a set of atoms $S \subseteq \mathsf{KA}(\mathcal{K})$, the following transformations, yielding positive knowledge bases, are defined: the *MKNF transformation* $\mathcal{K}/S$ is $\langle \mathcal{O}, \mathcal{P}/S \rangle$ where $\mathcal{P}/S$ is the set of rules $h \leftarrow a_1, \ldots, a_m$ such that there exists in $\mathcal{P}$ a rule $a \leftarrow a_1, \ldots, a_m, \sim$

$b_1, \ldots, \sim b_n$ with $\{b_1, \ldots, b_n\} \cap S = \emptyset$, and the *MKNF-coherent transformation* $\mathcal{K}//S$ is $\langle \mathcal{O}, \mathcal{P}//S \rangle$ where $\mathcal{P}//S$ is the set of rules $h \leftarrow a_1, \ldots, a_m$ such that there exists in $\mathcal{P}$ a rule $h \leftarrow a_1, \ldots, a_m, \sim b_1, \ldots, \sim b_n$ with $\{b_1, \ldots, b_m\} \cap S = \emptyset$ and $\mathsf{OB}_{\mathcal{K},S} \not\models \neg h$.

The KBs transformations induce transformations of sets of positive atoms, respectively: $\Gamma_{\mathcal{K}}(S) = \mathsf{lfp}(T_{\mathcal{K}/S})$ and $\Gamma'_{\mathcal{K}}(S) = \mathsf{lfp}(T_{\mathcal{K}//S})$. Using these transformations, the sequences of positive atoms $\mathbf{P}$ and $\mathbf{N}$ are defined as follows: $\mathbf{P}_0 = \emptyset$, $\mathbf{N}_0 = \mathsf{KA}(\mathcal{K})$, $\mathbf{P}_{n+1} = \Gamma_{\mathcal{K}}(\mathbf{N}_n)$ and $\mathbf{N}_{n+1} = \Gamma'_{\mathcal{K}}(\mathbf{P}_n)$, $\mathbf{P}_\omega = \bigcup \mathbf{P}_i$, $\mathbf{N}_\omega = \bigcap \mathbf{N}_i$. $\mathbf{P}_\omega$ contains everything that is necessarily true, while $\mathbf{N}_\omega$ contains everything that is not false. A KB $\mathcal{K}$ is *MKNF-inconsistent* iff $\mathcal{O}$ is inconsistent or $\Gamma'_{\mathcal{K}}(\mathbf{P}_\omega) \subset \Gamma_{\mathcal{K}}(\mathbf{P}_\omega)$ or $\Gamma'_{\mathcal{K}}(\mathbf{N}_\omega) \subset \Gamma_{\mathcal{K}}(\mathbf{N}_\omega)$.

The *well-founded model* of a MKNF-consistent $\mathcal{K}$ is $M_{WF} = \mathbf{P}_\omega \cup \pi(\mathcal{O}) \cup \{\sim B \mid B \in \mathsf{KA}(\mathcal{K}) \setminus \mathbf{N}_\omega\}$. The pair $(\mathbf{P}_\omega, \mathsf{KA}(\mathcal{K}) \setminus \mathbf{N}_\omega)$ is the *well-founded partition*, which establishes a property of all the MKNF models of the HKB: if $a \in \mathbf{P}_\omega$, then $\mathbf{K}\, a$ is true in all three-valued models (so, a is true in all the interpretations that compose each model), while if $a \in \mathbf{N}_\omega$ then **not** a is true in all three-valued models (a is false in at least one of the interpretations that compose each model).

Example 1 shows the WFM of a simple HKB, according to the semantics recalled above and to the one presented in this paper.

3 Iterated fixpoint definition of the well-founded semantics for HKB

In this section, we give the WFS for HKB in a bottom-up way, as in [14], and prove that it is equivalent to the one proposed in [11].

Definition 3. *For a HKB $\mathcal{K}$, sets Tr and Fa of ground atoms, and a 3-valued interpretation $\mathcal{I} = \langle I_T; I_F \rangle$ we define the operators $OpTrue^{\mathcal{K}}_{\mathcal{I}} : Int_2 \rightarrow Int_2$ and $OpFalse^{\mathcal{K}}_{\mathcal{I}} : Int_2 \rightarrow Int_2$ as*

$OpTrue^{\mathcal{K}}_{\mathcal{I}}(Tr) =$ *$\{a | a$ is not true in $\mathcal{I}$; and there is a clause $a \leftarrow a_1, ..., a_n, \sim b_1, \ldots, \sim b_r$ in the grounding of $\mathcal{P}$ and for every $1 \leq i \leq n$ either a_i is true in $\mathcal{I}$ or $a_i \in Tr$ and for every $1 \leq j \leq r$ b_j is false in $\mathcal{I}\} \cup \{a \in \mathsf{KA}(\mathcal{K}) | \mathsf{OB}_{\mathcal{K}, I_T \cup Tr} \models a\}$;*

$OpFalse^{\mathcal{K}}_{\mathcal{I}}(Fa) =$ *$\{a | a$ is not false in $\mathcal{I}$; and either $\mathsf{OB}_{\mathcal{K}, I_T} \models \neg a$, or for every clause $a \leftarrow a_1, ..., a_n, \sim b_1, \ldots, \sim b_r$ in the grounding of $\mathcal{P}$ there is some i $(1 \leq i \leq n)$ such that a_i is false in $\mathcal{I}$ or $a_i \in Fa$, or there is some j $(1 \leq i \leq r)$ such that b_j is true in $\mathcal{I}\ \} \cap \{a \in \mathsf{KA}(\mathcal{K}) | \mathsf{OB}_{\mathcal{K}, \mathsf{KA}(\mathcal{K}) \setminus (I_F \cup Fa)} \not\models a\}$*

In words, the operator $OpTrue^{\mathcal{K}}_{\mathcal{I}}(Tr)$ extends the interpretation $\mathcal{I}$ to add the new true atoms that can be derived from $\mathcal{K}$ knowing $\mathcal{I}$ and true atoms Tr, while $OpFalse^{\mathcal{K}}_{\mathcal{I}}(Fa)$ computes new false atoms in P by knowing $\mathcal{I}$ and false atoms Fa.

Theorem 1. *$OpTrue^{\mathcal{K}}_{\mathcal{I}}$ and $OpFalse^{\mathcal{K}}_{\mathcal{I}}$ are both monotonic.*

Proof. Proving monotonicity of $OpTrue^{\mathcal{K}}_{\mathcal{I}}$ is the same of proving that if $Tr \subseteq Tr'$, then $OpTrue^{\mathcal{K}}_{\mathcal{I}}(Tr) \subseteq OpTrue^{\mathcal{K}}_{\mathcal{I}}(Tr')$. Analogously, monotonicity of $OpFalse^{\mathcal{K}}_{\mathcal{I}}$ means that if $Fa \subseteq Fa'$, then $OpFalse^{\mathcal{K}}_{\mathcal{I}}(Fa) \subseteq OpFalse^{\mathcal{K}}_{\mathcal{I}}(Fa')$.

Regarding $OpTrue^{\mathcal{K}}_{\mathcal{I}}$, if $a \in OpTrue^{\mathcal{K}}_{\mathcal{I}}(Tr)$, Definition 3 ensures that either there is a clause $a \leftarrow l_1, ..., l_n$ in P's grounding and for every $1 \leq i \leq n$ either l_i is true in $\mathcal{I}$, or $l_i \in Tr$, or $\mathsf{OB}_{\mathcal{K},Tr} \models a$, i.e., $\pi(\mathcal{O}) \cup Tr \models a$. Since $Tr \subseteq Tr'$, if $l_i \in Tr$, then also $l_i\theta \in Tr'$, and if $\pi(\mathcal{O}) \cup Tr \models a$, then also $\pi(\mathcal{O}) \cup Tr' \models a$ for the monotonicity of first order logic. So $a \in OpTrue^{\mathcal{K}}_{\mathcal{I}}(Tr')$.

Regarding $OpFalse^{\mathcal{K}}_{\mathcal{I}}$, if $a \in OpFalse^{\mathcal{K}}_{\mathcal{I}}(Fa)$, then either

- $\mathsf{OB}_{\mathcal{K},\mathsf{KA}(\mathcal{K})\setminus I_F} \models \neg a$, or
- for every clause $a \leftarrow l_1, ..., l_n$ in P's grounding there is some i $(1 \leq i \leq n)$ such that either l_i is a literal false in $\mathcal{I}$, or $l_i \in Fa$, and since $Fa \subseteq Fa'$, $l_i \in Fa'$.

Also, $\mathsf{OB}_{\mathcal{K},\mathsf{KA}(\mathcal{K})\setminus(I_F \cup Fa)} \not\models a$, so $\mathsf{OB}_{\mathcal{K},\mathsf{KA}(\mathcal{K})\setminus(I_F \cup Fa')} \not\models a$ by the monotonicity of first order logic. So $a \in OpFalse^{\mathcal{K}}_{\mathcal{I}}(Fa')$. □

Since $OpTrue^{\mathcal{K}}_{\mathcal{I}}$ and $OpFalse^{\mathcal{K}}_{\mathcal{I}}$ are monotonic, they both have least and greatest fixpoints. An iterated fixpoint operator builds up *dynamic strata* by constructing successive three-valued interpretations as follows.

Definition 4 (Iterated Fixed Point). *For a normal program P, let $IFP^{\mathcal{K}} : Int_3 \rightarrow Int_3$ be defined as $IFP^{\mathcal{K}}(\mathcal{I}) = \mathcal{I} \cup \langle lfp(OpTrue^{\mathcal{K}}_{\mathcal{I}}), gfp(OpFalse^{\mathcal{K}}_{\mathcal{I}})\rangle$.*

Theorem 2. *$IFP^{\mathcal{K}}$ is monotonic.*

Proof. Let $\mathcal{I} = \langle I_T; I_F\rangle$, $\mathcal{I}' = \langle I'_T; I'_F\rangle$, $I_T \subseteq I'_T$ and $I_F \subseteq I'_F$. We now have to show that $I_T \cup lfp(OpTrue^{\mathcal{K}}_{\mathcal{I}}) \subseteq I'_T \cup lfp(OpTrue^{\mathcal{K}}_{\mathcal{I}'})$. We can do this by proving that $OpTrue^{\mathcal{K}}_{\mathcal{I}} \uparrow n \subseteq I'_T \cup OpTrue^{\mathcal{K}}_{\mathcal{I}'} \uparrow n$ for all n. For $n = 0$ $OpTrue^{\mathcal{K}}_{\mathcal{I}} \uparrow 0 \subseteq I'_T \cup OpTrue^{\mathcal{K}}_{\mathcal{I}'} \uparrow 0$ because $OpTrue^{\mathcal{K}}_{\mathcal{I}} \uparrow 0 = \emptyset$. Now, suppose it holds for some n and let $a \in OpTrue^{\mathcal{K}}_{\mathcal{I}} \uparrow (n+1)$, then there is a clause $a \leftarrow l_1, ..., l_m$ in P's grounding such that for every $1 \leq i \leq m$ either l_i is true in $\mathcal{I}$, or $l_i \in OpTrue^{\mathcal{K}}_{\mathcal{I}} \uparrow n$, or $\mathsf{OB}_{\mathcal{K},OpTrue^{\mathcal{K}}_{\mathcal{I}}\uparrow n} \models a$. If l_i is true in $\mathcal{I}$, it is also true in $\mathcal{I}'$, if $l_i \in OpTrue^{\mathcal{K}}_{\mathcal{I}} \uparrow n$ then either $l_i \in OpTrue^{\mathcal{K}}_{\mathcal{I}'} \uparrow n$ or $l \in I'_T$, and if $\mathsf{OB}_{\mathcal{K},OpTrue^{\mathcal{K}}_{\mathcal{I}}\uparrow n} \models a$, it also holds that $\mathsf{OB}_{\mathcal{K},I'_T \cup OpTrue^{\mathcal{K}}_{\mathcal{I}'}\uparrow n} \models a$, thanks to the monotonicity of first order logic. Therefore, $a \in I'_T \cup OpTrue^{\mathcal{K}}_{\mathcal{I}'} \uparrow (n+1)$.

We now have to show that $I_F \cup gfp(OpFalse^{\mathcal{K}}_{\mathcal{I}}) \subseteq I'_F \cup gfp(OpFalse^{\mathcal{K}}_{\mathcal{I}'})$. We prove that $OpFalse^{\mathcal{K}}_{\mathcal{I}} \downarrow n \subseteq OpFalse^{\mathcal{K}}_{\mathcal{I}'} \downarrow n$ for all n. For $n = 0$ $OpFalse^{\mathcal{K}}_{\mathcal{I}} \downarrow 0 = \mathsf{KA}(\mathcal{K}) = OpFalse^{\mathcal{K}}_{\mathcal{I}'} \downarrow 0$. Suppose it also holds for some n and let $a \in OpFalse^{\mathcal{K}}_{\mathcal{I}} \downarrow (n+1)$; then either $\mathsf{OB}_{\mathcal{K},I_T} \models \neg a$ (and also $\mathsf{OB}_{\mathcal{K},I'_T} \models \neg a$, because $I_T \subseteq I'_T$ and first order logic is monotonic), or for every clause $a \leftarrow l_1, ..., l_n$ in P's grounding there is some i $(1 \leq i \leq n)$ such that l_i is false in $\mathcal{I}$, or $l_i \in OpFalse^{\mathcal{K}}_{\mathcal{I}} \downarrow n$, or $\mathsf{OB}_{\mathcal{K},\mathsf{KA}(\mathcal{K})\setminus OpFalse^{\mathcal{K}}_{\mathcal{I}}\downarrow n} \not\models a$. If l_i is false in $\mathcal{I}$, it is also false in $\mathcal{I}'$, if $l_i \in OpFalse^{\mathcal{K}}_{\mathcal{I}} \downarrow n$ then either $l_i \in OpFalse^{\mathcal{K}}_{\mathcal{I}'} \downarrow n$ or $l \in I'_F$, and if $\mathsf{OB}_{\mathcal{K},\mathsf{KA}(\mathcal{K})\setminus OpFalse^{\mathcal{K}}_{\mathcal{I}}\downarrow n} \not\models a$, it also holds that $\mathsf{OB}_{\mathcal{K},\mathsf{KA}(\mathcal{K})\setminus OpFalse^{\mathcal{K}}_{\mathcal{I}'}\downarrow n} \not\models a$ for the inductive hypothesis and the monotonicity of first order logic. Therefore, $a \in OpFalse^{\mathcal{K}}_{\mathcal{I}'} \downarrow (n+1)$. □

So $IFP^{\mathcal{K}}$ has a least fixed point $lfp(IFP^{\mathcal{K}})$. The main result of the paper is the following theorem that our definition of the well-founded model is equivalent to that of Section 2.4.

Theorem 3. *The well-founded model $WFM(\mathcal{K})$ of $\mathcal{K}$ is in fact $lfp(IFP^{\mathcal{K}})$.*

Proof. We prove this by double induction. We show by induction that $IFP^{\mathcal{K}} \uparrow n = \langle \mathbf{P}_n; \mathsf{KA}(\mathcal{K}) \setminus \mathbf{N}_n \rangle$. For $n = 0$, $IFP^{\mathcal{K}} \uparrow 0 = \langle \emptyset, \emptyset \rangle$, while $\mathbf{P}_0 = \emptyset$ and $\mathbf{N}_0 = \mathsf{KA}(\mathcal{K})$, thus $\langle \mathbf{P}_0; \mathsf{KA}(\mathcal{K}) \setminus \mathbf{N}_0 \rangle = \langle \emptyset, \emptyset \rangle = IFP^{\mathcal{K}} \uparrow 0$.

For $n+1$, let $IFP^{\mathcal{K}} \uparrow n = \langle \mathbf{P}_n; \mathsf{KA}(\mathcal{K}) \setminus \mathbf{N}_n \rangle = \mathcal{I} = \langle I_T; I_F \rangle$.

We now prove that (1) $I_T \cup lfp(OpTrue^{\mathcal{K}}_{\mathcal{I}}) = lfp(T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F})$ and that (2) $\mathsf{KA}(\mathcal{K}) \setminus I_F \cup gfp(OpFalse^{\mathcal{K}}_{\mathcal{I}}) = lfp(T_{\mathcal{K}//I_T})$.

To prove (1), we first show by induction that $T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F} \uparrow m \subseteq I_T \cup OpTrue^{\mathcal{K}}_{\mathcal{I}} \uparrow m$.

For $m = 0$ $T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F} \uparrow 0 = \emptyset$ so $T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F} \uparrow 0 \subseteq I_T \cup OpTrue^{\mathcal{K}}_{\mathcal{I}} \uparrow 0$. For $m+1$, let Tr be $OpTrue^{\mathcal{K}}_{\mathcal{I}} \uparrow m = T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F} \uparrow m$.

If $a \in T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F}(Tr)$, suppose $a \in R_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F}(Tr)$. Then there is a rule $a \leftarrow l_1, ..., l_n$ in $\mathcal{P}/\mathsf{KA}(\mathcal{K}) \setminus I_F$ with each l_i a positive literal belonging to I_T or Tr. This means that the grounding of $\mathcal{P}$ contains a rule $a \leftarrow l_1, ..., l_n, \sim b_1, \ldots, \sim b_r$ with $b_1, \ldots, b_r$ in I_F. So $a \in OpTrue^{\mathcal{K}}_{\mathcal{I}} \uparrow m+1$ for the definition of $OpTrue^{\mathcal{K}}_{\mathcal{I}}$. If $a \in D_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F}(Tr)$ then $\mathsf{OB}_{\mathcal{K},Tr} \models a$ so $a \in OpTrue^{\mathcal{K}}_{\mathcal{I}}(Tr)$ for the definition of $OpTrue^{\mathcal{K}}_{\mathcal{I}}$.

Since $T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F} \uparrow m \subseteq I_T \cup OpTrue^{\mathcal{K}}_{\mathcal{I}} \uparrow m$, for all m, $lfp(T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F}) \subseteq I_T \cup lfp(OpTrue^{\mathcal{K}}_{\mathcal{I}})$, so to prove (1) it is sufficient to show that $I_T \cup lfp(OpTrue^{\mathcal{K}}_{\mathcal{I}}) \subseteq lfp(T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F})$. Note that $I_T \subseteq lfp(T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F})$ because $I_T = \mathbf{P}_n$, $\mathbf{P}_n$ is the least fixpoint of a $T_{\mathcal{K}'}$ operator where $\mathcal{K}'$ is a subset of $\mathcal{K}/\mathsf{KA}(\mathcal{K}) \setminus I_F$ and positive programs are monotonic.

If $a \in lfp(OpTrue^{\mathcal{K}}_{\mathcal{I}})$, suppose $a \notin lfp(T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F})$. Then for each rule $a \leftarrow a_1, ..., a_n, \sim b_1, \ldots, \sim b_r$ in the grounding of $\mathcal{P}$ there exits an a_i that is not in $lfp(T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F})$ or a $b_j \notin I_F$, and $\mathsf{OB}_{\mathcal{K},lfp(T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F})} \not\models a$. Since $lfp(T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F}) \subseteq I_T \cup lfp(OpTrue^{\mathcal{K}}_{\mathcal{I}})$ and $I_T \subseteq lfp(T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F})$, then $a_i \notin lfp(OpTrue^{\mathcal{K}}_{\mathcal{I}})$ or $b_j \notin I_F$. In both cases $a \notin lfp(OpTrue^{\mathcal{K}}_{\mathcal{I}})$ against the hypothesis. So we have $I_T \cup lfp(OpTrue^{\mathcal{K}}_{\mathcal{I}}) = lfp(T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F})$.

We prove (2) by proving that, for all m, $T_{\mathcal{K}//I_T} \uparrow m = \mathsf{KA}(\mathcal{K}) \setminus (I_F \cup OpFalse^{\mathcal{K}}_{\mathcal{I}} \downarrow m)$.

For $m = 0$ $T_{\mathcal{K}//I_T} \uparrow 0 = \emptyset$ and $OpFalse^{\mathcal{K}}_{\mathcal{I}} \downarrow 0 = \mathsf{KA}(\mathcal{K})$, so $T_{\mathcal{K}//I_T} \uparrow 0 = \mathsf{KA}(\mathcal{K}) \setminus (I_F \cup OpFalse^{\mathcal{K}}_{\mathcal{I}} \downarrow 0)$. For $m+1$, let S be $T_{\mathcal{K}//I_T} \uparrow m$ and let Fa be $OpFalse^{\mathcal{K}}_{\mathcal{I}} \downarrow m$; by the inductive hypothesis, $S = \mathsf{KA}(\mathcal{K}) \setminus (I_F \cup Fa)$. We now show that, for all $a \in \mathsf{KA}(\mathcal{K})$, $a \in T_{\mathcal{K}//I_T}(S)$ if and only if $a \notin (I_F \cup OpFalse^{\mathcal{K}}_{\mathcal{I}}(Fa))$. If $a \in T_{\mathcal{K}//I_T}(S)$, then $a \notin I_F$, because otherwise $a \in \mathbf{N}_{n+1}$ and, since $\mathbf{N}_{n+1} \subseteq \mathbf{N}_n$, $a \in \mathbf{N}_n$, but by the external inductive hypotheses I_F and $\mathbf{N}_n$ are disjoint. So we need to prove $a \notin OpFalse^{\mathcal{K}}_{\mathcal{I}}(Fa)$. As just proved, $a \notin I_F$; if $\mathsf{OB}_{\mathcal{K},S} \models a$, this means that $\mathsf{OB}_{\mathcal{K},\mathsf{KA}(\mathcal{K})\setminus(I_F \cup Fa)} \models a$, so $a \notin OpFalse^{\mathcal{K}}_{\mathcal{I}}(Fa)$; otherwise $\mathsf{OB}_{\mathcal{K},I_T} \not\models \neg a$ and there exists a rule $a \leftarrow a_1, \ldots, a_m, b_1, \ldots, b_n$ in $\mathcal{P}$'s grounding such that $\mathsf{OB}_{\mathcal{K},I_T} \not\models \neg a$, $\{a_1, \ldots, a_m\} \subseteq S$ and $\{b_1, \ldots, b_n\} \cap I_T = \emptyset$, which, by De Morgan's laws and because $S = \mathsf{KA}(\mathcal{K}) \setminus (I_F \cup Fa)$, is the negation of the fact that $\mathsf{OB}_{\mathcal{K},I_T} \models \neg a$ or, for each rule $a \leftarrow a_1, \ldots, a_m, b_1, \ldots, b_n$ in $\mathcal{P}$'s grounding, $\{a_1, \ldots, a_m\} \cup (I_F \cap Fa) \neq \emptyset$ or $\{b_1, \ldots, b_n\} \cap I_T \neq \emptyset$; so again $a \notin \mathit{OpFalse}^{\mathcal{K}}_{\mathcal{I}}(Fa)$. On the other hand, if $a \notin I_F \cup \mathit{OpFalse}^{\mathcal{K}}_{\mathcal{I}}(Fa)$, then $a \notin \mathit{OpFalse}^{\mathcal{K}}_{\mathcal{I}}(Fa)$, so either (i) $\mathsf{OB}_{\mathcal{K},\mathsf{KA}(\mathcal{K})\setminus(I_F\cup Fa)} \models a$ (and, since $\mathsf{KA}(\mathcal{K}) \setminus (I_F \cup Fa) = S$, $\mathsf{OB}_{\mathcal{K},S} \models a$, so $a \in T_{\mathcal{K}//I_T}(S)$), or (ii) $\mathsf{OB}_{\mathcal{K},I_T} \not\models \neg a$ and there exists a rule $a \leftarrow a_1, \ldots, a_m, b_1, \ldots, b_n$ in $\mathcal{P}$'s grounding such that $\{a_1, \ldots, a_m\} \cup (I_F \cap Fa) = \emptyset$ (i.e., $\{a_1, \ldots, a_m\} \subseteq S$) and $\{b_1, \ldots, b_n\} \cap I_T = \emptyset$, so again $a \in T_{\mathcal{K}//I_T}(S)$. $\square$

Let δ be the smallest ordinal such that $\mathit{WFM}(P) = \mathit{IFP}^{\mathcal{K}} \uparrow \delta$. We refer to δ as the *depth* of $\mathcal{K}$. The *stratum* of atom a is the least ordinal β such that $a \in \mathit{IFP}^{\mathcal{K}} \uparrow \beta$ (where a may be either in the true or false component of $\mathit{IFP}^{\mathcal{K}} \uparrow \beta$). Undefined atoms of the well-founded model do not belong to any stratum – i.e. they are not added to $\mathit{IFP}^{\mathcal{K}} \uparrow \delta$ for any ordinal δ.

Table 1. Computation of the WFM for the KB in Example 1 according to the semantics in [11] and the one presented here. For each k, columns 3 and 5 show the computation of $\mathbf{P}_k$ and $\mathbf{N}_k$ as the sequence of sets of atoms added by iteration m of the T_K operator, where K is the transformation of $\mathcal{K}_G$ as defined by the semantics, shown in columns 2 and 4, respectively. Columns 6 and 7 of line k show the computation of I_k as the fixpoint of the $\mathit{OpTrue}^{\mathcal{K}}_{I_{k-1}}$ and $\mathit{OpFalse}^{\mathcal{K}}_{I_{k-1}}$ operators, respectively.

k	$\mathcal{K}_G/\mathbf{N}_{k-1}$	New atoms in $T_{\mathcal{K}_G/\mathbf{N}_{k-1}} \uparrow m$	$\mathcal{K}_G//\mathbf{P}_{k-1}$	New atoms in $T_{\mathcal{K}_G//\mathbf{P}_{k-1}} \uparrow m$	$\mathit{OpTrue}^{\mathcal{K}}_{I_{k-1}} \uparrow m$	$\mathit{OpFalse}^{\mathcal{K}}_{I_{k-1}} \downarrow m$	m
1	$o(a) \leftarrow$		$e(a) \leftarrow o(a)$		$\emptyset$	$\mathsf{KA}(\mathcal{K})$	0
	$o(b) \leftarrow$	$\{o(a), o(b)\}$	$e(b) \leftarrow o(b)$	$\{o(a), o(b)\}$	$\{o(a), o(b)\}$	$\{d(b), f(a)\}$	1
		$\emptyset$	$d(a) \leftarrow o(a)$	$\{e(a), e(b), d(a)\}$	$\{o(a), o(b)\}$	$\{d(b), f(a)\}$	2
			$o(a) \leftarrow$	$\{f(b)\}$			3
			$o(b) \leftarrow$	$\emptyset$			4
2	$e(b) \leftarrow o(b)$		$e(a) \leftarrow o(a)$		$\emptyset$	$\mathsf{KA}(\mathcal{K})$	0
	$d(a) \leftarrow o(a)$	$\{o(a), o(b)\}$	$e(b) \leftarrow o(b)$	$\{o(a), o(b)\}$	$\{d(a), e(b)\}$	$\emptyset$	1
	$o(a) \leftarrow$	$\{e(b), d(a)\}$	$d(a) \leftarrow o(a)$	$\{e(a), e(b), d(a)\}$	$\{d(a), e(b), f(b)\}$	$\emptyset$	2
	$o(b) \leftarrow$	$\{f(b)\}$	$o(a) \leftarrow$	$\{f(b)\}$	$\{d(a), e(b), f(b)\}$		3
		$\emptyset$	$o(b) \leftarrow$	$\emptyset$			4
3	$e(b) \leftarrow o(b)$		$e(b) \leftarrow o(b)$		$\emptyset$	$\mathsf{KA}(\mathcal{K})$	0
	$d(a) \leftarrow o(a)$	$\{o(a), o(b)\}$	$d(a) \leftarrow o(a)$	$\{o(a), o(b)\}$	$\emptyset$	$\{e(a)\}$	1
	$o(a) \leftarrow$	$\{e(b), d(a)\}$	$o(a) \leftarrow$	$\{e(b), d(a)\}$		$\{e(a)\}$	2
	$o(b) \leftarrow$	$\{f(b)\}$	$o(b) \leftarrow$	$\{f(b)\}$			3
		$\emptyset$		$\emptyset$			4

Example 1. Consider the following HKB:

$$
\begin{aligned}
c &\sqsubseteq \neg d \\
c \sqcap d &\sqsubseteq f \\
b &: c \\
e(X) &\leftarrow o(X), {\sim} d(X) \\
d(X) &\leftarrow o(X), {\sim} f(X) \\
o(a) &\leftarrow \\
o(b) &\leftarrow
\end{aligned}
$$

Table 1 shows the computation of the WFM according to the semantics defined in [11] and recalled in Section 2.4, and the semantics presented here.

4 Alternative formulations

Consider the following alternative definition of $\mathit{IFP}^{\mathcal{K}}$

Definition 5. *Define* $\mathit{OpTrue}'^{\mathcal{K}}_{\mathcal{I}} : \mathit{Int}_2 \rightarrow \mathit{Int}_2$ *and* $\mathit{OpFalse}'^{\mathcal{K}}_{\mathcal{I}} : \mathit{Int}_2 \rightarrow \mathit{Int}_2$ *as*

$\mathit{OpTrue}'^{\mathcal{K}}_{\mathcal{I}}(\mathit{Tr}) = \{a |$ *there is a clause* $a \leftarrow a_1, ..., a_n, {\sim} b_1, \ldots, {\sim} b_r$ *in the grounding of* $\mathcal{P}$ *and for every* $1 \leq i \leq n$ *either* a_i *is true in* $\mathcal{I}$ *or* $a_i \in \mathit{Tr}$ *and for every* $1 \leq j \leq r$ b_j *is false in* $\mathcal{I}\} \cup \{a \in \mathsf{KA}(\mathcal{K}) | \mathsf{OB}_{\mathcal{K}, I_T \cup Tr} \models a\}$*;*

$\mathit{OpFalse}'^{\mathcal{K}}_{\mathcal{I}}(\mathit{Fa}) = \{a |$ *either* $\mathsf{OB}_{\mathcal{K}, I_T} \models \neg a$ *or for every clause* $a \leftarrow a_1, ..., a_n, {\sim} b_1, \ldots, {\sim} b_r$ *in the grounding of* $\mathcal{P}$ *there is some* i $(1 \leq i \leq n)$ *such that* a_i *is false in* $\mathcal{I}$ *or* $a_i \in \mathit{Fa}$*, or there is some* j $(1 \leq i \leq r)$ *such that* b_j *is true in* $\mathcal{I}\} \cap \{a \in \mathsf{KA}(\mathcal{K}) | \mathsf{OB}_{\mathcal{K}, \mathsf{KA}(\mathcal{K}) \setminus (I_F \cup Fa)} \not\models a\}$

Let $\mathit{IFP}'^{\mathcal{K}} : \mathit{Int}_3 \rightarrow \mathit{Int}_3$ *be* $\mathit{IFP}'^{\mathcal{K}}(\mathcal{I}) = \langle \mathit{lfp}(\mathit{OpTrue}'^{\mathcal{K}}_{\mathcal{I}}), \mathit{gfp}(\mathit{OpFalse}'^{\mathcal{K}}_{\mathcal{I}}) \rangle$*.*

$\mathit{IFP}'^{\mathcal{K}}$ differs from Definition 3 because $\mathit{OpTrue}'^{\mathcal{K}}_{\mathcal{I}}$ and $\mathit{OpFalse}'^{\mathcal{K}}_{\mathcal{I}}$ do not check whether the atoms are already true or false in $\mathcal{I}$. As a consequence, in each iteration of $\mathit{IFP}'^{\mathcal{K}}$ the set of true and false atoms is rebuilt and includes $\mathcal{I}$. Therefore, we do not need to add $\mathcal{I}$ to $\langle \mathit{lfp}(\mathit{OpTrue}'^{\mathcal{K}}_{\mathcal{I}}), \mathit{gfp}(\mathit{OpFalse}'^{\mathcal{K}}_{\mathcal{I}}) \rangle$. It is possible to prove the equivalence of $\mathit{IFP}'^{\mathcal{K}}$ with Definition 3.

It is even possible to check the truth of positive literals in $\mathit{OpTrue}^{\mathcal{K}}_{\mathcal{I}}$ and the falsity of positive literals in $\mathit{OpFalse}^{\mathcal{K}}_{\mathcal{I}}$ only with respect to the set of atoms that is changing in the inner fixpoint, without referring to interpretation $\mathcal{I}$, obtaining the following definition that can be proved to lead to the same semantics for HKBs.

Definition 6. *Define* $\mathit{OpTrue}''^{\mathcal{K}}_{\mathcal{I}} : \mathit{Int}_2 \rightarrow \mathit{Int}_2$ *and* $\mathit{OpFalse}''^{\mathcal{K}}_{\mathcal{I}} : \mathit{Int}_2 \rightarrow \mathit{Int}_2$ *as*

$\mathit{OpTrue}''^{\mathcal{K}}_{\mathcal{I}}(\mathit{Tr}) = \{a |$ *there is a clause* $a \leftarrow a_1, ..., a_n, {\sim} b_1, \ldots, {\sim} b_r$ *in the grounding of* $\mathcal{P}$ *and for every* $1 \leq i \leq n$ $a_i \in \mathit{Tr}$ *and for every* $1 \leq j \leq r$ b_j *is false in* $\mathcal{I}\} \cup \{a \in \mathsf{KA}(\mathcal{K}) | \mathsf{OB}_{\mathcal{K}, I_T \cup Tr} \models a\}$*;*

$OpFalse''^{\mathcal{K}}_{\mathcal{I}}(Fa) = \{a | either\ \mathsf{OB}_{\mathcal{K},I_T} \models \neg a\ or\ for\ every\ clause\ a \leftarrow a_1, ..., a_n, \sim b_1, \ldots, \sim b_r\ in\ the\ grounding\ of\ \mathcal{P}\ there\ is\ some\ i\ (1 \leq i \leq n)\ such\ that\ a_i \in Fa,\ or\ there\ is\ some\ j\ (1 \leq i \leq r)\ such\ that\ b_j\ is\ true\ in\ \mathcal{I}\} \cap \{a \in \mathsf{KA}(\mathcal{K}) | \mathsf{OB}_{\mathcal{K},\mathsf{KA}(\mathcal{K})\setminus(I_F \cup Fa)} \not\models a\}$

Let $IFP''^{\mathcal{K}} : Int_3 \rightarrow Int_3$ *be* $IFP''^{\mathcal{K}}(\mathcal{I}) = \langle lfp(OpTrue''^{\mathcal{K}}_{\mathcal{I}}), gfp(OpFalse''^{\mathcal{K}}_{\mathcal{I}})\rangle$.

In this case it is possible to prove that $T_{\mathcal{K}/\mathsf{KA}(\mathcal{K})\setminus I_F} \uparrow m = OpTrue''^{\mathcal{K}}_{\mathcal{I}} \uparrow m$ and $\mathsf{KA}(\mathcal{K}) \setminus gfp(OpFalse''^{\mathcal{K}}_{\mathcal{I}}) = lfp(T_{\mathcal{K}//I_T})$, obtaining operators that are closer to those of Knorr et al. [11].

5 Related work

In [3] the authors formalized a new Logic for Non-Monotone Inductive Definitions (ID-logic), which allows also second-order variables and quantification. This logic integrates classical logic and monotone with non-monotone inductive definitions. This work aims at demonstrating that non-monotonic inductive definitions, such as iterated inductive definitions and definitions over well-orders, can play an important role in knowledge representation since it can relate remote non-monotonic reasoning, logic programming, description logics, deductive databases and fixpoint logics.

Along this line, an extension of DL with rules is FO(ID) [19] which integrates inductive definitions into FOL. To achieve this integration they defined two new connectives, one similar to equivalence of DL concepts (and roles) which can work also with inductive definitions, and one which models definitional rules. This second operator allows defining set of rules which stands for a single inductive definition for a concept (role). This definition can also be modeled using the first connective to define the concept (role) as the union of the rules' body. However, keeping separated rules, and thus single definitions of the concept (role), makes adding and removing such definitions easier. In a second work, the authors defined FO(FD) [10] which can be seen as an extension of FO(ID) since all its inductive definitions can be translated into FO(FD). The inverse is not true, because FO(FD) allows the definition of *fixpoint definitions*, which are either a least fixpoint definition or a greatest fixpoint definition, and FO(ID) cannot represents such definitions.

While the FO(ID) language bears an obvious similarity with MKNF-based HKBs, in that both combine first-order formulas and rules, in order to compare their semantics one must first observe that FO(ID) models are two-valued interpretations, while MKNF models are sets of interpretations, so a comparison directly involving the MKNF semantics is not possible. However, one may try to establish a connection between the two semantics by giving a different understanding of an HKB $\mathcal{K} = \langle \mathcal{O}, \mathcal{P} \rangle$'s well-founded partition: the two-valued interpretation built by adding to the positive atoms in the well-founded partition some of the DL-atoms is a model (let us call it, for the present discussion, an HKB model) if it entails, in the classical logic sense, the HKB. The question is then whether the models so defined are the same as the FO(ID) models of the FO(ID) KB obtained from $\mathcal{K}$ by adding to $\mathcal{O}$ one definition composed of all the clauses in $\mathcal{P}$. In general, this is not the case. For instance, consider $\mathcal{K} = \langle \emptyset, \{p(a) \leftarrow q(a)\}\rangle$: the only model is $\emptyset$, while $\{p(a), q(a)\}$ is a FO(ID) model, so there is a FO(ID) model that is not a HKB model. The opposite can also occur: for $\mathcal{K} = \langle \{p \sqsubseteq q\}, \{p(a); p(a) \leftarrow q(a)\}\rangle$ has the HKB model $\{p(a), q(a)\}$ which is not a FO(ID) model. In conclusion, there is no obvious mapping between HKBs and FO(ID) KBs.

Another similar approach is represented by SWRL [9], which extends DLs with Horn clauses but limits their use so that they cannot be used to define concepts. Many fragments of this language have been defined, one of them, for example, makes use of DL-safety.

Other proposals combines DLs with Answer Set Programming [5,8,16,17] or with LP under the well-founded semantics [4]. These proposals, differently from HKBs, keep the two parts separated by allowing the logic program to query the DL part or by adding the ASP program on top of a DL KB and considering first predicates appearing in the DL KB only in isolation and then considering them in the ASP program.

6 Conclusions

We think that HKBs provide a powerful tool for modeling real world domains, allowing the use of the closed world and open world assumptions for different predicates in the same theory. When compared with other approaches for combining LP and DL, they have several advantages, the most important of which is the tighter integration of the two components. As such, it is important to study HKBs and develop in-depth analysis of their semantics. In this paper we provide an alternative formulation of the well-founded semantics of HKBs that is based on iterating fixpoints in a fully bottom-up way. In this way we aim at presenting a new viewpoint for the semantics that can highlight new relationships with other formalisms together with proposing an approach for computing the semantics that is possibly easier to implement.

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