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A THEORY OF ENRICHED SKETCHES

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ABSTRACT. The theory of enriched accessible categories over a suitable base category $\mathcal{V}$ is developed. It is proved that these enriched accessible categories coincide with the categories of flat functors, but also with the categories of models of enriched sketches. A particular attention is devoted to enriched locally presentable categories and enriched functors.

Introduction

The theory of locally presentable categories became popular after the work of Gabriel–Ulmer (see [7]), and the theory of accessible categories, after the work of Makkai–Paré (see [15]). Nevertheless, one should not forget that the notions of presentability and, in a slightly different form, accessibility, are due to Grothendieck and his school (see [8]). The study of general sketches and their relations with accessible categories is due to Ehresmann and his school (see [2] and [13]).

Now let us work over a symmetric monoidal base category $\mathcal{V}$. What can it mean for an enriched category to be accessible as an enriched category? Certainly not the single fact that the underlying ordinary category is accessible in the ordinary sense. For example, for a 2-category to be accessible as a 2-category, the ordinary underlying category of objects and 1-morphisms should certainly be accessible in the ordinary sense, but some additional requirement on 2-cells is obviously needed. The same question can be asked with respect to a Grothendieck topos $\mathcal{E}$ as base category. A category $\mathcal{C}$ enriched in the topos $\mathcal{E}$ has sheaves $\mathcal{C}(F,G) \in \mathcal{E}$ of morphisms, and its accessibility as $\mathcal{E}$-category should not be reduced to the fact that the underlying ordinary category, with the global sections of $\mathcal{C}(F,G) \in \mathcal{E}$ as morphisms, is accessible in the ordinary sense.

Let us recall here that the fundamental categorical results concerning locally presentable categories show the equivalence between the following notions, for a fixed regular cardinal α:

1. the locally α-presentable categories;

2. the categories of α-continuous presheaves;

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3. the categories of models of a projective α-sketch.

These equivalences have been generalized to the enriched context by G.M. Kelly, in [10], when the base category $\mathcal{V}$ is locally finitely presentable and the tensor product of finitely many presentable objects is still finitely presentable. Moreover, the proof in the case $\alpha = \aleph_0$ transposes immediately to the general case. To achieve this, G.M. Kelly has introduced a powerful notion of finite (or α-small) enriched limit, which we adopt here.

The case of accessible categories is substantially different. Here one has equivalences between

1. the α-accessible categories, for some α,
2. the categories of α-flat presheaves, for some α,
3. the categories of models of a sketch,

with α running through the regular cardinals. In fact the equivalence of 1. and 2. remains rather straightforward and holds for a fixed cardinal α: it is entirely based on the classical equivalence between being flat, and being a filtered colimit of representable functors. But as far as condition 3. is concerned, it is well known that the category of models of a sketch involving only finite limit and colimit specifications is accessible, but not finitely accessible, and on the other hand that a category of finitely flat functors is sketchable, but is not the category of models of a sketch with finite specifications. So every theory involving condition 3. generally requires some heavy cardinal arithmetic.

In [10], G.M. Kelly introduces the notion of flat functor in the enriched context, and we adopt it here: it is a functor whose Kan extension along the Yoneda embedding preserves finite limits, all this in the enriched sense. G.M. Kelly presents as an open problem the possible equivalence between being flat and being a filtered colimit of representable functors, in the enriched context. In [4], it is observed that in the case of sets, when colimits are seen as weighted colimits, a colimit is filtered precisely when its weight is flat. So, in the enriched context, over a base as in [10], the generalization of the notion of filtered colimit should be that of a weighted colimit, whose weight is flat in the sense of Kelly. With that definition, it is proved in [4] that the equivalence between 1. and 2. holds, in the case of enriched accessible categories and for a fixed regular cardinal α. Of course with that revised generalization of the concept of filtered colimit, Kelly's open problem becomes almost a tautology.

For what concerns Kelly's open problem in its original form, we produce a counterexample (our example 9.2) showing that an enriched flat functor is generally not a conical filtered colimit of representables. But our theorem 7.10 throws a more subtle light on this question. Every category of enriched flat functors is equivalent to some category of enriched α-flat functors, in which every α-flat functor is a conical α-filtered colimit of representables. Thus if one decides to define differently accessible categories, using Kelly's conical filtered colimits, some heavy cardinal arithmetic will allow proving that every category of enriched α-flat functors is accessible, but in fact, β-accessible, for some (much) bigger β.

Let us add a word on the terminology. We give evidence that the correct generalization of the notion of filtered colimit is "a colimit with a flat weight": indeed, it is with that definition that all classical theorems on flat functors and accessible categories generalize, in the same form, to the enriched case. Some people suggested to call these "flat colimits", reserving the term "filtered" for Kelly's filtered conical colimits. But this would mean that in the classical case of sets, which supports the intuition of most readers, our definitions and theorems would not reduce to the historically classical and universally accepted forms. Therefore we reserve the term "filtered" for colimits with a flat weight, while Kelly's colimits are called "conical filtered", what they are after all. Of course, over the category of sets, both notions coincide.

To recapture a wide range of interesting examples – like Banach spaces and all Grothendieck toposes – as possible base categories, we make a strong point that the base category $\mathcal{V}$ should only be locally presentable, not necessarily locally finitely presentable. The reader should be aware that this is not a straightforward generalization of the finite case. Developing the theory of α-accessible categories over a base $\mathcal{V}$ requires often transfinite constructions of the type

$$A_0 = A, \quad A_{\gamma+1} = \Gamma(A_\gamma), \quad A_\lambda = \operatorname{colim}_{\gamma<\lambda} A_\gamma \quad \text{for } \lambda \text{ limit ordinal},$$

followed by a presentability argument to show that the construction stops at some level, generally at the level α, and finally a proof by transfinite induction to show that the various A_γ have the property one expects. The definition at a limit ordinal λ is given by a λ-filtered colimit, and when $\mathcal{V}$ is finitely presentable, all these colimits have good properties, for example they are universal and – more important – they are preserved by representable functors $\mathcal{V}(G_i, -)$ for a dense family $(G_i)_{i\in I}$ of generators. But if $\mathcal{V}$ is only locally α_0-presentable for some α_0, those good properties hold only for $\lambda \geq \alpha_0$, which ruins often the argument, since already at the level $\aleph_0$ one is stucked by the absence of good properties of the corresponding conical filtered colimit $A_{\aleph_0} = \operatorname{colim}_{n\in\mathbb{N}} A_n$. So our choice of working over a general locally presentable base prevents us, most of the time, from using transfinite inductions, which would be available if the base was locally finitely presentable.

Our sections 1 to 4 introduce our concepts of locally presentable base $\mathcal{V}$ and enriched filtered colimit, and then study their relations with the notions of presentable object and flat functor. Section 5 proves the equivalence between being accessible and being a category of flat functors, while section 6 particularizes those results to recapture Kelly's theory of locally presentable categories. Of course, all this in the enriched context. In particular, we prove that in the presence of small enriched colimits, that is, in the locally presentable case, presentability in our sense is equivalent to presentability in Kelly's sense. This explains why Kelly's concepts, which do not allow to handle the accessible case, suffice nevertheless to treat the locally presentable case. For the convenience of the reader, we have chosen to give an integrated and self-contained treatment of these questions, instead of saying – when possible – that inspiration for a proof can be found by comparing with some arguments in [10] or [4], which do not imply our results but have some common flavour.

Now the core of this paper is section 7, which proves that enriched accessible categories are exactly the categories of models of enriched sketches. We use for this some sophisticated techniques inherited from [1], presenting in particular an accessible category as intersection of locally presentable categories. To achieve this, it is necessary to prove that enriched accessible categories are accessible in the ordinary sense. But our example 9.2 again shows that an enriched α-accessible category is generally not α-accessible in the ordinary sense. So some original clever jump in cardinal arithmetic is needed to prove that enriched accessibility implies accessibility, that is, an enriched α-accessible category is β-accessible in the ordinary sense, for some (much) bigger β. The necessity of this jump in cardinals is probably the most unexpected fact in this paper. A direct consequence of all this is the possibility of increasing as much as desirable the degree of accessibility of a given family of enriched accessible categories. This is our best substitute to the classical "sharply less" relation between cardinals.

We pursue the paper with a section on accessible functors in the enriched context. We focus on an adjoint functor theorem for accessible functors and the expected consequence that an accessible category is complete iff it is cocomplete, in which case it is locally presentable. Here again, the classical argument for proving that an accessible functor satisfies the solution set condition fails and it is again via a jump through cardinal arithmetic that we can nevertheless conclude to the validity of the result. We close the paper with some examples.

The present introduction shows that we have been able to generalize – in the enriched context – all fundamental results of the theory of accessible categories, locally presentable categories and sketches. Nevertheless many problems remain open. For example, we are able to raise the degree of accessibility of every specified enriched accessible category, but we do not have an enriched "sharply less" relation working at once for all enriched accessible categories. In the same spirit, we have managed to write down various proofs in terms of α-filtered colimits instead of α-filtered conical unions, avoiding so the study of a Löwenheim–Skolem theorem for enriched sketches; but one would nevertheless appreciate to discover an enriched version of the Löwenheim–Skolem theorem. And finally we can provide many interesting examples to which our theory applies, but we would appreciate, in these examples, to have explicit characterizations of the various notions we introduce.

We use freely the notation of [11]. In particular we write respectively $\{G, F\}$ and $G \star F$ for the enriched limit and the enriched colimit of F weighted by G.

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CONVENTION:
For the sake of brevity, through the whole paper,
(1) α_0 denotes a fixed regular cardinal;
(2) $\mathcal{V}$ denotes a fixed locally α_0-presentable base as in definition 1.1;
(3) α denotes a regular cardinal $\alpha \geq \alpha_0$.

1. Locally presentable bases

This section introduces the base categories which are relevant to our purpose.

1.1. Definition. *Let α_0 be a regular cardinal. A locally α_0-presentable base is a symmetric monoidal closed category $\mathcal{V}$ which is locally α_0-presentable and satisfies the following conditions:*
(1) the unit I of the tensor product is α_0-presentable;
(2) the tensor product of two α_0-presentable objects of $\mathcal{V}$ is again α_0-presentable.
By a locally presentable base we mean a locally α_0-presentable base, for some regular cardinal α_0.

1.2. Proposition. *Let $\alpha \leq \beta$ be regular cardinals. Every locally α-presentable base is also locally β-presentable.*

Proof. The unit I is α-presentable, thus β-presentable. Next if $V, W \in \mathcal{V}$ are β-presentable, they are β-colimits of α-presentable objects. Since the tensor product has a right adjoint, it preserves colimits; therefore $V \otimes W$ can be written as a β-colimit of objects $V_i \otimes W_j$, with V_i, W_j α-presentable. By 1.1.(2), each $V_i \otimes W_j$ is α-presentable and therefore $V \otimes W$ is β-presentable. ∎

1.3. Proposition. *Let α_0 be a regular cardinal and $\mathcal{V}$ a symmetric monoidal closed category. The following conditions are equivalent:*
(1) $\mathcal{V}$ is an α_0-base;
(2) $\mathcal{V}$ admits a strongly generating family constituted of α_0-presentable objects; this family contains the unit I and is stable for binary tensor products.

Proof. An arbitrary α_0-presentable object is an α_0-colimit of the generators in condition (2). One concludes as in proposition 1.2. ∎

1.4. Examples.
1.4.a. First of all, we pick up from [10] a series of examples of locally finitely (i.e. $\aleph_0$-) presentable bases.
(1) The category of models of a Lawvere theory, with the symmetric monoidal closed structure defined in [14].
(2) The categories of graded modules and differential graded modules, with their classical closed structures.
(3) The cartesian closed categories of small graphs, small categories, small groupoids, preordered sets, ordered sets, sets with an equivalence relation.
1.4.b. The category of Banach spaces and linear contractions is a locally $\aleph_1$-presentable base. It is symmetric monoidal closed (see [6]) and locally $\aleph_1$-presentable (see [15]). The $\aleph_1$-presentable Banach spaces are the separable ones, and the projective tensor product of two separable spaces is again separable.
1.4.c. Every Grothendieck topos is a locally presentable base. One can present the Grothendieck topos as a topos of sheaves on a site with finite limits, each representable functor being itself a sheaf (see [5], volume 3, 3.6.2). Considering the proof of 3.4.16 in

[5], volume 3, one concludes by 1.3.
1.4.d. Combining previous examples, one sees easily that the models of a commutative algebraic theory in a Grothendieck topos constitute a locally presentable base.
1.4.e. The category $[0, \infty]$ with $a \otimes b = a + b$ as tensor product (see [12]) is a locally presentable base. Indeed, if α is the cardinal of the real line, every object of $[0, \infty]$ is α-presentable. Assuming the continuum hypothesis, one can even choose $\alpha = \aleph_1$.

2. Enriched filtered colimits

We introduce and study a notion of enriched filtered colimit, borrowed from [4], and which differs significantly from that used in [10]. Our definition of α-smallness is nevertheless borrowed from that paper.

2.1. Definition. *A weight $F: \mathcal{X} \to \mathcal{V}$ for a $\mathcal{V}$-limit or $\mathcal{V}$-colimit is α-small when*
(1) $\mathcal{X}$ has stricly less than α objects;
(2) for all objects $X, Y \in \mathcal{X}$, the object $\mathcal{X}(X, Y) \in \mathcal{V}$ is α-presentable;
(3) for every object $X \in \mathcal{X}$, the object $F(X) \in \mathcal{V}$ is α-presentable.
An α-small $\mathcal{V}$-limit or colimit is one indexed by an α-small weight. An α-complete or cocomplete $\mathcal{V}$-category is one having all α-small $\mathcal{V}$-limits or colimits.

2.2. Lemma. *In a $\mathcal{V}$-category, every α-small conical limit or colimit is an α-small $\mathcal{V}$-limit or $\mathcal{V}$-colimit.*

Proof. If $\mathcal{X}$ is an α-small ordinary category, it is immediate that the free $\mathcal{V}$-category $\overline{\mathcal{X}}$ on $\mathcal{X}$ is an α-small $\mathcal{V}$-category, because an α-copower of the unit I is α-presentable. And then, of course, the constant $\mathcal{V}$-functor on I is α-small. ∎

2.3. Definition. *A $\mathcal{V}$-functor $G: \mathcal{C}^* \to \mathcal{V}$ is an α-filtered weight when its left $\mathcal{V}$-Kan extension along the Yoneda embedding*

$$\mathsf{Lan}_Y G: [\mathcal{C}, \mathcal{V}] \longrightarrow \mathcal{V}$$

preserves α-small $\mathcal{V}$-limits. An α-filtered $\mathcal{V}$-colimit is one indexed by an α-filtered weight.

2.4. Proposition. *In $\mathcal{V}$, α-small $\mathcal{V}$-limits commute with α-filtered $\mathcal{V}$-colimits.*

Proof. In definition 2.3, the Kan extension is pointwise, thus given by $G\star-$, the "colimit weighted by G" functor. ∎

2.5. Proposition. *A $\mathcal{V}$-category $\mathcal{C}$ has all α-small $\mathcal{V}$-limits iff*
(1) $\mathcal{C}$ has all α-small conical limits;
(2) for all $C \in \mathcal{C}$ and α-presentable $V \in \mathcal{V}$, the cotensor $V \pitchfork C$ exists in $\mathcal{C}$.

Proof. (1)⇒(2) is immediate and (2)⇒(1) follows at once from the construction of a $\mathcal{V}$-limit in terms of an equalizer and cotensors. ∎

2.6. Lemma. *For an object $V \in \mathcal{V}$, the following conditions are equivalent:*
(1) $\mathsf{Hom}(V,-)\colon \mathcal{V} \to \mathsf{Set}$ *preserves α-filtered conical colimits;*
(2) $[V,-]\colon \mathcal{V} \to \mathcal{V}$ *preserves α-filtered conical colimits.*

Proof. One has $\mathsf{Hom}(V,-) = \mathsf{Hom}(I,-) \circ [V,-]$, thus (2)⇒(1) since I is α-presentable. Conversely if $W = \operatorname{colim} W_i$ is an α-filtered conical colimit in $\mathcal{V}$, for every α-presentable $P \in \mathcal{V}$, $P \otimes V$ is α-presentable as well and thus

$$\begin{aligned}\mathsf{Hom}(P,[V,\operatorname{colim} W_i]) &\cong \mathsf{Hom}(P\otimes V,\operatorname{colim} W_i) \cong \operatorname{colim}\mathsf{Hom}(P\otimes V, W_i)\\ &\cong \operatorname{colim}\mathsf{Hom}(P,[V,W_i]) \cong \mathsf{Hom}(P,\operatorname{colim}[V,W_i])\end{aligned}$$

from which the conclusion follows, since the α-presentable P's constitute a strongly generating family. ∎

2.7. Lemma. *In a $\mathcal{V}$-category, every α-filtered conical colimit is an α-filtered $\mathcal{V}$-colimit.*

Proof. Let $\mathcal{X}$ be an ordinary α-cofiltered category and $\Delta_1\colon \mathcal{X} \to \mathsf{Set}$ the contravariant constant functor on the singleton; it is thus a conical α-filtered colimit of representable functors. Applying the "free $\mathcal{V}$-category" functor – which is a left adjoint – , we present the $\mathcal{V}$-functor $\Delta_I\colon \overline{\mathcal{X}}^* \to \mathcal{V}$, constant on the unit I, as a conical α-filtered colimit of $\mathcal{V}$-representable functors. We must prove it is a $\mathcal{V}$-filtered weight, in the sense of 2.3.

But as a left adjoint, the $\mathcal{V}$-Kan extension process along the Yoneda embedding preserves colimits. Since the $\mathcal{V}$-Kan extension of the representable $\mathcal{V}$-functor $\mathcal{X}(-,X)$ is evaluation at X, it preserves all $\mathcal{V}$-limits. So it suffices to prove that in the category of $\mathcal{V}$-functors from $[\mathcal{C},\mathcal{V}]$ to $\mathcal{V}$, an α-filtered conical colimit $F = \operatorname{colim} F_i$ of $\mathcal{V}$-functors F_i preserving (α-small) $\mathcal{V}$-limits is again a $\mathcal{V}$-functor F preserving α-small $\mathcal{V}$-limits.

Since in $\mathcal{V}$, α-small conical limits commute with α-filtered conical colimits, F preserves α-small conical limits. By 2.5, it remains to check that F preserves cotensors with α-presentable objects. Given $H \in [\mathcal{C},\mathcal{V}]$ and $V \in \mathcal{V}$, α-presentable, we have

$$\begin{aligned}\operatorname{colim} F_i(V \pitchfork H) &\cong \operatorname{colim} V \pitchfork F_i(H) \cong \operatorname{colim}[V,F_i(H)]\\ &\cong [V,\operatorname{colim} F_i(H)] \cong V \pitchfork \operatorname{colim} F_i(H)\end{aligned}$$

since each F_i preserves cotensors with V and $[V,-]$ preserves α-filtered conical colimits by 2.6. ∎

3. Enriched presentability

We introduce the notion of $\mathcal{V}$-presentable object and prove that it coincides in $\mathcal{V}$ with the ordinary notion of presentability.

3.1. Definition. *An object C of a $\mathcal{V}$-category $\mathcal{C}$ is α-presentable when the corresponding representable $\mathcal{V}$-functor $\mathcal{C}(C,-)\colon \mathcal{C} \to \mathcal{V}$ preserves α-filtered $\mathcal{V}$-colimits.*

3.2. Proposition. *In a $\mathcal{V}$-category, if an α-small $\mathcal{V}$-colimit of α-presentable objects exists, it is still α-presentable.*

PROOF. In a $\mathcal{V}$-category $\mathcal{C}$, consider an α-small $\mathcal{V}$-colimit $F \star G$ with each $G(X)$ α-presentable. Given an α-filtered colimit $H \star K$ in $\mathcal{C}$,

$$\begin{aligned}\mathcal{C}(F \star G, H \star K) &\cong \{F-, \mathcal{C}(G-, H \star K)\} \cong \{F-, H \star \mathcal{C}(G-, K)\} \\ &\cong H \star \{F-, \mathcal{C}(G-, K)\} \cong H \star \mathcal{C}(F \star G, K),\end{aligned}$$

by 2.4, the various assumptions and the fact that each $\mathcal{C}(-, C)$ transforms $\mathcal{V}$-colimits in $\mathcal{V}$-limits. ∎

3.3. COROLLARY. *In $\mathcal{V}$, an object is α-presentable in the ordinary sense iff it is α-presentable in the enriched sense.*

PROOF. The unit I is certainly α–presentable, since the functor $[I, -]$ is the identity on $\mathcal{V}$. An α-presentable object V can be written as $V \otimes I$, thus as $V \star I$ when V and I are seen as $\mathcal{V}$-valued $\mathcal{V}$-functors on the unit $\mathcal{V}$-category. This presents V as an α-small $\mathcal{V}$-colimit of α-presentable objects, proving that V is α-presentable (see 3.2). The converse follows at once from 2.7 and 2.6. ∎

4. Enriched flatness

Our notion of flatness is that of [10] and coincides with our notion of filtered weight (see 2.3)! Nevertheless, to extend a highly standard terminology, we prefer to keep two different names for this same notion.

4.1. DEFINITION. *A $\mathcal{V}$-functor $F: \mathcal{C} \to \mathcal{V}$ is α-flat when its left $\mathcal{V}$-Kan extension along the Yoneda embedding*

$$\mathsf{Lan}_Y F: [\mathcal{C}^*, \mathcal{V}] \longrightarrow \mathcal{V}$$

preserves α-small $\mathcal{V}$-colimits.

4.2. PROPOSITION. *A $\mathcal{V}$-functor $F: \mathcal{C} \to \mathcal{V}$ is α-flat iff it is an α-filtered $\mathcal{V}$-colimit of $\mathcal{V}$-representable functors.*

PROOF. Since for every $\mathcal{V}$-functor $F: \mathcal{C} \to \mathcal{V}$, one has $F = F \star Y$, where Y is the Yoneda embedding. ∎

4.3. COROLLARY. *Let $F: \mathcal{X} \to \mathcal{Y}$ be a $\mathcal{V}$-functor between small $\mathcal{V}$-categories. The left $\mathcal{V}$-Kan extension along F of an α-flat $\mathcal{V}$-functor $G: \mathcal{X} \to \mathcal{V}$ is again α-flat.*

PROOF. The left $\mathcal{V}$-Kan extension of the representable $\mathcal{V}$-functor $\mathcal{X}(X, -)$ is the representable $\mathcal{V}$-functor $\mathcal{Y}(FX, -)$. As a $\mathcal{V}$-left adjoint, the $\mathcal{V}$-Kan extension process preserves $\mathcal{V}$-colimits. One concludes by 4.2. ∎

4.4. PROPOSITION. *The category of α-flat functors on a small $\mathcal{V}$-category $\mathcal{C}$ is stable in $[\mathcal{C}, \mathcal{V}]$ under α-filtered $\mathcal{V}$-colimits.*

PROOF. By 2.4. ∎

4.5. Proposition. *For a $\mathcal{V}$-functor $F: \mathcal{C} \to \mathcal{V}$ on a small α-complete $\mathcal{C}$, the following conditions are equivalent:*
(1) F is α-flat;
(2) F preserves α-small $\mathcal{V}$-limits;
(3) F is an α-filtered conical colimit of representable $\mathcal{V}$-functors.
Under these conditions, F is often called α-continuous *instead of α-flat.*

Proof. (1)⇒(2) follows at once from 2.4, 4.4 and the fact that representable functors preserve all $\mathcal{V}$-limits. (3)⇒(1) is a direct consequence of 2.7 and 4.2

It remains to prove (2)⇒(3). The category of representable $\mathcal{V}$-functors over F is, by the $\mathcal{V}$-Yoneda lemma, the dual of the category of elements of the set valued functor $\mathsf{Hom}(I, F-)$. In particular, one has

$$\mathsf{Hom}_{\mathcal{V}}(I, F-) \cong \operatorname{colim}_{(C,c)} \mathsf{Hom}_{\mathcal{C}}(C, -) \cong \operatorname{colim}_{(C,c)} \mathsf{Hom}_{\mathcal{V}}(I, -) \circ \mathcal{C}(C, -)$$

where the colimit is computed on the category of elements of $\mathsf{Hom}(I, F-)$. Since both F and $\mathsf{Hom}(I, -)$ preserve conical α-limits, so does the composite and this category of elements is α-cofiltered in the ordinary sense (see [5], volume 1, chapter 6). Writing G for the α-filtered conical colimit of the representable $\mathcal{V}$-functors over F, we must prove that the canonical comparison morphism $\gamma: G \Rightarrow F$ is an isomorphism. For every α-presentable $V \in \mathcal{V}$ and $D \in \mathcal{C}$, the cotensors $V \pitchfork D$ exist by 2.5; therefore

$$\begin{aligned}
\mathsf{Hom}_{\mathcal{V}}(V, GD) &\cong \mathsf{Hom}_{\mathcal{V}}(V, \operatorname{colim}_{(C,c)} \mathcal{C}(C, D)) \\
&\cong \operatorname{colim}_{(C,c)} \mathsf{Hom}_{\mathcal{V}}(V, \mathcal{C}(C, D)) \\
&\cong \operatorname{colim}_{(C,c)} \mathsf{Hom}_{\mathcal{V}}(I, [V, \mathcal{C}(C, D)]) \\
&\cong \operatorname{colim}_{(C,c)} \mathsf{Hom}_{\mathcal{V}}(I, \mathcal{C}(C, V \pitchfork D)) \\
&\cong \mathsf{Hom}_{\mathcal{V}}(I, F(V \pitchfork D)) \\
&\cong \mathsf{Hom}_{\mathcal{V}}(I, V \pitchfork FD) \\
&\cong \mathsf{Hom}_{\mathcal{V}}(I, [V, FD]) \\
&\cong \mathsf{Hom}_{\mathcal{V}}(V, FD).
\end{aligned}$$

Since the α-presentable V constitute a strongly generating family, each γ_D is an isomorphism. ∎

5. Enriched accessibility

First we define $\mathcal{V}$-accessible categories and characterize them as the categories of flat $\mathcal{V}$-functors.

5.1. Definition. *Let $\mathcal{V}$ be a locally α_0-presentable base and $\alpha \geq \alpha_0$. A $\mathcal{V}$-category $\mathcal{C}$ is α-accessible when*
(1) $\mathcal{C}$ has α-filtered $\mathcal{V}$-colimits;
(2) $\mathcal{C}$ has a family $(G_i)_{i \in I}$ of α-presentable objects such that every object $C \in \mathcal{C}$ can be written as an α-filtered $\mathcal{V}$-colimit of these objects G_i.
A $\mathcal{V}$-category is accessible when it is α-accessible for some $\alpha \geq \alpha_0$.

5.2. Lemma. *Every object of an accessible $\mathcal{V}$-category is presentable.*

Proof. With the notation of 5.1, write $\mathcal{G} \subseteq \mathcal{C}$ for the full subcategory generated by the G_i's. Every object $C \in \mathcal{C}$ can be written as $C = G \star F$ where

$$G: \mathcal{X}^* \to \mathcal{V}, \quad \mathcal{X} \xrightarrow{F} \mathcal{G} \overset{i}{\hookrightarrow} \mathcal{C}$$

with $\mathcal{X}$ a small $\mathcal{V}$-category and F, G two $\mathcal{V}$-functors, with G an α-filtered weight. By smallness of $\mathcal{X}$, we can choose a cardinal $\beta \geq \alpha$ such that G is β-small. Then $C = G \star F$ presents C as a β-small $\mathcal{V}$-colimit of β-presentable objects. One concludes by 3.2. ∎

5.3. Theorem. *For a $\mathcal{V}$-category $\mathcal{C}$, the following conditions are equivalent.*
(1) $\mathcal{C}$ is α-accessible;
(2) $\mathcal{C}$ is equivalent to the category α-$\mathsf{Flat}(\mathcal{D}, \mathcal{V})$ of α-flat $\mathcal{V}$-functors on a small $\mathcal{V}$-category $\mathcal{D}$.

Proof. Let $\mathcal{C}$ be accessible; write $\mathcal{G}$ for the full $\mathcal{V}$-subcategory of $\mathcal{C}$ generated by the objects G_i, as in definition 5.1; we put $\mathcal{D} = \mathcal{G}^*$, the dual of $\mathcal{G}$. Since each $C \in \mathcal{C}$ is an α-filtered $\mathcal{V}$-colimit of a diagram in $\mathcal{G}$, while each $\mathcal{C}(G_i, -)$ preserves α-filtered colimits, we know that $\mathcal{G}$ is dense in $\mathcal{C}$ (see [11], 5.19). This means that

$$\Gamma: \mathcal{C} \longrightarrow [\mathcal{G}^*, \mathcal{V}] = [\mathcal{D}, \mathcal{V}], \quad C \mapsto (\mathcal{C}(-, C): \mathcal{G}^* \to \mathcal{V})$$

is $\mathcal{V}$-full and faithful.

Since each $G_i \in \mathcal{G}$ yields $\mathcal{C}(G_i, -)$ preserving α-filtered $\mathcal{V}$-colimits, and since moreover each $C \in \mathcal{C}$ is an α-filtered $\mathcal{V}$-colimit of G_k's, it follows at once that $\mathcal{C}(-, C): \mathcal{G}^* \to \mathcal{V}$ is an α-filtered $\mathcal{V}$-colimit of $\mathcal{C}(-, G_k) = \mathcal{G}^*(G_k, -)$, thus is α-flat by 4.2.

On the other hand if $F: \mathcal{G}^* \to \mathcal{V}$ is α-flat, then $F = F \star Y$ where $Y: \mathcal{G} \to [\mathcal{G}^*, \mathcal{V}]$ indicates the Yoneda embedding. Writing $i: \mathcal{G} \hookrightarrow \mathcal{C}$ for the canonical inclusion, the $\mathcal{V}$-colimit $F \star i \in \mathcal{C}$ exists by assumption on $\mathcal{C}$. Now Γ preserves α-filtered $\mathcal{V}$-colimits since each $\mathcal{C}(G_i, -)$ does; therefore

$$\Gamma(F \star i) = F \star (\Gamma \circ i) = F \star Y = F.$$

This concludes the proof that Γ induces a $\mathcal{V}$-equivalence between the categories $\mathcal{C}$ and α-$\mathsf{Flat}(\mathcal{G}^*, \mathcal{V})$.

Conversely, α-$Flat(\mathcal{D}, \mathcal{V})$ has α-filtered $\mathcal{V}$-colimits by 4.4. By the $\mathcal{V}$-Yoneda lemma, saying that the α-filtered colimits are pointwise means precisely that the representable $\mathcal{V}$-functors are α-presentable objects in α-$\mathsf{Flat}(\mathcal{D}, \mathcal{V})$. The last condition in definition 5.1 follows from 4.2. ∎

The rest of this section is devoted to proving that a canonical choice can be made for the G_i's in 5.1, namely, all the α-presentable objects.

5.4. Lemma. *Every absolute weight for a colimit is an α-filtered weight.*

PROOF. We recall that a weight G is absolute when $\mathcal{V}$-colimits weighted by G are preserved by all $\mathcal{V}$-functors (see [17]). If F is an α-small weight, G-weighted $\mathcal{V}$-colimits commute with F-weighted $\mathcal{V}$-limits precisely when F-weighted $\mathcal{V}$-limits commute with G-weighted $\mathcal{V}$-colimits, that is, if the right $\mathcal{V}$-Kan extension of F along the Yoneda embedding preserves G-weighted colimits. This is the case, by absoluteness of G. ∎

5.5. COROLLARY. *Every accessible $\mathcal{V}$-category is Cauchy complete.*

PROOF. Being Cauchy complete is having all absolute $\mathcal{V}$-colimits (see [17]). ∎

5.6. PROPOSITION. *In an α-accessible $\mathcal{V}$-category $\mathcal{C}$, the full subcategory $\mathcal{C}_\alpha$ of α-presentable objects is Cauchy complete.*

PROOF. Consider the situation

$$\mathcal{X}^* \xrightarrow{F} \mathcal{V}, \quad \mathcal{X} \xrightarrow{G} \mathcal{C}_\alpha \overset{i}{\hookrightarrow} \mathcal{C}$$

where F is an absolute weight, and on the other hand

$$\mathcal{Y} \xrightarrow{H} \mathcal{V}, \quad \mathcal{Y} \xrightarrow{K} \mathcal{C}$$

where H is an α-filtered weight. We get at one

$$\begin{aligned} \mathcal{C}(F \star iG, H \star K) &\cong \{F, \mathcal{C}(iG, H \star K)\} && (1)\\ &\cong \{F, H \star \mathcal{C}(iG, K)\} && (2)\\ &\cong H \star \{F, \mathcal{C}(iG, K)\} && (3)\\ &\cong H \star \mathcal{C}(F \star iG, K) && (4) \end{aligned}$$

where $F \star iG$ exists by 5.5 and $H \star K$ exists by definition of α-accessibility. The isomorphisms (1), (4) hold because each $\mathcal{V}$-functor $\mathcal{C}(-, C)$ transforms $\mathcal{V}$-colimits in $\mathcal{V}$-limits, while (2) holds because each GX is α-presentable and the $\mathcal{V}$-colimit $H \star K$ is α-filtered. Finally the isomorphism (3) holds because the absolute limits $\{F, -\}$ are preserved by the functor $H \star -$. This proves that $F \star iG \in \mathcal{C}_\alpha$. ∎

5.7. PROPOSITION. *The Cauchy completion of a small $\mathcal{V}$-category $\mathcal{X}$ is equivalent to the full subcategory of α-presentable objects in the $\mathcal{V}$-category α-$\mathsf{Flat}(\mathcal{X}^*, \mathcal{V})$ of α-flat $\mathcal{V}$-functors.*

PROOF. A $\mathcal{V}$-functor $F \in [\mathcal{X}^*, \mathcal{V}]$ is in the Cauchy completion of $\mathcal{X}$ when the canonical morphism

$$F \star [F, Y] \longrightarrow [F, F \star Y] \cong [F, F]$$

is an isomorphism, where Y is the $\mathcal{V}$-Yoneda embedding of $\mathcal{X}$(see [9]). The α-presentable objects in α-$\mathsf{Flat}(\mathcal{X}^*, \mathcal{V})$ constitute a Cauchy complete category $\mathcal{G}$ containing the representable $\mathcal{V}$-functors (see 5.6), thus the Cauchy completion of $\mathcal{X}$ is contained in $\mathcal{G}$.

Conversely if $F \in \mathcal{G}$, the functor

$$[F, -]\colon \alpha\text{-}\mathsf{Flat}(\mathcal{G}^*, \mathcal{V}) \longrightarrow \mathcal{V}$$

preserves α-filtered colimits, which are computed in α-$\mathsf{Flat}(\mathcal{G}^*, \mathcal{V})$ as in $[\mathcal{G}^*, \mathcal{V}]$ (see 4.4). Since F is α-flat, $F \star Y$ is a special instance of an α-filtered colimit in α-$\mathsf{Flat}(\mathcal{G}^*, \mathcal{V})$, from which follows the isomorphism at the beginning of the proof. ∎

5.8. Corollary. *If $\mathcal{C}$ is an α-accessible $\mathcal{V}$-category, the full $\mathcal{V}$-subcategory $\mathcal{C}_\alpha$ of α-presentable objects is equivalent to a small $\mathcal{V}$-category.*

Proof. When $\mathcal{V}$ is a locally presentable base, the essence of [9] is proving that the Cauchy completion of a small $\mathcal{V}$-category is still small. One concludes by 5.7. ∎

5.9. Theorem. *Every α-accessible $\mathcal{V}$-category $\mathcal{C}$ is equivalent to the category $\alpha\text{-}\mathsf{Flat}(\mathcal{C}^*_\alpha, \mathcal{V})$ of α-flat functors on the dual of the full subcategory $\mathcal{C}_\alpha \subseteq \mathcal{C}$ of α-presentable objects.*

Proof. By 5.8, the α-presentable objects can be chosen as generators $(G_i)_{i\in I}$ in definition 5.1. The proof of theorem 5.3 allows to conclude. ∎

6. Locally presentable categories

First of all, we generalize a classical definition in [7].

6.1. Definition. *Let $\mathcal{V}$ be a locally α_0-presentable base and $\alpha \geq \alpha_0$. A $\mathcal{V}$-category $\mathcal{C}$ is locally α-presentable when it is $\mathcal{V}$-cocomplete and admits a strongly $\mathcal{V}$-generating family $(G_i)_{i\in I}$, with each G_i α-presentable. A $\mathcal{V}$-category is locally presentable when it is locally α-presentable, for some $\alpha \geq \alpha_0$.*

6.2. Example. *Given a small $\mathcal{V}$-category $\mathcal{X}$, the category $[\mathcal{X}, \mathcal{V}]$ of $\mathcal{V}$-functors is locally α-presentable for every $\alpha \geq \alpha_0$.*

It is well-known that $[\mathcal{X}, \mathcal{V}]$ is $\mathcal{V}$-cocomplete, while $\mathcal{V}$-colimits are computed pointwise. By the $\mathcal{V}$-Yoneda lemma, $\mathsf{Nat}(\mathcal{X}(X,-),-)$ is evaluation at X, thus preserves all $\mathcal{V}$-colimits. So the representable $\mathcal{V}$-functors are α-presentable in $[\mathcal{X}, \mathcal{V}]$. But the representable $\mathcal{V}$-functors consitute a dense – thus strong – family of $\mathcal{V}$-generators in $[\mathcal{X}, \mathcal{V}]$. (see [11])

The next theorem shows both that our notion of local presentability coincides with that of [10] (condition 3) and that locally presentable $\mathcal{V}$-categories are accessible (condition 2).

6.3. Theorem. *For a $\mathcal{V}$-category $\mathcal{C}$, the following conditions are equivalent:*
(1) $\mathcal{C}$ is locally α-presentable;
(2) $\mathcal{C}$ is α-accessible and $\mathcal{V}$-cocomplete;
(3) $\mathcal{C}$ is equivalent to a $\mathcal{V}$-category $\alpha\text{-}\mathsf{Cont}(\mathcal{D}, \mathcal{V})$ of α-continuous functors on a small α-complete $\mathcal{V}$-category $\mathcal{D}$.
Moreover, every locally presentable $\mathcal{V}$-category is also $\mathcal{V}$-complete.

Proof. (1)⇒(2). Write $\mathcal{G}$ for the full $\mathcal{V}$-subcategory of $\mathcal{C}$ generated by a family $(G_i)_{i\in I}$ as in 6.1. Write $\mathcal{G} \subseteq \overline{\mathcal{G}} \subseteq \mathcal{C}$ for the closure of $\mathcal{G}$ in $\mathcal{C}$ under α-small $\mathcal{V}$-colimits. Since $\mathcal{G}$ is small and – up to isomorphisms – there is only a set of α-small weights, it follows that $\overline{\mathcal{G}}$ is equivalent to a small $\mathcal{V}$-category (see [11], section 3.5). On the other hand $\overline{\mathcal{G}}$ is obviously α-cocomplete and its objects are α-presentable in $\mathcal{C}$, by proposition 3.2.

Choose now $C \in \mathcal{C}$. Since $\overline{\mathcal{G}}$ is stable in $\mathcal{C}$ under α-small $\mathcal{V}$-colimits, the composite $\overline{\mathcal{G}} \overset{i}{\hookrightarrow} \mathcal{C} \xrightarrow{\mathcal{C}(-,C)} \mathcal{V}$ transforms α-small $\mathcal{V}$-colimits in α-small $\mathcal{V}$-limits. Thus

$$\mathcal{C}(i-, C) \colon \overline{\mathcal{G}} \longrightarrow \mathcal{V}$$

preserves α-small $\mathcal{V}$-limits and therefore (see 4.5 and 4.2) can be written as an α-filtered $\mathcal{V}$-colimit of representable $\mathcal{V}$-functors:

$$\mathcal{X}^* \xrightarrow{F} \mathcal{V}, \quad \mathcal{X} \xrightarrow{H} \overline{\mathcal{G}} \xrightarrow{Y} \left[\overline{\mathcal{G}}^*, \mathcal{V}\right]$$

where F is α-flat and $\mathcal{C}(i-, C) \cong F \star YH$. But the Yoneda embedding of $\overline{\mathcal{G}}$ factors as

$$\overline{\mathcal{G}} \overset{i}{\hookrightarrow} \mathcal{C} \xrightarrow{Z} \left[\overline{\mathcal{G}}^*, \mathcal{V}\right]$$

where $Z(D) = \mathcal{C}(i-, D)$ is conservative since the objects of $\mathcal{G}$– and thus of $\overline{\mathcal{G}}$ – constitute a strongly $\mathcal{V}$-generating family. The α-filtered $\mathcal{V}$-colimit $F \star iH$ exists in $\mathcal{C}$ and

$$\begin{aligned} Z(F \star iH) &\cong \mathcal{C}(i-, F \star iH) \cong F \star \mathcal{C}(i-, iH) \\ &\cong F \star \overline{\mathcal{G}}(-, H) \cong F \star YH \cong \mathcal{C}(i-, C) \cong Z(C) \end{aligned}$$

since each object iD is α-presentable and $F \star iH$ is α-filtered. Since Z is conservative, $F \star iH \cong C$, presenting C as an α-filtered $\mathcal{V}$-colimit of objects in $\overline{\mathcal{G}}$.

(2)⇒(3). By 5.9, we can write $\mathcal{C} \cong \alpha\text{-}\mathsf{Flat}(\mathcal{C}_\alpha, \mathcal{V})$ where $\mathcal{C}_\alpha \subseteq \mathcal{C}$ is the full $\mathcal{V}$-subcategory of α-presentable objects. Since $\mathcal{C}$ is $\mathcal{V}$-cocomplete, α-small $\mathcal{V}$-colimits of α-presentable objects exist and are α-presentable (see 3.2); thus $\mathcal{C}_\alpha$ is α-cocomplete. Therefore $\mathcal{C}_\alpha^*$ is α-complete and

$$\mathcal{C} \cong \alpha\text{-}\mathsf{Flat}(\mathcal{C}_\alpha^*, \mathcal{V}) \cong \alpha\text{-}\mathsf{Cont}(\mathcal{C}_\alpha^*, \mathcal{V})$$

by proposition 4.5.

(3)⇒(1). Again by proposition 4.5

$$\mathcal{C} \cong \alpha\text{-}\mathsf{Cont}(\mathcal{D}, \mathcal{V}) \cong \alpha\text{-}\mathsf{Flat}(\mathcal{D}, \mathcal{V})$$

proving, by theorem 5.3, that $\mathcal{C}$ is α-accessible. By definition, an α-accessible $\mathcal{V}$-category has a dense – thus strong – $\mathcal{V}$-generating family constituted of α-presentable objects. It remains to prove that $\mathcal{C}$ is cocomplete. But $\alpha\text{-}\mathsf{Cont}(\mathcal{D}, \mathcal{V})$ is $\mathcal{V}$-reflective in $[\mathcal{D}, \mathcal{V}]$ (see [11], section 6.3), thus is $\mathcal{V}$-complete and $\mathcal{V}$-cocomplete. ∎

6.4. Corollary. *Let $\mathcal{C}$ be a locally α-presentable $\mathcal{V}$-category. The full $\mathcal{V}$-subcategory $\mathcal{C}_\alpha$ of α-presentable objects is α-cocomplete and $\mathcal{C}$ is equivalent to the $\mathcal{V}$-category* $\alpha\text{-}\mathsf{Cont}(\mathcal{C}_\alpha^*, \mathcal{V})$ *of α-continous $\mathcal{V}$-functors.*

Proof. This was observed in the proof of 6.3. ∎

In general, our notion of presentable object (condition (1) in 6.5) differs from that of [10] (condition (2) in 6.5) and from the notion of α-presentable object in the ordinary underlying category (condition (3) in 6.5). But in a locally presentable $\mathcal{V}$-category, all three notions coincide.

6.5. LEMMA. *Let $\mathcal{C}$ be a locally α-presentable $\mathcal{V}$-category; write $\mathcal{C}_0$ for the underlying ordinary category. For a regular cardinal $\beta \geq \alpha$ and an object $C \in \mathcal{C}$, the following conditions are equivalent:*

(1) the $\mathcal{V}$-functor $\mathcal{C}(C,-)\colon \mathcal{C} \to \mathcal{V}$ preserves β-filtered $\mathcal{V}$-colimits;

(2) the ordinary functor $\mathcal{C}(C,-)\colon \mathcal{C}_0 \to \mathcal{V}_0$ preserves β-filtered conical colimits;

(3) the functor $\mathcal{C}_0(C,-)\colon \mathcal{C}_0 \to \mathsf{Set}$ preserves β-filtered conical colimits.

PROOF. By 1.2 and the fact that every locally α-presentable category is also locally β-presentable, there is no restriction in assuming $\alpha = \beta$.

(1) implies (2) by 2.7 and (2) implies (3) since $\mathcal{C}_0(C,-)$ is the composite functor

$$\mathcal{C} \xrightarrow{\mathcal{C}(C,-)} \mathcal{V} \xrightarrow{\mathcal{V}(I,-)} \mathsf{Set}$$

while the unit I is α-presentable in the ordinary sense.

Applying 6.4, an object $C \in \mathcal{C}$ correspond to the α-continuous functor $\mathcal{C}(-,C) \in \alpha\text{-}\mathsf{Cont}(\mathcal{C}^*_\alpha, \mathcal{V})$, thus by the proof of 4.5 can be written as an α-filtered conical colimit of representable $\mathcal{V}$-functors $\mathcal{C}_\alpha(-,C_i)$. In other words, every object $C \in \mathcal{C}$ is an α-filtered conical colimit $C = \operatorname{colim} C_i$ of α-presentable objects in the enriched sense. Assuming condition (3), C is α-presentable in the ordinary sense and the identity on $\mathcal{C}$ factors throught some C_i. Since $\mathcal{C}_\alpha$ is Cauchy complete (see 5.6) and C is a retract of an object in $\mathcal{C}_\alpha$, one concludes that $C \in \mathcal{C}_\alpha$. ∎

The previous proposition measures precisely the difference between ordinary local presentability and enriched local presentability. We make this more precise in the two following propositions.

6.6. PROPOSITION. *Let $\mathcal{C}$ be a $\mathcal{V}$-cocomplete $\mathcal{V}$-category and write $\mathcal{C}_0$ for the underlying ordinary category. The following conditions are equivalent:*

(1) $\mathcal{C}$ is locally α-presentable in the enriched sense;

(2) $\mathcal{C}_0$ is locally α-presentable in the ordinary sense and every object $C \in \mathcal{C}_0$ which is α-presentable in the ordinary sense is also an object $C \in \mathcal{C}$ which is α-presentable in the enriched sense.

PROOF. Applying 6.5, (1)⇒(2) reduces to proving that the α-presentable objects $C \in \mathcal{C}_0$ constitute a strongly generating family in the ordinary sense. Consider $f\colon C \to D$ in $\mathcal{C}_0$ such that $\mathcal{C}_0(X,f)$ is a bijection for every $X \in \mathcal{C}_\alpha$. For every α-presentable $V \in \mathcal{V}$, $V \otimes X \in \mathcal{C}_\alpha$ by 6.4 and 2.5, thus each $\mathcal{C}_0(V \otimes X, f) = \mathcal{V}_0(V, \mathcal{C}(X,f))$ is a bijection. Therefore each $\mathcal{C}(X,f)$ is an isomorphism since the α-presentable V's strongly generate $\mathcal{V}$. Finally f itself is an isomorphism since the $X \in \mathcal{C}_\alpha$ strongly $\mathcal{V}$-generate $\mathcal{C}$.

Assuming (2), it remains to observe that in $\mathcal{C}$, every strongly generating family in the ordinary sense is a fortiori strongly $\mathcal{V}$-generating. ∎

6.7. PROPOSITION. *A $\mathcal{V}$-category $\mathcal{C}$ is locally presentable in the enriched sense iff*

(1) $\mathcal{C}$ is $\mathcal{V}$-cocomplete;

(2) the underlying category $\mathcal{C}_0$ is locally presentable in the ordinary sense;

(3) every object $C \in \mathcal{C}$ is presentable in the enriched sense.

PROOF. The necessity of the conditions follows from 6.6, 6.5 and 5.2.

Conversely, assume conditions (1), (2), (3). The underlying category $\mathcal{C}_0$ is α-presentable for some α, which we can choose larger than α_0. Via condition (3), choose $\beta \geq \alpha$ such that each α-presentable object in the ordinary sense is β-presentable in the enriched sense. The α-presentable objects in the ordinary sense constitute a strongly generating family in the ordinary sense, thus a fortiori a strongly $\mathcal{V}$-generating family. Therefore the bigger family of β-presentable objects in the enriched sense is also strongly $\mathcal{V}$-generating and $\mathcal{C}$ is β-presentable in the enriched sense. ■

6.8. COROLLARY. *Let $\mathcal{D}$ be a $\mathcal{V}$-colimit closed full $\mathcal{V}$-subcategory of a locally presentable $\mathcal{V}$-category $\mathcal{C}$. The following conditions are equivalent:*

(1) $\mathcal{D}$ is locally presentable in the enriched sense;

(2) the underlying category $\mathcal{D}_0$ is locally presentable in the ordinary sense.

PROOF. (1) implies (2) by 6.7. Conversely every object $D \in \mathcal{D}$ is presentable in the enriched sense in $\mathcal{C}$, thus also in $\mathcal{D}$ because $\mathcal{D}$ is $\mathcal{V}$-colimit closed in $\mathcal{C}$. One concludes again by 6.7. ■

7. Enriched sketches

Let us recall that given

$$F: \mathcal{X} \to \mathcal{V}, \quad G: \mathcal{X} \to \mathcal{C}$$

the $\mathcal{V}$-limit $\{F, G\}$ of G weighted by F yields natural isomorphisms

$$\mathcal{V} - \mathsf{Nat}(F, \mathcal{C}(C, G-)) \cong \mathcal{C}(C, \{F, G\})$$

for all objects $C \in \mathcal{C}$. The corresponding "$\mathcal{V}$-limit cone" is the $\mathcal{V}$-natural transformation

$$F \Longrightarrow \mathcal{C}(\{F, G\}, G-)$$

corresponding to the identity on $\{F, G\}$. The case of $\mathcal{V}$-colimits is analogous.

The reader will observe that our definition of sketch does not refer at all to the cardinal α_0 appearing in definition 1.1.

7.1. DEFINITION. *Let $\mathcal{V}$ be a locally presentable base. A $\mathcal{V}$-sketch is a triple $(\mathcal{S}, \mathcal{P}, \mathcal{I})$ where*

(1) $\mathcal{S}$ is a small $\mathcal{V}$-category;

(2) $\mathcal{P}$ is a set of projective $\mathcal{V}$-cones, that is, quintuples $(\mathcal{X}, F, G, S, \gamma)$ with

$$F: \mathcal{X} \to \mathcal{V}, \quad G: \mathcal{X} \to \mathcal{S}, \quad \gamma: F \Longrightarrow \mathcal{S}(S, G-),$$

where $\mathcal{X}$ is a small $\mathcal{V}$-category, F, G are $\mathcal{V}$-functors, $S \in \mathcal{S}$ and γ is a $\mathcal{V}$-natural transformation.

(3) $\mathcal{I}$ is a set of inductive $\mathcal{V}$-cones, that is, quintuples $(\mathcal{X}, F, G, S, \gamma)$ with

$$F: \mathcal{X}^* \to \mathcal{V}, \quad G: \mathcal{X} \to \mathcal{S}, \quad \gamma: F \Longrightarrow \mathcal{S}(G-, S),$$

where $\mathcal{X}$ is a small $\mathcal{V}$-category, F, G are $\mathcal{V}$-functors, $S \in \mathcal{S}$ and γ is a $\mathcal{V}$-natural transformation.

In the previous definition, γ can be thought as a cone of vertex S, with weight F, on the functor G.

7.2. Definition. *A model of the $\mathcal{V}$-sketch in definition 7.1 is a $\mathcal{V}$-functor $M: \mathcal{S} \to \mathcal{V}$ which transforms the projective $\mathcal{V}$-cones of $\mathcal{P}$ in $\mathcal{V}$-limit cones and the inductive $\mathcal{V}$-cones of $\mathcal{I}$ in $\mathcal{V}$-colimit cones. The $\mathcal{V}$-category of these models is the full $\mathcal{V}$-subcategory of $[\mathcal{S}, \mathcal{V}]$ generated by the models. A $\mathcal{V}$-category which is equivalent to the $\mathcal{V}$-category of models of a $\mathcal{V}$-sketch is called $\mathcal{V}$-sketchable.*

First, we want to prove that the category of models of a $\mathcal{V}$-sketch is an accessible $\mathcal{V}$-category. To achieve this, we treat separately the special cases of "projective" and "inductive" $\mathcal{V}$-sketches.

7.3. Proposition. *The category of models of a $\mathcal{V}$-sketch $\mathcal{S}$ having only projective specifications is a locally presentable $\mathcal{V}$-category, closed in $[\mathcal{S}, \mathcal{V}]$ under α-filtered $\mathcal{V}$-colimits, for some $\alpha \geq \alpha_0$.*

Proof. Proposition 6.21 in [11] indicates that the category of models of such a sketch $\mathcal{S}$ is also the category of models of a sketch $\mathcal{T}$ with only projective specifications which are already $\mathcal{V}$-limit cones. The models of this new sketch $\mathcal{T}$ are thus the functors $\mathcal{T} \to \mathcal{V}$ preserving the specified $\mathcal{V}$-limits: they constitute a $\mathcal{V}$-reflective subcategory of the $\mathcal{V}$-category $[\mathcal{T}, \mathcal{V}]$ (see theorem 6.11 in [11]). Therefore, the category of models of $\mathcal{T}$, and thus of $\mathcal{S}$, is $\mathcal{V}$-cocomplete.

The representables are $\mathcal{T}$-models and they strongly $\mathcal{V}$-generate $[\mathcal{T}, \mathcal{V}]$, thus also $\mathsf{Mod}(\mathcal{T})$. Choosing $\alpha \geq \alpha_0$ such that all specified $\mathcal{V}$-cones in $\mathcal{T}$ have α-small weights, commutation of α-small $\mathcal{V}$-limits with α-filtered $\mathcal{V}$-colimits implies that α-filtered $\mathcal{V}$-colimits are pointwise in $\mathsf{Mod}(\mathcal{T})$. The $\mathcal{V}$-functor represented by a representable $\mathcal{T}(T, -)$ is evaluation at T, thus preserves α-filtered $\mathcal{V}$-colimits. Therefore the representable $\mathcal{V}$-functors are α-presentable and $\mathsf{Mod}(\mathcal{T})$ is a locally α-presentable $\mathcal{V}$-category. ■

7.4. Corollary. *A $\mathcal{V}$-category is locally presentable iff it is the $\mathcal{V}$-category of models of a $\mathcal{V}$-sketch with only projective specifications.*

Proof. By 7.3 and 6.4. ■

7.5. Proposition. *The category of models of a $\mathcal{V}$-sketch $\mathcal{S}$ having only inductive specifications is a locally presentable $\mathcal{V}$-category, $\mathcal{V}$-colimit closed in $[\mathcal{S}, \mathcal{V}]$.*

Proof. The $\mathcal{V}$-category of models of the inductive $\mathcal{V}$-sketch $\mathcal{S}$ is $\mathcal{V}$-colimit closed in the category $[\mathcal{S}, \mathcal{V}]$ of all $\mathcal{V}$-functors, by the classical interchange property of $\mathcal{V}$-colimits. On the other hand $[\mathcal{S}, \mathcal{V}]$ is a locally α-presentable $\mathcal{V}$-category for every $\alpha \geq \alpha_0$ (see example 6.2). By 6.8, it remains to check that the ordinary category of models is locally presentable in the ordinary sense.

We begin with the special case where the $\mathcal{V}$-sketch $\mathcal{S}$ has a single inductive cocone, namely

$$F: \mathcal{X}^* \to \mathcal{V}, \quad G: \mathcal{X} \to \mathcal{S}, \quad S \in \mathcal{S}, \quad \gamma: F \Rightarrow \mathcal{S}(G-, S).$$

For every $\mathcal{V}$-functor $H: \mathcal{S} \to \mathcal{V}$ we consider the canonical factorisation

$$\theta_H: F \star HG \longrightarrow H(F \star G).$$

This yields at once a functor

$$\Theta: [\mathcal{S}, \mathcal{V}] \longrightarrow \mathcal{V}^{\to}, \quad H \mapsto \theta_H$$

where $\mathcal{V}^{\to}$ is the ordinary category of arrows of $\mathcal{V}$. The functor Θ could easily be made a $\mathcal{V}$-functor, but we do not need this fact. For every conical colimit $H = \operatorname{colim} H_i$ in $[\mathcal{S}, \mathcal{V}]$, one has

$$F \star HG \cong F \star (\operatorname{colim} H_i)G \cong F \star \operatorname{colim} H_i G \cong \operatorname{colim}(F \star H_i G)$$
$$H(F \star G) \cong (\operatorname{colim} H_i)(F \star G) \cong \operatorname{colim} H_i(F \star G)$$

by commutation of $\mathcal{V}$-colimits with conical colimits. This implies at once that the functor Θ preserves conical colimits. In particular, $[\mathcal{S}, \mathcal{V}]$ and $\mathcal{V}^{\to}$ are locally presentable categories and Θ is an accessible functor (see [15]).

Let us write $\mathcal{V}^{\cong}$ for the full subcategory of $\mathcal{V}^{\to}$ whose objects are isomorphisms in $\mathcal{V}$. This category $\mathcal{V}^{\cong}$ is trivially equivalent to $\mathcal{V}$, thus is locally presentable. It is also colimit closed in $\mathcal{V}^{\to}$, so that the inclusion $i: \mathcal{V}^{\cong} \hookrightarrow \mathcal{V}^{\to}$ is an accessible functor between locally presentable categories. The inverse image of this inclusion along the functor Θ is precisely the category of models of our sketch $\mathcal{S}$ with a single $\mathcal{V}$-cocone. By remark 2.50 in [1], $\mathsf{Mod}(\mathcal{S})$ is an accessible category and the inclusion $\mathsf{Mod}(\mathcal{S}) \hookrightarrow [\mathcal{S}, \mathcal{V}]$ is accessible. Since $\mathsf{Mod}(\mathcal{S})$ is also cocomplete, it is locally presentable.

In the case of a $\mathcal{V}$-sketch $\mathcal{S}$ with an arbitrary set of inductive specifications, $\mathsf{Mod}(\mathcal{S})$ is the intersection of all the categories of models of the various sketches given by the $\mathcal{V}$-category $\mathcal{S}$ and only one of the inductive specifications. Therefore $\mathsf{Mod}(\mathcal{S})$ is accessible, as an intersection of a family of accessible, accessibly embedded full subcategories of $[\mathcal{S}, \mathcal{V}]$ (see [1], corollary 2.37). Since $\mathsf{Mod}(\mathcal{S})$ is also cocomplete, it is locally presentable in the ordinary sense. ∎

7.6. Theorem. *The category of models of a $\mathcal{V}$-sketch is an accessible $\mathcal{V}$-category.*

Proof. Given a $\mathcal{V}$-sketch $(\mathcal{S}, \mathcal{P}, \mathcal{I})$, we split it in two $\mathcal{V}$-sketches $\mathcal{S}_l = (\mathcal{S}, \mathcal{P}, \emptyset)$ and $\mathcal{S}_r = (\mathcal{S}, \emptyset, \mathcal{I})$ with only projective specifications or only inductive specifications. Clearly one has

$$\mathsf{Mod}(\mathcal{S}) = \mathsf{Mod}(\mathcal{S}_l) \cap \mathsf{Mod}(\mathcal{S}_r) \subseteq [\mathcal{S}, \mathcal{V}].$$

Applying 7.5, 7.3 and corollary 2.37 in [1], we conclude that $\mathsf{Mod}(\mathcal{S})$ is closed in $[\mathcal{S}, \mathcal{V}]$ under α-filtered $\mathcal{V}$-colimits, for some $\alpha \geq \alpha_0$, while the ordinary category of models of $\mathcal{S}$ is accessible in the ordinary sense. There is no restriction in choosing α sufficiently large, so that $\mathsf{Mod}(\mathcal{S})$ is α-accessible in the ordinary sense. In particular $\mathsf{Mod}(\mathcal{S})$ has only (up to isomorphisms) a set of objects which are α-presentable in the ordinary sense.

Every object in $[\mathcal{S}, \mathcal{V}]$ is β-presentable in the enriched sense, for some $\beta \geq \alpha$, while $\mathsf{Mod}(\mathcal{S})$ is closed in $[\mathcal{S}, \mathcal{V}]$ under β-filtered $\mathcal{V}$-colimits. Therefore every object in $\mathsf{Mod}(\mathcal{S})$

is presentable in the enriched sense. Let us choose a regular cardinal $\alpha \lhd \gamma$ ("sharply less" relation, see [15]) such that all objects of $\mathsf{Mod}(\mathcal{S})$ which are α-presentable in the ordinary sense are also γ-presentable in the enriched sense. Since $\alpha \lhd \gamma$ and $\mathsf{Mod}(\mathcal{S})$ is α-accessible, $\mathsf{Mod}(\mathcal{S})$ is also γ-accessible in the ordinary sense.

Since $\alpha \lhd \gamma$, it follows from 2.3.11 in [15] that every object $M \in \mathsf{Mod}(\mathcal{S})$ which is γ-presentable in the ordinary sense is an α-filtered γ-small conical colimit of objects which are α-presentable in the ordinary sense. But α-presentable objects in the ordinary sense are γ-presentable in the enriched sense. Moreover a diagram which is γ-small in the ordinary sense is also γ-small in the enriched sense (see 2.2). This shows already that in $\mathsf{Mod}(\mathcal{S})$, an object which is γ-presentable in the ordinary sense is also a γ-small $\mathcal{V}$-colimit of objects which are γ-presentable in the enriched sense, thus by 3.2, is γ-presentable in the enriched sense. Combining this with 2.7 and 2.6, we have proved that in $\mathcal{C}$, γ-presentable objects in the enriched sense coincide with γ-presentable objects in the ordinary sense.

Thus $\mathsf{Mod}(\mathcal{S})$ has γ-filtered $\mathcal{V}$-colimits computed as in $[\mathcal{S}, \mathcal{V}]$. Since $\mathsf{Mod}(\mathcal{S})$ is γ-accessible in the ordinary sense, every $M \in \mathsf{Mod}(\mathcal{S})$ can be written as a γ-filtered conical colimit of objects which are γ-presentable in the ordinary sense. By 2.7 and what we have just proved, this is also a γ-filtered $\mathcal{V}$-colimit of objects which are γ-presentable in the enriched sense. ∎

Next we prove that, conversely, accessible $\mathcal{V}$-categories are $\mathcal{V}$-sketchable. As a lemma, we relate models of a $\mathcal{V}$-sketch with what could be called "α-geometric morphisms".

7.7. Lemma. *Let $\mathcal{C}$ be an α-accessible $\mathcal{V}$-category; write $\mathcal{C}_\alpha$ for the full $\mathcal{V}$-subcategory of α-presentable objects. The $\mathcal{V}$-category $\mathcal{C}$ is equivalent to the $\mathcal{V}$-category of those functors*

$$F\colon [\mathcal{C}_\alpha, \mathcal{V}] \longrightarrow \mathcal{V}$$

which preserve α-small $\mathcal{V}$-limits and small $\mathcal{V}$-colimits.

Proof. The left $\mathcal{V}$-Kan extension along the $\mathcal{V}$-Yoneda embedding $\mathcal{C}_\alpha^* \to [\mathcal{C}_\alpha, \mathcal{V}]$ induces a $\mathcal{V}$-equivalence

$$[\mathcal{C}_\alpha^*, \mathcal{V}] \cong \mathcal{V}\text{-}\mathsf{Cocont}[[\mathcal{C}_\alpha, \mathcal{V}], \mathcal{V}]$$

between the $\mathcal{V}$-category of $\mathcal{V}$-functors $\mathcal{C}_\alpha^* \to \mathcal{V}$ and that of $\mathcal{V}$-functors $[\mathcal{C}_\alpha, \mathcal{V}] \to \mathcal{V}$ which preserve small $\mathcal{V}$-colimits (see [11], 4.51). By 4.1, this equivalence induces that of the statement. ∎

7.8. Theorem. *Let $\mathcal{V}$ be a locally presentable base. Every accessible $\mathcal{V}$-category is equivalent to the category of models of a $\mathcal{V}$-sketch.*

Proof. We keep the notation of 7.7; thus $\mathcal{C}$ is an α-accessible $\mathcal{V}$-category. By 5.8, $\mathcal{C}_\alpha$ is small (up to equivalence). Therefore there is only a set (up to isomorphisms) of situations of the type

$$H\colon \mathcal{X} \to \mathcal{V}, \quad \mathcal{X} \xrightarrow{F} \mathcal{C}_\alpha^* \xrightarrow{Y} [\mathcal{C}_\alpha, \mathcal{V}]$$

where $\mathcal{X}$ is a small $\mathcal{V}$-category, H, F are $\mathcal{V}$-functors with H an α-small weight and Y is the Yoneda embedding. For each such situation we consider the corresponding $\mathcal{V}$-limit in

$[\mathcal{C}_\alpha, \mathcal{V}]$, with canonical $\mathcal{V}$-cone

$$\lambda \colon H(-) \Longrightarrow \mathsf{Nat}(L, (Y \circ F)(-)).$$

As $\mathcal{V}$-category $\mathcal{S}$ underlying our sketch we take the full $\mathcal{V}$-subcategory $\mathcal{S} \subseteq [\mathcal{C}_\alpha, \mathcal{V}]$ generated by all the representable $\mathcal{V}$-functors and all the functors L we have just indicated; up to equivalence, this is a small $\mathcal{V}$-category. As family $\mathcal{P}$ of projective $\mathcal{V}$-cones we consider all the $\mathcal{V}$-cones λ we have just indicated. Now each $S \in \mathcal{S}$ can be canonically expressed as a $\mathcal{V}$-colimit of representable $\mathcal{V}$-functors on $\mathcal{C}_\alpha$, $S = S \star Y$, yielding a corresponding $\mathcal{V}$-cocone

$$\sigma \colon S(-) \Longrightarrow \mathsf{Nat}(Y(-), S);$$

we choose all these $\mathcal{V}$-cocones σ as family $\mathcal{I}$ of inductive $\mathcal{V}$-cones for our sketch.

We shall prove that the α-accessible $\mathcal{V}$-category $\mathcal{C}$ is equivalent to the category of models of this $\mathcal{V}$-sketch $(\mathcal{S}, \mathcal{P}, \mathcal{I})$. Observe at once that the $\mathcal{V}$-category $\mathcal{S}$ is by definition constituted of the α-small $\mathcal{V}$-limits of representable $\mathcal{V}$-functors on $\mathcal{C}_\alpha$: therefore it is an α-complete $\mathcal{V}$-category (see [10], 4.2). In the same way every model M preserves the α-small $\mathcal{V}$-limits of representables, thus it is α-continuous as a $\mathcal{V}$-functor.

Let us fix some notation. Since $\mathcal{S}$ contains the representables, the Yoneda embedding of $\mathcal{C}_\alpha$ factors as

$$\mathcal{C}^*_\alpha \xrightarrow{Z} \mathcal{S} \xhookrightarrow{i} [\mathcal{C}_\alpha, \mathcal{V}], \quad Z(C) = \mathcal{C}(C, -).$$

By definition, the projective $\mathcal{V}$-cones of $\mathcal{P}$ are $\mathcal{V}$-limit cones both in $\mathcal{S}$ and in $[\mathcal{C}_\alpha, \mathcal{V}]$ and the inductive $\mathcal{V}$-cones of $\mathcal{I}$ are $\mathcal{V}$-colimit cones both in $\mathcal{S}$ and in $[\mathcal{C}_\alpha, \mathcal{V}]$; thus i preserves those $\mathcal{V}$-limits and $\mathcal{V}$-colimits. On the other hanbd let us write α-$\mathsf{Geom}([\mathcal{C}_\alpha, \mathcal{V}], \mathcal{V})$ for the $\mathcal{V}$-category of "α-geometric morphisms" appearing in the statement of lemma 7.7. By lemma 7.7, it remains to prove that $\mathsf{Mod}(\mathcal{S})$ is equivalent to α-$\mathsf{Geom}([\mathcal{C}_\alpha, \mathcal{V}], \mathcal{V})$.

Let us first prove that the Kan extension along i restricts as a functor

$$\theta_i \colon \mathsf{Mod}(\mathcal{S}) \longrightarrow \alpha\text{-}\mathsf{Geom}([\mathcal{C}_\alpha, \mathcal{V}], \mathcal{V}); \quad M \mapsto \mathsf{Lan}_i M.$$

Since Y is $\mathcal{V}$-full and faithful, given $M \in \mathsf{Mod}(\mathcal{S})$,

$$M \circ Z = \mathsf{Lan}_Y(M \circ Z) \circ Y = \mathsf{Lan}_Y(M \circ Z) \circ i \circ Z$$

proving that M and $\mathsf{Lan}_Y(M \circ Z)$ coincide on the representables. By interchange property of $\mathcal{V}$-colimits, the functor

$$\mathsf{Lan}_Y(M \circ Z) = (M \circ Z) \star -$$

preserves the canonical $\mathcal{V}$-colimit $S = S \star Y$, for every $S \in \mathcal{S}$. But M, as a model of $\mathcal{S}$, preserves also this $\mathcal{V}$-colimit. Therefore $\mathsf{Lan}_Y(M \circ Z)(S) \cong M(S)$ and finally $\mathsf{Lan}_Y(M \circ Z) \circ i \cong M$.

Next one has

$$\begin{aligned}
(\mathsf{Lan}_Z(M \circ Z))(S) &\cong \mathsf{Nat}(Z(-), S) \star (M \circ Z)(-) && (1)\\
&\cong S \star (M \circ Z) && (2)\\
&\cong M(S \star Z) && (3)\\
&\cong M(S) && (4)
\end{aligned}$$

where (1) is the pointwise formula for $\mathcal{V}$-Kan extensions, (2) is an application of the $\mathcal{V}$-Yoneda lemma, (4) and (3) hold because $S = S\star Y = S\star Z$ and this $\mathcal{V}$-colimit is preserved by the model M. It follows at once

$$\mathsf{Lan}_i M = \mathsf{Lan}_i \mathsf{Lan}_Z(M \circ Z) = \mathsf{Lan}_{i\circ Z}(M \circ Z) = \mathsf{Lan}_Y(M \circ Z).$$

Since M is α-continous, by 4.3 and 4.5, $\mathsf{Lan}_i M$ is α-continous. So $\mathsf{Lan}_Y(M \circ Z)$ is α-continous and $M \circ Z$ is α-flat (see 4.1). By 7.7,

$$\theta_i(M) \cong \mathsf{Lan}_i M \cong \mathsf{Lan}_Y(M \circ Z) \in \alpha\text{-}\mathsf{Geom}([\mathcal{C}_\alpha, \mathcal{V}], \mathcal{V}).$$

Now since i preserves all $\mathcal{V}$-limits in $\mathcal{P}$ and all $\mathcal{V}$-colimits in $\mathcal{I}$, composing with i restricts also as a $\mathcal{V}$-functor

$$\tau_i \colon \alpha\text{-}\mathsf{Geom}([\mathcal{C}_\alpha, \mathcal{V}], \mathcal{V}) \longrightarrow \mathsf{Mod}(\mathcal{S}), \quad H \mapsto H \circ i.$$

But since i is full and faithful, $(- \circ i) \circ (\mathsf{Lan}_i -) \cong \mathsf{id}$, yielding $\tau_i \circ \theta_i \cong \mathsf{id}$. Moreover

$$\begin{aligned}(\mathsf{Lan}_i -) \circ (- \circ i) \circ (\mathsf{Lan}_Y -) &\cong (\mathsf{Lan}_i -) \circ (- \circ i) \circ (\mathsf{Lan}_i -) \circ (\mathsf{Lan}_Z -)\\ &\cong (\mathsf{Lan}_i -) \circ (\mathsf{Lan}_Z -)\\ &\cong \mathsf{Lan}_Y - .\end{aligned}$$

Since by 7.7 $\mathsf{Lan}_Y -$ is in particular essentially surjective on α-$\mathsf{Geom}([\mathcal{C}_\alpha, \mathcal{V}], \mathcal{V})$, this implies $\theta_i \circ \tau_i \cong \mathsf{id}$. ∎

7.9. Corollary. *Let $\mathcal{V}$ be a locally presentable base. A $\mathcal{V}$-category is accessible iff it is $\mathcal{V}$-sketchable.*

Proof. By 7.6 and 7.8. ∎

Our next theorem is quite amazing: it indicates that – if one agrees to change the degree of accessibility – α-filtered conical colimits become sufficient to treat the various problems studied in this paper. One should be very clear about that result. For example a finitely flat $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{V}$ has no reason to be a filtered conical colimit of representable $\mathcal{V}$-functors. But the category $\mathsf{Flat}(\mathcal{C}, \mathcal{V})$ is equivalent to a category $\alpha - \mathsf{Flat}(\mathcal{D}, \mathcal{V})$ and on $\mathcal{D}$, every α-flat $\mathcal{V}$-functor is an α-filtered conical colimit of representable $\mathcal{V}$-functors.

7.10. Theorem. *Let $\mathcal{V}$ be a locally presentable base. For every accessible $\mathcal{V}$-category $\mathcal{C}$, there exist arbitrarily large regular cardinals γ such that:*
(1) an object $C \in \mathcal{C}$ is γ-presentable in the enriched sense iff, in the ordinary category underlying $\mathcal{C}$, the object C is γ-presentable in the ordinary sense;
(2) every object $C \in \mathcal{C}$ is a γ-filtered conical colimit of γ-presentable objects;
(3) the ordinary category underlying $\mathcal{C}$ is γ-accessible in the ordinary sense.
(4) the $\mathcal{V}$-category $\mathcal{C}$ is γ-accessible in the enriched sense.

Proof. An accessible $\mathcal{V}$-category is $\mathcal{V}$-sketchable, thus it suffices to choose γ as in the proof of theorem 7.6. The properties of the "sharply less" relation $\lhd$ (see [15]) indicate the existence of arbitrarily large such cardinals γ. ∎

The previous theorem indicates in particular that we can always raise the degree of accessibility of a given accessible $\mathcal{V}$-category $\mathcal{C}$. Nevertheless, one should observe that the cardinal γ depends on $\mathcal{C}$, not just on the original degree α of accessibility. This is a striking difference with the "sharply less" relation which, in the ordinary case, allows raising the degree of accessibility for all accessible categories. The following proposition will be a sufficient substitute for this fact.

7.11. PROPOSITION. *If $(\mathcal{C}_i)_{i\in I}$ is a family of accessible $\mathcal{V}$-categories, there exist arbitrarily large regular cardinals γ such that each $\mathcal{C}_i$ is a γ-accessible $\mathcal{V}$-category and satisfies for this γ the requirements of theorem 7.10.*

PROOF. Each accessible $\mathcal{V}$-category $\mathcal{C}_i$ is $\mathcal{V}$-sketchable, yielding a corresponding regular cardinal α_i as in the proof of 7.6. There exist arbitrarily large regular cardinals γ such that $\alpha_i \lhd \gamma$ for each index i (see [15]). The proof of 7.6 yields the conclusion. ■

8. Enriched accessible functors

We follow with the study of accessible $\mathcal{V}$-functors, which can be thought as the "morphisms of accessible $\mathcal{V}$-categories".

8.1. DEFINITION. *A $\mathcal{V}$-functor $F: \mathcal{C} \to \mathcal{D}$ between α-accessible $\mathcal{V}$-categories is α-accessible when it preserves α-filtered $\mathcal{V}$-colimits. An accessible $\mathcal{V}$-functor is an α-accessible one, for some α.*

8.2. PROPOSITION. *Let $\mathcal{C}$ be an accessible $\mathcal{V}$-category. A $\mathcal{V}$-functor $F: \mathcal{C} \to \mathcal{V}$ is accessible iff it is a small $\mathcal{V}$-colimit of representable $\mathcal{V}$-functors.*

PROOF. Representable $\mathcal{V}$-functors are accessible by 5.2. By the $\mathcal{V}$-Yoneda lemma,

$$\mathsf{Nat}(\mathcal{C}(C,-),F) \cong F(C)$$

exists for every $\mathcal{V}$-functor $F: \mathcal{C} \to \mathcal{V}$. Since in its first variable, Nat transforms $\mathcal{V}$-colimits in $\mathcal{V}$-limits, it follows that $\mathsf{Nat}(F', F)$ exists in $\mathcal{V}$ for every F' which is a small $\mathcal{V}$-colimit of representables. Let us write $\mathsf{Acc}[\mathcal{C},\mathcal{V}]$ for the $\mathcal{V}$-category of those $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{V}$ which are small $\mathcal{V}$-colimits of representables.

Suppose first that $F = G \star (Y \circ H)$ where

$$G: \mathcal{X}^* \longrightarrow \mathcal{V}, \quad \mathcal{X} \xrightarrow{H} \mathcal{C}^* \xrightarrow{Y} \mathsf{Acc}[\mathcal{C},\mathcal{V}]$$

with $\mathcal{X}$ a small $\mathcal{V}$-category, G, H two $\mathcal{V}$-functors and Y the $\mathcal{V}$-Yoneda embedding. Choose a regular cardinal α large enough for having (see in particular 7.10 and 5.2)

(1) $\mathcal{C}$ is an α-caccessible $\mathcal{V}$-category;

(2) $G: \mathcal{X}^* \to \mathcal{V}$ is an α-small $\mathcal{V}$-functor;

(3) each $\mathcal{V}$-functor $\mathcal{C}(HX,-): \mathcal{C} \to \mathcal{V}$ is α-accessible.

By interchange property of colimits, $F = G \star (Y \circ H)$ preserves α-filtered $\mathcal{V}$-colimits since each $(\mathcal{Y} \circ H)(X) = \mathcal{C}(HX,-)$ does. This proves that F is α-accessible.

Conversely, suppose $\mathcal{C}$ is an α-accessible $\mathcal{V}$-category and $F: \mathcal{C} \to \mathcal{V}$, an α-accessible $\mathcal{V}$-functor. With the notation of 5.9, we consider

$$\mathcal{C}_\alpha \xhookrightarrow{i} \mathcal{C} \xrightarrow{F} \mathcal{V}, \quad \mathcal{C}^*_\alpha \xrightarrow{i^*} \mathcal{C}^* \xrightarrow{Y} \mathsf{Acc}[\mathcal{C}, \mathcal{V}].$$

For each $C \in \mathcal{C}_\alpha$, the Yoneda isomorphism $F(C) \cong \mathsf{Nat}(\mathcal{C}(C, -), F)$ induces a $\mathcal{V}$-natural transformation

$$(F \circ i)(-) \Longrightarrow \mathsf{Nat}((Y \circ i^*)(-), F),$$

from which a corresponding morphism

$$\varphi: (F \circ i) \star (Y \circ i^*) \Longrightarrow F$$

since in $\mathsf{Acc}[\mathcal{C}, \mathcal{V}]$ small $\mathcal{V}$-colimits are computed pointwise, and therefore keep their universal property with respect to $\mathcal{V}$-cocones with vertex an arbitrary $\mathcal{V}$-functor $F: \mathcal{C} \to \mathcal{V}$. We shall prove it is an isomorphism, which will conclude the proof since $\mathcal{C}_\alpha$ is small (see 5.8).

Given $C \in \mathcal{C}_\alpha$, the $\mathcal{V}$-functor $\mathcal{C}(C, -) = (Y \circ i^*)(C)$ preserves α-filtered $\mathcal{V}$-colimits. By the interchange property of $\mathcal{V}$-colimits, $(F \circ i) \star (Y \circ i^*)$ preserves α-filtered colimits as well. On the other hand, F preserves α-filtered colimits. Since every object $C \in \mathcal{C}$ is an α-filtered $\mathcal{V}$-colimit of objects in $\mathcal{C}_\alpha$, it suffices to prove that φ_C is an isomorphism for each $C \in \mathcal{C}_\alpha$. This reduces to proving

$$(F \circ i)(C) \cong (F \circ i) \star \mathcal{C}(C, i-) \cong (F \circ i) \star \mathcal{C}_\alpha(C, -)$$

which is the classical formula expressing $F \circ i$ as $\mathcal{V}$-colimit of representable $\mathcal{V}$-functors. ■

8.3. PROPOSITION. *Let $F: \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-functor between accessible $\mathcal{V}$-categories. The following conditions are equivalent:*

(1) F is an accessible $\mathcal{V}$-functor;

(2) for every object $D \in \mathcal{D}$, the $\mathcal{V}$-functor $\mathcal{D}(D, F-): \mathcal{C} \to \mathcal{V}$ is accessible.

PROOF. (1)⇒(2). Given $D \in \mathcal{D}$, by 7.11 choose α such that $\mathcal{C}$, $\mathcal{D}$ and F are α-accessible and D is α-presentable. Then both F and $\mathcal{D}(D, -)$ preserve α-filtered $\mathcal{V}$-colimits, thus also their composite $\mathcal{D}(D, F-)$.

(2)⇒(1). Again choose α such that $\mathcal{C}$ and $\mathcal{D}$ are α-accessible. Write $\mathcal{D}_\alpha \subseteq \mathcal{D}$ for the full subcategory of α-presentable objects. By assumption, each $\mathcal{D}(D, F-)$ preserves β_D-filtered $\mathcal{V}$-colimits for some cardinal β_D. Choose a regular cardinal γ such that $\mathcal{C}$, $\mathcal{D}$ are γ-accessible, while γ is larger than α and all the β_D, for $D \in \mathcal{D}_\alpha$. It follows at once that $\mathcal{D}(D, F-)$ preserves γ-filtered $\mathcal{V}$-colimits, for all $D \in \mathcal{D}_\alpha$. To prove that F preserves γ-filtered $\mathcal{V}$-colimits, it remains to observe that the family of functors $\mathcal{D}(D, -)$, with $D \in \mathcal{D}_\alpha$, reflects collectively γ-filtered $\mathcal{V}$-colimits. This is indeed the case: each $D \in \mathcal{D}_\alpha$ is γ-presentable, thus $\mathcal{D}(D, -)$ preserves γ-filtered $\mathcal{V}$-colimits; on the other hand these D's constitute a dense – thus strong – $\mathcal{V}$-generating family, thus the $\mathcal{D}(D, -)$ with $D \in \mathcal{D}_\alpha$ reflect collectively isomorphisms. ■

Next we proceed towards a $\mathcal{V}$-adjoint functor theorem for accessible $\mathcal{V}$-functors.

8.4. Proposition. *Let $F: \mathcal{C} \to \mathcal{D}$ be a $\mathcal{V}$-functor between accessible $\mathcal{V}$-categories. If F has a $\mathcal{V}$-left adjoint, it is accessible as a $\mathcal{V}$-functor.*

Proof. Write $G: \mathcal{D} \to \mathcal{C}$ for the $\mathcal{V}$-left adjoint of F. By 7.11, choose α such that $\mathcal{C}$ and $\mathcal{D}$ are α-accessible. Write $\mathcal{D}_\alpha \subseteq \mathcal{D}$ for the full $\mathcal{V}$-subcategory of α-presentable objects. Choose now a regular cardinal $\beta \geq \alpha$ such that each object $G(D)$, for $D \in \mathcal{D}_\alpha$, is β-presentable. We shall prove that F is a β-accessible $\mathcal{V}$-functor.

Consider a β-filtered $\mathcal{V}$-colimit $K \star H$ in $\mathcal{C}$:

$$K: \mathcal{X}^* \longrightarrow \mathcal{V}, \quad H: \mathcal{X} \longrightarrow \mathcal{C}$$

with $\mathcal{X}$ small and K a β-filtered weight. For every $D \in \mathcal{D}_\alpha$, both D and $G(D)$ are β-presentable, thus

$$\begin{aligned}\mathcal{D}(D, F(K \star H)) &\cong \mathcal{C}(G(D), K \star H)\\ &\cong K \star \mathcal{C}(G(D), H)\\ &\cong K \star \mathcal{D}(D, F \circ H)\\ &\cong \mathcal{D}(D, K \star (F \circ H))\end{aligned}$$

from which $F(K \star H) \cong K \star (F \circ H)$, since the $D \in \mathcal{D}_\alpha$ strongly $\mathcal{V}$-generate $\mathcal{D}$. ■

8.5. Proposition. *Let $F: \mathcal{C} \to \mathcal{D}$ be an accessible $\mathcal{V}$-functor between accessible $\mathcal{V}$-categories. The underlying ordinary functor satisfies the solution set condition.*

Proof. Fix $D \in \mathcal{D}$ and choose a regular cardinal γ such that D is γ-presentable and F is γ-accessible, while $\mathcal{C}$ and $\mathcal{D}$ satisfy for this γ the conditions of 7.10 and 7.11.

Given $C \in \mathcal{C}$, write it as a γ-filtered conical colimit $C = \operatorname{colim} C_i$ of γ-presentable objects. By 2.7, $FC = \operatorname{colim} FC_i$. Therefore given $d: D \to FC$ in the ordinary category underlying $\mathcal{D}$, the ordinary γ-presentability of D implies that d factors through some $F(C_{i_0})$. This means precisely that the γ-presentable objects in $\mathcal{C}$ constitute a solution set for D (see 5.8). ■

It is worth insisting on the fact that the conical character of γ-colimits is essential in the previous argument. Given $d: D \to K \star H$, with D γ-presentable and $K \star H$ a γ-filtered $\mathcal{V}$-colimit, there is no way in general to factor d through some "term" of this $\mathcal{V}$-colimit, whatever this means.

8.6. Theorem. *Let $\mathcal{V}$ be a locally presentable base and $F: \mathcal{C} \to \mathcal{D}$, a $\mathcal{V}$-functor between accessible $\mathcal{V}$-categories. When $\mathcal{C}$ is $\mathcal{V}$-complete, the following conditions are equivalent:*

(1) F has a $\mathcal{V}$-left adjoint;

(2) F is accessible and preserves small $\mathcal{V}$-limits.

PROOF. (1)⇒(2) follows at once from 8.4 and the classical properties of $\mathcal{V}$-adjoint functors (see [11] or [5], volume 2).

(2)⇒(1). The $\mathcal{V}$-category $\mathcal{C}$ is $\mathcal{V}$-complete, thus cotensored. Therefore a classical result on $\mathcal{V}$-adjunctions (see 6.7.6 in [5], volume 2) reduces the problem to proving that F preserves cotensors, while its underlying ordinary functor has a left adjoint in the ordinary sense. But F preserves cotensors since it preserves all small $\mathcal{V}$-limits, while applying 8.5, its underlying functor admits a left adjoint by the classical adjoint functor theorem . ∎

Let us conclude this section by generalizing a classical result: for an accessible category, completeness is equivalent to cocompleteness.

8.7. THEOREM. *Let $\mathcal{V}$ be a locally presentable base and $\mathcal{C}$, an accessible $\mathcal{V}$-category. The following conditions are equivalent:*

(1) $\mathcal{C}$ is a locally presentable $\mathcal{V}$-category;

(2) $\mathcal{C}$ is $\mathcal{V}$-cocomplete;

(3) $\mathcal{C}$ is $\mathcal{V}$-complete.

PROOF. (1)⇔(2) and (2)⇒(3) are proved in 6.3. So it remains to prove (3)⇒(2).

By 7.10, the underlying category $\mathcal{C}_0$ is accessible in the ordinary sense; since by assumption it is complete in the ordinary sense, it is also cocomplete in the ordinary sense (see [15]). So it remains to prove that $\mathcal{C}$ is tensored, that is, each $\mathcal{V}$-functor $\mathcal{C}(-, C): \mathcal{C} \to \mathcal{V}$ has a $\mathcal{V}$-left adjoint. This follows at once from theorem 8.6 since $\mathcal{V}$-representable $\mathcal{V}$-functors are accessible (see 8.2 and preserve small $\mathcal{V}$-limits. ∎

9. Two examples

The previous theory suggests to investigate numerous examples, among which the case of 2-categories should be given a particular emphasize. We intend to study this in a further publication. For the time being, we focus our attention on the preordered classes, that is, categories enriched in the two element chain **2**, and the preadditive categories, that is categories enriched in abelian groups.

9.1. EXAMPLE. *A preordered class $\mathcal{C}$ is finitely accessible as a **2**-category iff it is algebraic, that is, iff it has directed joins and a set of compact elements such that every element of $\mathcal{C}$ is a directed join of elements from this set.*

Let P be a preordered set. We will identify functors $F: P \to \mathbf{2}$ with the up-closed subsets $\{x \mid F(x) = 1\}$. Flat functors correspond then to filters, that is, to down-directed, up-closed subsets. Dually $[P^*, \mathbf{2}]$ can be identified with down-closed subsets of P and

$$(\mathsf{Lan}_Y F)(G) = \begin{cases} 0 & \text{if } G \wedge F = \emptyset \\ 1 & \text{else.} \end{cases}$$

Hence F is flat iff

$$G_1 \wedge F \neq \emptyset \neq G_2 \wedge F \Rightarrow (G_1 \wedge G_2) \wedge F \neq \emptyset$$

for every down-closed subsets $G_1, G_2 \subseteq P$.

Since representable functors correspond to principal filters $\uparrow x = \{y \in P \mid x \leq y\}$, every flat functor is a filtered conical colimit of representable functors. Hence finitely presentable flat functors are representable and therefore absolute colimits are trivial, following proposition 5.7. ∎

9.2. Example. *A preadditive category is finitely accessible, as a category enriched in abelian groups, iff it has sums and the underlying ordinary category is finitely accessible.*

Since sums are filtered conical colimits of finite sums and finite sums are absolute (as biproducts), every finitely accessible preadditive category has arbitrary small sums. Let $\mathcal{X}$ be a small preadditive category. Following [16], flat functors $\mathcal{X} \to \mathsf{Ab}$ coincide with filtered conical colimits of finite sums of representable functors. Hence $\mathsf{Flat}(\mathcal{X}, \mathsf{Ab})$ is finitely accessible in the ordinary sense.

Conversely, let $\mathcal{C}$ be a preadditive category with sums and such that $\mathcal{C}_0$ is finitely accessible in the ordinary sense. Following the just mentioned results of Oberst and Röhrl, $\mathcal{C}$ has all filtered colimits. Moreover, by the same result, every object C of $\mathcal{C}$ which is finitely presentable in the ordinary sense is a finite sum of representable functors. Hence, following proposition 5.7, C is finitely presentable in the enriched sense. Thus $\mathcal{C}$ is finitely accessible. ∎

Example 9.1 could be easily extended to cover accessible preordered classes. On the other hand example 9.2 shows that filtered colimits and filtered conical colimits do not coincide, and therefore answers the question posed in [10], 6.4. For an explicit easy counter-example, choose for $\mathcal{C}$ the ring $\mathbf{Z_2}$ of integers modulo 2: the linear endomorphisms of $\mathbf{Z_2}$ are the identity and the zero morphism, from which the conical filtered colimit closure of $\mathbf{Z_2}$ in its category of modules is reduced to $\mathbf{Z_2}$ and (0), and does not have sums. Still, filtered colimits over Ab are filtered conical colimits of absolute colimits, namely, finite sums. Analogously, if a locally finitely presentable base $\mathcal{V}$ has this property, then a $\mathcal{V}$-category $\mathcal{C}$ is finitely accessible iff $\mathcal{C}$ has absolute colimits and $\mathcal{C}_0$ is finitely accessible.

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