Mathematics

SECTION 1 (Maximum Marks: 12)

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If **ONLY** the correct option is chosen;
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : −1 In all other cases.

Q.1 Let $f:[1,\infty)\to\mathbb{R}$ be a differentiable function such that $f(1)=\frac{1}{3}$ and $3\int_1^x f(t)dt = x\,f(x)-\frac{x^3}{3}, x\in[1,\infty)$. Let e denote the base of the natural logarithm. Then the value of $f(e)$ is

(A) $\frac{e^2+4}{3}$ (B) $\frac{\log_e 4+e}{3}$ (C) $\frac{4e^2}{3}$ (D) $\frac{e^2-4}{3}$

Answer: C

Q.2 Consider an experiment of tossing a coin repeatedly until the outcomes of two consecutive tosses are same. If the probability of a random toss resulting in head is $\frac{1}{3}$, then the probability that the experiment stops with head is

(A) $\frac{1}{3}$ (B) $\frac{5}{21}$ (C) $\frac{4}{21}$ (D) $\frac{2}{7}$

Answer: B

Q.3 For any $y\in\mathbb{R}$, let $\cot^{-1}(y)\in(0,\pi)$ and $\tan^{-1}(y)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then the sum of all the solutions of the equation $\tan^{-1}\left(\frac{6y}{9-y^2}\right)+\cot^{-1}\left(\frac{9-y^2}{6y}\right)=\frac{2\pi}{3}$ for $0<|y|<3$, is equal to

(A) $2\sqrt{3}-3$ (B) $3-2\sqrt{3}$ (C) $4\sqrt{3}-6$ (D) $6-4\sqrt{3}$

Answer: C

Q.4 Let the position vectors of the points P, Q, R and S be $\vec{a} = \hat{i} + 2\hat{j} - 5\hat{k}$, $\vec{b} = 3\hat{i} + 6\hat{j} + 3\hat{k}$, $\vec{c} = \frac{17}{5}\hat{i} + \frac{16}{5}\hat{j} + 7\hat{k}$ and $\vec{d} = 2\hat{i} + \hat{j} + \hat{k}$, respectively. Then which of the following statements is true?

(A) The points P, Q, R and S are **NOT** coplanar

(B) $\frac{\vec{b} + 2\vec{d}}{3}$ is the position vector of a point which divides PR internally in the ratio $5:4$

(C) $\frac{\vec{b} + 2\vec{d}}{3}$ is the position vector of a point which divides PR externally in the ratio $5:4$

(D) The square of the magnitude of the vector $\vec{b} \times \vec{d}$ is 95

Answer: B

SECTION 2 (Maximum Marks: 12)

- This section contains **THREE (03)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 **ONLY** if (all) the correct option(s) is(are) chosen;
Partial Marks : +3 If all the four options are correct but **ONLY** three options are chosen;
Partial Marks : +2 If three or more options are correct but **ONLY** two options are chosen, both of which are correct;
Partial Marks : +1 If two or more options are correct but **ONLY** one option is chosen and it is a correct option;
Zero Marks : 0 If unanswered;
Negative Marks : −2 In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
choosing ONLY (A), (B) and (D) will get +4 marks;
choosing ONLY (A) and (B) will get +2 marks;
choosing ONLY (A) and (D) will get +2marks;
choosing ONLY (B) and (D) will get +2 marks;
choosing ONLY (A) will get +1 mark;
choosing ONLY (B) will get +1 mark;
choosing ONLY (D) will get +1 mark;
choosing no option(s) (i.e. the question is unanswered) will get 0 marks and
choosing any other option(s) will get −2 marks.

Q.5 Let $M=(a_{ij}),\ i,j\in\{1,2,3\}$, be the 3×3 matrix such that $a_{ij}=1$ if $j+1$ is divisible by i, otherwise $a_{ij}=0$. Then which of the following statements is(are) true?

(A) M is invertible

(B) There exists a nonzero column matrix $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ such that $M\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}=\begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}$

(C) The set $\{X\in\mathbb{R}^3 : MX=\mathbf{0}\}\neq\{\mathbf{0}\}$, where $\mathbf{0}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(D) The matrix $(M-2I)$ is invertible, where I is the 3×3 identity matrix

Answer:B, C

Q.6

Let $f:(0,1)\rightarrow\mathbb{R}$ be the function defined as $f(x)=[4x]\left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right)$, where $[x]$ denotes the greatest integer less than or equal to x. Then which of the following statements is(are) true?

(A) The function f is discontinuous exactly at one point in $(0,1)$

(B) There is exactly one point in $(0,1)$ at which the function f is continuous but **NOT** differentiable

(C) The function f is **NOT** differentiable at more than three points in $(0,1)$

(D) The minimum value of the function f is $-\frac{1}{512}$

Answer: A, B

Q.7

Let S be the set of all twice differentiable functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $\frac{d^2 f}{dx^2}(x)>0$ for all $x\in(-1,1)$. For $f\in S$, let X_f be the number of points $x\in(-1,1)$ for which $f(x)=x$. Then which of the following statements is(are) true?

(A) There exists a function $f\in S$ such that $X_f=0$

(B) For every function $f\in S$, we have $X_f\leq 2$

(C) There exists a function $f\in S$ such that $X_f=2$

(D) There does **NOT** exist any function f in S such that $X_f=1$

Answer: A, B, C

SECTION 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If **ONLY** the correct integer is entered;
Zero Marks : 0 In all other cases.

Q.8 For $x \in \mathbb{R}$, let $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the minimum value of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \int_0^{x \tan^{-1} x} \frac{e^{(t - \cos t)}}{1 + t^{2023}} \, dt$ is 0.

Q.9 For $x \in \mathbb{R}$, let $y(x)$ be a solution of the differential equation $(x^2 - 5)\frac{dy}{dx} - 2xy = -2x(x^2 - 5)^2$ such that $y(2) = 7$.
Then the maximum value of the function $y(x)$ is 16.

Q.10 Let X be the set of all five digit numbers formed using 1,2,2,2,4,4,0. For example, 22240 is in X while 02244 and 44422 are not in X. Suppose that each element of X has an equal chance of being chosen. Let p be the conditional probability that an element chosen at random is a multiple of 20 given that it is a multiple of 5. Then the value of $38p$ is equal to 31.

Q.11 Let $A_1, A_2, A_3, \ldots, A_8$ be the vertices of a regular octagon that lie on a circle of radius 2. Let P be a point on the circle and let PA_i denote the distance between the points P and A_i for $i = 1, 2, \ldots, 8$. If P varies over the circle, then the maximum value of the product $PA_1 \cdot PA_2 \cdots PA_8$, is 512.

Q.12 Let $R = \left\{ \begin{pmatrix} a & 3 & b \\ c & 2 & d \\ 0 & 5 & 0 \end{pmatrix} : a, b, c, d \in \{0, 3, 5, 7, 11, 13, 17, 19\} \right\}$. Then the number of invertible matrices in R is 3780.

Q.13 Let C_1 be the circle of radius 1 with center at the origin. Let C_2 be the circle of radius r with center at the point $A=(4,1)$, where $1<r<3$. Two distinct common tangents PQ and ST of C_1 and C_2 are drawn. The tangent PQ touches C_1 at P and C_2 at Q. The tangent ST touches C_1 at S and C_2 at T. Mid points of the line segments PQ and ST are joined to form a line which meets the x-axis at a point B. If $AB=\sqrt{5}$, then the value of r^2 is ___2___.

SECTION 4 (Maximum Marks: 12)

- This section contains **TWO (02)** paragraphs.
- Based on each paragraph, there are **TWO (02)** questions.
- The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If ONLY the correct numerical value is entered in the designated place;
Zero Marks : 0 In all other cases.

PARAGRAPH "I"

Consider an obtuse angled triangle ABC in which the difference between the largest and the smallest angle is $\frac{\pi}{2}$ and whose sides are in arithmetic progression. Suppose that the vertices of this triangle lie on a circle of radius 1.

(There are two questions based on PARAGRAPH "I", the question given below is one of them)

Q.14 Let a be the area of the triangle ABC. Then the value of $(64a)^2$ is 1008.00.
Range (1007.99 to 1008.01)

PARAGRAPH "I"

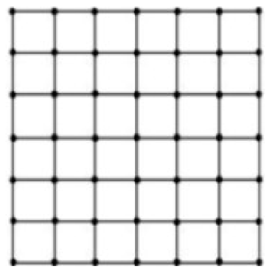
Consider an obtuse angled triangle ABC in which the difference between the largest and the smallest angle is $\frac{\pi}{2}$ and whose sides are in arithmetic progression. Suppose that the vertices of this triangle lie on a circle of radius 1.

(There are two questions based on PARAGRAPH "I", the question given below is one of them)

Q.15 Then the inradius of the triangle ABC is 0.25. Range(0.24 to 0.26)

PARAGRAPH "II"

Consider the 6×6 square in the figure. Let $A_1, A_2, \ldots, A_{49}$ be the points of intersections (dots in the picture) in some order. We say that A_i and A_j are friends if they are adjacent along a row or along a column. Assume that each point A_i has an equal chance of being chosen.

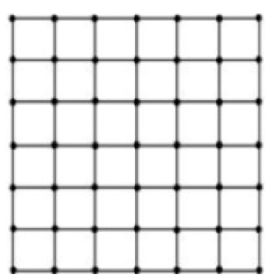


(There are two questions based on PARAGRAPH "II", the question given below is one of them)

Q.16 Let p_i be the probability that a randomly chosen point has i many friends, $i = 0, 1, 2, 3, 4$. Let X be a random variable such that for $i = 0, 1, 2, 3, 4$, the probability $P(X = i) = p_i$. Then the value of $7E(X)$ is 24.00. Range (23.99 to 24.01)

PARAGRAPH "II"

Consider the 6×6 square in the figure. Let $A_1, A_2, \ldots, A_{49}$ be the points of intersections (dots in the picture) in some order. We say that A_i and A_j are friends if they are adjacent along a row or along a column. Assume that each point A_i has an equal chance of being chosen.



(There are two questions based on PARAGRAPH "II", the question given below is one of them)

Q.17 Two distinct points are chosen randomly out of the points $A_1, A_2, \ldots, A_{49}$. Let p be the probability that they are friends. Then the value of $7p$ is 0.50. Range (0.49 to 0.51)

END OF THE QUESTION PAPER