Well-matchedness in Euler Diagrams

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Abstract. Euler diagrams are used for visualizing set-based information. Closed curves represent sets and the relationship between the curves correspond to relationships between sets. A notation is well-matched to meaning when its syntactic relationships are reflected in the semantic relationships being represented. Euler diagrams are said to be well-matched to meaning because, for example, curve containment corresponds to the subset relationship. In this paper we explore the concept of well-matchedness in Euler diagrams, considering different levels of well-matchedness. We also discuss how the properties, sometimes called well-formedness conditions, of an Euler diagram relate to the levels of well-matchedness.

1 Introduction

An Euler diagram is a collection of closed curves in the plane. Curves represent sets and the spatial relationships between the curves represent relationships between the sets. The lefthand diagram in fig. 1 represents the set-theoretic relationships C is a subset of A and A is disjoint from B. The righthand diagram in fig. 1 is a Venn diagram and has the same semantics as this Euler diagram. A Venn diagram contains all possible intersections of curves and shading is used to indicate empty sets. For example, the intersection between the curves labelled A and B is shaded indicating that the sets A and B are disjoint. The only part of the curve labelled C that is unshaded is that part within A and outside B indicating that C is a subset of A.

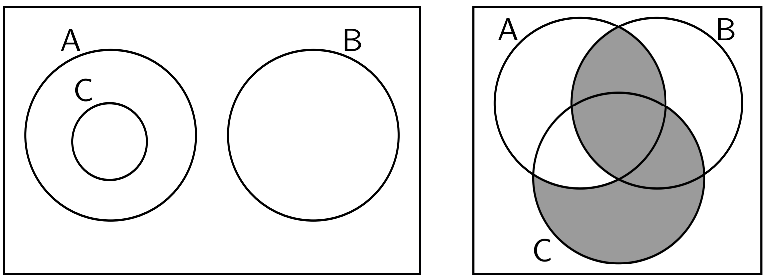

Fig. 1. An Euler diagram and an equivalent Venn diagram.

Peirce classified syntactic elements into three categories: *icon*, *index* and *symbol* [2]. In Euler diagrams, closed curves partition the plane into two regions: that within the curve and that outside; just as a set partitions elements into those in the set and those not in the set. For this reason, closed curves are considered to be icons within Euler diagrams, although some people argue that the iconicity is present only in the relationship between curves [4]. Curve labels indicate the names of the sets the curves represent. In this sense, a label is an index. In Venn diagrams, shading is a symbol used to indicate that the set represented is empty. Peirce thought that 'A diagram ought to be as iconic as possible' [2] page 433.

Closely related to iconicity is the notion of *well-matched to meaning*. A notation is well-matched to meaning when its syntactic relationships reflect the semantic relationships being represented [1]. Euler diagrams are considered to be a well-matched visualization of sets because subset, set disjointness and set intersection are represented by enclosure, disjointness, and overlap respectively. In fig. 1, the curve labelled C is enclosed by the curve labelled A, reflecting the semantics that C is a subset of A, and the curves labelled A and B are disjoint, reflecting that A and B are disjoint sets. For this reason, Euler diagrams are considered to be well-matched to meaning. However, in a Venn diagram all the curves overlap and so the notation is not well-matched to meaning.

As well as iconicity and well-matchedness, some notations contain *free rides* [3]. The semantics of the Euler diagram in fig. 1, in symbolic notation, is $C \subseteq A$ and $A \cap B = \emptyset$. From this information, we can deduce that $C \cap B = \emptyset$. However, in the Euler diagram this information comes 'for free'; we can just read it off from the diagram.

In this paper, we are mainly concerned with the notion of well-matchedness, which will be considered in section 3. In the next section we will give an informal definition of Euler diagrams.

2 Euler Diagrams

An Euler diagram comprises a set of closed curves drawn in the plane, where each curve has a label. Curve labels can be repeated. The set of curves with the same label is called a *contour*. The closed curves partition the plane into *minimal regions*. A *basic region* is a set of minimal regions that are all contained by the same curves. A *zone* is a set of basic regions for which the containing curves for each basic region have the same labels. The diagram D_6 in fig. 2 has four curves, three contours, eight minimal regions, six basic regions and five zones.

A range of properties have been defined for Euler diagrams, which are sometimes called well-formedness conditions:

1. All of the curves are simple (they do not self-intersect).
2. No pair of curves runs concurrently.
3. There are no triple points of intersection between the curves (i.e. three or more curves do not meet at the same point).
4. Whenever two curves intersect, they cross.

5. Each zone is connected (i.e. consists of exactly one minimal region).
6. Each curve label is used on at most one curve.

Definitions of these properties can be found in [5]. The Euler diagram in fig. 1 satisfies all of these properties. An Euler diagram satisfying all these properties is said to be "well-formed".

The semantics of an Euler diagram can be concisely presented as "missing zones represent the empty set". The "missing zones" are all the zones that would be included in a Venn diagram on the given curves of an Euler diagram that are not in the Euler diagram. We can extend Euler diagrams to include shading. This can be useful in allowing us to represent some set theoretic statements as well-formed Euler diagrams with extra zones that are shaded; an alternative way of representing an empty set. In this sense, all Venn diagrams are Euler diagrams (possibly with shading). In the next section we will consider the relationship between these properties and the concept of a diagram being well-matched to meaning.

3 Well-matchedness

Consider the statement 'C is a subset of A and C is disjoint from B'. We will use different Euler diagram representations of this statement, such as all of those given in fig. 2, to explore and illustrate the concept of well-matchedness in Euler diagrams and how it relates to the properties or well-formedness conditions given in section 2.

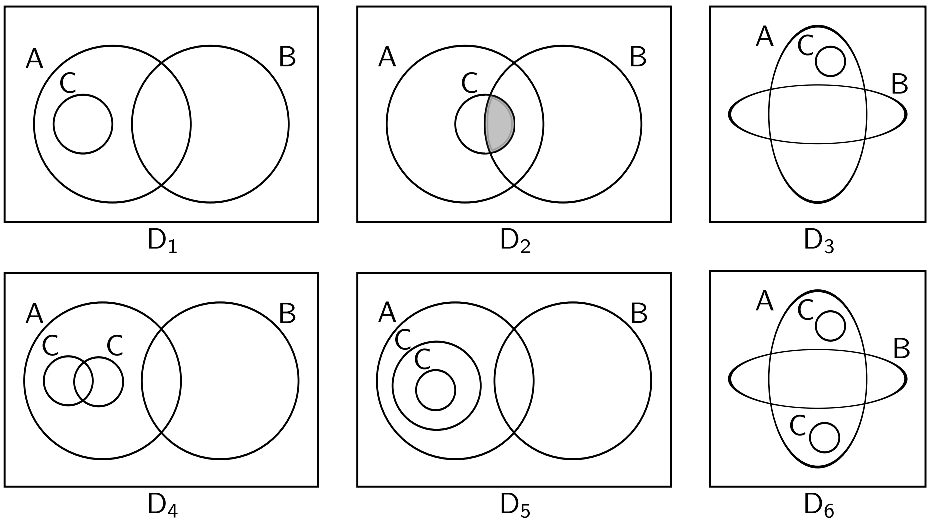

Fig. 2. Six Euler diagrams with equivalent semantics.

Diagram D_1 in fig. 2 satisfies all the properties listed in section 2. The curve labelled C is enclosed by the curve labelled A reflecting the first part of the statement 'C is a subset of A', and the curves labelled C and B are disjoint, reflecting the second part of the statement 'A and B are disjoint sets'. Hence D_1 is well-matched to meaning. The diagram D_2 contains shading but is well-formed (it satisfies all the properties listed in in section 2). In this diagram, the curve labelled C is still enclosed by the curve labelled A, but the curves labelled B and C are no longer disjoint. The diagram still represents C is disjoint from B, but this is achieved by shading a zone (symbolic notation) rather than disjoint curves (iconic notation). So, at best, D_2 is only partially well-matched to meaning. In general, an Euler diagram that contains extra zones that are shaded is not (fully) well-matched to meaning. This gives rise to the concept of well-matchedness at the zone level.

Well-matchedness Principle 1: An Euler diagram is **well-matched at the zone level** if it does not contain any extra zones (zones that must be shaded to preserve semantics).

We now consider well-matchedness when we break some of the well-formedness conditions. Diagram D_3 in fig. 2 contains two disconnected zones (breaking well-formedness condition 5). The zone inside A and outside B is represented by two minimal regions as is the zone inside B and outside A. Is this diagram well-matched to meaning? It contains all the appropriate zones and none extra. So, at the zone level, it is well-matched. Now the curve labelled C is enclosed by the curve labelled A and the curves labelled C and B are disjoint, so the diagram is well-matched as far as the curves are concerned. This gives rise to the concept of well-matchedness at the curve level.

Well-matchedness Principle 2: An Euler diagram is **well-matched at the curve level** if the subset, intersection and disjointness relationships between sets are matched by containment, overlap and disjointness of the curves representing the sets.

However, diagram D_3 seems to suggest that some parts of the set $A - B$ are disjoint from other parts of the same set; this set is represented by a zone consisting of two (disjoint) minimal regions. Having two disjoint regions both representing the same set is disconcerting and appears to go against the injective (one-to-one) nature of a well-matchedness relation. This gives rise to another well-matchedness principle.

Well-matchedness Principle 3: An Euler diagram is **well-matched at the minimal region level** if it is well-matched at the zone level and does not contain a disconnected zone.

Diagram D_4 in fig. 2 contains two curves with the same label (breaking well-formedness condition 6). When a diagram contains multiple curves with the same label, regions within an odd number of curves with the same label are deemed to be within the set and regions within an even number of curves with that label are outside the set. D_4 is well-matched at the zone level. However, the zone within A and C consists of two minimal regions, as does the zone within A but outside B and C – the minimal region inside both curves labelled C is part of this zone. So D_4 is not well-matched at the minimal region level. At the curve level, D_4 has two curves labelled C which intersect. Having two curves representing the same set again goes against the injective nature of well-matchedness. So D_4 is not well-matched at the curve level. However, the *contour* labelled C, consisting of the two curves labelled C, is enclosed by the contour (a single curve in this case) labelled A and disjoint from the contour (again a single curve) labelled B. So at the contour level this diagram is well-matched. This gives rise to the next well-matchedness principle.

Well-matchedness Principle 4: An Euler diagram is **well-matched at the contour level** if the subset, intersection and disjointness relationships between sets are matched by containment, overlap and disjointness of the contours representing the sets.

Diagram D_5 in fig. 2 again contains two curves with the same label, with one enclosing the other. This diagram is again well-matched at the zone level and at the contour level. However, it is not at the curve level – the two curves labelled C appear to be in a subset relation to each other – or at the minimal region level – the zone within A but outside B and C consists of two minimal regions, one of which is the region within both the curves labelled C.

Diagram D_6 in fig. 2 has two curves labelled C, which are placed in different minimal regions of the zone that is inside A and outside B. This diagram is well-matched at zone level and contour level. However, it seems to indicate that C is disjoint from itself, and is not well-matched at the curve or minimal region levels.

There is a well-formed Euler diagram without shading that represents the statement 'C is a subset of A and C is disjoint from B', that is diagram D_1 in fig. 2. Now consider the statement 'C is a subset of the disjoint union of A and B'. There is no well-formed Euler diagram without shading that represents this statement. The diagrams in fig. 3 represent this statement but include shading or break some of the well-formedness conditions stated in section 2. Diagram D_1 in fig. 3 is a well-formed diagram but contains shading, so it is not well-matched at any level. Diagram D_2 in fig. 3 contains two curves with label C; it is well-matched at the zone, minimal region and contour level, but not at the curve level.

Diagram D_3 in fig. 3 contains a non-simple curve; the curve labelled C intersects itself. D_3 also contains triple point where all three curves meet at a point. There are no extra zones and each zone is a minimal region, so the diagram

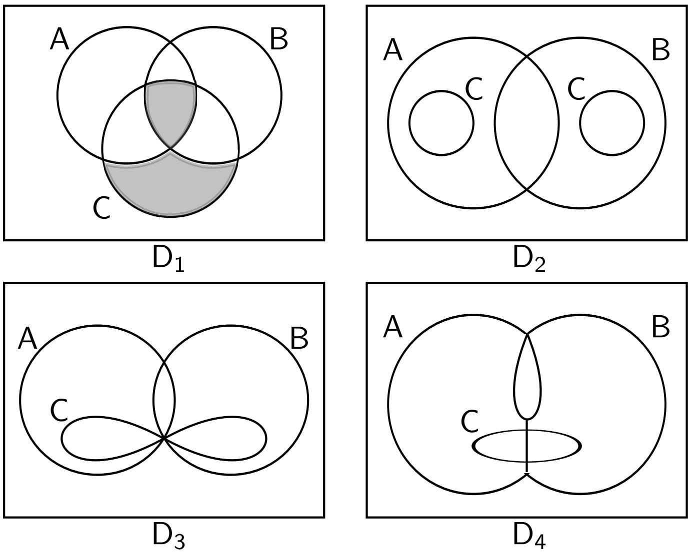

Fig. 3. Four Euler diagrams with equivalent semantics.

is well-matched at the zone and minimal region level. The curve C is enclosed exactly in the region inside A but outside B or inside B but outside A. So the diagram is well-matched at the curve level. Each curve is a contour as there are no repeated labels, so D is also well-matched at the contour level. This diagram is a fairly natural way to try and represent the statement 'C is a subset of the disjoint union of A and B' even though it contains the very unnatural non-simple curve. It is interesting that including a rather counter-intuitive feature allows the diagram to be presented in a well-matched form.

In diagram D_4 in fig. 3, the curves labelled A and B run concurrently for a part of their route allowing the curve labelled C to be placed in the region inside A but outside B or inside B but outside A without it also passing through the region within both A and B. It also contains two triple points where all three curves intersect. This diagram is well-matched at all levels, although it might be difficult to work out exactly the relationship between curves A and B.

Finally, we will consider two more examples to complete our analysis of the relationship between well-formedness and well-matchedness in Euler diagrams. Diagram D_1 in fig. 4 contains a non-simple curve; the curve labelled C intersects itself. As this curve does not intersect any other curves, the zone within the curve is divided into two minimal regions. This diagram is well-matched at the zone, curve and contour levels but not at the minimal region level. In diagram D_2 in fig. 4, curves A and B touch but do not cross, breaking well-formedness property 4, as do curves C and D. The diagram represents the statement 'A and B are disjoint and C and D are disjoint' and is well-matched at all levels. There is no well-formed diagram without shading that represents this statement.

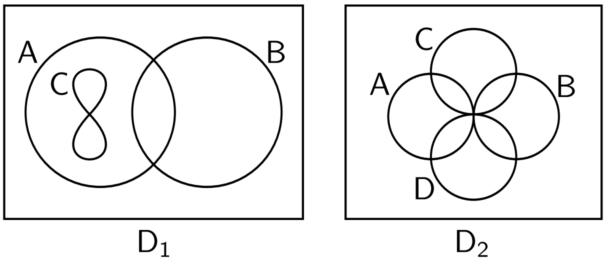

Fig. 4. Two non-well-formed Euler diagrams.

4 Conclusion

We have considered the notion of well-matchedness in Euler diagrams. In particular, we have considered well-matchedness in Euler diagrams that break some of the well-formedness properties. From this analysis, we have identified four levels of well-matchedness. Two of these concern curves: the curve and contour levels; and two concern regions: the zone and minimal region levels. Putting these four levels together we can state a general well-matchedness property.

Well-matchedness Principle 5: An Euler diagram is **fully well-matched** if it well-matched at the zone, minimal region, curve and contour levels.

As indicated in the text above, some set relationships cannot be represented as well-formed Euler diagrams without shading. The analysis in this paper shows that it is possible to break some of the "well-formedness" conditions but still have well-matched diagrams at some levels; this could be useful for visualizing these set relationships, perhaps by balancing well-formedness with well-matchedness. In well-formed Euler diagrams, each contour is a single simple closed curve and each zone is a single minimal region. A well-formed Euler diagram without shading contains no extra zones. Hence, each well-formed Euler diagram without shading is (fully) well-matched to meaning.

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