Introduction to Categorical Logic

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Introduction

Once upon a time, there was Logic, and there was Category Theory. Traditional logic once consisted of:

- Propositional calculus, first-order logic, formal systems of deduction, Tarski-style semantics, Gödel's completeness and incompleteness theorems.
- On that basis were erected model theory, set theory, computability theory, and proof theory.
- Logic was considered the study of the foundations of mathematics, but it was largely unrealted to other branches of mathematics.

And category theory originally consisted of:

- Homological algebra, homotopy theory, the study of various kinds of limits,
- Universal constructions like free algebras and tensor products,
- Duality theories such as that of Gelfand and Stone,
- Grothendieck's algebraic geometry and sheaf theory,
- The theory of monads and universal algebra, like Birkhoff's theorems.

Then along came F.W. Lawvere and noticed how the basic framework of Stone duality could be applied to algebraic theories, inventing *functorial semantics*. From this, the basic ideas of categorical logic followed:

- An equational theory is represented as a category $\mathbb{T}$ with finite products that's "freely generated as such by the signature"; a model of the theory, or $\mathbb{T}$-algebra, is then a finite product preserving functor $A : \mathbb{T} \to \mathsf{Set}$. The completeness of equational reasoning (one of Birkhoff's theorems) is then the fact that we have a contravariant embedding,
$$\mathbb{T}^{\mathsf{op}} \hookrightarrow \mathrm{Mod}(\mathbb{T}) = \mathrm{FP}(\mathbb{T}, \mathsf{Set}) \,,$$
so that the "syntax" is a (dual) subcategory of the "semantics".

- Following Rasiowa-Sikorski, propositional logic can be treated as Boolean algebra: formal deduction is a way to specify a free algebra, truth-table semantics is a description of the Boolean homomorphisms into $\{0, 1\}$, and Stone's representation theorem is the completenes theorem for propositional logic.

- First-order logic can be understoood as a Boolean algebra indexed over an algebraic theory, with the quantifiers as adjoints to the reindexing functors (Lawvere's hyperdoctrines). More generally, one can define the notion of a "Boolean category" as a solution to the analogy: "propositional logic is to Boolean algebra as first-order logic is to X", generalizing from posets to (proper) categories. Gödel completeness can be formulated as a (sheaf) representation theorem for Boolean (or Heyting) categories.

- The same ideas also apply to various fragments of first-order logic to relate diffferent kinds of logical theories (syntax) and their categories of models (semantics) via the general framework of functorial semantics.

- Finally, topos theory subsumes and generalizes logical duality, unifying the "algebraic" (syntactic) and "geometric" (semantic) aspects in the single category of Grothendieck toposes and geometric morphisms. A topos can also be seen a forcing model of set theory, Kripke model of intuitionistic or modal logic, a model of infinitary first-order logic, a model of higher-order (predicate) logic, or even a realizability model of computability.

There is also another, "constructive" tradition in logic, more closely related to proof theory and influenced by theoretical computer science.

- The *Curry-Howard correspondence* is a somewhat mysterious connection between propositional logic and type theory, according to which the "meaning" of a propositional formula is not just a truth-value, but rather the collection of its proofs. Propositions-as-Types, Proofs-as-Terms (or -Programs) is a proof-theoretic (or computational) alterative to Tarskian, truth-value semantics. It also extends to first-order logic and dependent type theory.

- Associated to this perspective, one also has categorical semantics of type theories like the λ-calculus in (locally) cartesian closed categories, like the category of Scott domains, rather than in Boolean and Heyting algebras (for propositional logic) and (pre)toposes (for first-order and higher-order predicate logic).

- The algebra used for the truth-value semantics of propositional or predicate logic (e.g. the Boolean algebra $\{0, 1\}$) is then seen to be the poset reflection of a proper category (e.g. Set) modeling the type theory that is the "proof-relevant" version of the logic. The general scheme can be represented as follows, with the righthand side the proof-relevant version of the left:

Logic	Algebra	Type Theory	Category
Propositional	Boolean algebra	Simple	CCC
Predicate	Boolean category	Dependent	LCCC

- The relationship between *validity* and *provability* classically described by that between logic and type theory, is described categorically by the relations of generalization and "poset reflection" between (structured) posets and categories. In this way, the Curry-Howard correspondence relates to the idea of "categorification": a structured category whose poset reflection is a given structured poset. For example, the categorification of a $\wedge$-semilattice is a category with finite products, and the categorification of the Boolean algebra $\{0, 1\}$ is the category Set.

Such was the state of Categorical Logic when these notes were begun, around the turn of the century. In the meantime, some new ideas have shifted the focus: the Curry-Howard paradigm relating truth-value semantics (model theory) and type-theoretic syntax (proof theory)—viewed as an instance of categorification—has turned out to capture only the first two levels of an infinite hierarchy of levels of structure, related by inclusion, truncation, (co-)reflection, and other operations. The importance of "proof-relevance" that underlies the Propositions-as-Types idea is essentially just a special case of the coherence issue that arises everywhere in higher category theory. And the once-bold replacement of both *truth-values* and *sets* by *types* in constructive logic and the foundations of computation parallels the replacement of discrete structures (sheaves) by "higher" ones (stacks) in algebra and geometry, except that we have now learned that the gap between the levels is not just a single step, but rather an infinite hierarchy of levels of structure, each just as significant as the first step. These insights are reflected in current categorical logic in the recent extension from algebraic logic (level 0) and topos theory (level 1) to higher topos theory and homotopy type theory (level ∞). The latter are the focus of much current research, but the unification of the various earlier topics that has been achieved already shows how much we have learned about what happens in passing from 0 to 1, by passing from the finite to the infinite.

For instance, the dualities of Stone, Lawvere, and Makkai that spurred the early development of categorical logic now fall into place from the standpoint shown in table 1, that focuses on typing, variance, and h-level, rather than the traditional distinction between syntax and semantics.

	0-types	1-types	n-types
Simple types	Positive PL	Alg. Theories	HITs
Dependent types	Coherent FOL	Gen. Alg. Theories	Universes

Table 1: Covariant fragments

Appendix A

Category Theory

A.1 Categories

Definition A.1.1. A *category* $\mathcal{C}$ consists of classes

$\mathcal{C}_0$ of objects A, B, C, ...
$\mathcal{C}_1$ of morphisms f, g, h, ...

such that:

- Each morphism f has uniquely determined *domain* $\mathsf{dom}\, f$ and *codomain* $\mathsf{cod}\, f$, which are objects. This is written:
$$f : \mathsf{dom}\, f \to \mathsf{cod}\, f$$
- For any morphisms $f : A \to B$ and $g : B \to C$ there exists a uniquely determined *composition* $g \circ f : A \to C$. Composition is associative:
$$h \circ (g \circ f) = (h \circ g) \circ f \ ,$$
where domains are codomains are as follows:
$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \xrightarrow{\ h\ } D$$
- For every object A there exists the *identity* morphism $1_A : A \to A$ which is a unit for composition,
$$1_A \circ f = f \ , \qquad\qquad g \circ 1_A = g \ ,$$
where $f : B \to A$ and $g : A \to C$.

Morphisms are also called *arrows* or *maps*. Note that morphisms do not actually have to be functions, and objects need not be sets or spaces of any sort. We often write $\mathcal{C}$ instead of $\mathcal{C}_0$.

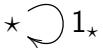
Definition A.1.2. A category $\mathcal{C}$ is *small* when the objects $\mathcal{C}_0$ and the morphisms $\mathcal{C}_1$ are sets (as opposed to proper classes). A category is *locally small* when for all objects $A, B \in \mathcal{C}_0$ the class of morphisms with domain A and codomain B, written $\mathsf{Hom}(A, B)$ or $\mathcal{C}_0(A, B)$, is a set.

We normally restrict attention to locally small categories, so unless we specify otherwise all categories are taken to be locally small. Next we consider several examples of categories.

A.1.1 Examples

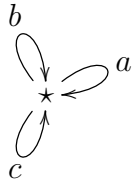
The empty category $\mathsf{0}$ The empty category has no objects and no arrows.

The unit category $\mathsf{1}$ The unit category, also called the terminal category, has one object $\star$ and one arrow $\mathsf{1}_\star$:



Other finite categories There are other finite categories, for example the category with two objects and one (non-identity) arrow, and the category with two parallel arrows:

Groups as categories Every group $(G, \cdot)$, is a category with a single object $\star$ and each element of G as a morphism:



$$a, b, c, \ldots \in G$$

The composition of arrows is given by the group operation:

$$a \circ b = a \cdot b$$

The identity arrow is the group unit e. This is indeed a category because the group operation is associative and the group unit is the unit for the composition. In order to get a category, we do not actually need to know that every element in G has an inverse. It suffices to take a *monoid*, also known as *semigroup*, which is an algebraic structure with an associative operation and a unit.

We can turn things around and *define* a monoid to be a category with a single object. A group is then a category with a single object in which every arrow is an *isomorphism* (in the sense of definition A.1.5 below).

Posets as categories Recall that a *partially ordered set*, or *poset* $(P, \leq)$, is a set with a reflexive, transitive, and antisymmetric relation:

$$\begin{aligned} x &\leq x && \text{(reflexive)} \\ x \leq y \;\&\; y \leq z &\;\Rightarrow\; x \leq z && \text{(transitive)} \\ x \leq y \;\&\; y \leq x &\;\Rightarrow\; x = y && \text{(antisymmetric)} \end{aligned}$$

Each poset is a category whose objects are the elements of P, and there is a single arrow $p \to q$ between $p, q \in P$ if, and only if, $p \leq q$. Composition of $p \to q$ and $q \to r$ is the unique arrow $p \to r$, which exists by transitivity of $\leq$. The identity arrow on p is the unique arrow $p \to p$, which exists by reflexivity of $\leq$.

Antisymmetry tells us that any two isomorphic objects in P are equal.[1] We do not need antisymmetry in order to obtain a category, i.e., a *preorder* would suffice.

Again, we may *define* a preorder to be a category in which there is at most one arrow between any two objects. A poset is a skeletal preorder, i.e. one in which the only isomorphisms are the identity arrows. We allow for the possibility that a preorder or a poset is a proper class rather than a set.

A particularly important example of a poset category is the poset of open sets $\mathcal{O}X$ of a topological space X, ordered by inclusion.

Sets as categories Any set S is a category whose objects are the elements of S and whose only arrows are identity arrows. Such a category, in which the only arrows are the identity arrows, is called a *discrete category*.

A.1.2 Categories of structures

In general, structures like groups, topological spaces, posets, etc., determine categories in which the maps are structure-preserving functions, composition is composition of functions, and identity morphisms are identity functions:

- Group is the category whose objects are groups and whose morphisms are group homomorphisms.
- Top is the category whose objects are topological spaces and whose morphisms are continuous maps.
- Set is the category whose objects are sets and whose morphisms are functions.[2]
- Graph is the category of (directed) graphs an graph homomorphisms.
- Poset is the category of posets and monotone maps.

[1] A category in which isomorphic object are equal is a *skeletal* category.

[2] A function between sets A and B is a relation $f \subseteq A \times B$ such that for every $x \in A$ there exists a unique $y \in B$ for which $\langle x, y \rangle \in f$. A morphism in Set is a triple $\langle A, f, B \rangle$ such that $f \subseteq A \times B$ is a function.

Such categories of structures are generally *large*, but locally small. Note that it is not necessary to check the associative and unit laws for such categories of functions (why?), unlike the following example.

Exercise A.1.3. The *category of relations* Rel has as objects all sets $A, B, C, \ldots$ and as arrows $A \to B$ the relations $R \subseteq A \times B$. The composite of $R \subseteq A \times B$ and $S \subseteq B \times C$, and the identity arrow on A, are defined by:

$$S \circ R = \left\{ \langle x, z \rangle \in A \times C \;\middle|\; \exists\, y \in B \,.\, xRy \;\&\; ySz \right\},$$
$$1_A = \left\{ \langle x, x \rangle \;\middle|\; x \in A \right\}.$$

Show that this is indeed a category!

A.1.3 Basic notions

We recall some further basic notions from category theory.

Definition A.1.4. A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ is given by a subclass of objects $\mathcal{C}'_0 \subseteq \mathcal{C}_0$ and a subclass of morphisms $\mathcal{C}'_1 \subseteq \mathcal{C}_1$ such that $f \in \mathcal{C}'_1$ implies $\mathsf{dom}\, f, \mathsf{cod}\, f \in \mathcal{C}'_0$, $1_A \in \mathcal{C}'_1$ for every $A \in \mathcal{C}'_0$, and $g \circ f \in \mathcal{C}'_1$ whenever $f, g \in \mathcal{C}'_1$ are composable.

A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is *full* if for all $A, B \in \mathcal{C}'_0$, we have $\mathcal{C}'(A, B) = \mathcal{C}(A, B)$, i.e. every $f : A \to B$ in $\mathcal{C}_1$ is also in $\mathcal{C}'_1$.

Definition A.1.5. An *inverse* of a morphism $f : A \to B$ is a morphism $f^{-1} : B \to A$ such that

$$f \circ f^{-1} = 1_B \qquad \text{and} \qquad f^{-1} \circ f = 1_A \,.$$

A morphism that has an inverse is an *isomorphism*, or *iso*. If there exists a pair of mutually inverse morphisms $f : A \to B$ and $f^{-1} : B \to A$ we say that the objects A and B are *isomorphic*, written $A \cong B$.

The notation f^{-1} is justified because an inverse, if it exists, is unique. A *left inverse* is a morphism $g : B \to A$ such that $g \circ f = 1_A$, and a *right inverse* is a morphism $g : B \to A$ such that $f \circ g = 1_B$. A left inverse is also called a *retraction*, whereas a right inverse is called a *section*.

Definition A.1.6. A *monomorphism*, or *mono*, is a morphism $f : A \to B$ that can be cancelled on the left: for all $g : C \to A$, $h : C \to A$,

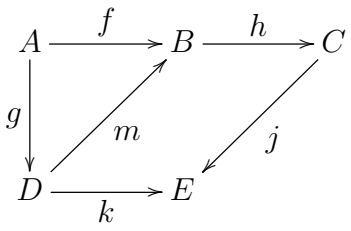
$$f \circ g = f \circ h \Rightarrow g = h \,.$$

An *epimorphism*, or *epi*, is a morphism $f : A \to B$ that can be cancelled on the right: for all $g : B \to C$, $h : B \to A$,

$$g \circ f = h \circ f \Rightarrow g = h \,.$$

In Set monomorphisms are the injective functions and epimorphisms are the surjective functions. Isomorphisms in Set are the bijective functions. Thus, in Set a morphism is iso if, and only if, it is both mono and epi. However, this example is misleading! In general, a morphism can be mono and epi without being an iso. For example, the non-identity morphism in the category consisting of two objects and one morphism between them is both epi and mono, but it has no inverse. A more interesting example of morphisms that are both epi and mono but are not iso occurs in the category Top of topological spaces and continuous maps, where not every continuous bijection is a homeomorphism.

A *diagram* of objects and morphisms is a directed graph whose vertices are objects of a category and edges are morphisms between them, for example:



Such a diagram is said to *commute* when the composition of morphisms along any two paths with the same beginning and end gives equal morphisms. Commutativity of the above diagram is equivalent to the following two equations:

$$f = m \circ g \ , \qquad\qquad k = j \circ h \circ m \ .$$

From these we can derive $k \circ g = j \circ h \circ f$ by a *diagram chase*.

A.2 Functors

Definition A.2.1. A *functor* $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of functions

$$F_0 : \mathcal{C}_0 \to \mathcal{D}_0 \qquad \text{and} \qquad F_1 : \mathcal{C}_1 \to \mathcal{D}_1$$

such that, for all $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$:

$$\begin{aligned} F_1 f : F_0 A \to F_0 B \ , \\ F_1(g \circ f) = (F_1 g) \circ (F_1 f) \ , \\ F_1(1_A) = 1_{F_0 A} \ . \end{aligned}$$

We usually write F for both F_0 and F_1.

A functor is thus a homomorphism of the category structure; note that it maps commutative diagrams to commutative diagrams because it preserves composition.

We may form the "category of categories" Cat whose objects are small categories and whose morphisms are functors. Composition of functors is composition of the corresponding functions, and the identity functor is one that is identity on objects and on morphisms. The category Cat is large but locally small.

Definition A.2.2. A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* when it is "locally injective on morphisms", in the sense that for all $f, g : A \to B$, if $Ff = Fg$ then $f = g$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* when it is "locally surjective on morphisms": for every $g : FA \to FB$ there exists $f : A \to B$ such that $g = Ff$.

We consider several examples of functors.

A.2.1 Functors between sets, monoids and posets

When sets, monoids, groups, and posets are regarded as categories, the functors turn out to be the *usual morphisms*, for example:

- A functor between sets S and T is a function from S to T.
- A functor between groups G and H is a group homomorphism from G to H.
- A functor between posets P and Q is a monotone function from P to Q.

Exercise A.2.3. Verify that the above claims are correct.

A.2.2 Forgetful functors

For categories of structures Group, Top, Graph, Poset, ..., there is a *forgetful* functor U which maps an object to the underlying set and a morphism to the underlying function. For example, the forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$ maps a group $(G, \cdot)$ to the set G and a group homomorphism $f : (G, \cdot) \to (H, \star)$ to the function $f : G \to H$.

There are also forgetful functors that forget only part of the structure, for example the forgetful functor $U : \mathsf{Ring} \to \mathsf{Group}$ which maps a ring $(R, +, \times)$ to the additive group $(R, +)$ and a ring homomorphism $f : (R, +_R, \cdot_S) \to (S, +_S, \cdot_S)$ to the group homomorphism $f : (R, +_R) \to (S, +_S)$. Note that there is another forgetful functor $U' : \mathsf{Ring} \to \mathsf{Mon}$ from rings to monoids.

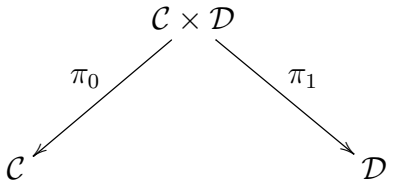
Exercise A.2.4. Show that taking the graph $\Gamma(f) = \left\{ \langle x, f(x) \rangle \;\middle|\; x \in A \right\}$ of a function $f : A \to B$ determines a functor $\Gamma : \mathsf{Set} \to \mathsf{Rel}$, from sets and functions to sets and relations, which is the identity on objects. Is this a forgetful functor?

A.3 Constructions of Categories and Functors

A.3.1 Product of categories

Given categories $\mathcal{C}$ and $\mathcal{D}$, we form the *product category* $\mathcal{C} \times \mathcal{D}$ whose objects are pairs of objects $\langle C, D \rangle$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$, and whose morphisms are pairs of morphisms $\langle f, g \rangle : \langle C, D \rangle \to \langle C', D' \rangle$ with $f : C \to C'$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$. Composition is given by $\langle f, g \rangle \circ \langle f', g' \rangle = \langle f \circ f', g \circ g' \rangle$.

There are evident *projection* functors



which act as indicated in the following diagrams:

Exercise A.3.1. Show that, for any categories $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, there are distinguished isos:

$$\mathbf{1} \times \mathbb{C} \cong \mathbb{C}$$
$$\mathbb{B} \times \mathbb{C} \cong \mathbb{C} \times \mathbb{B}$$
$$\mathbb{A} \times (\mathbb{B} \times \mathbb{C}) \cong (\mathbb{A} \times \mathbb{B}) \times \mathbb{C}$$

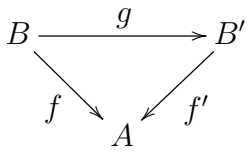
Does this make Cat a (commutative) monoid?

A.3.2 Slice categories

Given a category $\mathcal{C}$ and an object $A \in \mathcal{C}$, the *slice* category $\mathcal{C}/A$ has as objects, morphisms into A,

$$\begin{array}{c} B \\ \downarrow f \\ A \end{array} \tag{A.1}$$

and as morphisms, commutative diagrams over A:



(A.2)

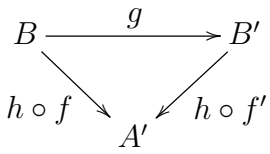
That is, a morphism from $f : B \to A$ to $f' : B' \to A$ is a morphism $g : B \to B'$ such that $f = f' \circ g$. Composition of morphisms in $\mathcal{C}/A$ is composition of morphisms in $\mathcal{C}$.

There is a forgetful functor $U_A : \mathcal{C}/A \to \mathcal{C}$ which maps an object (A.1) to its domain B, and a morphism (A.2) to the morphism $g : B \to B'$.

Furthermore, for each morphism $h : A \to A'$ in $\mathcal{C}$ there is a functor "composition by h",

$$\mathcal{C}/h : \mathcal{C}/A \to \mathcal{C}/A'$$

which maps an object (A.1) to the object $h \circ f : B \to A'$ and a morphisms (A.2) to the morphism

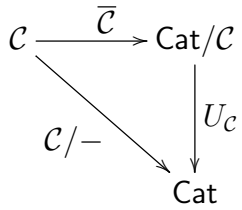


The construction of slice categories is itself a functor

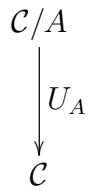
$$\mathcal{C}/- : \mathcal{C} \to \mathsf{Cat}$$

provided that $\mathcal{C}$ is small. This functor maps each $A \in \mathcal{C}$ to the category $\mathcal{C}/A$ and each morphism $h : A \to A'$ to the composition functor $\mathcal{C}/h : \mathcal{C}/A \to \mathcal{C}/A'$.

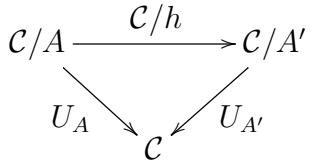
Since Cat is itself a category, we may form the slice category $\mathsf{Cat}/\mathcal{C}$ for any small category $\mathcal{C}$. The slice functor $\mathcal{C}/-$ then factors through the forgetful functor $U_{\mathcal{C}} : \mathsf{Cat}/\mathcal{C} \to \mathsf{Cat}$ via a functor $\overline{\mathcal{C}} : \mathcal{C} \to \mathsf{Cat}/\mathcal{C}$,



where for $A \in \mathcal{C}$, the object part $\overline{\mathcal{C}}A$ is



and for $h : A \to A'$ in $\mathcal{C}$, the morphism part $\overline{\mathcal{C}}h$ is



A.3.3 Arrow categories

Similar to the slice categories, an arrow category has arrows as objects, but without a fixed codomain. Given a category $\mathcal{C}$, the *arrow* category $\mathcal{C}^{\to}$ has as objects the morphisms of $\mathcal{C}$,

(A.3)

and as morphisms $f \to f'$ the commutative squares,

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow[g']{} & B'. \end{array} \tag{A.4}$$

That is, a morphism from $f : A \to B$ to $f' : A' \to B'$ is a pair of morphisms $g : A \to A'$ and $g' : B \to B'$ such that $g' \circ f = f' \circ g$. Composition of morphisms in $\mathcal{C}^{\to}$ is just componentwise composition of morphisms in $\mathcal{C}$.

There are two evident forgetful functors $U_1, U_2 : \mathcal{C}^{\to} \to \mathcal{C}$, given by the domain and codomain operations. (Can you find a common section for these?)

A.3.4 Opposite categories

For a category $\mathcal{C}$ the *opposite category* $\mathcal{C}^{\mathsf{op}}$ has the same objects as $\mathcal{C}$, but all the morphisms are turned around, that is, a morphism $f : A \to B$ in $\mathcal{C}^{\mathsf{op}}$ is a morphism $f : B \to A$ in $\mathcal{C}$. The identity arrows in $\mathcal{C}^{\mathsf{op}}$ are the same as in $\mathcal{C}$, but the order of composition is reversed. The opposite of the opposite of a category is clearly the original category.

A functor $F : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ is sometimes called a *contravariant functor* (from $\mathcal{C}$ to $\mathcal{D}$), and a functor $F : \mathcal{C} \to \mathcal{D}$ is a *covariant* functor.

For example, the opposite category of a preorder $(P, \leq)$ is the preorder P turned upside down, $(P, \geq)$.

Exercise A.3.2. Given a functor $F : \mathcal{C} \to \mathcal{D}$, can you define a functor $F^{\mathsf{op}} : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$ in such a way that $-^{\mathsf{op}}$ itself becomes a functor? On what category is it a functor?

A.3.5 Representable functors

Let $\mathcal{C}$ be a locally small category. Then for each pair of objects $A, B \in \mathcal{C}$ the collection of all morphisms $A \to B$ forms a set, written $\mathsf{Hom}_{\mathcal{C}}(A, B)$, $\mathsf{Hom}(A, B)$ or $\mathcal{C}(A, B)$. For every $A \in \mathcal{C}$ there is a functor

$$\mathcal{C}(A, -) : \mathcal{C} \to \mathsf{Set}$$

defined by

$$\begin{aligned} \mathcal{C}(A, B) &= \big\{ f \in \mathcal{C}_1 \;\big|\; f : A \to B \big\} \\ \mathcal{C}(A, g) &: f \mapsto g \circ f \end{aligned}$$

where $B \in \mathcal{C}$ and $g : B \to C$. In words, $\mathcal{C}(A, g)$ is composition by g. This is indeed a functor because, for any morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \tag{A.5}$$

we have

$$\mathcal{C}(A, h \circ g)f = (h \circ g) \circ f = h \circ (g \circ f) = \mathcal{C}(A, h)(\mathcal{C}(A, g)f) \ ,$$

and $\mathcal{C}(A, 1_B)f = 1_A \circ f = f = 1_{\mathcal{C}(A,B)}f$.

We may also ask whether $\mathcal{C}(-, B)$ is a functor. If we define its action on morphisms to be precomposition,

$$\mathcal{C}(f, B) : g \mapsto g \circ f \ ,$$

it becomes a *contravariant* functor,

$$\mathcal{C}(-, B) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set} \ .$$

The contravariance is a consequence of precomposition; for morphisms (A.5) we have

$$\mathcal{C}(g \circ f, D)h = h \circ (g \circ f) = (h \circ g) \circ f = \mathcal{C}(f, D)(\mathcal{C}(g, D)h) \ .$$

A functor of the form $\mathcal{C}(A, -)$ is a *(covariant) representable functor*, and a functor of the form $\mathcal{C}(-, B)$ is a *(contravariant) representable functor*.

It follows that the hom-set is a functor

$$\mathcal{C}(-, -) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}$$

which maps a pair of objects $A, B \in \mathcal{C}$ to the set $\mathcal{C}(A, B)$ of morphisms from A to B, and it maps a pair of morphisms $f : A' \to A$, $g : B \to B'$ in $\mathcal{C}$ to the function

$$\mathcal{C}(f, g) : \mathcal{C}(A, B) \to \mathcal{C}(A', B')$$

defined by

$$\mathcal{C}(f, g) : h \mapsto g \circ h \circ f \ .$$

(Why does it follow that this is a functor?)

A.3.6 Group actions

A group $(G, \cdot)$ is a category with one object $\star$ and elements of G as the morphisms. Thus, a functor $F : G \to \mathsf{Set}$ is given by a set $F\star = S$ and for each $a \in G$ a function $Fa : S \to S$ such that, for all $x \in S$, $a, b \in G$,

$$(Fe)x = x \ , \qquad\qquad (F(a \cdot b))x = (Fa)((Fb)x) \ .$$

Here e is the unit element of G. If we write $a \cdot x$ instead of $(Fa)x$, the above two equations become the familiar laws for a *left group action on the set* S:

$$e \cdot x = x \ , \qquad\qquad (a \cdot b) \cdot x = a \cdot (b \cdot x) \ .$$

Exercise A.3.3. A *right group action* by a group $(G, \cdot)$ on a set S is an operation $\cdot : S \times G \to S$ that satisfies, for all $x \in S$, $a, b \in G$,

$$x \cdot e = x \ , \qquad\qquad x \cdot (a \cdot b) = (x \cdot a) \cdot b \ .$$

Exhibit right group actions as functors.

A.4 Natural Transformations and Functor Categories

Definition A.4.1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \Longrightarrow G$ from F to G is a map $\eta : \mathcal{C}_0 \to \mathcal{D}_1$ which assigns to every object $A \in \mathcal{C}$ a morphism $\eta_A : FA \to GA$, called the *component of* η *at* A, such that for every $f : A \to B$ in $\mathcal{C}$ we have $\eta_B \circ Ff = Gf \circ \eta_A$, i.e., the following diagram in $\mathcal{D}$ commutes:

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[\eta_B]{} & GB \end{array}$$

A simple example is given by the "twist" isomorphism $t : A \times B \to B \times A$ (in Set). Given any maps $f : A \to A'$ and $g : B \to B'$, there is a commutative square:

$$\begin{array}{ccc} A \times B & \xrightarrow{t_{A,B}} & B \times A \\ {\scriptstyle f \times g}\downarrow & & \downarrow{\scriptstyle g \times f} \\ A' \times B' & \xrightarrow[t_{A',B'}]{} & B' \times A' \end{array}$$

Thus naturality means that the two *functors* $F(X, Y) = X \times Y$ and $G(X, Y) = Y \times X$ are related to each other (by $t : F \to G$), and not simply their individual values $A \times B$ and $B \times A$. As a further example of a natural transformation, consider groups G and H as categories and two homomorphisms $f, g : G \to H$ as functors between them. A natural transformation $\eta : f \Longrightarrow g$ is given by a single element $\eta_\star = b \in H$ such that, for every $a \in G$, the following diagram commutes:

$$\begin{array}{ccc} \star & \xrightarrow{b} & \star \\ {\scriptstyle fa}\downarrow & & \downarrow{\scriptstyle ga} \\ \star & \xrightarrow[b]{} & \star \end{array}$$

This means that $b \cdot fa = (ga) \cdot b$, that is $ga = b \cdot (fa) \cdot b^{-1}$. In other words, a natural transformation $f \Longrightarrow g$ is a *conjugation* operation $b^{-1} \cdot - \cdot b$ which transforms f into g.

For every functor $F : \mathcal{C} \to \mathcal{D}$ there exists the *identity transformation* $\mathbf{1}_F : F \Longrightarrow F$ defined by $(\mathbf{1}_F)_A = \mathbf{1}_A$. If $\eta : F \Longrightarrow G$ and $\theta : G \Longrightarrow H$ are natural transformations, then their composition $\theta \circ \eta : F \Longrightarrow H$, defined by $(\theta \circ \eta)_A = \theta_A \circ \eta_A$ is also a natural transformation. Composition of natural transformations is associative because it is composition in the codomain category $\mathcal{D}$. This leads to the definition of functor categories.

Definition A.4.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *functor category* $\mathcal{D}^{\mathcal{C}}$ is the category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations between them.

A functor category may be quite large, too large in fact. In order to avoid problems with size we normally require $\mathcal{C}$ to be a locally small category. The "hom-class" of all natural transformations $F \Longrightarrow G$ is usually written as

$$\mathsf{Nat}(F, G)$$

instead of the more awkward $\mathsf{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$.

Suppose we have functors F, G, and H with a natural transformation $\theta : G \Longrightarrow H$, as in the following diagram:

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \underset{H}{\overset{G}{\Downarrow \theta}} \mathbb{E}$$

Then we can form a natural transformation $\theta \circ F : G \circ F \Longrightarrow H \circ F$ whose component at $A \in \mathcal{C}$ is $(\theta \circ F)_A = \theta_{FA}$.

Similarly, if we have functors and a natural transformation

$$\mathcal{C} \underset{H}{\overset{G}{\Downarrow \theta}} \mathcal{D} \xrightarrow{F} \mathbb{E}$$

we can form a natural transformation $(F \circ \theta) : F \circ G \Longrightarrow F \circ H$ whose component at $A \in \mathcal{C}$ is $(F \circ \theta)_A = F\theta_A$. These operations are known as *whiskering*.

A *natural isomorphism* is an isomorphism in a functor category. Thus, if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ are two functors, a natural isomorphism between them is a natural transformation $\eta : F \Longrightarrow G$ whose components are isomorphisms. In this case, the inverse natural transformation $\eta^{-1} : G \Longrightarrow F$ is given by $(\eta^{-1})_A = (\eta_A)^{-1}$. We write $F \cong G$ when F and G are naturally isomorphic.

The definition of natural transformations is motivated in part by the fact that, for any small categories $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, we have

$$\mathsf{Cat}(\mathbb{A} \times \mathbb{B}, \mathbb{C}) \cong \mathsf{Cat}(\mathbb{A}, \mathbb{C}^{\mathbb{B}}) \; . \tag{A.6}$$

The isomorphism takes a functor $F : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ to the functor $\widetilde{F} : \mathbb{A} \to \mathbb{C}^{\mathbb{B}}$ defined on objects $A \in \mathbb{A}$, $B \in \mathbb{B}$ by

$$(\widetilde{F}A)B = F\langle A, B\rangle$$

and on a morphism $f : A \to A'$ by

$$(\widetilde{F}f)_B = F\langle f, 1_B\rangle \; .$$

The functor $\widetilde{F}$ is called the *transpose* of F.

The inverse isomorphism takes a functor $G : \mathbb{A} \to \mathbb{C}^{\mathbb{B}}$ to the functor $\widetilde{G} : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$, defined on objects by

$$\widetilde{G}\langle A, B\rangle = (GA)B$$

and on a morphism $\langle f, g\rangle : A \times B \to A' \times B'$ by

$$\widetilde{G}\langle f, g\rangle = (Gf)_{B'} \circ (GA)g = (GA')g \circ (Gf)_B \ ,$$

where the last equation holds by naturality of Gf:

$$\begin{array}{ccc} (GA)B & \xrightarrow{(Gf)_B} & (GA')B \\ {\scriptstyle (GA)g}\downarrow & & \downarrow{\scriptstyle (GA')g} \\ (GA)B' & \xrightarrow[(Gf)_{B'}]{} & (GA')B' \end{array}$$

A.4.1 Directed graphs as a functor category

Recall that a *directed graph* G is given by a set of vertices G_V and a set of edges G_E. Each edge $e \in G_E$ has a uniquely determined *source* $\mathsf{src}_G\, e \in G_V$ and *target* $\mathsf{trg}_G\, e \in G_V$. We write $e : a \to b$ when a is the source and b is the target of e. A *graph homomorphism* $\phi : G \to H$ is a pair of functions $\phi_0 : G_V \to H_V$ and $\phi_1 : G_E \to H_E$, where we usually write ϕ for both ϕ_0 and ϕ_1, such that whenever $e : a \to b$ then $\phi_1 e : \phi_0 a \to \phi_0 b$. The category of directed graphs and graph homomorphisms is denoted by Graph.

Now let $\cdot \rightrightarrows \cdot$ be the category with two objects and two parallel morphisms, depicted by the following "sketch":

$$E \underset{t}{\overset{s}{\rightrightarrows}} V$$

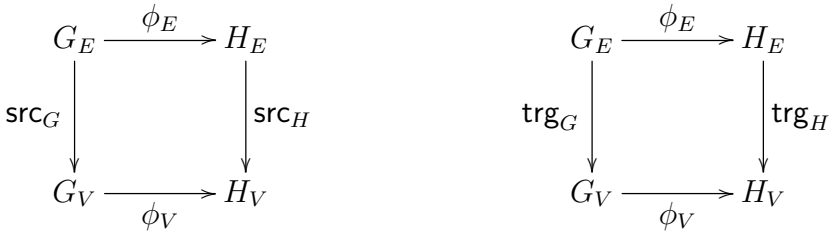
An object of the functor category $\mathsf{Set}^{\cdot\rightrightarrows\cdot}$ is a functor $G : (\cdot \rightrightarrows \cdot) \to \mathsf{Set}$, which consists of two sets GE and GV and two functions $Gs : GE \to GV$ and $Gt : GE \to GV$. But this is precisely a directed graph whose vertices are GV, the edges are GE, the source of $e \in GE$ is $(Gs)e$ and the target is $(Gt)e$. Conversely, any directed graph G is a functor $G : (\cdot \rightrightarrows \cdot) \to \mathsf{Set}$, defined by

$$GE = G_E \ , \qquad GV = G_V \ , \qquad Gs = \mathsf{src}_G \ , \qquad Gt = \mathsf{trg}_G \ .$$

Now category theory begins to show its worth, for the morphisms in $\mathsf{Set}^{\cdot\rightrightarrows\cdot}$ are precisely the graph homomorphisms. Indeed, a natural transformation $\phi : G \Longrightarrow H$ between graphs is a pair of functions,

$$\phi_E : G_E \to H_E \qquad \text{and} \qquad \phi_V : G_V \to H_V$$

whose naturality is expressed by the commutativity of the following two diagrams:



This is precisely the requirement that $e : a \to b$ implies $\phi_E e : \phi_V a \to \phi_V b$. Thus, in sum, we have,

$$\mathsf{Graph} = \mathsf{Set}^{\cdot \rightrightarrows \cdot}.$$

Exercise A.4.3. Exhibit the arrow category $\mathcal{C}^{\to}$ and the category of group actions $\mathsf{Set}(G)$ as functor categories.

A.4.2 The Yoneda embedding

The example $\mathsf{Graph} = \mathsf{Set}^{\cdot \rightrightarrows \cdot}$ leads one to wonder which categories $\mathcal{C}$ can be represented as functor categories $\mathsf{Set}^{\mathcal{D}}$ for a suitably chosen $\mathcal{D}$ or, when that is not possible, at least as full subcategories of $\mathsf{Set}^{\mathcal{D}}$.

For a locally small category $\mathcal{C}$, there is the hom-functor

$$\mathcal{C}(-,-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}\ .$$

By transposing as in (A.6) we obtain the functor

$$\mathsf{y} : \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$$

which maps an object $A \in \mathcal{C}$ to the representable functor

$$\mathsf{y}A = \mathcal{C}(-, A) : B \mapsto \mathcal{C}(B, A)$$

and a morphism $f : A \to A'$ in $\mathcal{C}$ to the natural transformation $\mathsf{y}f : \mathsf{y}A \Longrightarrow \mathsf{y}A'$ whose component at B is

$$(\mathsf{y}f)_B = \mathcal{C}(B, f) : g \mapsto f \circ g\ .$$

This functor y is called the *Yoneda embedding*.

Exercise A.4.4. Show that this *is* a functor.

Theorem A.4.5 (Yoneda embedding)**.** *For any locally small category $\mathcal{C}$ the Yoneda embedding*

$$\mathsf{y} : \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$$

is full and faithful and injective on objects. Therefore, $\mathcal{C}$ is a full subcategory of $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.

The proof of the theorem uses the famous Yoneda Lemma.

Lemma A.4.6 (Yoneda)**.** *Every functor $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ is naturally isomorphic to the functor $\mathsf{Nat}(\mathsf{y}-, F)$. That is, for every $A \in \mathcal{C}$,*

$$\mathsf{Nat}(\mathsf{y}A, F) \cong FA \ ,$$

and this isomorphism is natural in A.

Indeed, the displayed isomorphism is also natural in F.

Proof. The desired natural isomorphism θ_A maps a natural transformation $\eta \in \mathsf{Nat}(\mathsf{y}A, F)$ to $\eta_A \mathsf{1}_A$. The inverse ${\theta_A}^{-1}$ maps an element $x \in FA$ to the natural transformation $({\theta_A}^{-1}x)$ whose component at B maps $f \in \mathcal{C}(B, A)$ to $(Ff)x$. To summarize, for $\eta : \mathcal{C}(-, A) \Longrightarrow F$, $x \in FA$ and $f \in \mathcal{C}(B, A)$, we have

$$\begin{aligned} &\theta_A : \mathsf{Nat}(\mathsf{y}A, F) \to FA \ , && {\theta_A}^{-1} : FA \to \mathsf{Nat}(\mathsf{y}A, F) \ , \\ &\theta_A \eta = \eta_A \mathsf{1}_A \ , && ({\theta_A}^{-1}x)_B f = (Ff)x \ . \end{aligned}$$

To see that θ_A and ${\theta_A}^{-1}$ really are inverses of each other, observe that

$$\theta_A({\theta_A}^{-1}x) = ({\theta_A}^{-1}x)_A \mathsf{1}_A = (F\mathsf{1}_A)x = \mathsf{1}_{FA}x = x \ ,$$

and also

$$({\theta_A}^{-1}(\theta_A \eta))_B f = (Ff)(\theta_A \eta) = (Ff)(\eta_A \mathsf{1}_A) = \eta_B(\mathsf{1}_A \circ f) = \eta_B f \ ,$$

where the third equality holds by the following naturality square for η:

$$\begin{array}{ccc} \mathcal{C}(A, A) & \xrightarrow{\eta_A} & FA \\ {\scriptstyle \mathcal{C}(f, A)}\downarrow & & \downarrow{\scriptstyle Ff} \\ \mathcal{C}(B, A) & \xrightarrow[\eta_B]{} & FB \end{array}$$

It remains to check that θ is natural, which amounts to establishing the commutativity of the following diagram, with $g : A \to A'$:

$$\begin{array}{ccc} \mathsf{Nat}(\mathsf{y}A, F) & \xrightarrow{\theta_A} & FA \\ {\scriptstyle \mathsf{Nat}(\mathsf{y}g, F)}\uparrow & & \uparrow{\scriptstyle Fg} \\ \mathsf{Nat}(\mathsf{y}A', F) & \xrightarrow[\theta_{A'}]{} & FA' \end{array}$$

The diagram is commutative because, for any $\eta : \mathsf{y}A' \Longrightarrow F$,

$$(Fg)(\theta_{A'}\eta) = (Fg)(\eta_{A'}1_{A'}) = \eta_A(1_{A'} \circ g) = \\ \eta_A(g \circ 1_A) = (\mathsf{Nat}(\mathsf{y}g, F)\eta)_A 1_A = \theta_A(\mathsf{Nat}(\mathsf{y}g, F)\eta) \ ,$$

where the second equality is justified by naturality of η. □

Proof of Theorem A.4.5. That the Yoneda embedding is full and faithful means that for all $A, B \in \mathcal{C}$ the map

$$\mathsf{y} : \mathcal{C}(A, B) \to \mathsf{Nat}(\mathsf{y}A, \mathsf{y}B)$$

which maps $f : A \to B$ to $\mathsf{y}f : \mathsf{y}A \Longrightarrow \mathsf{y}B$ is an isomorphism. But this is just the Yoneda Lemma applied to the case $F = \mathsf{y}B$. Indeed, with notation as in the proof of the Yoneda Lemma and $g : C \to A$, we see that the isomorphism

$$\theta_A^{-1} : \mathcal{C}(A, B) = (\mathsf{y}B)A \to \mathsf{Nat}(\mathsf{y}A, \mathsf{y}B)$$

is in fact y:

$$(\theta_A{}^{-1} f)_C g = ((\mathsf{y}A)g)f = f \circ g = (\mathsf{y}f)_C g \ .$$

Furthermore, if $\mathsf{y}A = \mathsf{y}B$ then $1_A \in \mathcal{C}(A, A) = (\mathsf{y}A)A = (\mathsf{y}B)A = \mathcal{C}(B, A)$ which can only happen if $A = B$. Therefore, y is injective on objects. □

The following corollary is often useful.

Corollary A.4.7. *For $A, B \in \mathcal{C}$, $A \cong B$ if, and only if, $\mathsf{y}A \cong \mathsf{y}B$ in $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.*

Proof. Every functor preserves isomorphisms, and a full and faithful one also reflects them. (A functor $F : \mathcal{C} \to \mathcal{D}$ is said to *reflect* isomorphisms when $Ff : FA \to FB$ being an isomorphisms implies that $f : A \to B$ is an isomorphism.) □

Exercise A.4.8. Prove that a full and faithful functor reflects isomorphisms.

Functor categories $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ are important enough to deserve a name. They are called *presheaf categories*, and a functor $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ is called a *presheaf* on $\mathcal{C}$. We also use the notation $\widehat{\mathcal{C}} = \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.

A.4.3 Equivalence of categories

An isomorphism of categories $\mathcal{C}$ and $\mathcal{D}$ in Cat consists of functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

such that $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{\mathcal{D}}$. This is often too restrictive a notion. A more general notion which replaces the above identities with natural isomorphisms is more useful.

Definition A.4.9. An *equivalence of categories* is a pair of functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

such that there are natural isomorphisms

$$G \circ F \cong \mathsf{1}_{\mathcal{C}} \qquad \text{and} \qquad F \circ G \cong \mathsf{1}_{\mathcal{D}} \,.$$

We say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent categories* and write $\mathcal{C} \simeq \mathcal{D}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence functor* if there exists $G : \mathcal{D} \to \mathcal{C}$ such that F and G form an equivalence.

The point of equivalence of categories is that it preserves almost all categorical properties, but ignores those concepts that are not of interest from a categorical point of view, such as identity of objects.

The following proposition requires the Axiom of Choice as stated. However, in many specific cases a canonical choice can be made without appeal to that axiom.

Proposition A.4.10. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence functor if, and only if, F is full and faithful, and* essentially surjective *on objects, meaning that for every $B \in \mathcal{D}$ there exists $A \in \mathcal{C}$ such that $FA \cong B$.*

Proof. It is easily seen that the conditions are necessary, so we only show they are sufficient. Suppose $F : \mathcal{C} \to \mathcal{D}$ is full and faithful, and essentially surjective on objects. For each $B \in \mathcal{D}$, *choose* an object $GB \in \mathcal{C}$ and an isomorphism $\eta_B : F(GB) \to B$. If $f : B \to C$ is a morphism in $\mathcal{D}$, let $Gf : GB \to GC$ be the unique morphism in $\mathcal{C}$ for which

$$F(Gf) = {\eta_C}^{-1} \circ f \circ \eta_B \,. \tag{A.7}$$

Such a unique morphism exists because F is full and faithful. This defines a functor $G : \mathcal{D} \to \mathcal{C}$, as can be easily checked. In addition, (A.7) ensures that η is a natural isomorphism $F \circ G \Longrightarrow \mathsf{1}_{\mathcal{D}}$.

It remains to show that $G \circ F \cong \mathsf{1}_{\mathcal{C}}$. For $A \in \mathcal{C}$, let $\theta_A : G(FA) \to A$ be the unique morphism such that $F\theta_A = \eta_{FA}$. Naturality of θ_A follows from functoriality of F and naturality of η. Because F reflects isomorphisms, θ_A is an isomorphism for every A. □

Example A.4.11. As an example of equivalence of categories we consider the category of sets and partial functions and the category of pointed sets.

A *partial function* $f : A \rightharpoonup B$ is a function defined on a subset $\mathsf{supp}\, f \subseteq A$, called the *support*[3] of f, and taking values in B. Composition of partial functions $f : A \rightharpoonup B$ and $g : B \rightharpoonup C$ is the partial function $g \circ f : A \rightharpoonup C$ defined by

$$\mathsf{supp}\,(g \circ f) = \left\{ x \in A \;\middle|\; x \in \mathsf{supp}\, f \wedge fx \in \mathsf{supp}\, g \right\}$$
$$(g \circ f)x = g(fx) \quad \text{for } x \in \mathsf{supp}\,(g \circ f)$$

[3]The support of a partial function $f : A \rightharpoonup B$ is usually called its *domain*, but this terminology conflicts with A being the domain of f as a morphism.

Composition of partial functions is associative. This way we obtain a category Par of sets and partial functions.

A *pointed set* (A, a) is a set A together with an element $a \in A$. A *pointed function* $f : (A, a) \to (B, b)$ between pointed sets is a function $f : A \to B$ such that $fa = b$. The category $\mathsf{Set}_\bullet$ consists of pointed sets and pointed functions.

The categories Par and $\mathsf{Set}_\bullet$ are equivalent. The equivalence functor $F : \mathsf{Set}_\bullet \to \mathsf{Par}$ maps a pointed set (A, a) to the set $F(A, a) = A \setminus \{a\}$, and a pointed function $f : (A, a) \to (B, b)$ to the partial function $Ff : F(A, a) \rightharpoonup F(B, b)$ defined by

$$\mathsf{supp}\,(Ff) = \left\{x \in A \;\middle|\; fx \neq b\right\} , \qquad\qquad (Ff)x = fx .$$

The inverse equivalence functor $G : \mathsf{Par} \to \mathsf{Set}_\bullet$ maps a set $A \in \mathsf{Par}$ to the pointed set $GA = (A + \{\bot_A\}, \bot_A)$, where $\bot_A$ is an element that does not belong to A. A partial function $f : A \rightharpoonup B$ is mapped to the pointed function $Gf : GA \to GB$ defined by

$$(Gf)x = \begin{cases} fx & \text{if } x \in \mathsf{supp}\, f \\ \bot_B & \text{otherwise} . \end{cases}$$

A good way to think about the "bottom" point $\bot_A$ is as a special "undefined value". Let us look at the composition of F and G on objects:

$$\begin{aligned} G(F(A, a)) = G(A \setminus \{a\}) &= ((A \setminus \{a\}) + \bot_A, \bot_A) \cong (A, a) . \\ F(GA) = F(A + \{\bot_A\}, \bot_A) &= (A + \{\bot_A\}) \setminus \{\bot_A\} = A . \end{aligned}$$

The isomorphism $G(F(A, a)) \cong (A, a)$ is easily seen to be natural.

Example A.4.12. Another example of an equivalence of categories arises when we take the poset reflection of a preorder. Let $(P, \leq)$ be a preorder, If we think of P as a category, then $a, b \in P$ are isomorphic, when $a \leq b$ and $b \leq a$. Isomorphism $\cong$ is an equivalence relation, therefore we may form the quotient set $P/\!\cong$. The set $P/\!\cong$ is a poset for the order relation $\sqsubseteq$ defined by

$$[a] \sqsubseteq [b] \iff a \leq b .$$

Here $[a]$ denotes the equivalence class of a. We call $(P/\!\cong, \sqsubseteq)$ the *poset reflection* of P. The quotient map $q : P \to P/\!\cong$ is a functor when P and $P/\!\cong$ are viewed as categories. By Proposition A.4.10, q is an equivalence functor. Trivially, it is faithful and surjective on objects. It is also full because $qa \sqsubseteq qb$ in $P/\!\cong$ implies $a \leq b$ in P.

A.5 Adjoint Functors

The notion of adjunction is perhaps the most important concept revealed by category theory. It is a fundamental logical and mathematical concept that occurs everywhere and often marks an important and interesting connection between two constructions of interest. In logic, adjoint functors are pervasive, although this is only recognizable through the lens of category theory.

A.5.1 Adjoint maps between preorders

Let us begin with a simple situation. We have already seen that a preorder $(P, \leq)$ is a category in which there is at most one morphism between any two objects. A functor between preorders is a monotone map. Suppose we have preorders P and Q with monotone maps back and forth,

$$P \underset{g}{\overset{f}{\rightleftarrows}} Q\,.$$

We say that f and g are *adjoint*, and write $f \dashv g$, when for all $x \in P$, $y \in Q$,

$$fx \leq y \iff x \leq gy\,. \tag{A.8}$$

Note that adjointness is *not* a symmetric relation. The map f is the *left adjoint* and g is the *right adjoint* (note their positions with respect to $\leq$).

Equivalence (A.8) is more conveniently displayed as

$$\frac{fx \leq y}{\;\;x \leq gy\;\;}$$

The double line indicates the fact that this is a two-way rule: the top line implies the bottom line, and vice versa.

Let us consider two examples.

Conjunction is adjoint to implication Consider a propositional calculus with logical operations of conjunction $\wedge$ and implication $\Rightarrow$ (perhaps among others). The formulas of this calculus are built from variables x_0, x_1, x_2, ..., the truth values $\bot$ and $\top$, and the logical connectives $\wedge, \Rightarrow, \ldots$. The logical rules are given in natural deduction style:

$$\frac{}{\top} \qquad \frac{\bot}{A} \qquad \frac{A \qquad B}{A \wedge B} \qquad \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$$\frac{A \Rightarrow B \qquad A}{B} \qquad\qquad \frac{\begin{array}{c}[u : A]\\ \vdots \\ B\end{array}}{A \Rightarrow B}\, u$$

For example, we read the inference rules for $\Rightarrow$ as, respectively, "from $A \Rightarrow B$ and A we infer B" and "if from assumption A we infer B, then (without any assumptions) we infer $A \Rightarrow B$". Discharged assumptions are indicated by enclosing them in brackets, along with a label $[u : A]$ for the assumption, which is recorded along with the rule that discharges it, as above.

Logical entailment $\vdash$ between formulas of the propositional calculus is the relation $A \vdash B$ which holds if, and only if, from assuming A we can infer B (by using only the inference rules of the calculus). It is trivially the case that $A \vdash A$, and also

$$\text{if } A \vdash B \text{ and } B \vdash C \text{ then } A \vdash C \ .$$

In other words, $\vdash$ is a reflexive and transitive relation on the set P of all propositional formulas, so that $(\mathsf{P}, \vdash)$ is a preorder.

Let A be a propositional formula. Define $f : \mathsf{P} \to \mathsf{P}$ and $g : \mathsf{P} \to \mathsf{P}$ to be the maps

$$fB = (A \wedge B) \ , \qquad\qquad gB = (A \Rightarrow B) \ .$$

To see that the maps f and g are functors we need to show they respect entailment. Indeed, if $B \vdash B'$ then $A \wedge B \vdash A \wedge B'$ and $A \Rightarrow B \vdash A \Rightarrow B'$ by the following two derivations.

$$\dfrac{\dfrac{A \wedge B}{A} \qquad \begin{array}{c} \dfrac{A \wedge B}{B} \\ \vdots \\ B' \end{array}}{A \wedge B'} \qquad\qquad \dfrac{A \Rightarrow B \qquad [u : A]}{\begin{array}{c} B \\ \vdots \\ B' \end{array}}\ u$$

$$A \Rightarrow B' \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad A \Rightarrow B'$$

We claim that $f \dashv g$. For this we need to prove that $A \wedge B \vdash C$ if, and only if, $B \vdash A \Rightarrow C$. The following two derivations establish the required equivalence.

$$\dfrac{\dfrac{[u : A] \qquad B}{A \wedge B}}{\begin{array}{c} \vdots \\ C \end{array}}\ u \qquad\qquad \dfrac{\dfrac{A \wedge B}{B}}{\dfrac{\begin{array}{c} \vdots \\ A \Rightarrow C \end{array} \qquad \dfrac{A \wedge B}{A}}{C}}$$

$$A \Rightarrow C \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad C$$

Therefore, *conjunction is left adjoint to implication.*

Topological interior as an adjoint Recall that a *topological space* $(X, \mathcal{O}X)$ is a set X together with a family $\mathcal{O}X \subseteq \mathcal{P}X$ of subsets of X which contains $\emptyset$ and X, and is closed under finite intersections and arbitrary unions. The elements of $\mathcal{O}X$ are called the *open sets*.

The *topological interior* of a subset $S \subseteq X$ is the largest open set contained in S, namely,

$$\mathsf{int}\, S = \bigcup \big\{ U \in \mathcal{O}X \;\big|\; U \subseteq S \big\} \ .$$

Both $\mathcal{O}X$ and $\mathcal{P}X$ are posets ordered by subset inclusion. The inclusion $i : \mathcal{O}X \to \mathcal{P}X$ is thus a monotone map, and so indeed is the interior $\mathsf{int} : \mathcal{P}X \to \mathcal{O}X$, as follows immediately from its construction. So we have:

$$\mathcal{O}X \underset{\mathsf{int}}{\overset{i}{\rightleftarrows}} \mathcal{P}X$$

Moreover, for $U \in \mathcal{O}X$ and $S \in \mathcal{P}X$ we plainly also have

$$\frac{iU \subseteq S}{U \subseteq \mathsf{int}\, S}$$

since $\mathsf{int}\, S$ is the largest open set contained in S. Thus *topological interior is right adjoint* to the inclusion of $\mathcal{O}X$ into $\mathcal{P}X$.

A.5.2 Adjoint functors

Let us now generalize the notion of adjoint monotone maps from posets to the situation

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

with arbitrary categories and functors. For monotone maps $f \dashv g$, the adjunction condition is a bijection

$$\frac{fx \to y}{x \to gy}$$

between morphisms of the form $fx \to y$ and morphisms of the form $x \to gy$. This is the notion that generalizes the special case; for any $A \in \mathcal{C}$, $B \in \mathcal{D}$ we require a bijection between the sets $\mathcal{D}(FA, B)$ and $\mathcal{C}(A, GB)$:

$$\frac{FA \to B}{A \to GB}$$

Definition A.5.1. An *adjunction* $F \dashv G$ between the functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

is a natural isomorphism θ between functors

$$\mathcal{D}(F-,-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{D} \to \mathsf{Set} \qquad \text{and} \qquad \mathcal{C}(-,G-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{D} \to \mathsf{Set}\,.$$

This means that for every $A \in \mathcal{C}$ and $B \in \mathcal{D}$ there is a bijection

$$\theta_{A,B} : \mathcal{D}(FA, B) \cong \mathcal{C}(A, GB)\,,$$

and naturality of θ means that for $f : A' \to A$ in $\mathcal{C}$ and $g : B \to B'$ in $\mathcal{D}$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}(FA, B) & \xrightarrow{\theta_{A,B}} & \mathcal{D}(A, GB) \\ \Big\downarrow{\scriptstyle \mathcal{D}(Ff, g)} & & \Big\downarrow{\scriptstyle \mathcal{C}(f, Gg)} \\ \mathcal{D}(FA', B') & \xrightarrow[\theta_{A',B'}]{} & \mathcal{C}(A', GB') \end{array}$$

Equivalently, for every $h : FA \to B$ in $\mathcal{D}$,

$$Gg \circ (\theta_{A,B} h) \circ f = \theta_{A',B'}(g \circ h \circ Ff) \ .$$

We say that F is the *left adjoint* and G is the *right adjoint*.

We have already seen examples of adjoint functors. For any category $\mathbb{B}$ we have functors $(-) \times \mathbb{B}$ and $(-)^{\mathbb{B}}$ from Cat to Cat. Recall the isomorphism (A.6),

$$\mathsf{Cat}(\mathbb{A} \times \mathbb{B}, \mathbb{C}) \cong \mathsf{Cat}(\mathbb{A}, \mathbb{C}^{\mathbb{B}}) \ .$$

This isomorphism is in fact natural in $\mathbb{A}$ and $\mathbb{C}$, so that

$$(-) \times \mathbb{B} \dashv (-)^{\mathbb{B}} \ .$$

Similarly, for any set $B \in \mathsf{Set}$ there are functors

$$(-) \times B : \mathsf{Set} \to \mathsf{Set} \ , \qquad\qquad (-)^B : \mathsf{Set} \to \mathsf{Set} \ ,$$

where $A \times B$ is the cartesian product of A and B, and C^B is the set of all functions from B to C. For morphisms, $f \times B = f \times 1_B$ and $f^B = f \circ (-)$. We then indeed have a natural isomorphism, for all $A, C \in \mathsf{Set}$,

$$\mathsf{Set}(A \times B, C) \cong \mathsf{Set}(A, C^B) \ ,$$

which maps a function $f : A \times B \to C$ to the function $(\widetilde{f}x)y = f\langle x, y\rangle$. Therefore,

$$(-) \times B \dashv (-)^B \ .$$

Exercise A.5.2. Verify that the definition (A.8) of adjoint monotone maps between preorders is a special case of Definition A.5.1. What happened to the naturality condition?

For another example, consider the forgetful functor

$$U : \mathsf{Cat} \to \mathsf{Graph} \ ,$$

which maps a category to the underlying directed graph. It has a left adjoint $P \dashv U$. The functor P is the *free* construction of a category from a graph; it maps a graph G to the *category of paths* $P(G)$. The objects of $P(G)$ are the vertices of G. The morphisms of $P(G)$ are the finite paths

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} v_n$$

of edges in G, composition is concatenation of paths, and the identity morphism on a vertex v is the empty path starting and ending at v.

By using the Yoneda Lemma we can easily prove that adjoints are unique up to natural isomorphism.

Proposition A.5.3. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors, with $F \dashv G$. If also $G' : \mathcal{D} \to \mathcal{C}$ with $F \dashv G'$, then $G \cong G'$.*

Proof. Since the Yoneda embedding is full and faithful, we have $GB \cong G'B$ if, and only if, $\mathcal{C}(-, GB) \cong \mathcal{C}(-, G'B)$. But this indeed holds, because, for any $A \in \mathcal{C}$, we have

$$\mathcal{C}(A, GB) \cong \mathcal{D}(FA, B) \cong \mathcal{C}(A, G'B) \ ,$$

naturally in A. □

Left adjoints are of course also unique up to isomorphism, by duality.

A.5.3 The unit of an adjunction

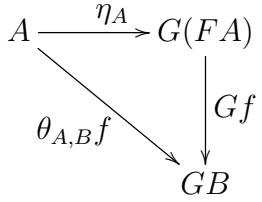
Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors, $F \dashv G$, and let $\theta : \mathcal{D}(F-, -) \to \mathcal{C}(-, G-)$ be the natural isomorphism witnessing the adjunction. For any object $A \in \mathcal{C}$ there is a distinguished morphism $\eta_A = \theta_{A,FA} 1_{FA} : A \to G(FA)$,

$$\frac{1_{FA} : FA \to FA}{\eta_A : A \to G(FA)}$$

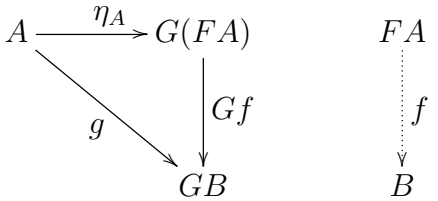
Since θ is natural in A, we have a natural transformation $\eta : 1_{\mathcal{C}} \Longrightarrow G \circ F$, which is called the *unit of the adjunction* $F \dashv G$. In fact, we can recover θ from η as follows. For $f : FA \to B$, we have

$$\theta_{A,B} f = \theta_{A,B}(f \circ 1_{FA}) = Gf \circ \theta_{A,FA}(1_{FA}) = Gf \circ \eta_A \ ,$$

where we used naturality of θ in the second step. Schematically, given any $f : FA \to B$, the following diagram commutes:



Since $\theta_{A,B}$ is a bijection, it follows that *every* morphism $g : A \to GB$ has the form $g = Gf \circ \eta_A$ for a *unique* $f : FA \to B$. We say that $\eta_A : A \to G(FA)$ is a *universal* morphism to G, or that η has the following *universal mapping property*: for every $A \in \mathcal{C}$, $B \in \mathcal{D}$, and $g : A \to GB$, there exists a *unique* $f : FA \to B$ such that $g = Gf \circ \eta_A$:



This means that an adjunction can be given in terms of its unit. The isomorphism $\theta : \mathcal{D}(F-,-) \to \mathcal{C}(-,G-)$ is then recovered by

$$\theta_{A,B} f = Gf \circ \eta_A \ .$$

Proposition A.5.4. *A functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ if, and only if, there exists a natural transformation*

$$\eta : 1_{\mathcal{C}} \Longrightarrow G \circ F \ ,$$

called the unit *of the adjunction, such that, for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$ the map $\theta_{A,B} : \mathcal{D}(FA, B) \to \mathcal{C}(A, GB)$, defined by*

$$\theta_{A,B} f = Gf \circ \eta_A \ ,$$

is an isomorphism.

Let us demonstrate how the universal mapping property of the unit of an adjunction appears as a well known construction in algebra. Consider the forgetful functor from monoids to sets,

$$U : \mathsf{Mon} \to \mathsf{Set} \ .$$

Does it have a left adjoint $F : \mathsf{Set} \to \mathsf{Mon}$? In order to obtain one, we need a "most economical" way of making a monoid FX from a given set X. Such a construction readily suggests itself, namely the *free monoid* on X, consisting of finite sequences of elements of X,

$$FX = \left\{ x_1 \dots x_n \;\middle|\; n \geq 0 \ \&\ x_1, \dots, x_n \in X \right\} \ .$$

The monoid operation is concatenation of sequences

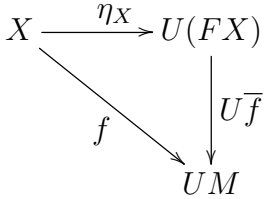
$$x_1 \dots x_m \cdot y_1 \dots y_n = x_1 \dots x_m y_1 \dots y_n \ ,$$

and the empty sequence is the unit of the monoid. In order for F to be a functor, it should also map morphisms to morphisms. If $f : X \to Y$ is a function, define $Ff : FX \to FY$ by

$$Ff : x_1 \dots x_n \mapsto (fx_1) \dots (fx_n) \ .$$

There is an inclusion $\eta_X : X \to U(FX)$ which maps every element $x \in X$ to the singleton sequence x. This gives a natural transformation $\eta : 1_{\mathsf{Set}} \Longrightarrow U \circ F$.

The monoid FX is "free" in the sense that it "satisfies only the equations required by the monoid laws"; we make this precise as follows. For every monoid M and function $f : X \to UM$ there exists a unique monoid homomorphism $\overline{f} : FX \to M$ such that the following diagram commutes:



This is precisely the condition required by Proposition A.5.4 for η to be the unit of the adjunction $F \dashv U$. In this case, the universal mapping property of η is just the usual characterization of the free monoid FX generated by the set X: a homomorphism from FX is uniquely determined by its values on the generators.

A.5.4 The counit of an adjunction

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors with $F \dashv G$, and let $\theta : \mathcal{D}(F-,-) \to \mathcal{C}(-,G-)$ be the natural isomorphism witnessing the adjunction. For any object $B \in \mathcal{D}$ we have a distinguished morphism $\varepsilon_B = \theta^{-1}_{GB,B} 1_{GB} : F(GB) \to B$ by:

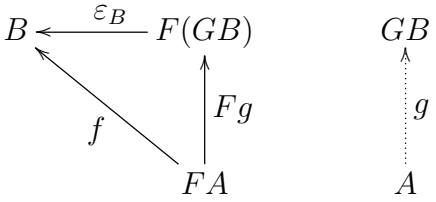
$$\frac{1_{GB} : GB \to GB}{\varepsilon_B : F(GB) \to B}$$

The natural transformation $\varepsilon : F \circ G \Longrightarrow 1_{\mathcal{D}}$ is called the *counit* of the adjunction $F \dashv G$. It is the dual notion to the unit of an adjunction. We state briefly the basic properties of the counit, which are easily obtained by "turning around" all the morphisms in the previous section and exchanging the roles of the left and right adjoints.

The bijection $\theta^{-1}_{A,B}$ can be recovered from the counit. For $g : A \to GB$ in $\mathcal{C}$, we have

$$\theta^{-1}_{A,B} g = \theta^{-1}_{A,B}(1_{GB} \circ g) = \theta^{-1}_{A,B} 1_{GB} \circ Fg = \varepsilon_B \circ Fg \ .$$

The universal mapping property of the counit is this: for every $A \in \mathcal{C}$, $B \in \mathcal{D}$, and $f : FA \to B$, there exists a *unique* $g : A \to GB$ such that $f = \varepsilon_B \circ Fg$:



The following is the dual of Proposition A.5.4.

Proposition A.5.5. *A functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ if, and only if, there exists a natural transformation*

$$\varepsilon : F \circ G \Longrightarrow 1_{\mathcal{D}} \ ,$$

called the counit *of the adjunction, such that, for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$ the map $\theta^{-1}_{A,B} : \mathcal{C}(A, GB) \to \mathcal{D}(FA, B)$, defined by*

$$\theta^{-1}_{A,B} g = \varepsilon_B \circ Fg \ ,$$

is an isomorphism.

Let us consider again the forgetful functor $U : \mathsf{Mon} \to \mathsf{Set}$ and its left adjoint $F : \mathsf{Set} \to \mathsf{Mon}$, the free monoid construction. For a monoid $(M, \star) \in \mathsf{Mon}$, the counit of the adjunction $F \dashv U$ is a monoid homomorphism $\varepsilon_M : F(UM) \to M$, defined by

$$\varepsilon_M(x_1 x_2 \dots x_n) = x_1 \star x_2 \star \cdots \star x_n \ .$$

It has the following universal mapping property: for $X \in \mathsf{Set}$, $(M, \star) \in \mathsf{Mon}$, and a homomorphism $f : FX \to M$ there exists a unique function $\overline{f} : X \to UM$ such that $f = \varepsilon_M \circ F\overline{f}$, namely

$$\overline{f}x = fx \ ,$$

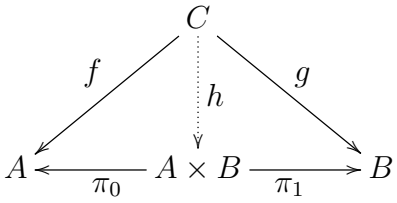
where in the above definition $x \in X$ is viewed as an element of the set X on the left-hand side, and as an element of the free monoid FX on the right-hand side. To summarize, the universal mapping property of the counit ε is the familiar piece of wisdom that a homomorphism $f : FX \to M$ from a free monoid is already determined by its values on the generators.

A.6 Limits and Colimits

The following limits and colimits are all special cases of adjoint functors, as we shall see.

A.6.1 Binary products

In a category $\mathcal{C}$, the *(binary) product* of objects A and B is an object $A \times B$ together with *projections* $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ such that, for every object $C \in \mathcal{C}$ and every pair of morphisms $f : C \to A$, $g : C \to B$ there exists a *unique* morphism $h : C \to A \times B$ for which the following diagram commutes:



We normally refer to the product $(A \times B, \pi_0, \pi_1)$ just by its object $A \times B$, but you should keep in mind that a product is given by an object *and* two projections. The arrow $h : C \to A \times B$ is denoted by $\langle f, g \rangle$. The property

$$\begin{aligned} &\text{for all } C, \text{ for all } f : C \to A, \text{ for all } g : C \to B, \\ &\text{there is a unique } h : C \to A \times B, \\ &\text{with } \pi_0 \circ h = f \ \& \ \pi_1 \circ h = g \end{aligned}$$

is the *universal mapping property* of the product $A \times B$. It characterizes the product of A and B uniquely up to isomorphism in the sense that if $(P, p_0 : P \to A, p_1 : P \to B)$ is

another product of A and B, then there is a unique isomorphism $r : P \xrightarrow{\sim} A \times B$ such that $p_0 = \pi_0 \circ r$ and $p_1 = \pi_1 \circ r$.

If in a category $\mathcal{C}$ every two objects have a product, we can turn binary products into an operation[4] by *choosing* a product $A \times B$ for each pair of objects $A, B \in \mathcal{C}$. In general this requires the Axiom of Choice, but in many specific cases a particular choice of products can be made without appeal to that axiom. When we view binary products as an operation, we say that "$\mathcal{C}$ has chosen products". The same holds for other instances of limits and colimits.

For example, in Set the usual cartesian product of sets is a product. In categories of structures, products are the usual construction: the product of topological spaces in Top is their topological product, the product of directed graphs in Graph is their cartesian product, the product of categories in Cat is their product category, and so on.

A.6.2 Terminal objects

A *terminal object* in a category $\mathcal{C}$ is an object $1 \in \mathcal{C}$ such that for every $A \in \mathcal{C}$ there exists a *unique* morphism $!_A : A \to 1$.

For example, in Set an object is terminal if, and only if, it is a singleton. The terminal object in Cat is the unit category $\mathsf{1}$ consisting of one object and one morphism.

Exercise A.6.1. Prove that if 1 and $1'$ are terminal objects in a category then they are isomorphic.

Exercise A.6.2. Let Field be the category whose objects are fields and morphisms are field homomorphisms.[5] Does Field have a terminal object? What about the category Ring of rings?

A.6.3 Equalizers

Given objects and morphisms

$$E \xrightarrow{\ e\ } A \underset{g}{\overset{f}{\rightrightarrows}} B$$

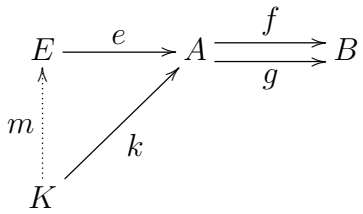
we say that e *equalizes* f and g when $f \circ e = g \circ e$.[6] An *equalizer* of f and g is a *universal* equalizing morphism; thus $e : E \to A$ is an equalizer of f and g when it equalizes them and, for all $k : K \to A$, if $f \circ k = g \circ k$ then there exists a unique morphism $m : K \to E$

[4] More precisely, binary product is a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$, cf. Section A.6.11.

[5] A field $(F, +, \cdot, ^{-1}, 0, 1)$ is a ring with a unit in which all non-zero elements have inverses. We also require that $0 \neq 1$. A homomorphism of fields preserves addition and multiplication, and consequently also 0, 1 and inverses.

[6] Note that this does *not* mean the diagram involving f, g and e is commutative!

such that $k = e \circ m$:



In Set the equalizer of parallel functions $f : A \to B$ and $g : A \to B$ is the set

$$E = \{x \in A \mid fx = gx\}$$

with $e : E \to A$ being the subset inclusion $E \subseteq A$, $ex = x$. In general, equalizers can be thought of as those subobjects (subsets, subgroups, subspaces, ...) that can be defined by an equation.

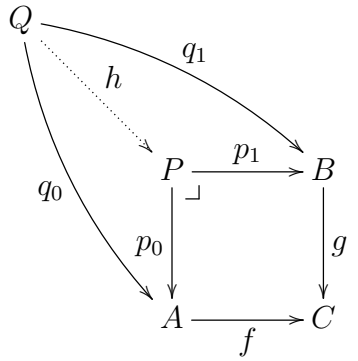
Exercise A.6.3. Show that an equalizer is a monomorphism, i.e., if $e : E \to A$ is an equalizer of f and g, then, for all $r, s : C \to E$, $e \circ r = e \circ s$ implies $r = s$.

Definition A.6.4. A morphism is a *regular mono* if it is an equalizer.

The difference between monos and regular monos is best illustrated in the category Top: a continuous map $f : X \to Y$ is mono when it is injective, whereas it is a regular mono when it is a topological embedding.[7]

A.6.4 Pullbacks

A *pullback* of $f : A \to C$ and $g : B \to C$ is an object P with morphisms $p_0 : P \to A$ and $p_1 : P \to B$ such that $f \circ p_0 = g \circ p_1$, and whenever Q, $q_0 : Q \to A$, and $q_1 : Q \to B$ are such that $f \circ q_0 = g \circ q_1$, there then exists a unique $h : Q \to P$ such that $q_0 = p_0 \circ h$ and $q_1 = p_1 \circ h$:



We indicate that P is a pullback by drawing a square corner next to it, as in the above diagram. The pullback is sometimes written $A \times_C B$, since it is indeed a product in the slice category over C.

[7] A continuous map $f : X \to Y$ is a topological embedding when, for every $U \in \mathcal{O}X$, the image $f[U]$ is an open subset of the image $\mathsf{im}(f)$; this means that there exists $V \in \mathcal{O}Y$ such that $f[U] = V \cap \mathsf{im}(f)$.

In Set, the pullback of $f : A \to C$ and $g : B \to C$ is the set

$$P = \big\{ \langle x, y \rangle \in A \times B \;\big|\; fx = gy \big\}$$

and the functions $p_0 : P \to A$, $p_1 : P \to B$ are the projections, $p_0\langle x, y\rangle = x$, $p_1\langle x, y\rangle = y$.

When we form the pullback of $f : A \to C$ and $g : B \to C$ we may also say that we *pull g back along f* and draw the diagram

$$\begin{array}{ccc} f^*B & \longrightarrow & B \\ {\scriptstyle f^*g}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

We think of $f^*g : f^*B \to A$ as the inverse image of B along f. This terminology is explained by looking at the pullback of a subset inclusion $u : U \hookrightarrow C$ along a function $f : A \to C$ in the category Set:

$$\begin{array}{ccc} f^*U & \longrightarrow & U \\ \downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ A & \xrightarrow[f]{} & C \end{array}$$

In this case the pullback is $\big\{ \langle x, y\rangle \in A \times U \;\big|\; fx = y \big\} \cong \big\{ x \in A \;\big|\; fx \in U \big\} = f^*U$, the inverse image of U along f.

Exercise A.6.5. Prove that in a category $\mathcal{C}$, a morphism $f : A \to B$ is mono if, and only if, the following diagram is a pullback:

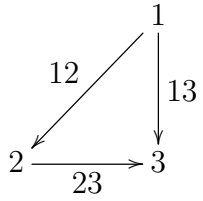
$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

A.6.5 Limits

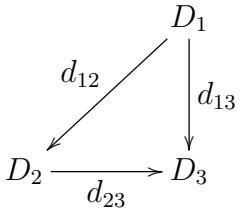
Let us now define the general notion of a limit.

A *diagram of shape $\mathcal{I}$* in a category $\mathcal{C}$ is a functor $D : \mathcal{I} \to \mathcal{C}$, where the category $\mathcal{I}$ is called the *index category*. We use letters i, j, k, ... for objects of an index category $\mathcal{I}$, call them *indices*, and write D_i, D_j, D_k, ... instead of Di, Dj, Dk, ...

For example, if $\mathcal{I}$ is the category with three objects and three morphisms



where $13 = 23 \circ 12$ then a diagram of shape $\mathcal{I}$ is a commutative diagram



(A.9)

For each object $A \in \mathcal{C}$, the *constant A-valued diagram* of shape $\mathcal{I}$ is given by the constant functor $\Delta_A : \mathcal{I} \to \mathcal{C}$, which maps every object to A and every morphism to $\mathsf{1}_A$.

Let $D : \mathcal{I} \to \mathcal{C}$ be a diagram of shape $\mathcal{I}$. A *cone* on D from an object $A \in \mathcal{C}$ is a natural transformation $\alpha : \Delta_A \Longrightarrow D$. This means that for every index $i \in \mathcal{I}$ there is a morphism $\alpha_i : A \to D_i$ such that whenever $u : i \to j$ in $\mathcal{I}$ then $\alpha_j = Du \circ \alpha_i$.

For a given diagram $D : \mathcal{I} \to \mathcal{C}$, we can collect all cones on D into a category $\mathsf{Cone}(D)$ whose objects are cones on D. A morphism between cones $f : (A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $\alpha_i = \beta_i \circ f$ for all $i \in \mathcal{I}$. Morphisms in $\mathsf{Cone}(D)$ are composed as morphisms in $\mathcal{C}$. A morphism $f : (A, \alpha) \to (B, \beta)$ is also called a factorization of the cone (A, α) through the cone (B, β).

A *limit* of a diagram $D : \mathcal{I} \to \mathcal{C}$ is a terminal object in $\mathsf{Cone}(D)$. Explicitly, a limit of D is given by a cone (L, λ) such that for every other cone (A, α) there exists a *unique* morphism $f : A \to L$ such that $\alpha_i = \lambda_i \circ f$ for all $i \in \mathcal{I}$. We denote (the object part of) a limit of D by one of the following:

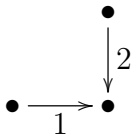
$$\lim D \qquad \lim_{i \in \mathcal{I}} D_i \qquad \varprojlim_{i \in \mathcal{I}} D_i \ .$$

Limits are also called *projective limits*. We say that a category *has limits of shape $\mathcal{I}$* when every diagram of shape $\mathcal{I}$ in $\mathcal{C}$ has a limit.

Products, terminal objects, equalizers, and pullbacks are all special cases of limits:

- a product $A \times B$ is the limit of the functor $D : \mathsf{2} \to \mathcal{C}$ where $\mathsf{2}$ is the discrete category on two objects 0 and 1, and $D_0 = A$, $D_1 = B$.
- a terminal object $\mathsf{1}$ is the limit of the (unique) functor $D : \mathsf{0} \to \mathcal{C}$ from the empty category.
- an equalizer of $f, g : A \to B$ is the limit of the functor $D : (\cdot \rightrightarrows \cdot) \to \mathcal{C}$ which maps one morphism to f and the other one to g.

- the pullback of $f : A \to C$ and $g : B \to C$ is the limit of the functor $D : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is the category



  with $D1 = f$ and $D2 = g$.

It is clear how to define the product of an arbitrary family of objects

$$\left\{ A_i \in \mathcal{C} \;\middle|\; i \in I \right\} .$$

Such a family is a diagram of shape I, where I is viewed as a discrete category. A *product* $\prod_{i \in I} A_i$ is then given by an object $P \in \mathcal{C}$ and morphisms $\pi_i : P \to A_i$ such that, whenever we have a family of morphisms $\left\{ f_i : B \to A_i \;\middle|\; i \in I \right\}$ there exists a *unique* morphism $\langle f_i \rangle_{i \in I} : B \to P$ such that $f_i = \pi_i \circ f$ for all $i \in I$.

A *finite product* is a product of a finite family. As a special case we see that a terminal object is the product of an empty family. It is not hard to show that a category has finite products precisely when it has a terminal object and binary products.

A diagram $D : \mathcal{I} \to \mathcal{C}$ is *small* when $\mathcal{I}$ is a small category. A *small limit* is a limit of a small diagram. A *finite limit* is a limit of a diagram whose index category is finite.

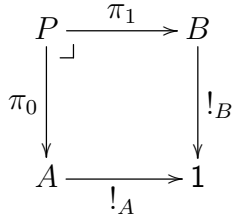
Exercise A.6.6. Prove that a limit, when it exists, is unique up to isomorphism.

The following proposition and its proof tell us how to compute arbitrary limits from simpler ones. We omit detailed proofs as they can be found in any standard textbook on category theory.

Proposition A.6.7. *The following are equivalent for a category $\mathcal{C}$:*

1. *$\mathcal{C}$ has a terminal object and all pullbacks.*
2. *$\mathcal{C}$ has equalizers and all finite products.*
3. *$\mathcal{C}$ has all finite limits.*

Proof. We only show how to get binary products from pullbacks and a terminal object. For objects A and B, let P be the pullback of $!_A$ and $!_B$:



Then (P, π_0, π_1) is a product of A and B because, for all $f : X \to A$ and $g : X \to B$, it is trivially the case that $!_A \circ f = !_B \circ g$. □

Proposition A.6.8. *The following are equivalent for a category $\mathcal{C}$:*

1. *$\mathcal{C}$ has equalizers and all small products.*
2. *$\mathcal{C}$ has all small limits.*

Proof. We indicate how to construct an arbitrary limit from a product and an equalizer. Let $D : \mathcal{I} \to \mathcal{C}$ be a small diagram of an arbitrary shape $\mathcal{I}$. First form an $\mathcal{I}_0$-indexed product P and an $\mathcal{I}_1$-indexed product Q

$$P = \prod_{i \in \mathcal{I}_0} D_i \ , \qquad\qquad Q = \prod_{u \in \mathcal{I}_1} D_{\mathsf{cod}\, u} \ .$$

By the universal property of products, there are unique morphisms $f : P \to Q$ and $g : P \to Q$ such that, for all morphisms $u \in \mathcal{I}_1$,

$$\pi^Q_u \circ f = Du \circ \pi^P_{\mathsf{dom}\, u} \ , \qquad\qquad \pi^Q_u \circ g = \pi^P_{\mathsf{cod}\, u} \ .$$

Let E be the equalizer of f and g,

$$E \xrightarrow{\ e\ } P \underset{g}{\overset{f}{\rightrightarrows}} Q$$

For every $i \in \mathcal{I}$ there is a morphism $\varepsilon_i : E \to D_i$, namely $\varepsilon_i = \pi^P_i \circ e$. We claim that (E, ε) is a limit of D. First, (E, ε) is a cone on D because, for all $u : i \to j$ in $\mathcal{I}$,

$$Du \circ \varepsilon_i = Du \circ \pi^P_i \circ e = \pi^Q_u \circ f \circ e = \pi^Q_u \circ g \circ e = \pi^P_j \circ e = \varepsilon_j \ .$$

If (A, α) is any cone on D there exists a unique $t : A \to P$ such that $\alpha_i = \pi^P_i \circ t$ for all $i \in \mathcal{I}$. For every $u : i \to j$ in $\mathcal{I}$ we have

$$\pi^Q_u \circ g \circ t = \pi^P_j \circ t = t_j = Du \circ t_i = Du \circ \pi^P_i \circ t = \pi^Q_u \circ f \circ t \ ,$$

therefore $g \circ t = f \circ t$. This implies that there is a unique factorization $k : A \to E$ such that $t = e \circ k$. Now for every $i \in \mathcal{I}$

$$\varepsilon_i \circ k = \pi^P_i \circ e \circ k = \pi^P_i \circ t = \alpha_i$$

so that $k : A \to E$ is the required factorization of the cone (A, α) through the cone (E, ε). To see that k is unique, suppose $m : A \to E$ is another factorization such that $\alpha_i = \varepsilon_i \circ m$ for all $i \in \mathcal{I}$. Since e is mono it suffices to show that $e \circ m = e \circ k$, which is equivalent to proving $\pi^P_i \circ e \circ m = \pi^P_i \circ e \circ k$ for all $i \in \mathcal{I}$. This last equality holds because

$$\pi^P_i \circ e \circ k = \pi^P_i \circ t = \alpha_i = \varepsilon_i \circ m = \pi^P_i \circ e \circ m \ .$$

□

A category is *(small) complete* when it has all small limits, and it is *finitely complete* (or *left exact*, briefly *lex*) when it has finite limits.

Limits of presheaves Let $\mathcal{C}$ be a locally small category. Then the presheaf category $\widehat{\mathcal{C}} = \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ has all small limits and they are computed pointwise, e.g., $(P \times Q)A = PA \times QA$ for $P, Q \in \widehat{\mathcal{C}}$, $A \in \mathcal{C}$. To see that this is really so, let $\mathcal{I}$ be a small index category and $D : \mathcal{I} \to \widehat{\mathcal{C}}$ a diagram of presheaves. Then for every $A \in \mathcal{C}$ the diagram D can be instantiated at A to give a diagram $DA : \mathcal{I} \to \mathsf{Set}$, $(DA)_i = D_i A$. Because Set is small complete, we can define a presheaf L by computing the limit of DA:

$$LA = \lim DA = \varprojlim_{i \in \mathcal{I}} D_i A \ .$$

We should keep in mind that $\lim DA$ is actually given by an object $(\lim DA)$ and a natural transformation $\delta A : \Delta_{(\lim DA)} \Longrightarrow DA$. The value of LA is supposed to be just the object part of $\lim DA$. From a morphism $f : A \to B$ we obtain for each $i \in \mathcal{I}$ a function $D_i f \circ (\delta A)_i : LA \to D_i B$, and thus a cone $(LA, Df \circ \delta A)$ on DB. Presheaf L maps the morphism $f : A \to B$ to the unique factorization $Lf : LA \Longrightarrow LB$ of the cone $(LA, Df \circ \delta A)$ on DB through the limit cone LB on DB.

For every $i \in \mathcal{I}$, there is a function $\Lambda_i = (\delta A)_i : LA \to D_i A$. The family $\{\Lambda_i\}_{i \in \mathcal{I}}$ is a natural transformation from Δ_{LA} to DA. This gives us a cone (L, Λ) on D, which is in fact a limit cone. Indeed, if (S, Σ) is another cone on D then for every $A \in \mathcal{C}$ there exists a unique function $\phi_A : SA \to LA$ because SA is a cone on DA and LA is a limit cone on DA. The family $\{\phi_A\}_{A \in \mathcal{C}}$ is the unique natural transformation $\phi : S \Longrightarrow L$ for which $\Sigma = \phi \circ \Lambda$.

A.6.6 Colimits

Colimits are the dual notion of limits. Thus, a *colimit* of a diagram $D : \mathcal{I} \to \mathcal{C}$ is a limit of the dual diagram $D^{\mathsf{op}} : \mathcal{I}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$ in the dual (i.e., opposite) category $\mathcal{C}^{\mathsf{op}}$:

$$\operatorname{colim}(D : \mathcal{I} \to \mathcal{C}) = \lim(D^{\mathsf{op}} : \mathcal{I}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}) \ .$$

Explicitly, the colimit of a diagram $D : \mathcal{I} \to \mathcal{C}$ is the initial object in the category of *cocones* $\mathsf{Cocone}(D)$ on D. A cocone (A, α) on D is a natural transformation $\alpha : D \Longrightarrow \Delta_A$. It is given by an object $A \in \mathcal{C}$ and, for each $i \in \mathcal{I}$, a morphism $\alpha_i : D_i \to A$, such that $\alpha_i = \alpha_j \circ Du$ whenever $u : i \to j$ in $\mathcal{I}$. A morphism between cocones $f : (A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $\beta_i = f \circ \alpha_i$ for all $i \in \mathcal{I}$.

A colimit of $D : \mathcal{I} \to \mathcal{C}$ is then given by a cocone (C, ζ) on D such that, for every cocone (A, α) on D there exists a unique morphism $f : C \to A$ such that $\alpha_i = f \circ \zeta_i$ for all $i \in D$. We denote a colimit of D by one of the following:

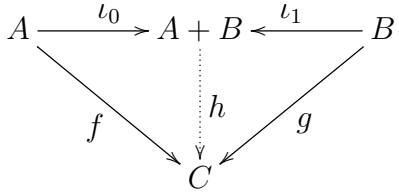
$$\operatorname{colim} D \qquad \operatorname{colim}_{i \in \mathcal{I}} D_i \qquad \varinjlim_{i \in \mathcal{I}} D_i \ .$$

Colimits are also called *inductive limits*.

Exercise A.6.9. Formulate the dual of Proposition A.6.7 and Proposition A.6.8 for colimits (coequalizers are defined in Section A.6.9).

A.6.7 Binary coproducts

In a category $\mathcal{C}$, the *(binary) coproduct* of objects A and B is an object $A + B$ together with *injections* $\iota_0 : A \to A + B$ and $\iota_1 : B \to A + B$ such that, for every object $C \in \mathcal{C}$ and all morphisms $f : A \to C$, $g : B \to C$ there exists a *unique* morphism $h : A + B \to C$ for which the following diagram commutes:



The arrow $h : A + B \to C$ is denoted by $[f, g]$.

The coproduct $A + B$ is the colimit of the diagram $D : 2 \to \mathcal{C}$, where $\mathcal{I}$ is the discrete category on two objects 0 and 1, and $D_0 = A$, $D_1 = B$.

In Set the coproduct is the disjoint union, defined by

$$X + Y = \big\{ \langle 0, x\rangle \;\big|\; x \in X \big\} \cup \big\{ \langle 1, y\rangle \;\big|\; x \in Y \big\} \ ,$$

where 0 and 1 are distinct sets, for example $\emptyset$ and $\{\emptyset\}$. Given functions $f : X \to Z$ and $g : Y \to Z$, the unique function $[f, g] : X + Y \to Z$ is the usual *definition by cases*:

$$[f, g]u = \begin{cases} fx & \text{if } u = \langle 0, x\rangle \\ gx & \text{if } u = \langle 1, x\rangle \ . \end{cases}$$

Exercise A.6.10. Show that the categories of posets and of topological spaces both have coproducts.

A.6.8 Initial objects

An *initial object* in a category $\mathcal{C}$ is an object $\mathsf{0} \in \mathcal{C}$ such that for every $A \in \mathcal{C}$ there exists a *unique* morphism $\mathsf{o}_A : \mathsf{0} \to A$.

An initial object is the colimit of the empty diagram.

In Set, the initial object is the empty set.

Exercise A.6.11. What is the initial and what is the terminal object in the category of groups?

A *zero object* is an object that is both initial and terminal.

Exercise A.6.12. Show that in the category of Abelian[8] groups finite products and coproducts agree, that is $\mathsf{0} \cong \mathsf{1}$ and $A \times B \cong A + B$.

Exercise A.6.13. Suppose A and B are Abelian groups. Is there a difference between their coproduct in the category Group of groups, and their coproduct in the category $\mathsf{AbGroup}$ of Abelian groups?

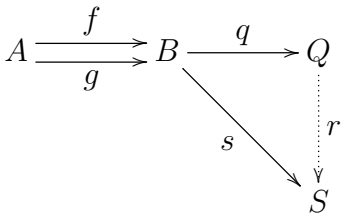
[8] An Abelian group is one that satisfies the commutative law $x \cdot y = y \cdot x$.

A.6.9 Coequalizers

Given objects and morphisms

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{q} Q$$

we say that q *coequalizes* f and g when $e \circ f = e \circ g$. A *coequalizer* of f and g is a *universal* coequalizing morphism; thus $q : B \to Q$ is a coequalizer of f and g when it coequalizes them and, for all $s : B \to S$, if $s \circ f = s \circ g$ then there exists a *unique* morphism $r : Q \to S$ such that $s = r \circ q$:



In Set the coequalizer of parallel functions $f : A \to B$ and $g : A \to B$ is the quotient set $Q = B/\!\sim$ where $\sim$ is the least equivalence relation on B satisfying

$$fx = gy \Rightarrow x \sim y \ .$$

The function $q : B \to Q$ is the canonical quotient map which assigns to each element $x \in B$ its equivalence class $[x] \in B/\!\sim$. In general, a coequalizer can be thought of as the quotient by the equivalence relation generated by the corresponding equation.

Exercise A.6.14. Show that a coequalizer is an epimorphism, i.e., if $q : B \to Q$ is a coequalizer of f and g, then, for all $u, v : Q \to T$, $u \circ q = v \circ q$ implies $u = v$. [Hint: use the duality between limits and colimits and Exercise A.6.3.]

Definition A.6.15. A morphism is a *regular epi* if it is a coequalizer.

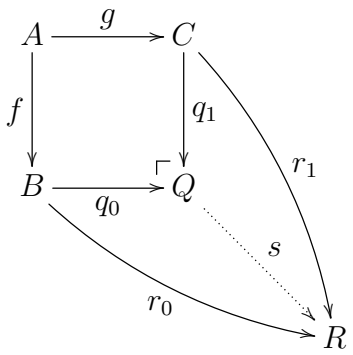
The difference between epis and regular epis is also illustrated in the category Top: a continuous map $f : X \to Y$ is epi when it is surjective, whereas it is a regular epi when it is a topological quotient map.[9]

A.6.10 Pushouts

A *pushout* of $f : A \to B$ and $g : A \to C$ is an object Q with morphisms $q_0 : B \to Q$ and $q_1 : C \to Q$ such that $q_0 \circ f = q_1 \circ g$, and whenever $r_0 : B \to R$, $r_1 : C \to R$ are such that

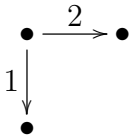
[9] A continuous map $f : X \to Y$ is a topological quotient map when it is surjective and, for every $U \subseteq Y$, U is open if, and only if, f^*U is open.

$r_0 \circ f = r_1 \circ g$, then there exists a unique $s : Q \to R$ such that $r_0 = s \circ q_0$ and $r_1 = s \circ q_1$:



We indicate that Q is a pushout by drawing a square corner next to it, as in the above diagram. The above pushout Q is sometimes denoted by $B +_A C$.

A pushout as above is a colimit of the diagram $D : \mathcal{I} \to \mathcal{C}$ where the index category $\mathcal{I}$ is



and $D1 = f$, $D2 = g$.

In Set, the pushout of $f : A \to C$ and $g : B \to C$ is the quotient set

$$Q = (B + C)/\!\sim$$

where $B + C$ is the disjoint union of B and C, and $\sim$ is the least equivalence relation on $B + C$ such that, for all $x \in A$,

$$fx \sim gx \ .$$

The functions $q_0 : B \to Q$, $q_1 : C \to Q$ are the injections, $q_0 x = [x]$, $q_1 y = [y]$, where $[x]$ is the equivalence class of x.

A.6.11 Limits as adjoints

Limits and colimits can be defined as adjoints to certain very simple functors.

First, observe that an object $A \in \mathcal{C}$ can be viewed as a functor from the terminal category $\mathsf{1}$ to $\mathcal{C}$, namely the functor which maps the only object $\star$ of $\mathsf{1}$ to A. Since $\mathsf{1}$ is the terminal object in Cat, there exists a unique functor $!_{\mathcal{C}} : \mathcal{C} \to \mathsf{1}$, which maps every object of $\mathcal{C}$ to $\star$.

Now we can ask whether this simple functor $!_{\mathcal{C}} : \mathcal{C} \to \mathsf{1}$ has any adjoints. Indeed, it has a right adjoint just if $\mathcal{C}$ has a terminal object $\mathsf{1}_{\mathcal{C}}$, for the corresponding functor $\mathsf{1}_{\mathcal{C}} : \mathsf{1} \to \mathcal{C}$ has the property that, for every $A \in \mathcal{C}$ we have a (trivially natural) bijective correspondence:

$$\frac{!_A : A \to \mathsf{1}_{\mathcal{C}}}{\mathsf{1}_\star : !_{\mathcal{C}} A \to \star}$$

Similarly, an initial object is a left adjoint to $!_{\mathcal{C}}$:

$$0_{\mathcal{C}} \dashv !_{\mathcal{C}} \dashv 1_{\mathcal{C}} \ .$$

Now consider the diagonal functor,

$$\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C},$$

defined by $\Delta A = \langle A, A \rangle$, $\Delta f = \langle f, f \rangle$. When does this have adjoints?

If $\mathcal{C}$ has all binary products, then they determine a functor

$$- \times - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

which maps $\langle A, B \rangle$ to $A \times B$ and a pair of morphisms $\langle f : A \to A', g : B \to B' \rangle$ to the unique morphism $f \times g : A \times B \to A' \times B'$ for which $\pi_0 \circ (f \times g) = f \circ \pi_0$ and $\pi_1 \circ (f \times g) = g \circ \pi_1$,

$$\begin{array}{ccccc} A & \xleftarrow{\pi_0} & A \times B & \xrightarrow{\pi_1} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f \times g} & & \downarrow{\scriptstyle g} \\ A' & \xleftarrow[\pi_0]{} & A' \times B' & \xrightarrow[\pi_1]{} & B' \end{array}$$

The product functor $\times$ is right adjoint to the diagonal functor Δ. Indeed, there is a natural bijective correspondence:

$$\frac{\langle f, g \rangle : \langle A, A \rangle \to \langle B, C \rangle}{f \times g : A \to B \times C}$$

Similarly, binary coproducts are easily seen to be left adjoint to the diagonal functor,

$$+ \dashv \Delta \dashv \times \ .$$

Now in general, consider limits of shape $\mathcal{I}$ in a category $\mathcal{C}$. There is the constant diagram functor

$$\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$$

that maps $A \in \mathcal{C}$ to the constant diagram $\Delta_A : \mathcal{I} \to \mathcal{C}$. The limit construction is a functor

$$\varprojlim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$$

that maps each diagram $D \in \mathcal{C}^{\mathcal{I}}$ to its limit $\varprojlim D$. These two are adjoint, $\Delta \dashv \varprojlim$, because there is a natural bijective correspondence between cones $\alpha : \Delta_A \Longrightarrow D$ on D, and their factorizations through the limit of D,

$$\frac{\alpha : \Delta_A \Longrightarrow D}{A \to \varprojlim D}$$

An analogous correspondence holds for colimits, so that we obtain a pair of adjunctions,

$$\varinjlim \dashv \Delta \dashv \varprojlim \ ,$$

which, of course, subsume all the previously mentioned cases.

Exercise A.6.16. How are the functors $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$, $\varinjlim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$, and $\varprojlim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ defined on morphisms?

A.6.12 Preservation of limits

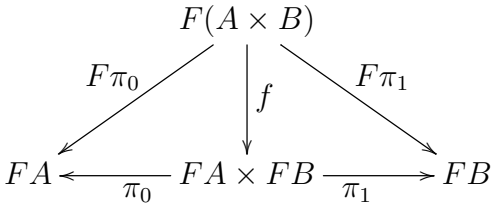
We say that a functor $F : \mathcal{C} \to \mathcal{D}$ *preserves products* when, given a product

$$A \xleftarrow{\pi_0} A \times B \xrightarrow{\pi_1} B$$

its image in $\mathcal{D}$,

$$FA \xleftarrow{F\pi_0} F(A \times B) \xrightarrow{F\pi_1} FB$$

is a product of FA and FB. If $\mathcal{D}$ has chosen binary products, F preserves binary products if, and only if, the unique morphism $f : F(A \times B) \to FA \times FB$ which makes the following diagram commutative is an isomorphism: [10]



In general, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to *preserve limits* of shape $\mathcal{I}$ when it maps limit cones to limit cones: if (L, λ) is a limit of $D : \mathcal{I} \to \mathcal{C}$ then $(FL, F \circ \lambda)$ is a limit of $F \circ D : \mathcal{I} \to \mathcal{D}$.

Analogously, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to *preserve colimits* of shape $\mathcal{I}$ when it maps colimit cocones to colimit cocones: if (C, ζ) is a colimit of $D : \mathcal{I} \to \mathcal{C}$ then $(FC, F \circ \zeta)$ is a colimit of $F \circ D : \mathcal{I} \to \mathcal{D}$.

Proposition A.6.17. *(a) A functor preserves finite (small) limits if, and only if, it preserves equalizers and finite (small) products. (b) A functor preserves finite (small) colimits if, and only if, it preserves coequalizers and finite (small) coproducts.*

Proof. This follows from the fact that limits are constructed from equalizers and products, cf. Proposition A.6.8, and that colimits are constructed from coequalizers and coproducts, cf. Exercise A.6.9. □

Proposition A.6.18. *For a locally small category $\mathcal{C}$, the Yoneda embedding $\mathsf{y} : \mathcal{C} \to \widehat{\mathcal{C}}$ preserves all limits that exist in $\mathcal{C}$.*

[10] Products are determined up to isomorphism only, so it would be too restrictive to require $F(A \times B) = FA \times FB$. When that is the case, however, we say that the functor F *strictly* preserves products.

Proof. Suppose (L, λ) is a limit of $D : \mathcal{I} \to \mathcal{C}$. The Yoneda embedding maps D to the diagram $\mathsf{y} \circ D : \mathcal{I} \to \widehat{\mathcal{C}}$, defined by

$$(\mathsf{y} \circ D)_i = \mathsf{y}D_i = \mathcal{C}(-, D_i) \ .$$

and it maps the limit cone (L, λ) to the cone $(\mathsf{y}L, \mathsf{y} \circ \lambda)$ on $\mathsf{y} \circ D$, defined by

$$(\mathsf{y} \circ \lambda)_i = \mathsf{y}\lambda_i = \mathcal{C}(-, \lambda_i) \ .$$

To see that $(\mathsf{y}L, \mathsf{y} \circ \lambda)$ is a limit cone on $\mathsf{y} \circ D$, consider a cone (M, μ) on $\mathsf{y} \circ D$. Then $\mu : \Delta_M \Longrightarrow D$ consists of a family of functions, one for each $i \in \mathcal{I}$ and $A \in \mathcal{C}$,

$$(\mu_i)_A : MA \to \mathcal{C}(A, D_i) \ .$$

For every $A \in \mathcal{C}$ and $m \in MA$ we get a cone on D consisting of morphisms

$$(\mu_i)_A m : A \to D_i \ . \qquad (i \in \mathcal{I})$$

There exists a unique morphism $\phi_A m : A \to L$ such that $(\mu_i)_A m = \lambda_i \circ \phi_A m$. The family of functions

$$\phi_A : MA \to \mathcal{C}(A, L) = (\mathsf{y} \circ L)A \qquad (A \in \mathcal{C})$$

forms a factorization $\phi : M \Longrightarrow \mathsf{y}L$ of the cone (M, μ) through the cone (L, λ). This factorization is unique because each $\phi_A m$ is unique. □

In effect we showed that a covariant representable functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathsf{Set}$ preserves existing limits,

$$\mathcal{C}(A, \varprojlim_{i \in \mathcal{I}} D_i) \cong \varprojlim_{i \in \mathcal{I}} \mathcal{C}(A, D_i) \ .$$

By duality, the contravariant representable functor $\mathcal{C}(-, A) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ maps existing colimits to limits,

$$\mathcal{C}(\varinjlim_{i \in \mathcal{I}} D_i, A) \cong \varprojlim_{i \in \mathcal{I}} \mathcal{C}(D_i, A) \ .$$

Exercise A.6.19. Prove the above claim that a contravariant representable functor $\mathcal{C}(-, A) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ maps existing colimits to limits. Use duality between limits and colimits. Does it also follow *by a simple duality argument* that a contravariant representable functor $\mathcal{C}(-, A)$ maps existing limits to colimits? How about a covariant representable functor $\mathcal{C}(A, -)$ mapping existing colimits to limits?

Exercise A.6.20. Prove that a functor $F : \mathcal{C} \to \mathcal{D}$ preserves monos if it preserves limits. In particular, the Yoneda embedding preserves monos. Hint: Exercise A.6.5.

Proposition A.6.21. *Right adjoints preserve limits, and left adjoints preserve colimits.*

Proof. Suppose we have adjoint functors

$$\mathcal{C} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{D}$$

and a diagram $D : \mathcal{I} \to \mathcal{D}$ whose limit exists in $\mathcal{D}$. We would like to use the following slick application of Yoneda Lemma to show that G preserves limits: for every $A \in \mathcal{C}$,

$$\begin{aligned}\mathcal{C}(A, G(\varprojlim D)) \cong \mathcal{D}(FA, \varprojlim D) &\cong \varprojlim_{i\in\mathcal{I}} \mathcal{D}(FA, D_i) \\ &\cong \varprojlim_{i\in\mathcal{I}} \mathcal{C}(A, GD_i) \cong \mathcal{C}(A, \varprojlim(G \circ D)) \ .\end{aligned}$$

Therefore $G(\lim D) \cong \lim(G \circ D)$. However, this argument only works if we already know that the limit of $G \circ D$ exists.

We can also prove the stronger claim that whenever the limit of $D : \mathcal{I} \to \mathcal{D}$ exists then the limit of $G \circ D$ exists in $\mathcal{C}$ and its limit is $G(\lim D)$. So suppose (L, λ) is a limit cone of D. Then $(GL, G \circ \lambda)$ is a cone on $G \circ D$. If (A, α) is another cone on $G \circ D$, we have by adjunction a cone (FA, γ) on D,

$$\frac{\alpha_i : A \to GD_i}{\gamma_i : FA \to D_i}$$

There exists a unique factorization $f : FA \to L$ of this cone through (L, λ). Again by adjunction, we obtain a unique factorization $g : A \to GL$ of the cone (A, α) through the cone $(GL, G \circ \lambda)$:

$$\frac{f : FA \to L}{g : A \to GL}$$

The factorization g is unique because γ is uniquely determined from α, f uniquely from α, and g uniquely from f.

By a dual argument, a left adjoint preserves colimits. □

Appendix B

Logic

B.1 Concrete and abstract syntax

By *syntax* we generally mean manipulation of finite strings of symbols according to given *grammatical rules*. For instance, the strings "$7)6+/(8$" and "$(6+8)/7$" both consist of the same symbols but you will recognize one as junk and the other as *well formed* because you have (implicitly) applied the grammatical rules for arithmetical expressions.

Grammatical rules are usually quite complicated, as they need to prescribe associativity of operators (does "$5+6+7$" mean "$(5+6)+7$" or "$5+(6+7)$"?) and their precedence (does "$6+8/7$" mean "$(6+8)/7$" or "$6+(8/7)$"?), the role of *white space* (empty space between symbols and line breaks), rules for nesting and balancing parentheses, etc. It is not our intention to dwell on such details, but rather to focus on the mathematical nature of well-formed expressions, namely that they represent inductively generated finite trees.[1] Under this view the string "$(6+8)/7$" is just a concrete representation of the tree depicted in Figure B.1.

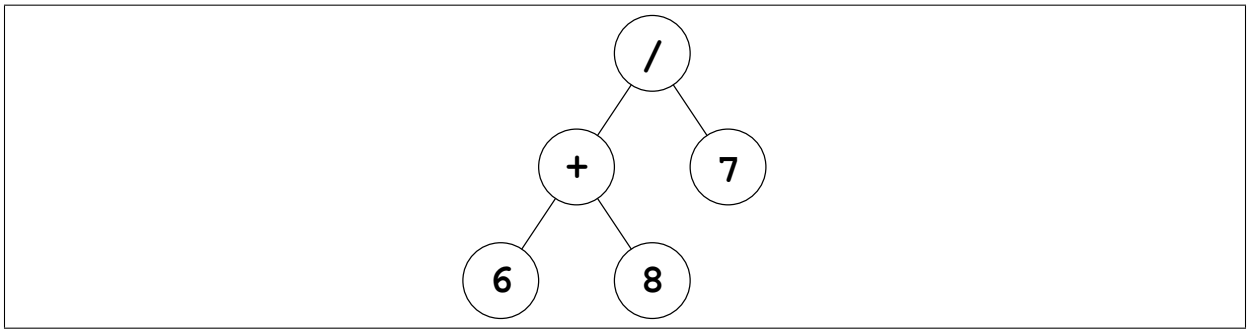


Figure B.1: The tree represented by $(6+8)/7$

Concrete representation of expressions as finite strings of symbols is called *concrete syntax*, while in *abstract syntax* we view expressions as finite trees. The passage from the

[1] We are limiting attention to the so-called *context-free* grammar, which are sufficient for our purposes. More complicated grammars are rarely used to describe formal languages in logic and computer science.

former to the latter is called *parsing* and is beyond the scope of this book. We will always specify only abstract syntax and assume that the corresponding concrete syntax follows the customary rules for parentheses, associativity and precedence of operators.

As an illustration we give rules for the (abstract) syntax of propositional calculus in *Backus-Naur* form:

$$\text{Propositional variable } p ::= \mathtt{p}_1 \mid \mathtt{p}_2 \mid \mathtt{p}_3 \mid \cdots$$
$$\text{Propositional formula } \phi ::= p \mid \bot \mid \top \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \neg\phi$$

The vertical bars should be read as "or". The first rule says that a propositional variable is the constant $\mathtt{p}_1$, or the constant $\mathtt{p}_2$, or the constant $\mathtt{p}_3$, etc.[2] The second rule tells us that there are seven inductive rules for building a propositional formula:

- a propositional variable is a formula,
- the constants $\bot$ and $\top$ are formulas,
- if ϕ_1, ϕ_2, and ϕ are formulas, then so are $\phi_1 \wedge \phi_2$, $\phi_1 \vee \phi_2$, $\phi_1 \Rightarrow \phi_2$, and $\neg\phi$.

Even though abstract syntax rules say nothing about parentheses or operator associativity and precedence, we shall rely on established conventions for mathematical notation and write down concrete representations of propositional formulas, e.g., $\mathtt{p}_4 \wedge (\mathtt{p}_1 \vee \neg \mathtt{p}_1) \wedge \mathtt{p}_4 \vee \mathtt{p}_2$.

A word of warning: operator associativity in syntax is not to be confused with the usual notion of associativity in mathematics. We say that an operator $\star$ is *left associative* when an expression $x \star y \star z$ represents the left-hand tree in Figure B.2, and *right associative* when it represents the right-hand tree. Thus the usual operation of subtraction $-$ is left

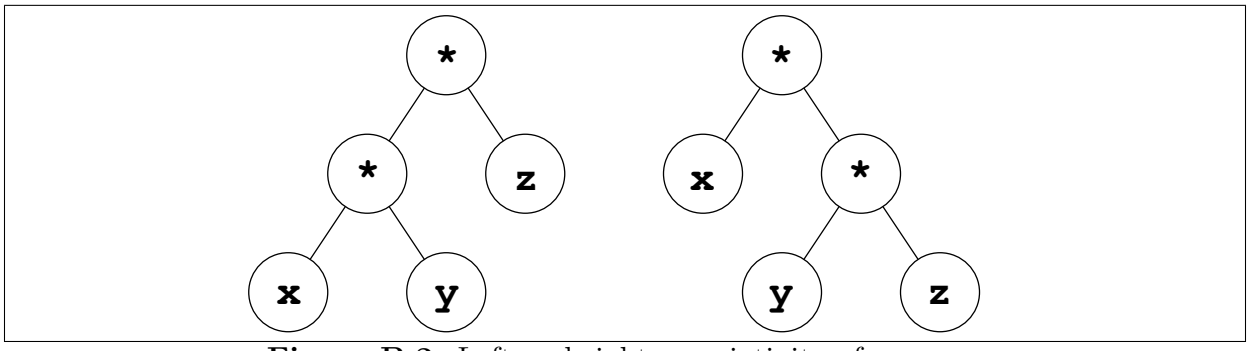


Figure B.2: Left and right associativity of $x \star y \star z$

associative, but is not associative in the usual mathematical sense.

[2] In an actual computer implementation we would allow arbitrary finite strings of letters as propositional variables. In logic we only care about the fact that we can never run out of fresh variables, i.e., that there are countably infinitely many of them.

B.2 Free and bound variables

Variables appearing in an expression may be *free* or *bound*. For example, in expressions

$$\int_0^1 \sin(a \cdot x)\, dx, \qquad x \mapsto ax^2 + bx + c, \qquad \forall\, x\,.\,(x < a \;\vee\; b < x)$$

the variables a, b and c are free, while x is bound by the integral operator $\int$, the function formation $\mapsto$, and the universal quantifier $\forall$, respectively. To be quite precise, it is an *occurrence* of a variable that is free or bound. For example, in expression $\phi(x) \vee \exists\, x\,.\, A\psi(x, x)$ the first occurrence of x is free and the remaining ones are bound.

In this book the following operators bind variables:

- quantifiers $\exists$ and $\forall$, cf. **??**,
- λ-abstraction, cf. **??**,
- search for others **??**.

When a variable is bound we may always rename it, provided the renaming does not confuse it with another variable. In the integral above we could rename x to y, but not to a because the binding operation would *capture* the free variable a to produce the unintended $\int_0^1 \sin(a^2)\, da$. Renaming of bound variables is called *α-renaming*.

We consider two expressions *equal* if they only differ in the names of bound variables, i.e., if one can be obtained from the other by α-renaming. Furthermore, we adhere to *Barendregt's variable convention [**?**, p. 2]*, which says that bound variables are always chosen so as to differ from free variables. Thus we would never write $\phi(x) \vee \exists\, x\,.\, A\psi(x, x)$ but rather $\phi(x) \vee \exists\, y\,.\, A\psi(y, y)$. By doing so we need not worry about capturing or otherwise confusing free and bound variables.

In logic we need to be more careful about variables than is customary in traditional mathematics. Specifically, we always specify which free variables may appear in an expression.[3] We write

$$x_1 : A_1, \ldots, x_n : A_n \mid t$$

to indicate that expression t may contain only free variables $x_1, \ldots, x_n$ of types $A_1, \ldots, A_n$. The list

$$x_1 : A_1, \ldots, x_n : A_n$$

is called a *context* in which t appears. To see why this is important consider the different meaning that the expression $x^2 + y^2 \leq 1$ recevieves in different contexts:

- $x : \mathbb{Z}, y : \mathbb{Z} \mid x^2 + y^2 \leq 1$ denotes the set of tuples $\{(-1, 0), (0, 1), (1, 0), (0, -1)\}$,
- $x : \mathbb{R}, y : \mathbb{R} \mid x^2 + y^2 \leq 1$ denotes the closed unit disc in the plane, and

[3] This is akin to one of the guiding principles of good programming language design, namely, that all variables should be *declared* before they are used.

- $x : \mathbb{R}, y : \mathbb{R}, z : \mathbb{R} \mid x^2 + y^2 \leq 1$ denotes the infinite cylinder in space whose base is the closed unit disc.

In single-sorted theories there is only one type or sort A. In this case we abbreviate a context by listing just the variables, $x_1, \ldots, x_n$.

B.3 Substitution

Substitution is a basic syntactic operation which replaces (free occurrences of) distinct variables $x_1, \ldots, x_n$ in an expression t with expressions $t_1, \ldots, t_n$, which is written as

$$t[t_1/x_1, \ldots, t_n/x_n].$$

We sometimes abbreviate this as $t[\vec{t}/\vec{x}]$ where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{t} = (t_1, \ldots, t_n)$. Here are several examples:

$$\begin{aligned} (x^2 + x + y)[(2+3)/x] &= (2+3)^2 + (2+3) + y \\ (x^2 + y)[y/x, x/y] &= y^2 + x \\ \left(\forall\, x\,.\, \left(x^2 < y + x^3\right)\right)[x + y/y] &= \forall\, z\,.\, \left(z^2 < (x+y) + z^3\right). \end{aligned}$$

Notice that in the third example we first renamed the bound variable x to z in order to avoid a capture by $\forall$.

Substitution is simple to explain in terms of trees. Assuming Barendregt's convention, the substitution $t[u/x]$ means that in the tree t we replace the leaves labeled x by copies of the tree u. Thus a substitution never changes the structure of the tree–it only "grows" new subtrees in places where the substituted variables occur as leaves.

Substitution satisfies the distributive law

$$(t[u/x])[v/y] = (t[v/y])[u[v/y]/x],$$

provided x and y are distinct variables. There is also a corresponding multivariate version which is written the same way with a slight abuse of vector notation:

$$(t[\vec{u}/\vec{x}])[\vec{v}/\vec{y}] = (t[\vec{v}/\vec{y}])[\vec{u}[\vec{v}/\vec{y}]/\vec{x}].$$

B.4 Judgments and deductive systems

A formal system, such as first-order logic or type theory, concerns itself with *judgments*. There are many kinds of judgments, such as:

- The most common judgments are equations and other logical statements. We distinguish a formula ϕ and the judgment "ϕ holds" by writing the latter as

  $$\vdash \phi\,.$$

  The symbol $\vdash$ is generally used to indicate judgments.

- Typing judgments
$$\vdash t : A$$
expressing the fact that a term t has type A. This is not to be confused with the set-theoretic statement $t \in u$ which says that individuals t and u (of type "set") are in relation "element of" $\in$.

- Judgments expressing the fact that a certain entity is well formed. A typical example is a judgment
$$\vdash x_1 : A_1, \ldots, x_n : A_n \quad \texttt{ctx}$$
which states that $x_1 : A_1, \ldots, x_n : A_n$ is a well-formed context. This means that $x_1, \ldots, x_n$ are distinct variables and that $A_1, \ldots, A_n$ are well-formed types. This kind of judgement is often omitted and it is tacitly assumed that whatever entities we deal with are in fact well-formed.

A *hypothetical judgement* has the form

$$H_1, \ldots, H_n \vdash C$$

and means that hypotheses $H_1, \ldots, H_n$ entail consequence C (with respect to a given decuctive system). We may also add a typing context to get a general form of judgment

$$x_1 : A_1, \ldots, x_n : A_n \mid H_1, \ldots, H_m \vdash C.$$

This should be read as: "if $x_1, \ldots, x_n$ are variables of types $A_1, \ldots, A_n$, respectively, then hypotheses $H_1, \ldots, H_m$ entail conclusion C." For our purposes such contexts will suffice, but you should not be surprised to see other kinds of judgments in logic.

A *deductive system* is a set of inference rules for deriving judgments. A typical inference rule has the form

$$\frac{J_1 \quad J_2 \quad \cdots \quad J_n}{J}\, C$$

This means that we can infer judgment J if we have already derived judgments $J_1, \ldots, J_n$, provided that the optional side-condition C is satisfied. An *axiom* is an inference rule of the form

$$\frac{}{J}$$

A *two-way rule*

$$\frac{J_1 \quad J_2 \quad \cdots \quad J_n}{K_1 \quad K_2 \quad \cdots \quad K_m}$$

is a combination of $n+m$ inference rules stating that we may infer each K_i from $J_1, \ldots, J_n$ and each J_i from $K_1, \ldots, K_m$.

A *derivation* of a judgment J is a finite tree whose root is J, the nodes are inference rules, and the leaves are axioms. An example is presented in the next subsection.

The set of all judgments that hold in a given deductive system is generated inductively by starting with the axioms and applying inference rules.

B.5 Example: Equational reasoning

Equational reasoning is so straightforward that one almost doesn't notice it, consisting mainly, as it does, of "substituting equals for equals". The only judgements are equations between terms, $s = t$, which consist of function symbols, constants, and variables. The inference rules are just the usual ones making $s = t$ a congruence relation on the terms. More formally, we have the following specification of what may be called the *equational calculus*.

$$\begin{aligned} \text{Variable } v &::= x \mid y \mid z \mid \cdots \\ \text{Constant symbol } c &::= \mathtt{c}_1 \mid \mathtt{c}_2 \mid \cdots \\ \text{Function symbol } f^k &::= \mathtt{f}_1^{k_1} \mid \mathtt{f}_2^{k_2} \mid \cdots \\ \text{Term } t &::= v \mid c \mid f^k(t_1, \ldots, t_k) \end{aligned}$$

The superscript on the function symbol f^k indicates the arity.

The equational calculus has just one form of judgement

$$x_1, \ldots, x_n \mid t_1 = t_2$$

where $x_1, \ldots, x_n$ is a *context* consisting of distinct variables, and the variables in the equation must occur among the ones listed in the context.

There are four inference rules for the equational calculus. They may be assumed to leave the contexts unchanged, which may therefore be omitted.

$$\frac{}{t = t} \qquad \frac{t_1 = t_2}{t_2 = t_1} \qquad \frac{t_1 = t_2,\ t_2 = t_3}{t_1 = t_3} \qquad \frac{t_1 = t_2,\ t_3 = t_4}{t_1[t_3/x] = t_2[t_4/x]}$$

An *equational theory* $\mathbb{T}$ consists of a set of constant and function symbols (with arities), and a set of equations, called *axioms*. We write

$$\mathbb{T} \vdash t_1 = t_2$$

to mean that the equation $t_1 = t_2$ has a derivation from the axioms of $\mathbb{T}$ using the equational calculus.

B.6 Example: Predicate calculus

We spell out the details of single-sorted predicate calculus and first-order theories. This is the most common deductive system taught in classical courses on logic.

The predicate calculus has the following syntax:

$$\begin{aligned}
\text{Variable } v &::= x \mid y \mid z \mid \cdots \\
\text{Constant symbol } c &::= \mathtt{c}_1 \mid \mathtt{c}_2 \mid \cdots \\
\text{Function symbol}^4\ f^k &::= \mathtt{f}_1^{k_1} \mid \mathtt{f}_2^{k_2} \mid \cdots \\
\text{Term } t &::= v \mid c \mid f^k(t_1, \ldots, t_k) \\
\text{Relation symbol } R^m &::= \mathtt{R}_1^{m_1} \mid \mathtt{R}_2^{m_2} \mid \cdots \\
\text{Formula } \phi &::= \bot \mid \top \mid R^m(t_1, \ldots, t_m) \mid t_1 = t_2 \mid \\
&\qquad \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \neg\phi \mid \forall\, x \,.\, \phi \mid \exists\, x \,.\, \phi.
\end{aligned}$$

The variable x is bound in $\forall\, x \,.\, \phi$ and $\exists\, x \,.\, \phi$.

The predicate calculus has one form of judgement

$$x_1, \ldots, x_n \mid \phi_1, \ldots, \phi_m \vdash \phi$$

where $x_1, \ldots, x_n$ is a *context* consisting of distinct variables, $\phi_1, \ldots, \phi_m$ are *hypotheses* and ϕ is the *conclusion*. The free variables in the hypotheses and the conclusion must occur among the ones listed in the context. We abbreviate the context with Γ and Φ with hypotheses. Because most rules leave the context unchanged, we omit the context unless something interesting happens with it.

The following inference rules are given in the form of adjunctions. See Appendix **??** for the more usual formulation in terms of introduction an elimination rules.

$$\frac{}{\phi_1, \ldots, \phi_m \vdash \phi_i} \qquad \frac{}{\Phi \vdash \top} \qquad \frac{}{\Phi, \bot \vdash \phi}$$

$$\frac{\Phi \vdash \phi_1 \quad \Phi \vdash \phi_2}{\Phi \vdash \phi_1 \wedge \phi_2} \qquad \frac{\Phi, \phi_1 \vdash \psi \quad \Phi, \phi_2 \vdash \psi}{\Phi, \phi_1 \vee \phi_2 \vdash \psi} \qquad \frac{\Phi, \phi_1 \vdash \phi_2}{\Phi \vdash \phi_1 \Rightarrow \phi_2}$$

$$\frac{\Gamma, x, y \mid \Phi, x = y \vdash \phi}{\Gamma, x \mid \Phi \vdash \phi[x/y]} \qquad \frac{\Gamma, x \mid \Phi, \phi \vdash \psi}{\Gamma \mid \Phi, \exists\, x \,.\, \phi \vdash \psi} \qquad \frac{\Gamma, x \mid \Phi \vdash \phi}{\Gamma \mid \Phi \vdash \forall\, x \,.\, \phi}$$

The equality rule implicitly requires that y does not appear in Φ, and the quantifier rules implicitly require that x does not occur freely in Φ and ψ because the judgments below the lines are supposed to be well formed.

Negation $\neg\phi$ is defined to be $\phi \Rightarrow \bot$. To obtain *classical* logic we also need the law of excluded middle,

$$\frac{}{\Phi \vdash \phi \vee \neg\phi}$$

Comment on the fact that contraction and weakening are admissible.

Give an example of a derivation.

A *first-order theory* $\mathbb{T}$ consists of a set of constant, function and relation symbols with corresponding arities, and a set of formulas, called *axioms*.

Give examples of a first-order theories.

Bibliography

[ALR03] J. Adamek, F. W. Lawvere, and J. Rosiky. On the duality between varieties and algebraic theories. *Algebra Universalis*, 49:35–49, 2003.

[AR94] Jiri Adamek and Jiri Rosicky. *Locally Presentable and Accessible Categories*. Number 189 in London Mathematical Society Lecture Notes. Cambridge University Press, 1994.

[ARV10] J. Adamek, J. Rosicky, and E.M. Vitale. *Algebraic Theories*. Cambridge University Press, 2010.

[Bir35] Garrett Birkhoff. On the structure of abstract algebras. *Proc. Camb. Philos. Soc.*, 31:433–454, 1935.

[Bor94] F. Borceux. *Handbook of Categorical Algebra II. Categories and Structures*, volume 51 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, 1994.

[Lan71] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, New York, 1971.

[Law63a] F. W. Lawvere. Functorial semantics of algebraic theories. Ph.D. thesis, Columbia University, 1963.

[Law63b] F. W. Lawvere. Functorial semantics of algebraic theories. *Proc. Nat. Acad. Sci.*, 50:869–872, 1963.

[Law65] F. W. Lawvere. Algebraic theories, algebraic categories, and algebraic functors. In *The Theory of Models*, pages 413–418. North-Holland, 1965.

[Law69] F.W. Lawvere. Adjointness in foundations. *Dialectica*, 23:281–296, 1969.

[Makzz] Michael Makkai. Duality and definability. *Memiors of the AMS*, nn:xx–yy, zzzz.

[McC93] W. McCune. Single axioms for groups and abelian groups with various operations. *Journal of Automated Reasoning*, 10(1):1–13, 1993.

[MR91] Ieke Moerdijk and Gonzalo Reyes. *Models for Smooth Infinitesimal Analysis*. Springer-Verlag, New York, 1991.