PATH INTEGRATION: CONNECTING PURE JUMP AND WIENER PROCESSES

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1. Introduction.

The Feynman path integral is known to be a powerful tool in different domains of physics and various mathematical approaches to its construction were developed (see e.g. extensive reviews of the recent literature in [ABB], [SS], [K4], [K5]). However, most of them work only for very restrictive class of potentials. Moreover, they are often defined not as genuine integrals, but as some generalized functionals specified by some limiting procedure. In [K2], [K3] the author proposed a representation of the solutions to the Schrödinger equation in terms of the well defined infinite dimensional Feynman integral defined as a genuine integral over a bona fide σ-additive measure on an appropriate space of trajectories (usually the Cameron-Martin space). This construction covers very general equations. In [K5] it is extended to the Schrödinger equations with magnetic fields with even singular vector potentials defined as Radon measures. The construction uses the idea of the regularization by means of the introduction of continuous quantum observations or complex times and extends the approach of Maslov-Chebotarev (see [MCh]) which was based on the pure jump processes that appear naturally in the momentum representation of the Schrödinger equation, whose potential can be presented as a Fourier transform of a finite complex measure (Ito's complex measure condition). The present paper sketches the various connections between pure jump and Wiener processes that are relevant to the path integral study of the complex diffusion equations

$$\frac{\partial \psi}{\partial t} = G\left(\frac{1}{2}\Delta - V(x)\right)\psi, \qquad \psi \in L^2(\mathcal{R}^d), \tag{1.1}$$

where G is a complex constant with $ReG \geq 0$, the case $G = 1$ (respectively $G = i$) standing for diffusion (respectively Schrödinger) equation.

Section 2 is devoted to a unified construction of the Wiener and pure jump measures on a path space that exploits the notion of complex Markov chains (as introduced in [M]). It is based essentially on [K1] and extends the Nelson approach to the construction of the usual Wiener measure. Section 3 discusses the Fock space lifting of the pure-jump processes and the resulting possible interpretation of the pure jump path integral in terms of the Wiener integral based on the Wiener chaos decomposition. The last Section 4 presents a new asymptotic formula for the solutions of the Fourier transform of the diffusion equation which is obtained by passing to the Brownian motion limit in a representation of the solutions as expectations with respect to a pure jump Poisson process.

2. Infinitely divisible complex distributions and complex Markov processes.

We present here a general construction of complex measures on path spaces that can be used for the path integral representation of various evolutionary equations.

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Let $\mathcal{B}(\Omega)$ denote the class of all Borel sets of a topological space (i.e. it is the σ-algebra of sets generated by all open sets). If Ω is locally compact we denote (as usual) by $C_0(\Omega)$ the space of all continuous complex-valued functions on Ω vanishing at infinity. Equipped with the uniform norm $\|f\| = \sup_x |f(x)|$ this space is known to be a Banach space. It is also well known (Riesz-Markov theorem) that if Ω is a locally compact space, then the set $\mathcal{M}(\Omega)$ of all finite complex regular Borel measures on Ω equipped with the norm $\|\mu\| = \sup |\int_\Omega f(x)\mu(dx)|$, where sup is taken over all functions $f \in C_0(\Omega)$ with $\|f(x)\| \leq 1$, is a Banach space, which coincides with the set of all continuous linear functionals on $C_0(\Omega)$. Any complex σ-additive measure μ on $\mathcal{R}^d$ has a representation of form

$$\mu(dy) = f(y)M(dy) \tag{2.1}$$

with a positive measure M and a bounded complex-valued function f Moreover, the measure M in (2.1) is uniquely defined under additional assumption that $|f(y)| = 1$ for all y. If this condition holds, the positive measure M is called the total variation measure of the complex measure μ and is denoted by $|\mu|$. In general, if (2.1) holds, then $\|\mu\| = \int |f(y)|M(dy)$.

We say that a map ν from $\mathcal{R}^d \times \mathcal{B}(\mathcal{R}^d)$ into $\mathcal{C}$ is a *complex transition kernel*, if for every x, the map $A \mapsto \nu(x, A)$ is a (finite complex) measure on $\mathcal{R}^d$, and for every $A \in \mathcal{B}(\mathcal{R}^d)$, the map $x \mapsto \nu(x, A)$ is $\mathcal{B}$-measurable. A (time homogeneous) *complex transition function* (abbreviated CTF) on $\mathcal{R}^d$ is a family ν_t, $t \geq 0$, of complex transition kernels such that $\nu_0(x, dy) = \delta(y - x)$ for all x, where $\delta_x(y) = \delta(y - x)$ is the Dirac measure in x, and such that for every non-negative s, t, the Chapman-Kolmogorov equation

$$\int \nu_s(x, dy)\nu_t(y, A) = \nu_{s+t}(x, A)$$

is satisfied. (We consider only time homogeneous CTF for simplicity, the generalization to non-homogeneous case is straightforward).

A CTF is said to be (spatially) homogeneous, if $\nu_t(x, A)$ depends on x, A only through the difference $A - x$. If a CTF is homogeneous it is natural to denote $\nu_t(0, A)$ by $\nu_t(A)$) and to write the Chapman-Kolmogorov equation in the form

$$\int \nu_t(dy)\nu_s(A - y) = \nu_{t+s}(A).$$

A CTF will be called *regular*, if there exists a positive constant K such that for all x and $t > 0$, the norm $\|\nu_t(x, .)\|$ of the measure $A \mapsto \nu_t(x, A)$ does not exceed $\exp\{Kt\}$.

CTFs appear naturally in the theory of evolutionary equations: if T_t is a strongly continuous semigroup of bounded linear operators in $C_0(\mathcal{R}^d)$, then there exists a time-homogeneous CTF ν such that

$$T_t f(x) = \int \nu_t(x, dy)f(y). \tag{2.2}$$

In fact, the existence of a measure $\nu_t(x, .)$ such that (2.2) is satisfied follows from the Riesz- Markov theorem, and the semigroup identity $T_s T_t = T_{s+t}$ is equivalent to the Chapman-Kolmogorov equation. Since $\int \nu_t(x, dy)f(y)$ is continuous for all $f \in C_0(\mathcal{R}^d)$, it follows by the monotone convergence theorem (and the fact that each complex measure is a linear combination of four positive measures) that $\nu_t(x, A)$ is a Borel function of x.

We say that the semigroup T_t is *regular*, if the corresponding CTF is regular. Clearly, this is equivalent to the assumption that $\|T_t\| \leq e^{Kt}$ for all $t > 0$ and some constant K.

Now we construct a measure on the path space corresponding to each regular CTF, introducing first some (rather standard) notations. Let $\dot{\mathcal{R}}_d$ denote the one point compactification of the Euclidean space $\mathcal{R}^d$ (i.e. $\dot{\mathcal{R}}_d = \mathcal{R}^d \cup \{\infty\}$ and is homeomorphic to the sphere S^d). Let $\dot{\mathcal{R}}_d^{[s,t]}$ denote the infinite product of $[s, t]$ copies of $\dot{\mathcal{R}}_d$, i.e. it is the set of all functions from $[s,t]$ to $\dot{\mathcal{R}}_d$, the path space. As usual, we equip this set with the product topology, in which it is a compact space (Tikhonov's theorem). Let $Cyl^k_{[s,t]}$ denote the set of functions on $\dot{\mathcal{R}}_d^{[s,t]}$ having the form

$$\phi^f_{t_0,t_1,...t_{k+1}}(y(.)) = f(y(t_0),...,y(t_{k+1}))$$

for some bounded complex Borel function f on $(\dot{\mathcal{R}}^d)^{k+2}$ and some points t_j, $j = 0,...,k+1$, such that $s = t_0 < t_1 < t_2 < ... < t_k < t_{k+1} = t$. The union $Cyl_{[s,t]} = \cup_{k\in\mathcal{N}} Cyl^k_{[s,t]}$ is called the set of cylindrical functions (or functionals) on $\dot{\mathcal{R}}_d^{[s,t]}$. It follows from the Stone-Weierstrasse theorem that the linear span of all continuous cylindrical functions is dense in the space $C(\dot{\mathcal{R}}_d^{[s,t]})$ of all complex continuous functions on $\dot{\mathcal{R}}_d^{[s,t]}$. Any CTF ν defines a family of linear functionals $\nu^x_{s,t}$, $x \in \mathcal{R}^d$, on $Cyl_{[s,t]}$ by the formula

$$\begin{aligned} &\nu^x_{s,t}(\phi^f_{t_0...t_{k+1}}) \\ &= \int f(x,y_1,...,y_{k+1})\nu_{t_1-t_0}(x,dy_1)\nu_{t_2-t_1}(y_1,dy_2)...\nu_{t_{k+1}-t_k}(y_k,dy_{k+1}). \end{aligned} \tag{2.3}$$

Due to the Chapman-Kolmogorov equation, this definition is correct, i.e. if one considers an element from $Cyl^k_{[s,t]}$ as an element from $Cyl^{k+1}_{[s,t]}$ (any function of l variables $y_1,...,y_l$ can be considered as a function of $l+1$ variables $y_1,...,y_{l+1}$, which does not depend on y_{l+1}), then the two corresponding formulae (2.3) will be consistent.

PROPOSITION 2.1. *If the semigroup T_t in $C_0(\mathcal{R}^d)$ is regular and ν is its corresponding CTF, then the functional (2.3) is bounded. Hence, it can be extended by continuity to a unique bounded linear functional ν^x on $C(\dot{\mathcal{R}}_d^{[s,t]})$, and consequently there exists a (regular) complex Borel measure $D^{s,t}_x$ on the path space $\dot{\mathcal{R}}_d^{[s,t]}$ such that*

$$\nu^x_{s,t}(F) = \int F(y(.))D^{s,t}_x(dy(.)) \tag{2.4}$$

for all $F \in C(\dot{\mathcal{R}}_d^{[s,t]})$. In particular,

$$(T_t f)(x) = \int f(y(t))D^{s,t}_x(dy(.)).$$

Proof. It is a direct consequence of the Riesz-Markov theorem, because the regularity of CTF implies that the norm of the functional $\nu^x_{s,t}$ does not exceed $\exp\{K(t-s)\}$. □

Formula (2.3) defines the family of finite complex distributions on the path space, which gives rise to a finite complex measure on this path space (under the regularity assumptions). Therefore, this family of measures can be called a complex Markov process. Unlike the case of the standard Markov processes, the generator, say A, of the corresponding semigroup T_t and the corresponding bilinear "Dirichlet form" (Av, v) are complex.

The following simple fact can be used in proving the regularity of a semigroup.

PROPOSITION 2.2. *Let B and A be linear operators in $C_0(\mathcal{R}^d)$ such that A is bounded and B is the generator of a strongly continuous regular semigroup T_t. Then $A+B$ is also the generator of a regular semigroup, which we denote by $\tilde{T}_t$.*

Proof. Follows directly from the fact that $\tilde{T}_t$ can be presented as the convergent (in the sense of the norm) series of standard perturbation theory

$$\tilde{T}_t = T_t + \int_0^t T_{t-s}AT_s\,ds + \int_0^t ds\int_0^s d\tau T_{t-s}AT_{s-\tau}AT_\tau + ... \tag{2.5}$$

Of major importance for our purposes are the spatially homogeneous CTFs. Let us discuss them in greater detail, in particular, their connection with infinitely divisible characteristic functions.

Let $\mathcal{F}(\mathcal{R}^d)$ denote the Banach space of Fourier transforms of elements of $\mathcal{M}(\mathcal{R}^d)$, i.e. the space of (automatically continuous) functions on $\mathcal{R}^d$ of form

$$V(x) = V_\mu(x) = \int_{\mathcal{R}^d} e^{ipx}\,\mu(dp) \tag{2.6}$$

for some $\mu \in \mathcal{M}(\mathcal{R}^d)$, with the induced norm $\|V_\mu\| = \|\mu\|$. Since $\mathcal{M}(\mathcal{R}^d)$ is a Banach algebra with convolution as the multiplication, it follows that $\mathcal{F}(\mathcal{R}^d)$ is also a Banach algebra with respect to the standard (point-wise) multiplication. We say that an element $f \in \mathcal{F}(\mathcal{R}^d)$ is *infinitely divisible* if there exists a family $(f_t,\, t \geq 0,)$ of elements of $\mathcal{F}(\mathcal{R}^d)$ such that $f_0 = 1$, $f_1 = f$, and $f_{t+s} = f_t f_s$ for all positive s, t. Clearly if f is infinitely divisible, then it has no zeros and a continuous function $g = \log f$ is well defined (and is unique up to an imaginary shift). Moreover, the family f_t has the form $f_t = \exp\{tg\}$ and is defined uniquely up to a multiplier of the form $e^{2\pi ikt}$, $k \in \mathcal{N}$. Let us say that a continuous function g on $\mathcal{R}^d$ is a *complex characteristic exponent* (abbreviated CCE), if e^g is an infinitely divisible element of $\mathcal{F}(\mathcal{R}^d)$, or equivalently, if e^{tg} belongs to $\mathcal{F}(\mathcal{R}^d)$ for all $t > 0$. □

It follows from the definitions that the set of spatially homogeneous CTFs $\nu_t(dx)$ is in one-to-one correspondence with CCE g, in such a way that for any positive t the function e^{tg} is the Fourier transform of the transition measure $\nu_t(dx)$.

PROPOSITION 2.3. *If V is a CCE, then the solution to the Cauchy problem*

$$\frac{\partial u}{\partial t} = V\left(\frac{1}{i}\frac{\partial}{\partial y}\right)u \tag{2.7}$$

defines a strongly continuous and spatially homogeneous semigroup T_t of bounded linear operators in $C_0(\mathcal{R}^d)$ (i.e. $(T_t u_0)(y)$ is the solution to equation (2.7) with the initial function u_0). Conversely, each such semigroup is the solution to the Cauchy problem of an equation of type (2.7) with some CCE g.

Proof. This is straightforward. Since (2.7) is a pseudo-differential equation, it follows that the Fourier transform $\tilde{u}(t,x)$ of the function $u(t,y)$ satisfies the ordinary differential equation

$$\frac{\partial \tilde{u}}{\partial t}(t,x) = V(x)\tilde{u}(t,x),$$

whose solution is $\tilde{u}_0(x)\exp\{tV(x)\}$. Since e^{tV} is the Fourier transform of the complex transition measure $\nu_t(dy)$, it follows that the solution to the Cauchy problem of equation (2.7) is given by the formula $(T_t u_0)(y) = \int u_0(z)\nu_t(dz - y)$, which is as required. □

We say that a CCE is *regular*, if equation (2.7) defines a regular semigroup.

It would be very interesting to describe explicitly all regular CCE. We only give here two classes of examples. First of all, if a CCE is given by the Lévy- Khintchine formula (i.e. it defines a transition function consisting of probability measures), then this CCE is regular, because all CTF consisting of probability measures are regular. Another class is given by the following result.

PROPOSITION 2.4. *Let $V \in \mathcal{F}(\mathcal{R}^d)$, i.e. it is given by (2.6) with $\mu \in \mathcal{M}(\mathcal{R}^d)$. Then V is a regular CCE. Moreover, if the positive measure M in the representation (2.1) for μ has no atom at the origin, i.e. $M(\{0\}) = 0$, then the corresponding measure $D_x^{0,t}$ on the path space from Proposition 2.1 is concentrated on the set of piecewise-constant paths in $\dot{\mathcal{R}}_d^{[0,t]}$ with a finite number of jumps. In other words, $D_x^{0,t}$ is the measure of a jump-process.*

Proof. Let $W = W_M$ be defined as

$$W(x) = \int_{\mathcal{R}^d} e^{ipx}\,M(dp). \tag{2.8}$$

The function $\exp\{tV\}$ is the Fourier transform of the measure $\delta_0 + t\mu + \frac{t^2}{2}\mu\star\mu + \dots$ which can be denoted by $\exp^\star(t\mu)$ (it is equal to the sum of the standard exponential series, but with the convolution of measures instead of the standard multiplication). Clearly $\|\exp^\star(t\mu)\| \le \|\exp^\star(t\bar{f}M)\|$, where we denoted by $\bar{f}$ the supremum of the function f, and both these series are convergent series in the Banach algebra $\mathcal{M}(\mathcal{R}^d)$. Therefore $\|e^{Vt}\| \le \|e^{Wt}\| \le \exp\{t\bar{f}\|\mu\|\}$, and consequently V is a regular CCE. Moreover, the same estimate shows that the measure on the path space corresponding to the CCE V is absolutely continuous with respect to the measure on the path space corresponding to the CCE W. But the latter coincides up to a positive constant multiplier with the probability measure of the compound Poisson process with the Lévy measure M defined by the equation

$$\frac{\partial u}{\partial t} = \left(W\left(\frac{1}{i}\frac{\partial}{\partial y}\right) - \lambda_M \right) u, \tag{2.9}$$

where $\lambda_M = M(\mathcal{R}^d)$, or equivalently

$$\frac{\partial u}{\partial t} = \int (u(y+\xi) - u(y))\, M(d\xi). \tag{2.10}$$

It remains to note that as is well known the measures of compound Poisson processes are concentrated on piecewise-constant paths.

Therefore, we have two different classes (essentially different, because they obviously are not disjoint) of regular CCE: those given by the Lévy-Khintchine formula, and those given by Proposition 2.4. It is easy to prove that one can combine these regular CCEs, more precisely that the class of regular CCE is a convex cone, see [K1]. □

Let us apply the simple results obtained sofar to the case of the pseudo-differential equation of the Schrödinger type

$$\frac{\partial \tilde{u}}{\partial t} = -G(-\Delta)^\alpha \tilde{u} + \left(A, \frac{\partial}{\partial x} \right) \tilde{u} + V(x)\tilde{u}, \tag{2.11}$$

where G is a complex constant with a non-negative real part, α is any positive constant, A is a real-valued vector (if $Re\, G > 0$, then A can be also complex- valued), and V is a complex-valued function of form (2.6). The standard Schrödinger equation corresponds to the case $\alpha = 1$, $G = i$, $A = 0$ and V being purely imaginary. We consider a more general equation to include the Schrödinger equation, the heat equation with drifts and sources, and also their stable (when $\alpha \in (0,1)$) and complex generalizations in one formula. This general consideration also shows directly how the functional integral corresponding to the Schrödinger equation can be obtained by the analytic continuation from the functional integral corresponding to the heat equation, which gives a connection with other approaches to the path integration. The equation on the inverse Fourier transform

$$u(y) = (2\pi)^{-d} \int_{\mathcal{R}^d} e^{-iyx} \tilde{u}(x)\, dx$$

of $\tilde{u}$ (or equation (2.11) in momentum representation) clearly has the form

$$\frac{\partial u}{\partial t} = -G(y^2)^\alpha u + i(A, y)u + V(\frac{1}{i}\frac{\partial}{\partial y})u. \tag{2.12}$$

One easily sees that already in the trivial case $V = 0$, $A = 0$, $\alpha = 1$, equation (2.11) defines a regular semigroup only in the case of real positive G, i.e. only in the case of the heat equation. It turns out however that for equation (2.12) the situation is completely different. The following simple result (obtained from Proposition 2.3 and the Trotter formula, see [K2] for details) generalizes the corresponding result from [MCh] on the standard Schrödinger equation to equation (2.11).

PROPOSITION 2.5. *The solution to the Cauchy problem of equation (2.12) can be written in the form of a complex Feynman-Kac formula*

$$u(t,y) = \int \exp\left\{ -\int_0^t [G(q(\tau)^2)^\alpha - (A, q(\tau))]\, d\tau \right\} u_0(q(t)) D_y^{0,t}(dq(.)), \tag{2.13}$$

where D_y is the measure of the jump process corresponding to equation (2.7).

Examples including the Schrödinger equation for a anharmonic oscillator and also some stochastic Schrödinger equations can be found in [K1], [K3], see also Section 4. In principle, the method covers all Schrödinger equations, because in the spectral representation of an arbitrary self-adjoint operator A, this operator is the operator of multiplication by a continuous real function $f(x)$ in $L^2(X)$ with some locally compact space X. The solution to the equation $\dot{\phi} = iA\phi$ is given by the multiplication by $\exp\{itf(x)\}$, and this family of operators defines a regular semigroup (in the above sense) on $C_0(X)$. However in practice, finding a spectral representation is a difficult task, and moreover one often wishes to find a solution in some given (physically natural) representation (e.g. in the position, momentum, or the occupation number representation). Hence, it is worth noticing that the method works generally for the equation of the type

$$\frac{\partial \phi}{\partial t} = i(A - B)\phi,$$

where A is a self-adjoint operator with the spectral representation in $L^2(X)$ and where B defines a bounded operator in $C_0(X)$ in this representation. Let us notice for conclusion that as any complex measure has a density with respect to its total variation measure, it is easy to rewrite the integral in (2.13) as an integral over a positive measure (see [ChQ], [PQ], [K2]).

Remark. Let us point out the connection with the well known infinite oscillatory integrals of Albeverio and Hoegh-Krohn. This approach works for potentials V of the Schrödinger equation belonging to the space $\mathcal{F}(\mathcal{R}^d)$, i.e. being given by (2.6), and is based on the possibility to represent the function $\exp\{\int_0^t V(y(s))ds\}$ (as a function of a curve $y(.)$) as the Fourier transform of a finite measure M_V on the Cameron-Martin Hilbert space of curves with square integrable derivatives. The theory developed above yields a precise description of this measure M_V. Namely, as was noted in [M] and is easy to show, the function $\exp\{\int_0^t V(y(s))ds\}$ is the Fourier-Feynman transform of the measure of the pure jump process generated by μ, i.e. of the measure from (2.13). Passing from momentums (velocities) to positions leads to the measure M_V on the Cameron-Martin space: it is concentrated on piecewise linear paths (denoted CPL in the next section), with the jumps of derivatives being distributed according to the measure μ from (2.6).

3. Regularization and the Fock space lifting. In this section we give an alternative direct and remarkably elementary construction of the measures on path spaces associated with pure jump processes bypassing the general theory of the previous section. In particular, no hard results as the Tikhonov or the Riesz-Markov theorems will be used.

Let CPL denote the set of continuous piecewise linear paths (broken lines) and let $CPL^{x,y}(0,t)$ denote the class of paths $q : [0,t] \mapsto \mathcal{R}^d$ from CPL joining x and y in time t, i.e. such that $q(0) = x$, $q(t) = y$. By $CPL_n^{x,y}(0,t)$ we denote the subclass consisting of all paths from $CPL^{x,y}(0,t)$ that have exactly n jumps of their derivative. Obviously,

$$CPL^{x,y}(0,t) = \cup_{n=0}^{\infty} CPL_n^{x,y}(0,t).$$

Notice also that the set $CPL^{x,y}(0,t)$ belongs to the Cameron-Martin space of curves that have derivatives in $L^2([0,t])$.

To any σ-finite measure M on $\mathcal{R}^d$ there corresponds a unique σ-finite measure M^{CPL} on $CPL^{x,y}(0,t)$, which is the sum of the measures M_n^{CPL} on $CPL_n^{x,y}(0,t)$, where M_0^{CPL} is just the unit measure on the one-point set $CPL_0^{x,y}(0,t)$ and each M_n^{CPL}, $n > 0$, is the direct product of the Lebesgue measure on the simplex Sim_t^n of the jump times $0 < s_1 < ... < s_n < t$ of the derivatives of the paths $q(.)$ and of n copies of the measure M on the values $q(s_j)$ of the paths at these times. In other words, if

$$q(s) = q_{\eta_1 \dots \eta_n}^{s_1 \dots s_n}(s) = \eta_j + (s - s_j)\frac{\eta_{j+1} - \eta_j}{s_{j+1} - s_j}, \quad s \in [s_j, s_{j+1}] \tag{3.1}$$

(where $s_0 = 0, s_{n+1} = t, \eta_0 = x, \eta_{n+1} = y$) is a typical path in $CPL_n^{x,y}(0,t)$ and Φ is a functional on $CPL^{x,y}(0,t)$, then

$$\begin{aligned}\int_{CPL^{x,y}(0,t)} &\Phi(q(.))M^{CPL}(dq(.)) \\ &= \sum_{n=0}^{\infty} \int_{CPL_n^{x,y}(0,t)} \Phi(q(.))M_n^{CPL}(dq(.)) \\ &= \sum_{n=0}^{\infty} \int_{Sim_t^n} ds_1...ds_n \int_{\mathcal{R}^d} ... \int_{\mathcal{R}^d} M(d\eta_1)...M(d\eta_n)\Phi(q(.)).\end{aligned} \tag{3.2}$$

Similarly (see [K2],[K3]) one can construct measures on piece-wise constant paths. Using these measures one easily gets a mathematically rigorous representation to the solutions to a rather general one-dimensional Schrödinger equation (see [K2]), but the class of finite dimensional Schrödinger equations that can be treated by these measures directly is rather restrictive. It includes,of course, the equations with potentials satisfying the Ito's complex measure conditions (see e.g. previous section). In general, a regularization is required. Seemingly, the most physically natural regularization consists in the introduction of a continuous quantum observation (see [K2], [K3] and for physical discussion also [Me]) leading to the path integral representation of the Belavkin quantum filtering equation (see [AKS]). However, for simplicity we shall use a technically much more transparent regularization by complex times, e.g. instead of the Schrödinger equation ((1.1) with $G = 1$) we shall consider the equation

$$\frac{\partial \psi}{\partial t} = \frac{1}{2}(i+\epsilon)\Delta\psi - (i+\epsilon)V(x)\psi. \tag{3.3}$$

To stress the wide applicability of the method we shall work directly with singular potentials that are given by Radon measures.

Following essentially [A] we shall introduce the dimensionality of a Borel measure V on $\mathcal{R}^d$ as the least upper bound of all positive numbers α such that there exists a constant $C = C(\alpha)$ such that

$$V(B_r(x)) \leq Cr^\alpha \tag{3.4}$$

for all $x \in \mathcal{R}^d$ and all $r > 0$. The dimensionality will be denoted by $dim(V)$. The following result is proved in [K3] and in a more general situation (with magnetic fields) in [K5], the main ingredient of the proof being the observation that the representation of the path integral as the sum of finite-dimensional integrals (3.2) corresponds in this context to the usual series of perturbation theory solving the Schrödinger equation.

PROPOSITION 3.1. *Let V be a finite Borel measure on $\mathcal{R}^d$ with $dim(V) > d-2$. Then*

(i) to the operator $-\Delta + V$ one can give a meaning as a rigorously defined self-adjoint operator in $L^2(\mathcal{R}^d)$;

(ii) for arbitrary $\epsilon > 0$ there exists a unique solution $G_\epsilon(t,x,y)$ to the Cauchy problem of equation (3.3) with Dirac initial data $\delta(x-y)$. This solution (i.e. the Green functions for (3.3)) is uniformly bounded for all (x,y) and t in any compact interval of the open half-line and is expressed in terms of path integrals as

$$G_\epsilon(t,x,y) = \int_{CPL^{x,y}(0,t)} \Phi_\epsilon(q(.))V^{CPL}(dq(.)), \tag{3.5}$$

with

$$\begin{aligned}
\Phi_\epsilon(q(.)) &= \prod_{j=1}^{n+1}(2\pi(s_j-s_{j-1})(i+\epsilon))^{-d/2}(-(\epsilon+i))^n \exp\{-\frac{1}{2(i+\epsilon)}\int_0^t \dot q^2(s)\,ds\} \\
&= \prod_{j=1}^{n+1}(2\pi(s_j-s_{j-1})(i+\epsilon))^{-d/2}(-(\epsilon+i))^n \\
&\quad\times \exp\{-\sum_{j=1}^{n+1}\frac{|\eta_j-\eta_{j-1}|^2}{2(i+\epsilon)(s_j-s_{j-1})}\};
\end{aligned} \tag{3.6}$$

(iii) for arbitrary $\psi_0 \in L^2(\mathcal{R}^d)$ the solution $\psi_0(t,s)$ of the Cauchy problem for equation (3.3) with the initial data ψ_0 and $\epsilon > 0$ has the form

$$\psi_\epsilon(t,x) = \int_{CPL^{x,y}(0,t)}\int_{\mathcal{R}^d}\psi_0(y)\Phi_\epsilon(q(.))V^{CPL}(dq(.))dy, \tag{3.7}$$

and the solution to (3.3) with $\epsilon = 0$ can be expressed as an improper (not absolutely convergent) path integral

$$\psi(t,x) = \lim_{\epsilon\to 0_+}\int_{CPL^{x,y}(0,t)}\int_{\mathcal{R}^d}\psi_0(y)\Phi_\epsilon(q(.))V^{CPL}(dq(.))dy, \tag{3.8}$$

where the limit is understood in L^2-sense.

The examples of interest (see relevant references and a more detailed discussion in [K3]) are given by measures V on $\mathcal{R}^3$ concentrated on a Brownian path, potentials being the finite sums of the Dirac measures of closed hypersurfaces in $\mathcal{R}^d$ and measures with densities $V \in L^\infty(\mathcal{R}^d) + L^p(\mathcal{R}^d)$ with $p > d/2$, which includes, in particular, the Coulomb potential in $d = 3$. In [K5] one can find also the two sided exponential estimates for the Green function (3.5).

We shall discuss now the Fock space lifting of the formulae above and the resulting representation of (3.5) in terms of the Wiener measure. The paths of the spaces CPL are parametrised by finite sequences $(s_1,x_1),...,(s_n,x_n)$ with $s_1 < ... < s_n$ and $x_j \in \mathcal{R}^d$, $j = 1,...,d$. Denote by $\mathcal{P}^d$ the set of all these sequences and by $\mathcal{P}^d_n$ its subset consisting of sequences of the length n. Thus, functionals on the path space CPL can be considered as functions on $\mathcal{P}^d$. To each measure ν on $\mathcal{R}^d$ there corresponds a measure $\nu_{\mathcal{P}}$ on $\mathcal{P}^d$ which is the sum of the measures ν^n on $\mathcal{P}^d_n$, where ν^n are the product measures $ds_1...ds_n d\nu(x_1)...d\nu(x_n)$. The Hilbert space $L^2(\mathcal{P}^d,\nu_{\mathcal{P}})$ is known to be isomorphic to the Fock space Γ^d_ν over the Hilbert space $L^2(\mathcal{R}_+\times\mathcal{R}^d, dt\times\nu)$ (which is isomorphic to the space of square integrable functions on $\mathcal{R}_+$ with values in $L^2(\mathcal{R}^d,\nu)$). Therefore, square integrable functionals on CPL can be considered as vectors in the Fock space $\Gamma^d_{V(dx)}$. It is well known that the Wiener, Poisson, general Lévy and many other interesting processes can be naturally realized in a Fock space: the corresponding probability space is defined as the spectrum of a commutative von Neumann algebra of bounded linear operators in this space. For example, the isomorphism between $\Gamma^0 = \Gamma(L^2(\mathcal{R}_+))$ and $L^2(W)$, where W is the Wiener space of continuous real functions on half-line is given by the Wiener chaos decomposition, and a construction of a Lévy process with the Lévy measure ν in the Fock space Γ_ν can be found in [Mey]. Therefore, using Fock space representation, one can give different stochastic representations for path integrals over CPL rewriting them as expectations with respect to different stochastic processes.

Of course, a straightforward probabilistic interpretation of the infinite-dimensional integral(3.5) is given in terms of an expectation with respect to a compound Poisson process. The following statement is a direct consequence of Proposition 3.1 and the standard properties of the Poisson processes.

Proposition 3.2. *Suppose a measure V is finite and satisfies the assumptions of Proposition 3.1. Let $\lambda_V = V(\mathcal{R}^d)$. Let paths of CPL are parametrized by (3.1) and let*

E denote the expectation with respect to the process of jumps η_j which are identically independently distributed according to the probability measure V/λ_V and which occur at times s_j from $[0,t]$ that are distributed according to the Poisson process of intensity λ_V. Then the function (3.7) can be written in the form

$$\psi_\epsilon(t,x) = e^{t\lambda_V} \int_{\mathcal{R}^d} \psi_0(y) E(\Phi_\epsilon(q(.)))\, dy.$$

A much more involved interpretation of our path integral can be given in terms of the Wiener measure. In order to obtain such a representation for the Green function (3.5) let us first rewrite it as the integral of an element of the Fock space $\Gamma^0 = L^2(Sim_t)$ with $Sim_t = \cup_{n=0}^\infty Sim_t^n$ (which was denoted $\mathcal{P}^0$ above), where Sim_t^n is as usual the simplex $\{0 < s_1 < ... < s_n < t\}$. Let

$$g_0^V = g_0^V(t;x,y) = (2\pi t(i+\epsilon))^{-d/2} \exp\left\{ -\frac{(x-y)^2}{2t(i+\epsilon)} \right\}$$

and let

$$g_n^V(s_1,...,s_n) = g_n^V(s_1,...,s_n;t;x,y) = \int_{\mathcal{R}^{nd}} \Phi_\epsilon(q_{\eta_1...\eta_n}^{s_1...s_n})\, d\eta_1...d\eta_n$$

for $n = 1,2,...$, where Φ_ϵ and $q_{\eta_1...\eta_n}^{s_1...s_n}$ are given by (3.6) and (3.1). Considering the series of functions $\{g_n^V\}$ as a single function g^V on Sim_t we shall rewrite the r.h.s. of (3.6) in the following concise notation:

$$\int_{Sim_t} g^V(s)\, ds = \sum_{n=0}^\infty \int_{Sim_t^n} g_n^V(s_1,...,s_n)\, ds_1...ds_n.$$

The Wiener chaos decomposition theorem states (see e.g. [Mey]) that, if $dW_{s_1}...dW_{s_n}$ denotes the n-dimensional stochastic Wiener differential, then to each $f = \{f_n\} \in L^2(Sim_t)$ there corresponds an element $\phi_f \in L^2(\Omega_t)$, where Ω_t is the Wiener space of continuous real functions on $[0,t]$, given by the formula

$$\phi_f(W) = \sum_{n=0}^\infty \int_{Sim_t^n} f_n(s_1,...,s_n)\, dW_{s_1}...dW_{s_n},$$

or in concise notations

$$\phi_f(W) = \int_{Sim_t} f(s)\, dW_s.$$

Moreover the mapping $f \mapsto \phi_f$ is an isometric isomorphism, i.e.

$$E_W(\phi_f(W)\bar{\phi}_g(W)) = \int_{Sim_t} f(s)\bar{g}(s)\, ds,$$

where E_W denotes the expectation with respect to the standard Wiener process. Since (see e.g. again [Mey])

$$\int_{Sim_t} dW_s = e^{W(t)-t/2},$$

it follows that if the function g^V belongs not only to $L^1(Sim_t)$ (as is the case in Proposition 3.1) but also to $L^2(Sim_t)$, then formula (3.5) can be rewritten as

$$G_\epsilon(t,x,x_0) = E_W(\phi_{g^V} \exp\{W(t) - t/2\}). \tag{3.9}$$

In particular, the following result holds.

Proposition 3.3. *Under the assumptions of Proposition 3.1 suppose additionally that V is finite and $\dim(V) > d-1$. Then the Green function (3.5) can be written in form (3.9).*

Proof. Due to the discussion above, one needs to show that the corresponding function g^V belongs not only to $L^1(Sim_t)$ (as follows from Proposition 3.1), but also to $L^2(Sim_t)$, i.e. that the series

$$\sum_{n=0}^{\infty} \int_{Sim_t^n} (g_n^V(s_1, ..., s_n))^2 \, ds_1 ... ds_n$$

converges. From the definition of g_n^V it follows that

$$g_n^V(s_1, ..., s_n; t; x, y) = (2\pi(t-s_n)(i+\epsilon))^{-d/2} \times \int \exp\{-\frac{(x-\eta)^2}{2(i+\epsilon)(t-s_n)}\} g_{n-1}^V(s_1, ..., s_{n-1}; s_n; \eta, y) V(d\eta)$$

for $n = 1, 2, ...$ In particular,

$$|g_1^V(s;t;x,y)| = (2\pi)^{-d}((1+\epsilon^2)(t-s)s)^{-d/2} \int \exp\left\{-\frac{\epsilon(x-\eta)^2}{2(1+\epsilon^2)(t-s)} - \frac{\epsilon(\eta-y)^2}{2(1+\epsilon^2)s}\right\} V(d\eta).$$

Simple manipulations imply

$$\begin{aligned} |g_1^V(s;t;x,y)| &= (2\pi)^{-d}((1+\epsilon^2)(t-s)s)^{-d/2} \\ &\times \int \exp\left\{-\frac{\epsilon t}{2(1+\epsilon^2)(t-s)s}\left(\eta - \frac{sx+(t-s)y}{t}\right)^2\right\} \\ &\times \exp\left\{-\frac{(x-y)^2}{2(i+\epsilon)t}\right\} V(d\eta) \\ &= (2\pi)^{-d}((1+\epsilon^2)(t-s)s)^{-d/2} \exp\left\{-\frac{\epsilon(x-y)^2}{2(1+\epsilon^2)t}\right\} \\ &\times \int \exp\left\{-\frac{t\epsilon\eta^2}{2(1+\epsilon^2)s(t-s)}\right\} \tilde{V}(d\eta), \end{aligned}$$

where $\tilde{V}$ is obtained from V by shifting on $(sx+(t-s)y)/t$. To estimate this integral, let us write the r.h.s. as $I_1 + I_2$ by dividing the domain of integration into two parts $D_1 \cup D_2$ with

$$D_1 = \left\{\eta : \frac{t\epsilon\eta^2}{2(1+\epsilon^2)s(t-s)} \le \left(\frac{t}{s(t-s)}\right)^{\omega} \Longleftrightarrow |\eta| \le \sqrt{2(1+\epsilon^2)/\epsilon}\left(\frac{s(t-s)}{t}\right)^{(1-\omega)/2}\right\}.$$

By the assumption $\dim(V) > d-1$ (and as the dimensionality is not changed by a shift) it follows that

$$I_1 \le C(2\pi)^{-d}((1+\epsilon^2)(t-s)s)^{-d/2} \exp\left\{-\frac{\epsilon(x-y)^2}{2(1+\epsilon^2)t}\right\}\left(\frac{s(t-s)}{t}\right)^{(1-\omega)\alpha/2}$$

with some $\alpha > d-1$ and $C > 0$, and consequently

$$I_1 \le C(2\pi)^{-d}((1+\epsilon^2)^{-d/2} t^{-(1-\omega)\alpha/2}[s(t-s)]^{((1-\omega)\alpha-d)/2} \exp\left\{-\frac{\epsilon(x-y)^2}{2(1+\epsilon^2)t}\right\}.$$

Moreover, since

$$\exp\left\{-\left(\frac{t}{s(t-s)}\right)^{\omega}\right\} \le C(\omega,\beta)\left(\frac{s(t-s)}{t}\right)^{\beta}$$

for arbitrary $\beta > 0$ and some constant $C(\omega, \beta)$ (and because V was supposed to be a finite measure), it follows that I_2 does not exceed I_1 (up to a constant) and hence the above estimate for I_1 is also an estimate for g_1^V. Consequently

$$|g_1^V(s;t;x,y)| \le C|g_0^V(t;x,y)|t^{-((1-\omega)\alpha-d)/2}[s(t-s)]^{((1-\omega)\alpha-d)/2}$$

with some constant C (depending on $d, \alpha, \omega, \epsilon$). Using the recursive formula for g_n^V one obtains by induction the estimate

$$|g_n^V(s_1,...,s_n;t;x,y)| \le C^n|g_0^V(t;x,y)|[s_1(s_2-s_1)...(s_n-s_{n-1})(t-s_n)/t]^{((1-\omega)\alpha-d)/2}.$$

It remains to show that the series

$$S = \sum_1^\infty C^n \int_{\{0<s_1<...<s_n<t\}} [s_1(s_2-s_1)...(s_n-s_{n-1})(t-s_n)]^{\beta-1} ds_1...ds_n$$

converges, where $\beta - 1 = (1-\omega)\alpha - d$. One can choose $\omega \in (0,1)$ in such a way that $\beta > 0$, because $\alpha > d-1$. Calculating the terms of this series by induction and using the Euler β-function

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx = \Gamma(p)\Gamma(q)/\Gamma(p+q),$$

yields

$$S = \sum_{n=1}^\infty C^n t^{(n+1)\beta-1} B(\beta,\beta)B(\beta,2\beta)...B(\beta,n\beta) = \sum_{n=1}^\infty C^n \frac{(t^\beta\Gamma(\beta))^{n+1}}{t\Gamma((n+1)\beta)}.$$

At last, Stirling's formula for $\Gamma((n+1)\beta)$ implies the convergence of this sum for all $t > 0$. This completes the proof of Proposition 3.3. □

Remarks. 1. The assumption that the measure V is finite was made for technical simplifications, and seemingly can be removed. 2. The assumption $dim(V) > d-1$ is essential for (3.9) to be true. However, one can regularize (3.9) appropriately to include more general V (see [K2]). 3. There is seemingly some overlap of ideas between our construction of the Wiener path integral for the Schrödinger equation and the theory of rough paths of T. Lyons [L]. 4. It is worth noting that the natural topology on the path space CPL is the one induced from the uniform topology of continuous paths (or from the Hilbert space topology of the Cameron-Martin space). This topology enjoys the following properties: (i) it is compatible with the measure, (ii) when reduced to any finite-dimensional simplex it yields its natural Euclidean topology, (iii) any simplex Sim_t^n is the boundary for the simplex Sim_t^{n+1}, (iv) the topology is not locally compact, but the whole space is a countable union of locally compact spaces.

4. Two remarks on parabolic equations in momentum representation. As was already mentioned, the first definition of the Feynman path integral representing solutions for the Schrödinger equation as a genuine Lebesgue integral arising from a pure jump Markov process was given in [M], [MCh]. This integral was defined for the Scgrödinger equation in momentum representation with potentials satisfying Ito's complex measure condition. This was an important breakthrough. As for the diffusion equation the familiar Feynman-Kac representation exists, the analogous result for the diffusion equation in momentum representation did not receive much attention. This is also due to the fact that unlike Schrödinger equation the momentum representation for the diffusion equation often does not seem very natural physically, though it does make sense in the study of tunnel effects in quantum mechanics. A recent paper [Ch] is devoted to an interesting detailed analysis of the underlying jump processes for diffusion equations in momentum representation under Ito's complex measure condition for sources (potentials) and drifts. In this section I like to point out two simple observations about this theory. Firstly, in some cases one can get meaningful path integral representation for diffusion equations in momentum representation even when the source does not satisfy Ito's condition (so that the underlying process is not of pure jump type) and when an unbounded source prevents the possibility of using the standard Feynman-Kac formula with the Wiener measure. Secondly, a curious asymptotic formula can be obtained by passing to a central limit in a pure jump path integral representation for the diffusion equation. Moreover, instead of just diffusion equations one can directly consider more general parabolic differential and even pseudo-differential equations without any increase in the complexity.

1. Stable laws for parabolic equations. Consider the pseudo-differential parabolic equation

$$\frac{\partial \psi}{\partial t} = -G|\Delta|^{\alpha}\psi + |x|^{\beta}\psi \tag{4.1}$$

in $\mathcal{R}^d$, where $\alpha > 0$, $G > 0$, $\beta \in (0,1)$ are given constants (in fact, instead of $|\Delta|^{\alpha}$ one can take even more general operators $f(|\Delta|)$ with non-negative continuous function f). Since

$$|x|^{\beta} = c\int_{\mathcal{R}^d} (e^{ix\xi} - 1)|\xi|^{-d-\beta}\, d\xi$$

with some c depending on d and β (see any book discussing stable processes, e.g. [K2]), in momentum representation, i.e. for $u(p) = \int e^{-ipx}\psi(x)\, dx$, the equation (4.1) takes the form

$$\frac{\partial u}{\partial t}(p) = -G|p|^{2\alpha}u(p) + c\int (u(p+\xi) - u(p))|\xi|^{-d-\beta}\, d\xi. \tag{4.2}$$

As the second operator in this equation generates the Feller semigroup of a β-stable Lévy process, the following result is straightforward.

PROPOSITION 4.1. *Solution to (4.2) with an arbitrary bounded initial function u_0 is given by the formula*

$$u(t,p) = E_p\left[\exp\{-G\int_0^t |y(s)|^{2\alpha}\, ds\}u_0(y(t))\right],$$

where E_p denotes the expectation with respect to the corresponding β-stable Lévy motion starting at p.

2. A central limit for pure jump path integrals. Consider now the equation

$$h^2\frac{\partial \psi}{\partial t} = -Gh^{2+2\alpha}|\Delta|^{\alpha}\psi + V(x)\psi, \tag{4.3}$$

with a small parameter $h > 0$, α, G are again positive constants and $V(x) = \int e^{ix\xi}M(d\xi)$ with some finite positive measure M, i.e. V satisfies Ito's condition. Suppose also that M is symmetric in the sense that $\int \xi M(d\xi) = 0$, and that it has finite moments at least up to the third order. Denote by $\nu(M) = \{\nu_{ij}(M)\}$ the matrix of the second moments $\int \xi_i\xi_j M(d\xi)$ of M. Performing the h-Fourier transform (which is usual in the theory of semiclassical asymptotics, see e.g. [MF]), i.e. passing to the function

$$u_h(t,p) = \int e^{-ipx/h}\psi(t,x)\, dx \Longleftrightarrow \psi_h(t,x) = (2\pi h)^{-d}\int e^{ipx/h}u(t,p)\, dp$$

yields for $u_h(t,p)$ the equation

$$\frac{\partial u}{\partial t} = -G|p|^{2\alpha}u + \frac{1}{h^2}V\left(\frac{h}{i}\frac{\partial}{\partial p}\right)u \tag{4.4}$$

for $u_h(t,p)$.

PROPOSITION 4.2. *For the solution of (4.4) with the initial condition u_0 the following asymptotic formula holds:*

$$\lim_{h\to 0} \exp\{th^{-2}\|M\|\}u_h(t,p) = E_W \exp\left\{-G\int_0^t |p+W(s)|^{2\alpha}\,ds\right\}u_0(p+W(t)), \tag{4.5}$$

where E_W denotes the expectation with respect to the d-dimensional Wiener process $W(s)$ with the covariance matrix $\nu(M)$, and where $\|M\|$ is, of course, the full measure $M(\mathcal{R}^d)$.

Proof. Using the properties of pure jump processes, one can write the solution to (4.4) with the initial condition u_0 as the path integral

$$u_h(t,p) = \exp\{th^{-2}\|M\|\}E_h^p \exp\left\{-G\int_0^t |y(s)|^\alpha\,ds\right\}, \tag{4.6}$$

where E_h^p denotes the expectation with respect to the process of jumps which are identically independently distributed according to the probability measure $\mu(dp) = M(d(p/h))/\|M\|$ and which occur at times s_j from $[0,t]$ that are distributed according to the Poisson process of intensity $h^{-2}\|M\|$. Observing that

$$\begin{aligned} h^{-2}V(hx) &= h^{-2}\int(e^{ihx\xi}-1)M(d\xi) + h^{-2}\|M\| \\ &= -\sum_{i,j=1}^d x_i x_j \nu_{ij} + O(h) + h^{-2}\|M\|, \end{aligned}$$

we conclude that the characteristic exponent of our compound Poisson process converges, as $h \to 0$, to the characteristic exponent $-(\nu(M)x,x)$ of the Wiener process indicated above. This implies the convergence of the corresponding measures on trajectories, and (4.5) follows. □

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