From intuitionistic to point-free topology: some remarks on the foundation of homotopy theory

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1 Introduction

Brouwer's pioneering results in topology, e.g. invariance of dimension, were developed within a classical framework of mathematics. Some years later he explained that "in his topological work he tried to use only methods which he expected could be made constructive" [10, p. XIV]. It seems that very little of algebraic topology, homotopy and homology, has actually been developed constructively in any detail, or, at any rate, found its way to publication. In the comprehensive treatise *Foundations of Constructive Mathematics,* Beeson [4, pp. 26 – 27] writes however

> "The classical results of algebraic topology do not require the general concept of a topological space. If we content ourselves to treat metric spaces, then the standard treatments of the homotopy and homology groups are quite straightforwardly constructive, e.g. Greenberg [1967][17]. One draws all the usual corollaries, e.g. $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic unless $n = m$; [...] It is quite essential to deal with uniformly continuous functions, and not just with continuous functions."

To be able to make certain quotient and glueing constructions it is necessary to have a constructive theory of more general topological spaces than metric spaces. As argued by many authors, locales, or point-free topologies, should provide a good constructive foundation for topological theories; see [2, 8, 3, 19, 20]. The point-free approach can also be regarded as a (re)interpretation of Brouwer's theory of choice sequences (Martin-Löf [18], Coquand [6, p. 31]), whereby one regains covering compactness results, e.g. the Heine-Borel theorem, but with a fully constructive interpretation. For some background and history of point-free topology from lattice-theoretic origins, see [13, 14]. It is interesting to note that such ideas were

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also introduced in the intuitionistic school, first by Freudenthal [9], and then further developed by Troelstra [23]. Already in [9], which treats compact Hausdorff spaces, the characteristics of point-free topology were clearly visible, some class of open sets form the fundamental objects, points are merely derived and covering relations are defined without reference to points. There is also a development of general topology [24] using non-effective principles, like Brouwer's Fan Theorem (which is valid in certain toposes).

Fundamental groups for locales have been constructed by W. He [12]. Various generalisations and equivalents of fundamental groups for locales and toposes are studied by Kennison [16]. Most prominently a point-free approach to paths and homotopy is developed. These papers are not explicitly concerned with constructivity, and certainly not predicativity. We shall here investigate a key element in the construction of the fundamental group for a formal topology, combining those foundational perspectives. Formal topology is a predicative version of locale theory, due to Martin-Löf and Sambin; see [22]. We shall therefore work within the framework of constructive mathematics in the style of Bishop (cf. [5]), which in turn can be formalised in constructive type theory, or constructive set theory. In fact some of the results, Theorem 3.4 and 4.1, do not use any kind of choice principle and are therefore valid in any topos (cf. [15]).

2 Paths in spaces

In the construction of the fundamental group of a space, joining and deformation of paths in a space are basic operations.

A space Y is said to have the *path joining property (PJP)* if $f : [a,b] \longrightarrow Y$ and $g : [b,c] \longrightarrow Y$ are continuous functions with $f(b) = g(b)$, then there exists a unique continuous function $h : [a,c] \longrightarrow Y$, with $h(t) = f(t)$ for $t \in [a,b]$, and $h(t) = g(t)$ for $t \in [b,c]$. Classically, every topological space satisfies PJP since we can define h by cases and then check its continuity. Constructively, case-wise definition is a priori not possible. Furthermore, uniform continuity, in some form, is the basic notion. In this section, the discussion is restricted to metric spaces.

Lemma 2.1 *Let X be a metric space and let Y be a complete metric space. If $D \subseteq X$ is a dense subset, and $f : D \longrightarrow Y$ is uniformly continuous, then there is a unique continuous function $h : X \longrightarrow Y$ so that $h(t) = f(t)$ for all $t \in D$.*

Proof. The existence follows from [5, Lemma 3.3.7]. Uniqueness is direct from continuity, using the inequality $|h_1(x) - h_2(x)| \leq |h_1(x) - h_1(t)| + |h_2(t) - h_2(x)|$, where $t \in D$. $\square$

A dense set in the interval $[a,b]$ is, for instance, $D_{a,b} = \{a + (b-a)k2^{-n} : n = 1,2,3,\ldots; k = 0,\ldots,2^n\}$.

Theorem 2.2 *PJP is valid for complete metric spaces Y.*

Proof. Let $f : [a,b] \longrightarrow Y$ and $g : [b,c] \longrightarrow Y$ be continuous functions. The set $E = D_{a,b} \cup D_{b,c}$ is dense in $[a,c]$. Since $D_{a,b} \cap D_{b,c}$ contains only the element b, and $f(b) = g(b)$, the following is indeed a definition of a function $k : E \longrightarrow Y$. Let $k(t) = y$ if, and only if,

$$t \in D_{a,b} \text{ and } f(t) = y, \text{ or } t \in D_{b,c} \text{ and } g(t) = y.$$

If ω_f and ω_g are the continuity moduli of f and g, respectively, then

$$\omega_k(\varepsilon) =_{\text{def}} \min(\omega_f(\frac{\varepsilon}{2}), \omega_g(\frac{\varepsilon}{2}))$$

is a modulus for k. Let h be the unique extension of k according to Lemma 2.1. Again by this lemma, h restricted to $[a,b]$ is f, and h restricted to $[b,c]$ is g. It is also the unique such h. $\square$

This method fails for general metric spaces.

Proposition 2.3 *There is a metric space Y, such that if the PJP is valid for Y, then for any real x*

$$x \leq 0 \textit{ or } x \geq 0.$$

Proof. Let Y be $[-1,0] \cup [0,1]$ as metric subspace of $\mathbb{R}$, and suppose that PJP is valid for Y. Let $f : [-1,0] \longrightarrow Y$ and $g : [0,1] \longrightarrow Y$ be inclusion maps. Take $h : [-1,1] \longrightarrow Y$ to be the unique common extension of those maps, as given by PJP. Thus $h_{\restriction[-1,0]} = f$ and $h_{\restriction[0,1]} = g$. We show that $h(x) = x$ for all $x \in [-1,1]$. Consider the inclusion $k : Y \longrightarrow [-1,1]$, and the composition $k \circ h : [-1,1] \longrightarrow [-1,1]$. Now $(k \circ h)_{\restriction[-1,0]} = k \circ f$, i.e. the inclusion of $[-1,0]$ into $[-1,1]$, and $(k \circ h)_{\restriction[0,1]} = k \circ g$, i.e. the inclusion of $[0,1]$ into $[-1,1]$. The identity map $\mathrm{id}_{[-1,1]}$ restricted similarly, gives obviously the same inclusions. By Theorem 2.2 for $[-1,1]$, we get $k \circ h = id_{[-1,1]}$. Thus $h(x) = k(h(x)) = x$ for all $x \in [-1,1]$. But $h(x) \in Y$, so for all $x \in [-1,1]$,

$$x \leq 0 \text{ or } 0 \leq x.$$

The desired conclusion follows, by noting that for any real x it holds that $x \in [-1,1]$, $x \leq -1/2$ or $1/2 \leq x$. $\square$

As the conclusion is non-constructive [5], the proposition provides a Brouwerian counterexample to the general validity of PJP.

A continuous deformation of a function into another function is a homotopy [1]. Composition of such two deformations can be regarded as generalised joining operation: It takes two continuous functions $f : X \times [a,b] \longrightarrow Y$ and $g : X \times [b,c] \longrightarrow Y$ which agree at the edges b as: $f(x,b) = g(x,b)$ for all $x \in X$. The operation should then join them by producing a continuous function $h : X \times [a,c] \longrightarrow Y$ so that h agrees with f and g on $X \times [a,b]$ respectively $X \times [b,c]$. It can be expressed in terms of maps only using a commutative diagram as follows. (This formulation will be useful in Section 3.) A pair of spaces (X,Y) has the *homotopy joining property* (HJP), if the diagram

$$\begin{array}{ccc} X & \xrightarrow{\langle 1_X, \hat{b}\rangle} & X \times [b,c] \\ {\scriptstyle \langle 1_X, \hat{b}\rangle}\downarrow & & \downarrow{\scriptstyle 1_X \times j_2} \\ X \times [a,b] & \xrightarrow[1_X \times j_1]{} & X \times [a,c] \end{array} \tag{1}$$

commutes, and for any uniformly continuous $F : X \times [a,b] \longrightarrow Y$ and $G : X \times [b,c] \longrightarrow Y$ with $F \circ \langle 1_X, \hat{b}\rangle = G \circ \langle 1_X, \hat{b}\rangle$, there exists a unique uniformly continuous $H : X \times [a,c] \longrightarrow Y$ so that $F = H \circ (1_X \times j_1)$ and $G = H \circ (1_X \times j_2)$. Here the $\hat{b}$'s are constant maps with value b on X, and the j's are inclusions of intervals. (The case where $X = 1$ (the one point metric space) is PJP for Y.)

Theorem 2.4 *HJP holds for (X,Y) when X is a metric space and Y is a complete metric space.*

Proof. Analogous to Theorem 2.2, using the dense set $X \times E$ in $X \times [a,c]$. □

Using this theorem it is possible to construct the fundamental group at a point of a complete metric space along standard lines [1]. Only the cases where X is a finite, closed interval are needed. HJP is used, in particular, when showing that the homotopy relation is transitive: Suppose that p and q are homotopic, and that q and r are homotopic. Then there are uniformly continuous $F : X \times [0,1] \longrightarrow Y$ and $G : X \times [1,2] \longrightarrow Y$ so that for all $x \in X$

$$F(x,0) = p(x) \quad F(x,1) = q(x) = G(x,1) \quad G(x,2) = r(x).$$

Then HJP gives the desired homotopy $H : X \times [0,2] \longrightarrow Y$ deforming p into r.

3 Formal topology

In contrast to the negative result of the previous section, the category of formal topologies satisfies HJP for all pairs of spaces (X,Y). Equivalently, it can be stated as: the diagram (1) is a pushout in this category, for every object X. This main result in shown in Section 4. In Section 3.1 we review of some of the basics of formal topology. Sections 3.2-3.4 prepares for the main result.

3.1 Basic definitions and results

Definition 3.1 A *formal topology* consists of a pre-order $X = (X, \leq)$ of *basic open neighbourhoods* and $\lhd \subseteq X \times \mathcal{P}(X)$, the *covering relation,* satisfying the four *covering conditions*

(Ref) $\quad a \in U \Longrightarrow a \lhd U,$

(Tra) $\quad a \lhd U, U \lhd V \Longrightarrow a \lhd V,$

(Loc) $\quad a \lhd U, a \lhd V \Longrightarrow a \lhd U_{\leq} \cap V_{\leq},$

(Ext) $\quad a \leq b \Longrightarrow a \lhd \{b\}.$

Here $U \lhd V \Leftrightarrow_{\text{def}} (\forall a \in U)\, a \lhd V$, and moreover, $Z_{\leq} =_{\text{def}} \{x \in X : (\exists z \in X)\, x \leq z\}$ is the *down-closure* of Z. Furthermore we require that the cover relation is *set-presented,* in the sense that there is a family $\{C(a,i)\}_{i \in I(a)}$ of subsets of X so that

$$a \lhd U \Longleftrightarrow (\exists i \in I(a))\, C(a,i) \subseteq U.$$

We write the components of a formal space $\mathcal{X}$ as $(X, \leq_{\mathcal{X}}, \lhd_{\mathcal{X}}, C_{\mathcal{X}})$, often omitting the set-presentation $C_{\mathcal{X}}$.

Define the mutual cover relation $U \sim V$ to hold iff $U \lhd V$ and $V \lhd U$. Let $Z_{\lhd} = \{x \in X : x \lhd Z\}$. A subset $Z \subseteq X$ is *saturated* if $Z_{\lhd} = Z$. The saturated subsets corresponds to elements in the associated locale. They may always be represented by subsets up to mutual covering, since $U \sim U_{\lhd}$. Any subset represents an open set in this way. A subset $Z \subseteq X$ is *down-closed* if $Z_{\leq} = Z$.

A pair (a, U), where $a \in X$ and $U \subseteq X$, is called a *covering axiom*. A formal topology $\mathcal{X}$ is *generated by a family of covering axioms* (a_i, U_i) $(i \in I)$, if $\lhd_{\mathcal{X}}$ is the smallest relation satisfying covering conditions and the axioms

$$a_i \lhd_{\mathcal{X}} U_i \quad (i \in I).$$

From the set-presentation of a formal topology $\mathcal{X}$ one can obtain a generating family of covering axioms $(b_j, V_j)_{j \in J}$ for $\mathcal{X}$ as follows. Let $J = \{(a,i) : a \in X, i \in I_{\mathcal{X}}(a)\}$ and put $b_{(a,i)} = a, V_{(a,i)} = C_{\mathcal{X}}(a,i)$. Conversely, one can show (see e.g. [21]) that if $\mathcal{X}$ satisfies axioms (Ref), (Tra), (Loc) and (Ext) and is generated by a set-indexed family of covering axioms, then $\mathcal{X}$ is a formal topology. It is often easier to exhibit a set of covering axioms, than a set-presentation.

3.1.1 Points

A *point* of $\mathcal{X}$ is a non-void subset $\alpha \subseteq S$ which is

(Fil) $\leq$-filtering, i.e. for $a, b \in \alpha$, there is $c \in \alpha$ with $c \leq a$ and $c \leq b$,

(Spl) such that α contains a neighbourhood from U, whenever $a \lhd U$ and $a \in \alpha$. (This is often expressed as: "a point splits any cover").

The points of a formal topology $\mathcal{X}$ form a class $\mathrm{Pt}(\mathcal{X})$, which, under certain conditions, is a set. For $a \in X$, let a^* denote the subclass of points in $\mathcal{X}$ satisfying $a \in \alpha$. For a subset $U \subseteq X$, let U^* denote the union of all the subclasses a^* for $a \in U$.

Lemma 3.2 *Any formal cover in $\mathcal{X}$ is a point-wise cover:*

$$a \lhd_{\mathcal{X}} U \Longrightarrow a^* \subseteq \cup U^*. \quad \square$$

The converse implication is not true in general, and this explains why results like the Heine-Borel Theorem are possible to prove in this setting. We say the covers of formal topology $\mathcal{X}$ are *order conservative,* if $a \leq_{\mathcal{X}} b$ whenever $a \lhd_{\mathcal{X}} \{b\}$. This notion is of course only interesting if $\leq$ has a simpler definition than $\lhd$. (Any formal topology is isomorphic to an order conservative one where $\lhd$ is the partial order.) The covers are *point-wise order conservative* if $a^* \subseteq b^*$ implies $a \leq_{\mathcal{X}} b$. In view of Lemma 3.2 the latter is a stronger property.

3.1.2 Continuous morphisms

Let $\mathcal{S} = (S, \leq, \lhd)$ and $\mathcal{T} = (T, \leq', \lhd')$ be formal topologies. A relation $F \subseteq S \times T$ is a *continuous mapping* $\mathcal{S} \longrightarrow \mathcal{T}$ if

(A1) $a F b, b \lhd' V \Longrightarrow a \lhd F^{-1} V$,

(A2) $a \lhd U, x F b$ for all $x \in U \Longrightarrow a F b$,

(A3) $S \lhd F^{-1} T$,

(A4) $a F b, a F c \Longrightarrow a \lhd F^{-1}(b_{\leq'} \cap c_{\leq'})$.

Here $F^{-1} Z = \{x \in S : (\exists y \in Z) x R y\}$ and $z_{\leq'}$ is $\{z\}_{\leq'}$. It is possible to replace the quantifications over the subsets U and V, with quantification over the set-presentations of $\mathcal{S}$ and $\mathcal{T}$ respectively.

Some equivalent versions of the above axioms are

(A1') $b \lhd' V \Longrightarrow F^{-1} b \lhd F^{-1} V$,

(A2') $a \lhd F^{-1} b \Longrightarrow a F b$,

(A4') $F^{-1} U \cap F^{-1} V \lhd F^{-1}(U_{\leq'} \cap V_{\leq'})$.

We have for any continuous F that $F^{-1} U \lhd F^{-1} V$, if $U \lhd' V$. Hence $F^{-1} U \sim F^{-1} V$ whenever $U \sim' V$. Also by (A1) $F^{-1}(U_{\lhd'}) \sim F^{-1} U$. By (A2) it follows that each $F^{-1} Z$ is down-closed.

Each continuous mapping induces a point function $f = \mathrm{Pt}(F)$ given by

$$\alpha \mapsto \{b : (\exists a \in \alpha) F(a,b)\} : \mathrm{Pt}(\mathcal{S}) \longrightarrow \mathrm{Pt}(\mathcal{T})$$

and which satisfies: $a F b \Rightarrow f[a^*] \subseteq b^*$.

Composition of two continuous morphisms $F : \mathcal{X} \longrightarrow \mathcal{Y}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ is given as follows

$$a (G \circ F) c \Longleftrightarrow a \lhd_{\mathcal{X}} F^{-1}[G^{-1}(c)].$$

The one-point formal topology is the terminal object in the category of formal topologies. It is constructed as $\mathbf{1} = (\{*\}, \leq_{\mathbf{1}}, \lhd_{\mathbf{1}})$, where $* \leq_{\mathbf{1}} *$ and $a \lhd_{\mathbf{1}} U$ iff U is inhabited. The terminal map $!_{\mathcal{Y}}$ from $\mathcal{Y}$ to $\mathbf{1}$ is defined by letting the relation $y\,!_{\mathcal{Y}}\,a$ be true for all y and a.

Now any point $\alpha \in \mathrm{Pt}(\mathcal{X})$ in a formal topology, gives a unique morphism $F_\alpha : \mathbf{1} \longrightarrow \mathcal{X}$, which is given by

$$aF_\alpha x \Longleftrightarrow x \in \alpha.$$

A map $\hat{\alpha} : \mathcal{Z} \longrightarrow \mathcal{X}$ which is constant α is defined by the composition $F_\alpha \circ !_{\mathcal{Z}}$. More explicitly, the map is given by the relation

$$z\hat{\alpha}x \Longleftrightarrow z \lhd_{\mathcal{Z}} \{u \in Z : x \in \alpha\}.$$

In particular, if z is covered by the empty set, then $z\,\hat{\alpha}x$ holds for any x.

3.2 Closed subspaces

Let $\mathcal{X} = (X, \leq, \lhd)$ be a formal topology. A subset $U \subseteq X$ defines an open set in the topology. It also defines a *closed subspace* by its formal complement as follows. Let $\mathcal{X} \dot{-} U = (X, \leq, \lhd')$ where

$$a \lhd' V \Longleftrightarrow a \lhd U \cup V.$$

(Note that $\lhd'$ is generated by the covering axioms for $\lhd$ and the pairs $(a, \emptyset)$ for $a \lhd U$.) By the definition of $\lhd'$ we see that

$$X \dot{-} U = X \dot{-} (U_\lhd). \tag{2}$$

Proposition 3.3 *Let $\mathcal{X}$ be a formal topology. For $S \subseteq X$, we have*

$$\alpha \in \mathrm{Pt}(\mathcal{X} \dot{-} S) \Longleftrightarrow \alpha \in \mathrm{Pt}(\mathcal{X}) \text{ and } \alpha \notin S^*.$$

We shall consider inclusion mappings between closed subspaces of a formal topology $\mathcal{X}$. For subsets $V \subseteq U \subseteq X$, let $E_{U,V} : \mathcal{X} \dot{-} U \longrightarrow \mathcal{X} \dot{-} V$ be defined by

$$xE_{U,V}\,y \Longleftrightarrow_{\mathrm{def}} x \lhd_{(\mathcal{X} \dot{-} U)} \{y\}.$$

The right hand side is thus equivalent to $x \lhd_{\mathcal{X}} U \cup \{y\}$, and hence we have

$$a \lhd_{\mathcal{X} \dot{-} U} E^{-1}_{U,V} W \Longleftrightarrow a \lhd_{\mathcal{X}} U \cup W. \tag{3}$$

Each morphism $E_{U,V}$ is a monomorphism in the category of formal topologies. Furthermore it follows that

$$E_{V,W} \circ E_{U,V} = E_{U,W} \tag{4}$$

for $W \subseteq V \subseteq U \subseteq X$. We shall write E_U for $E_{U,\emptyset} : (\mathcal{X} \dot{-} U) \longrightarrow \mathcal{X}$. Note that

$$E_{U,V} = E_{U_\lhd, V_\lhd}. \tag{5}$$

3.2.1 A glueing theorem

We generalise slightly the glueing theorem from [19] (there stated without proof).

Theorem 3.4 *Let $\mathcal{X}$ be a formal topology. Let I be a finite index set, and suppose that $U_i \subseteq X$ is down-closed for each $i \in I$. Suppose that $F_i : (\mathcal{X} \dot{-} U_i) \longrightarrow \mathcal{Y}$, $i \in I$, are continuous morphisms such that for all $i, j \in I$,*

$$F_i \circ E_{U_{ij},U_i} = F_j \circ E_{U_{ij},U_j} \tag{6}$$

where $U_{ij} = U_i \cup U_j$. Let $W = \cap_{i\in I} U_i$. Then there is a unique $F : (\mathcal{X} \dot{-} W) \longrightarrow \mathcal{Y}$ such that

$$F \circ E_{U_i,W} = F_i \tag{7}$$

for all $i \in I$.

Proof. Note that by definition of composition and (3) we have

$$\begin{aligned} a\,(F_i \circ E_{U_{ij},U_i})\,b &\iff a \lhd_{\mathcal{X} \dot{-} U_{ij}} E^{-1}_{U_{ij},U_i} F_i^{-1} b \\ &\iff a \lhd_{\mathcal{X}} U_{ij} \cup F_i^{-1} b \\ &\iff a \lhd_{\mathcal{X}} U_i \cup U_j \cup F_i^{-1} b \\ &\iff a \lhd_{\mathcal{X}} U_j \cup F_i^{-1} b. \end{aligned}$$

The last step follows since $U_i \subseteq F_i^{-1} b$. By a similar equivalence for $F_j \circ E_{U_{ij},U_j}$, the equation (6) can be read as the equivalence: for all a and b,

$$a \lhd_{\mathcal{X}} U_j \cup F_i^{-1} b \iff a \lhd_{\mathcal{X}} U_i \cup F_j^{-1} b. \tag{8}$$

Any saturated set, such as $F_i^{-1} b$, is down-closed as well. This property is used frequently when applying the localisation axiom (Loc).

We show uniqueness first, which gives an explicit definition of F. Suppose that F satisfies (7). Then for any $i \in I$, by expanding definitions and using transitivity,

$$a \lhd_{\mathcal{X}} U_i \cup F^{-1} b \iff a \lhd_{\mathcal{X}} U_i \cup E^{-1}_{U_i,W} F^{-1} b \iff a\,F_i\,b. \tag{9}$$

We use localisation to obtain

$$a \lhd_{\mathcal{X}} \bigcap_{i\in I} (U_i \cup F^{-1} b) \iff (\forall i \in I)\; a\,F_i\,b. \tag{10}$$

Note that finiteness of I is essential here. Now

$$\bigcap_{i\in I} (U_i \cup F^{-1} b) = (\bigcap_{i\in I} U_i) \cup F^{-1} b = W \cup F^{-1} b,$$

so the left hand side of (10) is equivalent to $a\,F\,b$. Thus we have

$$a\,F\,b \iff (\forall i \in I)\; a\,F_i\,b$$

which is an explicit definition of F, which is therefore uniquely determined.

Suppose now that F is given by this explicit definition. We show that it satisfies (7) and is continuous. To show (7), note first that we have by the definition of $\circ$

$$aF \circ E_{U_i,W}\, b \Longleftrightarrow a \lhd_X U_i \cup F^{-1}b. \tag{11}$$

Since $F \subseteq F_i$, its right hand side implies $a \lhd_X U_i \cup F_i^{-1}b$, that is $aF_i b$. Thus we have $F \circ E_{U_i,W} \subseteq F_i$. To show the reverse inclusion, suppose that $aF_i b$. From (8) we get, for any j, that $a \lhd_X U_i \cup F_j^{-1}b$. Then applying localisation,

$$a \lhd_X U_i \cup \bigcap_{j \in I} F_j^{-1}b.$$

But $\bigcap_{j\in I} F_j^{-1}b = F^{-1}b$, by definition of F, so this gives, via (11), $aF \circ E_{U_i,W}\, b$. This proves (7).

Finally, we show that F is continuous. The following lemma is then used.

Lemma 3.5 *Suppose that $a \lhd_X U_i \cup F_i^{-1}V$, for every $i \in I$. Then $a \lhd_X W \cup F^{-1}V$.*

Proof. By localisation, we have, since U_i is down-closed,

$$a \lhd_X \bigcap_{i \in I} (U_i \cup F_i^{-1}V).$$

Let x be an element in the right hand side. Using that I is finite, there are only two cases to consider:

Case 1: If $x \in U_i$ for all i, we have $x \in \bigcap_{j\in I} U_j = W$, and, trivially, $x \lhd_X W \cup F^{-1}V$.

Case 2: If $x \in F_i^{-1}V$, for some i, then V is inhabited and thus $U_j \subseteq F_j^{-1}V$ for any j, which implies that $x \in \bigcap_{j\in I} F_j^{-1}V$. Hence there are $b_j \in V$, $j \in J$, so that $xF_j b_j$ for each $j \in J$. Fix $j \in J$. Using (8) it follows from $xF_j b_j$, that for any k, $x \lhd_X U_j \cup F_k^{-1} b_j$. Thus by localization, and the definition of F,

$$x \lhd_X U_j \cup \bigcap_{k \in I} F_k^{-1} b_j = U_j \cup F^{-1} b_j \subseteq U_j \cup F^{-1}V$$

Again by localisation, we obtain $x \lhd_X (\bigcap_{j\in I} U_j) \cup F^{-1}V = W \cup F^{-1}V$ as desired. $\Box$

Using this lemma the conditions (A1) – (A4) for continuity are now straightforward to check. This finishes the proof of the glueing theorem. $\Box$

The theorem now gives the following corollary which appeared in [19].

Corollary 3.6 *Let $\mathcal{Y}$ be a formal topology and suppose that $U_1, U_2 \subseteq Y$ are saturated subsets. Write $V = U_1 \cup U_2$ and $W = U_1 \cap U_2$. Then the following diagram is* *a pushout*

$$\begin{tikzcd}
\mathcal{Y} \dot{-} V \arrow[r, "E_{V,U_2}"] \arrow[d, "E_{V,U_1}"'] & \mathcal{Y} \dot{-} U_2 \arrow[d, "E_{U_2,W}"] \\
\mathcal{Y} \dot{-} U_1 \arrow[r, "E_{U_1,W}"'] & \mathcal{Y} \dot{-} W
\end{tikzcd} \tag{12}$$

Proof. By (4) it follows that the diagram commutes. The pushout property is a direct consequence of Theorem 3.4 for $I = \{1,2\}$. $\Box$

We use this result to prove HJP for formal topology in Section 4.

3.3 Formal reals

The basic neighbourhoods of the formal reals $\mathcal{R}$ are $R = \{(a,b) \in \mathbb{Q}^2 : a < b\}$ given the inclusion order (as intervals), denoted by $\leq_{\mathcal{R}}$. The cover $\lhd_{\mathcal{R}}$ is generated by

(G1) $(a,b) \lhd \{(a',b') : a < a' < b' < b\}$ for all $a < b$,

(G2) $(a,b) \lhd \{(a,c),(d,b)\}$ for all $a < d < c < b$.

This means that $\lhd_{\mathcal{R}}$ is the smallest covering relation satisfying (G1) and (G2). The set of points $\mathrm{Pt}(\mathcal{R})$ form a structure isomorphic to the Cauchy reals $\mathbb{R}$. The order relation of points is given by $\alpha < \beta$ iff $b < c$ for some $(a,b) \in \alpha$ and $(c,d) \in \beta$. Define $\alpha \leq \beta$ iff $\neg(\beta < \alpha)$. The latter is equivalent to: $b \leq c$ for all $(a,b) \in \alpha$ and $(c,d) \in \beta$.

For a rational $q \in \mathbb{Q}$ define the corresponding real by $\check{q} = \{(a,b) \in R : a < q < b\}$. We write $\check{q}$ as q when no confusion can arise. It is easily seen that $(a,b)^* \subseteq (c,d)^*$ implies $(a,b) \leq (c,d)$. By Lemma 3.2 its follows that $\mathcal{R}$ is point-wise order conservative.

The following may be regarded as a point-free version of the trichotomy principle for real numbers.

Theorem 3.7 *For any point β of the formal reals $\mathcal{R}$ let*

$$T_\beta = \{(a,b) \in R : b < \beta \text{ or } (a,b) \in \beta \text{ or } \beta < a\}.$$

Then for any $U \subseteq R$ we have

$$U \sim_{\mathcal{R}} U_{\leq} \cap T_\beta.$$

Proof. The covering $U_{\leq} \cap T_\beta \lhd_{\mathcal{R}} U$ is clear by the axioms (Tra) and (Ext). To prove the converse covering, suppose that $(a,b) \in U$. Then it suffices by axiom (G1) to show $(c,d) \lhd_{\mathcal{R}} U_{\leq} \cap T_\beta$ for any $a < c < d < b$. For any such c,d we have

$\beta < c$ or $a < \beta$, by the co-transitivity principle for real numbers. In the former case, $(c,d) \in U_{\leq} \cap T_\beta$, so suppose $a < \beta$. Similarly comparing d,b to β we get $d < \beta$, in which case $(c,d) \in U_{\leq} \cap T_\beta$, or we get $\beta < b$. In the latter case we have $(a,b) \in U_{\leq} \cap T_\beta$, so in particular $(c,d) \lhd U_{\leq} \cap T_\beta$. $\square$

Next we define the closed interval $[\alpha,\beta]$, for $\alpha \leq \beta$, as the formal topology $\mathcal{R} \dot{-} U_{\alpha,\beta}$ where

$$U_{\alpha,\beta} = \{(a,b) : b < \alpha \text{ or } \beta < a\}.$$

Then $[\alpha,\alpha]$ is a terminal object in the category, but the proof of this is not as immediate as in point-set topology:

Lemma 3.8 *Let $\mathcal{X}$ be a formal topology. Let $\beta \in \mathrm{Pt}(\mathcal{R})$. Then $\hat{\beta}$ is the unique continuous map $\mathcal{X} \longrightarrow [\beta,\beta]$.*

Proof. The proof that $\hat{\beta}$ is a continuous map $\mathcal{X} \longrightarrow [\beta,\beta]$ is left to the reader. Suppose now that $F : \mathcal{X} \longrightarrow [\beta,\beta]$ is another continuous map. For any $I \in \beta$, we have $R \lhd_{[\beta,\beta]} I$, and hence $F^{-1}R \lhd_{[\beta,\beta]} F^{-1}I$. Thus for any $u \in R$, we have, using (A3) for F, $u \lhd_{[\beta,\beta]} F^{-1}I$, i.e. uFI. It has been shown that for any I,

$$\{u \in R : I \in \beta\} \subseteq \{u \in R : uFI\} = F^{-1}I. \tag{13}$$

Suppose $x\hat{\beta}I$, i.e. $x \lhd_{\mathcal{X}} \{u \in R : I \in \beta\}$. Hence xFI, by (13). This proves $\hat{\beta} \subseteq F$.

To prove $F \subseteq \hat{\beta}$, suppose that xFI. By Theorem 3.7,

$$I \sim_{\mathcal{R}} I_{\leq} \cap T_\beta.$$

Thus $x \lhd_{\mathcal{X}} F^{-1}(I_{\leq} \cap T_\beta)$. Take any $u \in F^{-1}(I_{\leq} \cap T_\beta)$. It suffices to prove $u\hat{\beta}I$. For some $J = (a,b) \leq I$ we have $J \in T_\beta$ and uFJ. There are three cases: (i) $b < \beta$, (ii) $\beta < a$ and (iii) $a < \beta < b$. For (i) we get $J \lhd_{[\beta,\beta]} \emptyset$, so

$$u \lhd_{\mathcal{X}} F^{-1}J \lhd_{\mathcal{X}} F^{-1}\emptyset = \emptyset = \{v \in R : J \in \beta\},$$

i.e. $u\hat{\beta}J$. Case (ii) is symmetric and yields $u\hat{\beta}J$ as well. In case (iii), $J \in \beta$, so $u\hat{\beta}J$ is immediate. Thus in all cases $u\hat{\beta}J$. This shows

$$F^{-1}(I_{\leq} \cap T_\beta) \subseteq \hat{\beta}^{-1}(I_{\leq} \cap T_\beta),$$

and therefore by transitivity $x \lhd_{\mathcal{X}} \hat{\beta}^{-1}(I_{\leq} \cap T_\beta)$. Again using $I \sim_{\mathcal{R}} I_{\leq} \cap T_\beta$, we get $x \lhd_{\mathcal{X}} \hat{\beta}^{-1}(I)$, i.e. $x\hat{\beta}I$. This shows $F \subseteq \hat{\beta}$. $\square$

3.4 Product topologies

We recall the construction of the product of two formal topologies $\mathcal{X}_1 = (X_1, \leq_1, \lhd_1)$ and $\mathcal{X}_2 = (X_2, \leq_2, \lhd_2)$. The product is $\mathcal{X} = (X_1 \times X_2, \leq, \lhd)$ where $(x_1,x_2) \leq (y_1,y_2)$ iff $x_1 \leq_1 y_1$ and $x_2 \leq y_2$, and where $\lhd$ is the least cover relation so that

(PC1) $x_1 \lhd_1 U$ implies $(x_1,x_2) \lhd U \times \{x_2\}$

(PC2) $x_2 \lhd_2 V$ implies $(x_1,x_2) \lhd \{x_1\} \times V$.

The projections $P_1 : \mathcal{X} \longrightarrow \mathcal{X}_1$ and $P_2 : \mathcal{X} \longrightarrow \mathcal{X}_2$ are given by

$$\begin{aligned}(x_1,x_2)\,P_1\,u \quad &\Leftrightarrow \quad (x_1,x_2) \lhd \{u\} \times X_2 \\ (x_1,x_2)\,P_2\,v \quad &\Leftrightarrow \quad (x_1,x_2) \lhd X_1 \times \{v\}.\end{aligned}$$

If $F : \mathcal{Z} \longrightarrow \mathcal{X}_1$ and $G : \mathcal{Z} \longrightarrow \mathcal{X}_2$ are continuous, then $\langle F,G\rangle : \mathcal{Z} \longrightarrow \mathcal{X}$ given by

$$z\,\langle F,G\rangle\,(x_1,x_2) \Longleftrightarrow_{\mathrm{def}} z\,F\,x_1 \text{ and } z\,G\,x_2$$

is the unique continuous map $\mathcal{Z} \longrightarrow \mathcal{X}$ such that $P_1 \circ \langle F,G\rangle = F$ and $P_2 \circ \langle F,G\rangle = G$. For $H_1 : \mathcal{Z}_1 \longrightarrow \mathcal{X}_1$ and $H_2 : \mathcal{Z}_2 \longrightarrow \mathcal{X}_2$ we write as usual $H_1 \times H_2$ for $\langle H_1 \circ P_1, H_2 \circ P_2\rangle$. The following lemma is easily proved by induction on the covers of the products.

Lemma 3.9 *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be formal topologies. Suppose that $(x_1,x_2) \lhd_{\mathcal{X}_1 \times \mathcal{X}_2} W$.*

(i) If $x_2 \in \beta$ and $\beta \in \mathrm{Pt}(\mathcal{X}_2)$, then $x_1 \lhd_{\mathcal{X}_1} \{y_1 \in X_1 : (\exists y_2 \in \beta)(y_1,y_2) \in W\}$.

(ii) If $x_1 \in \alpha$ and $\alpha \in \mathrm{Pt}(\mathcal{X}_1)$, then $x_2 \lhd_{\mathcal{X}_2} \{y_2 \in X_2 : (\exists y_1 \in \alpha)(y_1,y_2) \in W\}$.

In particular, if $(x_1,x_2)\,P_1\,y$ and x_2 belongs to some point of $\mathcal{X}_2$, then $x_1 \lhd_{\mathcal{X}_1} y$. On the other hand, if $(x_1,x_2)\,P_2\,y$ and x_1 belongs to some point of $\mathcal{X}_1$, then $x_2 \lhd_{\mathcal{X}_2} y$.

Corollary 3.10 *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be order conservative formal topologies, where every neighbourhood contains a point. Then the projection $P_k : \mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{X}_k$ satisfies*

$$(x_1,x_2)\,P_k\,y \Longleftrightarrow x_k \leq y.$$

Using Lemma 3.9 it is straightforward to prove:

Proposition 3.11 *Let $\mathcal{X}_1$ and $\mathcal{X}_2$. Then $\mathrm{Pt}(\mathcal{X}_1 \times \mathcal{X}_2)$ and $\mathrm{Pt}(\mathcal{X}_1) \times \mathrm{Pt}(\mathcal{X}_2)$ are homeomorphic via $\gamma \mapsto (\gamma_{(1)},\gamma_{(2)})$ and $(\alpha,\beta) \mapsto \langle\alpha,\beta\rangle$. Here $\gamma_{(1)} = \{a : (\exists b)\,(a,b) \in \gamma\}$, $\gamma_{(2)} = \{b : (\exists a)\,(a,b) \in \gamma\}$ and $\langle\alpha,\beta\rangle = \{(a,b) : a \in \alpha, b \in \beta\}$.*

On the other hand, it is well-known in classical locale theory that products are not preserved by the left adjoint Ω to Pt, unless at least one factor is locally compact [13, p. 61].

To calculate subspaces of a product space the following result is useful.

Lemma 3.12 *Let $\mathcal{X}$ and $\mathcal{Y}$ be formal topologies.*

(a) If V is a subset of Y, then the covers of the spaces $\mathcal{X} \times (\mathcal{Y} \dot{-} V)$ and $\mathcal{X} \times \mathcal{Y} \dot{-} X \times V$, are the same:

$$w \lhd_{\mathcal{X} \times (\mathcal{Y} \dot{-} V)} W \Longleftrightarrow w \lhd_{\mathcal{X} \times \mathcal{Y} \dot{-} X \times V} W.$$

(b) For $U \subseteq V \subseteq Y$ the maps

$$1_{\mathcal{X}} \times E_{V,U} : \mathcal{X} \times (\mathcal{Y} \dot{-} V) \longrightarrow \mathcal{X} \times (\mathcal{Y} \dot{-} U)$$

and

$$E_{X\times V, X\times U} : \mathcal{X} \times \mathcal{Y} \dot{-} X \times V \longrightarrow \mathcal{X} \times \mathcal{Y} \dot{-} X \times U$$

are identical.

Proof. Part (a): ($\Rightarrow$) is proved by induction on covers. It suffices to check the implication for generators of the product, which is straightforward.

($\Leftarrow$) is proved by first establishing the following implication by induction

$$w \lhd_{\mathcal{X}\times\mathcal{Y}} Z \Longrightarrow w \lhd_{\mathcal{X}\times(\mathcal{Y}\dot{-}V)} Z.$$

Using this, one gets from $w \lhd_{\mathcal{X}\times\mathcal{Y}\dot{-}X\times V} W$ that $w \lhd_{\mathcal{X}\times(\mathcal{Y}\dot{-}V)} (X\times V)\cup W$. But since $V \lhd_{\mathcal{Y}\dot{-}V} \emptyset$, we get by (PC2) in fact $w \lhd_{\mathcal{X}\times(\mathcal{Y}\dot{-}V)} W$.

Part (b): We have by the definitions of the two maps to be compared

$$(x,y)\, E_{X\times V, X\times U}\, (u,v) \Longleftrightarrow (x,y) \lhd_{\mathcal{X}\times\mathcal{Y}} X\times V \cup \{(u,v)\} \tag{14}$$

and

$$(x,y)\, 1_{\mathcal{X}} \times E_{V,U}\, (u,v) \Longleftrightarrow (x,y)\, P_1\, u \text{ and } (x,y)\, (E_{V,U}\circ P_2)\, v \tag{15}$$

Using Part (a) we get

$$(x,y)\, P_1\, u \Longleftrightarrow (x,y) \lhd_{\mathcal{X}\times\mathcal{Y}} X\times V \cup \{u\}\times Y. \tag{16}$$

Furthermore, a calculation gives

$$(x,y)\, (E_{V,U}\circ P_2)\, v \Longleftrightarrow (x,y) \lhd_{\mathcal{X}\times\mathcal{Y}} X\times V \cup X\times\{v\}. \tag{17}$$

By applying localisation to the right hand sides of (16) and (17) it follows that the right hand sides of (14) and (15) are equivalent. $\square$

4 HJP for formal topologies

Given the preparations of the previous section we can now quite straightforwardly prove the main result.

Theorem 4.1 *Let $\mathcal{X}$ be a formal topology. For $\alpha \le \beta \le \gamma$ in $\mathrm{Pt}(\mathcal{R})$, the diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\langle 1_{\mathcal{X}}, \hat{\beta}\rangle"] \arrow[d, "\langle 1_{\mathcal{X}}, \hat{\beta}\rangle"'] & \mathcal{X}\times[\beta,\gamma] \arrow[d, "1_{\mathcal{X}}\times E_2"] \\ \mathcal{X}\times[\alpha,\beta] \arrow[r, "1_{\mathcal{X}}\times E_1"'] & \mathcal{X}\times[\alpha,\gamma] \end{tikzcd} \tag{18}$$

is a pushout diagram. Here E_1 and E_2 are the obvious embeddings of subspaces.

Proof. We employ Corollary 3.6 with $\mathcal{Y} = \mathcal{X} \times \mathcal{R}$ as the formal topology, and with $U_1 = (X \times U_{\alpha,\beta})_{\lhd}$ and $U_2 = (X \times U_{\beta,\gamma})_{\lhd}$ as saturated subsets, writing $\lhd$ for $\lhd_{\mathcal{Y}}$. This gives diagram (12), where $V = U_1 \cup U_2$ and $W = U_1 \cap U_2$. We have

$$X \times U_{\beta,\beta} = X \times U_{\alpha,\beta} \cup X \times U_{\beta,\gamma}.$$

It follows easily that

$$V_{\lhd} = (X \times U_{\beta,\beta})_{\lhd} \tag{19}$$

Furthermore, we have

$$X \times U_{\alpha,\gamma} = X \times U_{\alpha,\beta} \cap X \times U_{\beta,\gamma}.$$

From which it follows that

$$W = (X \times U_{\alpha,\gamma})_{\lhd} \tag{20}$$

By (2) and (19) that

$$\mathcal{Y} \dot{-} V = \mathcal{Y} \dot{-} V_{\lhd} = \mathcal{Y} \dot{-} (X \times U_{\beta,\beta})_{\lhd} = \mathcal{Y} \dot{-} X \times U_{\beta,\beta}.$$

Then using Lemma 3.12.(a) the right hand side is seen to equal $\mathcal{X} \times (\mathcal{R} \dot{-} U_{\beta,\beta})$. Thus

$$\mathcal{Y} \dot{-} V = \mathcal{X} \times [\beta, \beta] \tag{21}$$

Similarly, using Lemma 3.12.(a) and (2) we get

$$\mathcal{Y} \dot{-} U_1 = \mathcal{X} \times [\alpha, \beta] \qquad \mathcal{Y} \dot{-} U_2 = \mathcal{X} \times [\beta, \gamma] \tag{22}$$

As for the final corner of (12), we get by (2), (20) and Lemma 3.12.(a)

$$\mathcal{Y} \dot{-} W = \mathcal{X} \times [\alpha, \gamma] \tag{23}$$

We now use (5) to rewrite the maps of the diagram. From Lemma 3.12.(b) and (19) follows that

$$E_{V,U_1} = E_{(X \times U_{\beta,\beta})_{\lhd}, (X \times U_{\beta,\gamma})_{\lhd}} = E_{X \times U_{\beta,\beta}, X \times U_{\beta,\gamma}} = 1_{\mathcal{X}} \times E_{U_{\beta,\beta}, U_{\beta,\gamma}}. \tag{24}$$

Symmetrically,

$$E_{V,U_2} = 1_{\mathcal{X}} \times E_{U_{\beta,\beta}, U_{\alpha,\beta}}. \tag{25}$$

Similarly, now using (20), we get

$$E_{U_1,W} = 1_{\mathcal{X}} \times E_{U_{\alpha,\beta}, U_{\alpha,\gamma}} \qquad E_{U_2,W} = 1_{\mathcal{X}} \times E_{U_{\beta,\gamma}, U_{\alpha,\gamma}}. \tag{26}$$

Abbreviating $E_1 = E_{U_{\alpha,\beta}, U_{\alpha,\gamma}}$, $E_2 = E_{U_{\beta,\gamma}, U_{\alpha,\gamma}}$, $E^1 = E_{U_{\beta,\beta}, U_{\alpha,\beta}}$ and $E^2 = E_{U_{\beta,\beta}, U_{\beta,\gamma}}$, we conclude from equations (21) – (26) above that the following is then a pushout diagram:

$$\begin{tikzcd}
\mathcal{X} \times [\beta, \beta] \arrow[r, "1_{\mathcal{X}} \times E^2"] \arrow[d, "1_{\mathcal{X}} \times E^1"'] & \mathcal{X} \times [\beta, \gamma] \arrow[d, "1_{\mathcal{X}} \times E_2"] \\
\mathcal{X} \times [\alpha, \beta] \arrow[r, "1_{\mathcal{X}} \times E_1"'] & \mathcal{X} \times [\alpha, \gamma]
\end{tikzcd} \tag{27}$$

To obtain the pushout diagram (18) from (27) it suffices to show that $\langle 1_{\mathcal{X}}, \hat{\beta}\rangle : \mathcal{X} \longrightarrow \mathcal{X} \times [\beta,\beta]$ is an isomorphism and that

$$(1_{\mathcal{X}} \times E^1) \circ \langle 1_{\mathcal{X}}, \hat{\beta}\rangle = \langle 1_{\mathcal{X}}, \hat{\beta}\rangle \qquad (1_{\mathcal{X}} \times E^2) \circ \langle 1_{\mathcal{X}}, \hat{\beta}\rangle = \langle 1_{\mathcal{X}}, \hat{\beta}\rangle. \tag{28}$$

The equations (28) are straightforward to check. As for the isomorphism note that $P_1 \circ \langle 1_{\mathcal{X}}, \hat{\beta}\rangle = 1_{\mathcal{X}}$. Moreover, $\langle 1_{\mathcal{X}}, \hat{\beta}\rangle \circ P_1 = \langle 1_{\mathcal{X}} \circ P_1, \hat{\beta} \circ P_1\rangle = \langle P_1, \hat{\beta}\rangle$. But $P_2 = \hat{\beta} : \mathcal{X} \times [\beta,\beta] \longrightarrow [\beta,\beta]$, by Lemma 3.8 ,so $\langle P_1, \hat{\beta}\rangle = \langle P_1, P_2\rangle = 1_{\mathcal{X}\times[\beta,\beta]}$. □

Remark 4.2 The special case of Theorem 4.1 when $\mathcal{X} = 1$ gives the PJP property for formal topologies. This property could as well be obtained more easily from Corollary 3.6 directly. The latter yields a simple alternative proof of Theorem 4.1, under the restrictive assumption that $\mathcal{X}$ is locally compact. It is known that then the exponentiation functor $(-)^{\mathcal{X}}$ exists; see Maietti [17]. Hence by adjointness, $\mathcal{X} \times (-)$ is functor which preserves colimits and, in particular, pushouts. Applying the functor to an arbitrary PJP diagram then gives the result.

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