A single proof of classical behaviour in da Costa's C_n systems

Mauricio Osorio, José Abel Castellanos

Universidad de las Américas, Sta. Catarina Martir, Cholula, Puebla, 72820 México
{osoriomauri,jose.castellanosjo}@gmail.com

Abstract. A strong negation in da Costa's C_n systems can be naturally extended from the strong negation ($\neg^*$) of C_1. In [1] Newton da Costa proved the conectives $\{\rightarrow, \wedge, \vee, \neg^*\}$ in C_1 satisfy all schemas and inference rules of classical logic. In the following paper we present a proof that all logics in the C_n herarchy also behave classically as C_1. This result tell us the existance of a common property among the paraconsistent family of logics created by da Costa.

Keywords: Paraconsistent logic, C_n systems, Strong negation

1 Introduction

According to the authors in [1] a paraconsistent logic is the underlying logic for inconsistent but non-trivial theories. In fact, many authors [2, 3] have pointed out *paraconsistency* is mainly due to the construction of a negation operator which satisfies some properties about classical logic, but at the same time do not hold the so called law of explosion $\alpha, \neg\alpha \vdash \beta$ for arbitrary formulas α, β, as well as others [1].

A common misconception related to paraconsistent logics is the confusion between triviality and contradiction. A theory T is *trivial* when any of the sentences in the language of T can be proven. We say that a theory T is *contradictory* if exists a sentence α in the language of T such that T proves α and $\neg\alpha$. Finally, a theory T is *explosive* if and only if T is *trivial* in the presence of a *contradiction.* We can see that *contradictoriness* and *triviality* are equivalent if and only if for the underlying logic the law of explosion is valid [4]. One of the greatest achievements of paraconsistent logic is to provide a general framework to the study of inconsistent theories based on the distinction of contradiction and triviality.

Paraconsistent logics were born in two different ways. In 1948, Jaskowski gave the following conditions that any paraconsistent logic should satisfy [5]:

J1. When applied to inconsistent systems it should not always entail their trivialization;
J2. It should be rich enough to enable practical inferences;

J3. It should have an intuitive justification.

Also, in 1963, we can find a new approach given by da Costa, who independently defined a set of conditions that a paraconsistent logic should satisfy. These conditions are the following:

dC1. In these calculi the principle of non-contradiction, in the form $\neg(\alpha \wedge \neg\alpha)$, should not be a valid schema;
dC2. From two contradictory formulae, α and $\neg\alpha$, it would not in general be possible to deduce any arbitrary formula β;
dC3. It should be simple to extend these calculi to corresponding predicate calculi;
dC4. They should contain the most part of the schemata and rules of the classical propositional calculus which do not interfere with the first conditions.

Nowadays we can find paraconsistent logics applications in many fields such as informatics, physics, medicine, etc. From Minsky's comment we can see that paraconsistent ideas are an approach in Artificial Intelligence [6]: *"But I do not believe that consistency is necessary or even desirable in a developing intelligent system. No one is ever completely consistent. What is important is how one handles paradox or conflict, how one learns from mistakes, how one turns aside from suspected inconsistencies"*.

In physics the authors in [7] have established an approach to formalize concepts in quantum mechanics, the so called principle of superposition, via paraconsistent methods. In general most of scientific knowledge as theories can have inconsistencies. Most of the time scientist do not throw away these theories if they are successful in predicting results and describing phenomena [4].

In the literature we can find many proper paraconsistent logics [8] in the sense of da Costa. The most known paraconsistent logic is C_1 which in [1] the author also introduces an increasingly weaker family/hierarchy of logics called the $C_n, 1 \leq n \leq \omega$. Also the authors mention that the strong negation defined in the da Costa's C_n systems has all properties of the propositional classical negation.

Finding a strong negation in the C_n herarchy is interesting because we can collapse a fragment of these logics into classical logic, that is, we can have a translation which provides an embedding of classical logic into any logic of this C_n system. This fact is mentioned in many papers [1, 9], where the proof does not explicitly appears. In this paper we present an inductive proof about the relation between strong negation and classical behaviour in the C_n systems. The proof follows from three lemmas and two theorems. From this proof we can see that many properties in C_1 can also hold in C_n, excluding the obvious ones.

The organization of this document is as follows: In Section 2 we present basic background in logic, including definitions of some basic properties (monotonicity, cut-elimination, deduction theorem) of the paraconsistent logic C_ω that we are going to work with. In Section 3 we present a inductive proof about the classical behavior of the strong negation defined in the C_n systems. Finally, in Section 4, we present some conclusions about the proof presented.

2 Background

We first introduce the syntax of logical formulas considered in this paper. Then we present a few basic definitions of how logics can be built to interpret the meaning of such formulas.

2.1 Logic Systems

We consider a formal (propositional) language built from: an enumerable set $\mathcal{L}$ of elements called *atoms* (denoted *a, b, c, ...*); the binary connectives $\wedge$ (*conjuntion*), $\vee$ (*disjunction*) and $\rightarrow$ (*implication*); and the unary connective $\neg$ (*negation*). Formulas (denoted $\alpha, \beta, \gamma, ...$) are constructed as usual by combining these basic connectives together with the help of parentheses. We also use $\alpha \leftrightarrow \beta$ to abbreviate $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. Finally, it is useful to agree on some conventions to avoid the use of many parenthesis when writing formulas in order to make easier the reading of complicated expressions. First, we may omit the outer pair of parenthesis of a formula. Second, the connectives are ordered as follows: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, and parentheses are eliminated according to the rule that, first, $\neg$ applies to the smallest formula following it, then $\wedge$ is to connect the smallest formulas surrounding it, and so on.

We consider a *logic* simply as a set of formulas that (i) is closed under Modus Ponens (i.e. if α and $\alpha \rightarrow \beta$ are in the logic, then so is β) and (ii) is closed under substitution (i.e. if a formula α is in the logic, then any other formula obtained by replacing all occurrences of an atom b in α with another formula β is also in the logic). The elements of a logic are called *theorems* and the notation $\vdash_X \alpha$ is used to state that the formula α is a theorem of X (i.e. $\alpha \in X$). We say that a logic X is *weaker than or equal to a logic* Y if $X \subseteq Y$, similarly we say that X is *stronger than or equal to* Y if $Y \subseteq X$.

Hilbert proof systems There are many different approaches that have been used to specify the meaning of logic formulas or, in other words, to define logics. In Hilbert style proof systems, also known as axiomatic systems, a logic is specified by giving a set of axioms (which is usually assumed to be closed under substitution). This set of axioms specifies, so to speak, the "kernel" of the logic. The actual logic is obtained when this "kernel" is closed with respect to some given inference rules which include Modus Ponens. The notation $\vdash_X \alpha$ for provability of a logic formula α in the logic X is usually extended within Hilbert style systems; given a theory Γ, we use $\Gamma \vdash_X \alpha$ to denote the fact that the formula α can be derived from the axioms of the logic and the formulas contained in Γ by a sequence of applications of the inference rules.

As a example of a Hilbert style system we present next a logic that is relevant for our work.

C_ω [1] is defined by the following set of axiom schemata:

Pos1: $\alpha \to (\beta \to \alpha)$
Pos2: $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$
Pos3: $\alpha \wedge \beta \to \alpha$
Pos4: $\alpha \wedge \beta \to \beta$
Pos5: $\alpha \to (\beta \to \alpha \wedge \beta)$
Pos6: $\alpha \to (\alpha \vee \beta)$
Pos7: $\beta \to (\alpha \vee \beta)$
Pos8: $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$
$C_\omega 1$: $\alpha \vee \neg\alpha$
$C_\omega 2$: $\neg\neg\alpha \to \alpha$

Note that the first 8 axiom schemata somewhat constrain the meaning of the $\to, \wedge$ and $\vee$ connectives to match our usual intuitions. It is a well known result that in any logic satisfying Pos1 and Pos2, and with Modus Ponens as its unique inference rule, the *deduction theorem* holds [10].

Theorem 1. *Let Γ and Δ be two sets of formulas. Let θ, θ_1, θ_2, α and ψ be arbitrary formulas. Let $\vdash$ be the deductive inference operator of C_ω. Then the following basic properties hold.*

1. *$\Gamma \vdash \alpha \to \alpha$ (identity theorem)*
2. *$\Gamma \vdash \alpha$ implies $\Gamma \cup \Delta \vdash \alpha$ (monotonicity)*
3. *$\Gamma \vdash \alpha$ and $\Delta, \alpha \vdash \psi$ then $\Gamma \cup \Delta \vdash \psi$ (cut)*
4. *$\Gamma, \theta \vdash \alpha$ if and only if $\Gamma \vdash \theta \to \alpha$ (deduction theorem)*
5. *$\Gamma \vdash \theta_1 \wedge \theta_2$ if and only if $\Gamma \vdash \theta_1$ and $\Gamma \vdash \theta_2$ ($\wedge$ - rules)*
6. *$\Gamma, \theta \vdash \alpha$ and $\Gamma, \neg\theta \vdash \alpha$ if and only if $\Gamma \vdash \alpha$ (strong proof by cases)*

3 Strong negation in C_n systems

We will start giving some basic definitions in order to understand concepts needed in the C_n hierarchy.

Definition 1. *([1]) $\alpha^o =_{def} \neg(\alpha \wedge \neg\alpha)$. We will refer to (^{o}) as the consistency operator.*

In fact α^o can be seen as a modal operator to the formula α that captures the idea of consistency/well - behavior in C_1.

Definition 2. *([9]) We recursively define α^n, $0 \leq n < \omega$ as follows:*

(i) $\alpha^0 =_{def} \alpha$
(ii) $\alpha^{n+1} =_{def} (\alpha^n)^o$

Definition 3. *([9]) We recursively define $\alpha^{(n)}$, $1 \leq n < \omega$ as follows:*

(i) $\alpha^{(1)} =_{def} \alpha^1$
(ii) $\alpha^{(n+1)} =_{def} \alpha^{(n)} \wedge \alpha^{n+1}$

For the careful reader should not confuse α^0 with α^o. Basically α^n represents n applications of the consistency operator (^{o}) to the formula α, and $\alpha^{(n)}$ represents a conjunction of $\alpha^1, \dots, \alpha^n$.

Definition 4. *([1]) We define C_n as an extension of C_ω which also includes the following axiom schemas:*

$$C_n1: \beta^{(n)} \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$$
$$C_n2: (\alpha^{(n)} \wedge \beta^{(n)}) \to ((\alpha \to \beta)^{(n)} \wedge (\alpha \vee \beta)^{(n)} \wedge (\alpha \wedge \beta)^{(n)})$$

Also, we can see that in C_n, the axiom C_n1 can be replaced by the axiom schema $(\beta \wedge \neg\beta \wedge \beta^{(n)}) \to \alpha$. Intuitively from C_n2 we see that $\alpha^{(n)}$ propagates the what we call *n-consisteny* in C_n. Finally we define a strong negation in both C_1 and C_n.

Definition 5. *([1]) The strong negations for C_1 and C_n are defined as:*

(i) For C_1: $\neg^*\alpha =_{def} \neg\alpha \wedge \alpha^o$
(ii) For C_n: $\neg^{(n)}\alpha =_{def} \neg\alpha \wedge \alpha^{(n)}$

Lemma 1. *For all $n \in \mathbb{N}$ we have that $\neg(\alpha^n) \vdash_{C_\omega} \alpha$*

Proof. By induction on n.

Base case (n = 1). By Definition 2 we have that $\vdash_{C_\omega} \neg(\alpha^1) \leftrightarrow \neg(\alpha^o)$. Also by Definition 1, $\vdash_{C_\omega} \neg(\alpha^o) \leftrightarrow \neg(\neg(\alpha \wedge \neg\alpha))$, we can expand the last formula to $\vdash_{C_\omega} \neg(\alpha^1) \leftrightarrow \neg(\neg(\alpha \wedge \neg\alpha))$. We can use axiom schema $\neg\neg\alpha \to \alpha$ to prove $\vdash_{C_\omega} \neg(\alpha^1) \to \alpha \wedge \neg\alpha$, which is by axiom schema Pos3 we have $\vdash_{C_\omega} \neg(\alpha^1) \to \alpha$. From this we apply *deduction theorem* to obtain $\neg(\alpha^1) \vdash_{C_\omega} \alpha$ as desired.

Inductive step. We assume by induction hypothesis that $\neg(\alpha^n) \vdash_{C_\omega} \alpha$ holds. Accordingly to Definition 2 we have that $\vdash_{C_\omega} \neg(\alpha^{n+1}) \leftrightarrow \neg(\alpha^n)^o$, which in fact is $\vdash_{C_\omega} \neg(\alpha^n)^o \leftrightarrow \neg\neg(\alpha^n \wedge \neg(\alpha^n))$. From this is easy to prove that $\neg(\alpha^{n+1}) \vdash_{C_\omega} \neg(\alpha^n)$, and with the inductive hypothesis we have that $\neg(\alpha^{n+1}) \vdash_{C_\omega} \alpha$. □

Lemma 2. *For all $n \in \mathbb{N}$ we have that $\vdash_{C_\omega} \alpha \vee \alpha^n$*

Proof. We can see that $\alpha^n \vdash_{C_\omega} \alpha \vee \alpha^n$. On the other hand, due to Lemma 1 we have that $\neg(\alpha^n) \vdash_{C_\omega} \alpha$, therefore $\neg(\alpha^n) \vdash_{C_\omega} \alpha \vee \alpha^n$. Applying strong proof by cases we have that $\vdash_{C_\omega} \alpha \vee \alpha^n$. □

Lemma 3. *For all $n \in \mathbb{N}$ we have that $\vdash_{C_\omega} \alpha \vee \alpha^{(n)}$*

Proof. By induction on n.

Base case (n = 1). From Lemma 1 we have that $\vdash_{C_\omega} \alpha \vee \alpha^o$ holds when n = 1.

Inductive step. We assume by induction hypothesis that $\vdash_{C_\omega} \alpha \vee \alpha^{(n)}$ holds. We know from Lemma 2 that $\vdash_{C_\omega} \alpha \vee \alpha^{n+1}$. Thus $\vdash_{C_\omega} (\alpha \vee \alpha^{(n)}) \wedge (\alpha \vee \alpha^{n+1})$. Applying the *distributive law* to the last formula we have that $\vdash_{C_\omega} \alpha \vee (\alpha^{n+1} \wedge \alpha^{(n)})$, which in fact it is by definition $\vdash_{C_\omega} \alpha \vee \alpha^{(n+1)}$. □

Theorem 2 (Excluded Middle). *In C_ω, we have that* $\vdash_{C_\omega} \alpha \vee \neg^{(n)}\alpha$

Proof. In C_ω we have the following:

$$\begin{aligned}
&\vdash_{C_\omega} (\alpha \vee \neg^{(n)}\alpha) \leftrightarrow (\alpha \vee (\alpha \wedge \alpha^{(n)}))\\
&\vdash_{C_\omega} (\alpha \vee \neg^{(n)}\alpha) \leftrightarrow (\alpha \vee \neg\alpha) \wedge (\alpha \vee \alpha^{(n)})\\
&\vdash_{C_\omega} (\alpha \vee \neg^{(n)}\alpha) \leftrightarrow \alpha \vee \alpha^{(n)}
\end{aligned}$$

Therefore it is only necessary to check that $\alpha \vee \alpha^{(n)}$ holds, but accordingly to the *Lemma 3* this is true. □

The next two theorems follows from a similar proof in [1] where the author proved the same theorems in C_1.

Theorem 3 (Reductio Ad Absurdum). *In C_n we have that:*

$$(\Gamma \cup \{\alpha\} \vdash_{C_n} \beta), (\Gamma \cup \{\alpha\} \vdash_{C_n} \neg\beta), (\Gamma \cup \{\alpha\} \vdash_{C_n} \beta^{(n)}) \Rightarrow \Gamma \vdash_{C_n} \neg\alpha$$

Proof. Using *Deduction Theorem* we can prove the following from the hypothesis given: $\Gamma \vdash_{C_n} \alpha \to \beta^{(n)}$, $\Gamma \vdash_{C_n} \alpha \to \beta$ and $\Gamma \vdash_{C_n} \alpha \to \neg\beta$. By the transitive rule and the axiom schema $\vdash_{C_n} \beta^{(n)} \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$ we have that $\Gamma \vdash_{C_n} \alpha \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$. By the application of *Modus Ponens* twice we have that $\Gamma \vdash_{C_n} \alpha \to \neg\alpha$. From this, using theorem $\vdash_{C_n} \neg\alpha \to \neg\alpha$ (as an instance of Identity theorem), and axiom schemas $\vdash_{C_n} \alpha \vee \neg\alpha$ and $\vdash_{C_n} (\alpha \to \neg\alpha) \to ((\neg\alpha \to \neg\alpha) \to ((\alpha \vee \neg\alpha) \to \neg\alpha))$ we can conclude that $\Gamma \vdash_{C_n} \neg\alpha$. □

Theorem 4 (Explosive Principle). *In C_n we have that:*

$$\vdash_{C_n} \alpha \to (\neg^{(n)}\alpha \to \beta)$$

Proof. According to the strong negation definition we have that: $\alpha, \neg^{(n)}\alpha, \neg\beta \vdash_{C_n} \neg\alpha \wedge \alpha^{(n)}$, therefore $\alpha, \neg^{(n)}\alpha, \neg\beta \vdash_{C_n} \neg\alpha$ and $\alpha, \neg^{(n)}\alpha, \neg\beta \vdash_{C_n} \alpha^{(n)}$. Also we have that $\alpha, \neg^{(n)}\alpha, \neg\beta \vdash_{C_n} \alpha$. By the theorem 3 is easy to prove that $\alpha, \neg^{(n)}\alpha \vdash_{C_n} \neg\neg\beta$. C_n contains the axiom schemata $\neg\neg\alpha \to \alpha$, which it let us prove that $\alpha, \neg^{(n)}\alpha \vdash_{C_n} \beta$. Finally, applying two times *deduction theorem* to the last formula we have that $\vdash_{C_n} \alpha \to (\neg^{(n)}\alpha \to \beta)$. □

Theorem 5. *The connectives $\{\to, \wedge, \vee, \neg^{(n)}\}$ in C_n satisfy all the axiom schemata and inference rules in classical propositional calculus.*

Proof. Any logic in C_n extends the positive logic axioms from C_ω. Then, it is only necessary observe that the following axiom $(\neg^{(n)}\alpha \to \neg^{(n)}\beta) \to (\beta \to \alpha)$ holds in C_n

1.	$\neg^{(n)}\alpha \to \neg^{(n)}\beta$	Hypothesis
2.	β	Hypothesis
3.	$\beta \to (\neg^{(n)}\beta \to \alpha)$	From *Theorem 4*

4. $\neg^{(n)}\beta \to \alpha$ — Modus Ponens (2, 3)
5. $\neg^{(n)}\alpha \to \alpha$ — Transitivity (1, 4)
6. $\alpha \to \alpha$ — Identity theorem
7. $(\alpha \to \alpha) \to ((\neg^{(n)}\alpha \to \alpha) \to ((\alpha \vee \neg^{(n)}\alpha) \to \alpha))$ — Axiom Pos8
8. $(\neg^{(n)}\alpha \to \alpha) \to ((\alpha \vee \neg^{(n)}\alpha) \to \alpha)$ — Modus Ponens (6, 7)
9. $(\alpha \vee \neg^{(n)}\alpha) \to \alpha$ — Modus Ponens (5, 8)
10. $\alpha \vee \neg^{(n)}\alpha$ — From Theorem 2
11. α — Modus Ponens (10, 9)
12. $(\neg^{(n)}\alpha \to \neg^{(n)}\beta), \beta \vdash_{C_n} \alpha$ — 1-11
13. $(\neg^{(n)}\alpha \to \neg^{(n)}\beta) \vdash_{C_n} \beta \to \alpha$ — Deduction Theorem(12)
14. $\vdash_{C_n} (\neg^{(n)}\alpha \to \neg^{(n)}\beta) \to (\beta \to \alpha)$ — Deduction Theorem(13)

□

4 Conclusions

The presented work gives general ideas how to possibly extend a property in C_1 to C_n, mainly using inductive proofs. We know that all logics in the C_n system are strictly weaker than C_1 [1], perhaps many of them share many things in common as a strong negation. In the future should be interesting to investigate how much these logics are related each other among relevant properties.

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