Mathematics

SECTION 1 (Maximum Marks: 12)

- This section contains **THREE (03)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : $+4$ **ONLY** if (all) the correct option(s) is(are) chosen;
  Partial Marks : $+3$ If all the four options are correct but **ONLY** three options are chosen;
  Partial Marks : $+2$ If three or more options are correct but **ONLY** two options are chosen, both of which are correct;
  Partial Marks : $+1$ If two or more options are correct but **ONLY** one option is chosen and it is a correct option;
  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
  Negative Marks : -2 In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  choosing ONLY (A), (B) and (D) will get $+4$ marks;
  choosing ONLY (A) and (B) will get $+2$ marks;
  choosing ONLY (A) and (D) will get $+2$ marks;
  choosing ONLY (B) and (D) will get $+2$ marks;
  choosing ONLY (A) will get +1 mark;
  choosing ONLY (B) will get +1 mark;
  choosing ONLY (D) will get +1 mark;
  choosing no option (i.e. the question is unanswered) will get 0 marks; and
  choosing any other combination of options will get -2 marks.

Q.1 Let $S=(0,1)\cup(1,2)\cup(3,4)$ and $T=\{0,1,2,3\}$. Then which of the following statements is(are) true?

(A) There are infinitely many functions from S to T
(B) There are infinitely many strictly increasing functions from S to T
(C) The number of continuous functions from S to T is at most 120
(D) Every continuous function from S to T is differentiable

Answer: A, C, D

Q.2 Let T_1 and T_2 be two distinct common tangents to the ellipse $E: \frac{x^2}{6}+\frac{y^2}{3}=1$ and the parabola $P: y^2=12x$. Suppose that the tangent T_1 touches P and E at the points A_1 and A_2, respectively and the tangent T_2 touches P and E at the points A_4 and A_3, respectively. Then which of the following statements is(are) true?

(A) The area of the quadrilateral $A_1A_2A_3A_4$ is 35 square units

(B) The area of the quadrilateral $A_1A_2A_3A_4$ is 36 square units

(C) The tangents T_1 and T_2 meet the x-axis at the point $(-3,0)$

(D) The tangents T_1 and T_2 meet the x-axis at the point $(-6,0)$

Answer: A, C

Q.3 Let $f:[0,1]\to[0,1]$ be the function defined by $f(x)=\frac{x^3}{3}-x^2+\frac{5}{9}x+\frac{17}{36}$. Consider the square region $S=[0,1]\times[0,1]$. Let $G=\{(x,y)\in S: y>f(x)\}$ be called the green region and $R=\{(x,y)\in S: y<f(x)\}$ be called the red region. Let $L_h=\{(x,h)\in S: x\in[0,1]\}$ be the horizontal line drawn at a height $h\in[0,1]$. Then which of the following statements is(are) true?

(A) There exists an $h\in\left[\frac{1}{4},\frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the green region below the line L_h

(B) There exists an $h\in\left[\frac{1}{4},\frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the red region below the line L_h

(C) There exists an $h\in\left[\frac{1}{4},\frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the red region below the line L_h

(D) There exists an $h\in\left[\frac{1}{4},\frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the green region below the line L_h

Answer: B, C, D

SECTION 2 (Maximum Marks: 12)

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If **ONLY** the correct option is chosen;
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : −1 In all other cases.

Q.4 Let $f:(0,1)\to\mathbb{R}$ be the function defined as $f(x)=\sqrt{n}$ if $x\in\left[\frac{1}{n+1},\frac{1}{n}\right)$ where $n\in\mathbb{N}$. Let $g:(0,1)\to\mathbb{R}$ be a function such that $\int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt<g(x)<2\sqrt{x}$ for all $x\in(0,1)$. Then $\lim_{x\to 0}f(x)g(x)$

(A) does **NOT** exist
(B) is equal to 1
(C) is equal to 2
(D) is equal to 3

Answer: C

Q.5 Let Q be the cube with the set of vertices $\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1,x_2,x_3\in\{0,1\}\}$. Let F be the set of all twelve lines containing the diagonals of the six faces of the cube Q. Let S be the set of all four lines containing the main diagonals of the cube Q; for instance, the line passing through the vertices $(0,0,0)$ and $(1,1,1)$ is in S. For lines ℓ_1 and ℓ_2, let $d(\ell_1,\ell_2)$ denote the shortest distance between them. Then the maximum value of $d(\ell_1,\ell_2)$, as ℓ_1 varies over F and ℓ_2 varies over S, is

(A) $\frac{1}{\sqrt{6}}$ (B) $\frac{1}{\sqrt{8}}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{1}{\sqrt{12}}$

Answer: A

Q.6 Let $X=\left\{(x,y)\in\mathbb{Z}\times\mathbb{Z} : \frac{x^2}{8}+\frac{y^2}{20}<1 \text{ and } y^2<5x\right\}$. Three distinct points P, Q and R are randomly chosen from X. Then the probability that P, Q and R form a triangle whose area is a positive integer, is

(A) $\frac{71}{220}$ (B) $\frac{73}{220}$ (C) $\frac{79}{220}$ (D) $\frac{83}{220}$

Answer: B

Q.7 Let P be a point on the parabola $y^2 = 4ax$, where $a > 0$. The normal to the parabola at P meets the x-axis at a point Q. The area of the triangle PFQ, where F is the focus of the parabola, is 120. If the slope m of the normal and a are both positive integers, then the pair (a, m) is

(A) $(2,3)$ (B) $(1,3)$ (C) $(2,4)$ (D) $(3,4)$

Answer: A

SECTION 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If **ONLY** the correct integer is entered;
Zero Marks : 0 In all other cases.

Q.8 Let $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, for $x \in \mathbb{R}$. Then the number of real solutions of the equation $\sqrt{1+\cos(2x)} = \sqrt{2}\tan^{-1}(\tan x)$ in the set $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is equal to 3.

Q.9 Let $n \geq 2$ be a natural number and $f : [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} n(1-2nx) & \text{if } 0 \leq x \leq \frac{1}{2n} \\ 2n(2nx-1) & \text{if } \frac{1}{2n} \leq x \leq \frac{3}{4n} \\ 4n(1-nx) & \text{if } \frac{3}{4n} \leq x \leq \frac{1}{n} \\ \frac{n}{n-1}(nx-1) & \text{if } \frac{1}{n} \leq x \leq 1 \end{cases}$$

If n is such that the area of the region bounded by the curves $x = 0$, $x = 1$, $y = 0$ and $y = f(x)$ is 4, then the maximum value of the function f is 8.

Q.10 Let $7\overbrace{5\cdots5}^{r}7$ denote the $(r+2)$ digit number where the first and the last digits are 7 and the remaining r digits are 5. Consider the sum $S = 77 + 757 + 7557 + \cdots + 7\overbrace{5\cdots5}^{98}7$. If $S = \frac{7\overbrace{5\cdots5}^{99}7 + m}{n}$, where m and n are natural numbers less than 3000, then the value of $m + n$ is 1219.

Q.11 Let $A=\left\{\dfrac{1967+1686i\ \sin\theta}{7-3i\cos\theta}\ :\ \theta\in\mathbb{R}\right\}$. If A contains exactly one positive integer n, then the value of n is __281__.

Q.12 Let P be the plane $\sqrt{3}x+2y+3z=16$ and let $S=\left\{\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}\ :\ \alpha^2+\beta^2+\gamma^2=1 \text{ and the distance of } (\alpha,\beta,\gamma) \text{ from the plane } P \text{ is } \dfrac{7}{2}\right\}$. Let $\vec{u},\vec{v}$ and $\vec{w}$ be three distinct vectors in S such that $|\vec{u}-\vec{v}|=|\vec{v}-\vec{w}|=|\vec{w}-\vec{u}|$. Let V be the volume of the parallelepiped determined by vectors $\vec{u},\vec{v}$ and $\vec{w}$. Then the value of $\dfrac{80}{\sqrt{3}}V$ is __45__.

Q.13 Let a and b be two nonzero real numbers. If the coefficient of x^5 in the expansion of $\left(ax^2+\dfrac{70}{27bx}\right)^4$ is equal to the coefficient of x^{-5} in the expansion of $\left(ax-\dfrac{1}{bx^2}\right)^7$, then the value of $2b$ is __3__.

SECTION 4 (Maximum Marks: 12)

- This section contains **FOUR (04)** Matching List Sets.
- Each set has **ONE** Multiple Choice Question.
- Each set has **TWO** lists: **List-I** and **List-II**.
- **List-I** has **Four** entries (P), (Q), (R) and (S) and **List-II** has **Five** entries (1), (2), (3), (4) and (5).
- **FOUR** options are given in each Multiple Choice Question based on **List-I** and **List-II** and **ONLY ONE** of these four options satisfies the condition asked in the Multiple Choice Question.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks :	+3	**ONLY** if the option corresponding to the correct combination is chosen;
Zero Marks :	0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks :	−1	In all other cases.

Q.14 Let α, β and γ be real numbers. Consider the following system of linear equations

$$x+2y+z=7$$
$$x+\alpha z=11$$
$$2x-3y+\beta z=\gamma$$

Match each entry in **List-I** to the correct entries in **List-II.**

List-I	**List-II**
(P) If $\beta=\frac{1}{2}(7\alpha-3)$ and $\gamma=28$, then the system has	(1) a unique solution
(Q) If $\beta=\frac{1}{2}(7\alpha-3)$ and $\gamma\neq 28$, then the system has	(2) no solution
(R) If $\beta\neq\frac{1}{2}(7\alpha-3)$ where $\alpha=1$ and $\gamma\neq 28$, then the system has	(3) infinitely many solutions
(S) If $\beta\neq\frac{1}{2}(7\alpha-3)$ where $\alpha=1$ and $\gamma=28$, then the system has	(4) $x=11$, $y=-2$ and $z=0$ as a solution
	(5) $x=-15$, $y=4$ and $z=0$ as a solution

The correct option is:

(A) $(P)\rightarrow(3)$ $(Q)\rightarrow(2)$ $(R)\rightarrow(1)$ $(S)\rightarrow(4)$
(B) $(P)\rightarrow(3)$ $(Q)\rightarrow(2)$ $(R)\rightarrow(5)$ $(S)\rightarrow(4)$
(C) $(P)\rightarrow(2)$ $(Q)\rightarrow(1)$ $(R)\rightarrow(4)$ $(S)\rightarrow(5)$
(D) $(P)\rightarrow(2)$ $(Q)\rightarrow(1)$ $(R)\rightarrow(1)$ $(S)\rightarrow(3)$

Answer: A

Q.15 Consider the given data with frequency distribution

x_i	3	8	11	10	5	4
f_i	5	2	3	2	4	4

Match each entry in **List-I** to the correct entries in **List-II.**

List-I	List-II
(P) The mean of the above data is	(1) 2.5
(Q) The median of the above data is	(2) 5
(R) The mean deviation about the mean of the above data is	(3) 6
(S) The mean deviation about the median of the above data is	(4) 2.7
	(5) 2.4

The correct option is:

(A) $(P)\rightarrow(3)$ $(Q)\rightarrow(2)$ $(R)\rightarrow(4)$ $(S)\rightarrow(5)$

(B) $(P)\rightarrow(3)$ $(Q)\rightarrow(2)$ $(R)\rightarrow(1)$ $(S)\rightarrow(5)$

(C) $(P)\rightarrow(2)$ $(Q)\rightarrow(3)$ $(R)\rightarrow(4)$ $(S)\rightarrow(1)$

(D) $(P)\rightarrow(3)$ $(Q)\rightarrow(3)$ $(R)\rightarrow(5)$ $(S)\rightarrow(5)$

Answer: A

Q.16 Let ℓ_1 and ℓ_2 be the lines $\vec{r}_1 = \lambda(\hat{i}+\hat{j}+\hat{k})$ and $\vec{r}_2 = (\hat{j}-\hat{k})+\mu(\hat{i}+\hat{k})$, respectively. Let X be the set of all the planes H that contain the line ℓ_1. For a plane H, let $d(H)$ denote the smallest possible distance between the points of ℓ_2 and H. Let H_0 be a plane in X for which $d(H_0)$ is the maximum value of $d(H)$ as H varies over all planes in X.

Match each entry in **List-I** to the correct entries in **List-II.**

List-I	**List-II**
(P) The value of $d(H_0)$ is	(1) $\sqrt{3}$
(Q) The distance of the point $(0,1,2)$ from H_0 is	(2) $\frac{1}{\sqrt{3}}$
(R) The distance of origin from H_0 is	(3) 0
(S) The distance of origin from the point of intersection of planes $y=z$, $x=1$ and H_0 is	(4) $\sqrt{2}$
	(5) $\frac{1}{\sqrt{2}}$

The correct option is:

(A) $(P)\rightarrow(2)$ $(Q)\rightarrow(4)$ $(R)\rightarrow(5)$ $(S)\rightarrow(1)$

(B) $(P)\rightarrow(5)$ $(Q)\rightarrow(4)$ $(R)\rightarrow(3)$ $(S)\rightarrow(1)$

(C) $(P)\rightarrow(2)$ $(Q)\rightarrow(1)$ $(R)\rightarrow(3)$ $(S)\rightarrow(2)$

(D) $(P)\rightarrow(5)$ $(Q)\rightarrow(1)$ $(R)\rightarrow(4)$ $(S)\rightarrow(2)$

Answer: B

Q.17 Let z be a complex number satisfying $|z|^3 + 2z^2 + 4\bar{z} - 8 = 0$, where $\bar{z}$ denotes the complex conjugate of z. Let the imaginary part of z be nonzero.

Match each entry in **List-I** to the correct entries in **List-II.**

List-I	**List-II**
(P) $|z|^2$ is equal to	(1) 12
(Q) $|z-\bar{z}|^2$ is equal to	(2) 4
(R) $|z|^2 + |z+\bar{z}|^2$ is equal to	(3) 8
(S) $|z+1|^2$ is equal to	(4) 10
	(5) 7

The correct option is:

(A) $(P)\rightarrow(1)$ $(Q)\rightarrow(3)$ $(R)\rightarrow(5)$ $(S)\rightarrow(4)$

(B) $(P)\rightarrow(2)$ $(Q)\rightarrow(1)$ $(R)\rightarrow(3)$ $(S)\rightarrow(5)$

(C) $(P)\rightarrow(2)$ $(Q)\rightarrow(4)$ $(R)\rightarrow(5)$ $(S)\rightarrow(1)$

(D) $(P)\rightarrow(2)$ $(Q)\rightarrow(3)$ $(R)\rightarrow(5)$ $(S)\rightarrow(4)$

Answer: B

END OF THE QUESTION PAPER