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QUASI LOCALLY CONNECTED TOPOSES

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ABSTRACT. We have shown [2, 4] that complete spreads (with a locally connected domain) over a bounded topos $\mathscr{E}$ (relative to $\mathscr{S}$) are 'comprehensive' in the sense that they are precisely the second factor of a factorization associated with an instance of the comprehension scheme [8, 12] involving $\mathscr{S}$-valued distributions on $\mathscr{E}$ [9, 10]. Lawvere has asked whether the 'Michael coverings' (or complete spreads with a definable dominance domain [3]) are comprehensive in a similar fashion. We give here a positive answer to this question. In order to deal effectively with the comprehension scheme in this context, we introduce a notion of an 'extensive topos doctrine,' where the extensive quantities (or distributions) have values in a suitable subcategory of what we call 'locally discrete' locales. In the process we define what we mean by a quasi locally connected topos, a notion that we feel may be of interest in its own right.

Introduction

Complete spreads over a bounded $\mathscr{S}$-topos $\mathscr{E}$ with locally connected domain [2, 4] are motivated by the complete spreads of R. H. Fox [6] in topology, and shown therein to be precisely the supports of $\mathscr{S}$-valued Lawvere distributions [9, 10] on $\mathscr{E}$. In particular, the pure, complete spread (with locally connected domain) factorization is 'comprehensive' in the sense of [12], (associated with a comprehension scheme [8]) with respect to distributions on $\mathscr{E}$ with values in discrete locales.

E. Michael [11] has generalized complete spreads to the general (non locally connected) case. We have likewise generalized complete spreads in topos theory over an arbitrary base topos $\mathscr{S}$ [3], under an assumption ('definable dominance' [5]) on the domains which essentially corresponds to the classical property of composability of complemented subobjects.

Our goal here is to explain in what sense the hyperpure, complete spread factorization of geometric morphisms [3] is 'comprehensive' with respect to distributions with values in 0-dimensional, rather than just discrete, locales. For this purpose we introduce what we shall call an 'extensive topos doctrine' in order to discuss the (restricted) comprehension scheme in topos theory and its associated factorization. There are several examples.

In the process, we define what we shall call a 'quasi locally connected topos,' to mean roughly the existence of a 0-dimensional locale reflection, by analogy with the existence of a discrete locale reflection in the case of a locally connected topos.

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An outline of the paper follows.

In § 1 we describe the 0-dimensional locale reflection for definable dominances in terms of a close analysis of the pure, entire factorization [5, 4] over an arbitrary base topos $\mathscr{S}$.

In § 2 we revisit the hyperpure, complete spread factorization [3], and compare it with the pure, entire factorization. This gives another perspective on the 0-dimensional locale reflection for definable dominance toposes, since it identifies the latter with the locale of quasicomponents of the topos. In the locally connected case, this gives the familiar discrete locale of components.

We are thus led, in § 3, to introduce 'locally discrete' locales and the concept of a '**V**-determined' topos for suitable subcategories **V** of locally discrete locales. For instance, the locally connected toposes are the same as $\mathscr{S}$-determined toposes (identifying $\mathscr{S}$ with the category of discrete locales).

In § 4 we introduce and study **V**- initial geometric morphisms, by analogy with initial functors [12].

The notion of a comprehensive factorization was modelled after logic and stated in the context of a hyperdoctrine, or of an eed (elementary existential doctrine) [8]. In § 5 we use the term 'extensive topos doctrine' (ETD) for a variant of a hyperdoctrine (or of an eed) that retains only the covariant aspects of the latter. An ETD consists (roughly) of a pair $(\mathbf{T}, \mathbf{V})$, where $\mathbf{T}$ is a 2-category of **V**-determined toposes.

In the framework of an ETD, a (restricted) 'comprehension scheme' can be stated. The 'support' of a **V**-distribution is constructed in § 6, and leads to a 'comprehensive factorization' in § 7. In § 8 we characterize those **V**-distributions on a topos $\mathscr{E}$ that are 'well-supported' in the sense that the support over $\mathscr{E}$ has a **V**-determined domain topos.

Finally, in § 9 we define the notion of a quasi locally connected topos and use it to establish the desired result, namely, that 'Michael coverings' [3] are comprehensive.

1. The 0-dimensional locale reflection

Fox [6] has introduced spreads in topology as a unifying concept encompassing all singular coverings, whether the singularities be branchings or folds. A continuous mapping $Y \xrightarrow{f} X$ is said to be a spread, or 0-dimensional, if the topology of Y is generated by the clopen subsets of inverse images $f^{-1}U$, for U ranging over the opens in X [6].

If Y is locally connected, we can rephrase the definition of a spread with respect to the connected components of the $f^{-1}U$. A point $x \in X$ is said to be an ordinary point if it has a neighborhood U in X that is evenly covered by f, that is, if $f^{-1}U$ is non-empty and each component of it is mapped topologically onto U by f. All other points of X are called singular points. For $Y \xrightarrow{f} X$ to be a spread it is necessary that $f^{-1}(x)$ be 0-dimensional for every point of $x \in f(Y)$. This may be expressed intuitively by saying that Y lies over the image space of f *in thin sheets*.

Here are some examples.

1. Any covering projection (locally constant) over X is a spread over X with no singular

points.

2. The shadow of a floating balloon over the earth, regarded as a map $S^2 \longrightarrow S^2$, is a spread whose singular set is a circle in the codomain 2-sphere.

3. A finitely punctured 2-sphere $U \hookrightarrow S^2$ has a universal covering projection $P \xrightarrow{p} U$. The composite $P \xrightarrow{p} U \hookrightarrow S^2$ is a spread (the composite of two spreads). Its completion $Y \xrightarrow{\varphi} S^2$ is a (complete) spread obtained by canonically providing fibers for the deleted points.

Let $\mathbf{Top}_{\mathscr{S}}$ denote the pseudo slice category of bounded toposes over a base topos $\mathscr{S}$. The objects of $\mathbf{Top}_{\mathscr{S}}$ may be thought of as generalized spaces, and the geometric morphisms between them as generalized continuous mappings. We next recall one way to define spreads in topos theory [2, 4].

Over an arbitrary base topos $\mathscr{S}$, definable subobjects [1] generalize clopen subsets. A morphism $X \xrightarrow{m} Y$ in a topos $\mathscr{F}$ is *definable* if it can be put in a pullback square as follows.

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ \downarrow & \lrcorner & \downarrow \\ f^*A & \xrightarrow[f^*n]{} & f^*B \end{array}$$

A *definable subobject* is a monomorphism that is definable. It is easy to see that definable morphisms (and subobjects) are pullback stable.

Let $\mathscr{F} \xrightarrow{f} \mathscr{S}$ be an object of $\mathbf{Top}_{\mathscr{S}}$. Denote by

$$\tau : f^*\Omega_{\mathscr{S}} \longrightarrow \Omega_{\mathscr{F}}$$

the characteristic map of $f^*1 \xrightarrow{f^*t} f^*\Omega_{\mathscr{S}}$. Then f is said to be *subopen* if τ is a monomorphism [7]. If f is subopen, then the pair $\langle f^*\Omega_{\mathscr{S}}, f^*t\rangle$ classifies definable subobjects in $\mathscr{F}$.

Consider a diagram

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\psi} & \mathscr{E} \\ {\scriptstyle f}\downarrow & & \\ \mathscr{S} & & \end{array}$$

of geometric morphisms. Let H denote $\psi_* f^*\Omega_{\mathscr{S}}$ in $\mathscr{E}$. It follows that H is a Heyting algebra in $\mathscr{E}$ since $\Omega_{\mathscr{S}}$ is a Heyting algebra in $\mathscr{S}$ and the functor $\psi_* f^* : \mathscr{S} \longrightarrow \mathscr{E}$ is left exact. $\mathscr{E}^{H^{\mathrm{op}}} \xrightarrow{\gamma} \mathscr{E}$ denotes the topos of presheaves associated with H regarded as a poset in $\mathscr{E}$.

Suppose now that $\mathscr{E} = Sh(\mathbb{C}, J)$, where $\langle \mathbb{C}, J\rangle$ is a site, so that $\mathscr{E} \rightarrowtail P(\mathbb{C})$ is a subtopos of the presheaf topos $P(\mathbb{C}) = \mathscr{S}^{\mathbb{C}^{\mathrm{op}}}$. Sometimes we notationally identify the objects C of $\mathbb{C}$ with the representable functors h_C in $\mathscr{E}$, after sheafification.

Associated with H and $\mathbb{C}$ is a category $\mathbb{H}$ whose objects are pairs (C, x), such that $C \xrightarrow{x} H$ is a morphism in $\mathscr{E}$. A morphism

$$(C, x) \xrightarrow{m} (D, y)$$

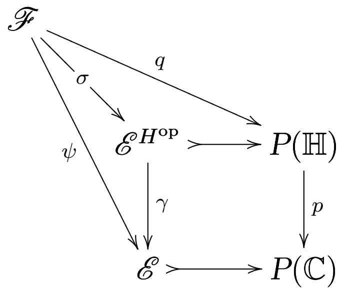
of $\mathbb{H}$ is a morphism $C \xrightarrow{m} D$ in $\mathbb{C}$ such that $x \leq y \cdot m$. The functor $\mathbb{H} \longrightarrow \mathbb{C}$ such that $(C, x) \mapsto C$ induces a geometric morphism $P(\mathbb{H}) \xrightarrow{p} P(\mathbb{C})$.

When f is subopen there is an alternative description of $\mathbb{H}$ in which the objects are pairs (C, U), such that $U \rightarrowtail \psi^* C$ is a definable subobject. The passage from one interpretation of H to the other uses the adjointness $\psi^* \dashv \psi_*$, via the following pullback diagram:

$$\begin{array}{ccc} U & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow t \\ \psi^* C & \xrightarrow{\hat{x}} & f^* \Omega_{\mathscr{S}} \end{array} \tag{1}$$

where $\psi^* C \xrightarrow{\hat{x}} f^* \Omega_{\mathscr{S}}$ is the transpose of $C \xrightarrow{x} H = \psi_* f^* \Omega_{\mathscr{S}}$. This determines a definable subobject $U \rightarrowtail \psi^* C$.

Consider the following diagram in $\mathbf{Top}_{\mathscr{S}}$. The inner square is a pullback.



(2)

We explain the rest of the diagram. There is a flat functor

$$Q : \mathbb{H} \longrightarrow \mathscr{F}$$

such that $Q(C, x) = U$, where U is the definable subobject (1) associated with x. The functor Q induces the geometric morphism q in (2). Since the inner square is a pullback, there is induced a geometric morphism σ as depicted.

A definable dominance [5] is a subopen topos $\mathscr{F} \xrightarrow{f} \mathscr{S}$ in which definable subobjects compose.

1.1. Remark. A locally connected topos is a definable dominance [1]. Any topos over a Boolean base topos $\mathscr{S}$ is a definable dominance, as in that case definable subobobject means complemented.

We say that a geometric morphism $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ over $\mathscr{S}$ is *a spread* if it has an $\mathscr{S}$-definable family that generates $\mathscr{F}$ relative to $\mathscr{E}$ [4]. It follows that if $\mathscr{F}$ is a definable dominance, then a geometric morphism $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ is a spread iff σ in (2) is an inclusion.

The notion of a spread in topos theory is the natural generalization of that of a spread in topology. In passing from spaces (and continuous maps) to more general toposes (and geometric morphisms), the intuitive description of spreads given earlier is often lost. For instance [4], any geometric morphism $\mathscr{F} \xrightarrow{\varphi} \mathscr{E}$ over the base topos $\mathscr{S}$, with $\mathscr{F} \xrightarrow{f} \mathscr{S}$ locally connected, gives rise to a (complete) spread over $\mathscr{E}$, namely, the support of the distribution $f_! \cdot \varphi^* : \mathscr{E} \longrightarrow \mathscr{S}$.

1.2. Definition. A locale X in $\mathscr{S}$ is said to be *0-dimensional* if its topos of sheaves $Sh(X) \longrightarrow \mathscr{S}$ is a spread with a definable dominance domain. We denote by $\mathbf{Loc}_0$ the category of 0-dimensional locales.

In order to construct the 0-dimensional localic reflection we recall some details regarding the pure, entire factorization of a geometric morphism whose domain is a definable dominance [5, 4].

A poset P in a topos $\mathscr{E}$ over $\mathscr{S}$ is said to be *$\Omega_{\mathscr{S}}$-cocomplete* if for any definable monomorphism $\alpha : B \rightarrowtail A$ in $\mathscr{E}$, the induced poset morphism

$$\mathscr{E}(\alpha, P) : \mathscr{E}(A, P) \longrightarrow \mathscr{E}(B, P)$$

has a left adjoint $\bigvee_\alpha$ satisfying the BCC.

1.3. Proposition. *Let $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ be a geometric morphism over $\mathscr{S}$, whose domain f is a definable dominance. Then Heyting algebra $H = \psi_* f^* \Omega_{\mathscr{S}}$, as a poset, is $\Omega_{\mathscr{S}}$-cocomplete.*

Proof. Let $m : E \rightarrowtail F$ be definable in $\mathscr{E}$. $\bigvee_m : H^E \longrightarrow H^F$ arises as follows. A generalized element $X \longrightarrow H^E$ is the same as a definable subobject $S \rightarrowtail \psi^*(X \times E)$ in $\mathscr{F}$. We compose this with the definable subobject $\psi^*(X \times m)$ to produce a definable subobject of $\psi^*(X \times F)$, which is the same as a generalized element $X \longrightarrow H^F$. ∎

An $\Omega_{\mathscr{S}}$-ideal of an $\Omega_{\mathscr{S}}$-cocomplete poset P in $\mathscr{E}$ is a subobject of P such that:

1. its classifying map $P \xrightarrow{\chi} \Omega_{\mathscr{E}}$ is order-reversing, in the sense that it satisfies $(p \leq q) \Rightarrow (\chi(q) \Rightarrow \chi(p))$, and
2. for any definable subobject $\alpha : X \rightarrowtail Y$ in $\mathscr{E}$, the diagram

$$\begin{array}{ccc} P^X & \xrightarrow{\bigvee_\alpha} & P^Y \\ {\scriptstyle \chi^X}\downarrow & & \downarrow{\scriptstyle \chi^Y} \\ {\Omega_{\mathscr{E}}}^X & \xrightarrow[\bigwedge_\alpha]{} & {\Omega_{\mathscr{E}}}^Y \end{array}$$

commutes.

1.4. Proposition. *If $\mathscr{F} \xrightarrow{f} \mathscr{S}$ is a definable dominance, then the canonical order-preserving map $\tau : f^*\Omega_{\mathscr{S}} \rightarrowtail \Omega_{\mathscr{F}}$ preserves finite infima. If the domain of a geometric morphism $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ is a definable dominance, then the Heyting algebra $H = \psi_* f^*\Omega_{\mathscr{S}}$ is a sub-Heyting algebra of the frame $\psi_*\Omega_{\mathscr{F}}$.*

Proof. The second statement follows easily from the first. To prove the first, recall that, since f is subopen, $\langle f^*(\Omega_{\mathscr{S}}), f^*(t)\rangle$ classifies definable subobjects in $\mathscr{F}$. Consider now any two definable subobjects $A \rightarrowtail C$ and $B \rightarrowtail C$ of an object C of $\mathscr{F}$. Then $A \wedge B \rightarrowtail C$ is definable, as follows from the pullback diagram

$$\begin{array}{ccc} A \wedge B & \rightarrowtail & A \\ \downarrow & & \downarrow \\ B & \rightarrowtail & C \end{array}$$

using that definable subobjects are pullback stable and compose. ■

We denote by $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)$ the subobject of ${\Omega_{\mathscr{E}}}^H$ in $\mathscr{E}$ of all $\Omega_{\mathscr{S}}$-ideals of the $\Omega_{\mathscr{S}}$-cocomplete poset H. Since H is an $\Omega_{\mathscr{S}}$-distributive lattice, then the poset $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)$ is a frame [5, 4], in fact, the free frame on H.

Suppose that the domain of $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ is a definable dominance. Let $H = \psi_* f^*\Omega_{\mathscr{S}}$. Then there is a commutative diagram

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\pi} & Sh(\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)) \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & \mathscr{E} & \end{array} \qquad (3)$$

where π is induced by the morphism $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H) \longrightarrow \psi_*\Omega_{\mathscr{F}}$ of frames, in turn the result of the freeness of $\mathrm{Idl}_{\Omega_{\mathscr{S}}}(H)$ on H, and the $\wedge$-preserving map $\psi_*\tau : \psi_* f^*\Omega_{\mathscr{S}} \rightarrowtail \psi_*\Omega_{\mathscr{F}}$.

A geometric morphism $\mathscr{F} \xrightarrow{\rho} \mathscr{E}$ (over $\mathscr{S}$) is said to be *pure* if the unit

$$\eta_{e^*\Omega_{\mathscr{S}}} : e^*\Omega_{\mathscr{S}} \longrightarrow \rho_*\rho^* e^*\Omega_{\mathscr{S}}$$

of adjointness $\rho^* \dashv \rho_*$ at $e^*\Omega_{\mathscr{S}}$ is an isomorphism.

1.5. Theorem. [5] *Any geometric morphism over $\mathscr{S}$ whose domain is a definable dominance admits a pure, entire factorization. Its construction is given by diagram (3).*

Factoring the pure factor of a geometric morphism ψ into its surjection, inclusion parts gives the pure surjection, spread factorization of ψ.

In particular, we may consider the pure surjection, spread factorization of a definable dominance as in the following diagram.

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\rho} & X \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & \mathscr{S} & \end{array}$$

It follows that X is again a definable dominance. We shall refer to X as the *0-dimensional locale reflection* of $\mathscr{F}$. More generally, for any object Y of $\mathscr{F}$, we may consider the 0-dimensional locale reflection of $\mathscr{F}/Y$, denoted X_Y.

We shall see in § 2 that the pure factor of a definable dominance f is already a surjection.

Notation: henceforth we shall usually write X in a topos diagram, as above, when of course we mean $Sh(X)$.

2. Complete spreads revisited

In this section we explain how the 0-dimensional locale reflection of a topos $\mathscr{F}$ may be understood as the locale of quasicomponents of $\mathscr{F}$ [3]. This explanation involves complete spreads.

Let $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ be a geometric morphism whose domain is subopen. As in § 1, if $\mathscr{E} = Sh(\mathbb{C}, J)$, then $\mathbb{H}$ denotes the category of pairs (C, U), such that $U \rightarrowtail \psi^*(C)$ is a definable subobject in $\mathscr{F}$. Consider the Grothendieck topology in $\mathbb{H}$ generated by the sieves

$$\{(C, U_a) \xrightarrow{1_C} (C, U)\}_{a \in A}$$

such that $U = \bigvee_A U_a$ in $Sub_{\mathscr{F}}(\psi^*(C))$. Such a sieve can be expressed with the following diagram in $\mathscr{F}$, in which the top horizontal morphism is an epimorphism, and the bottom square is pullback. Let V denote the coproduct $\coprod_A U_a$ in $\mathscr{F}$.

$$\begin{array}{ccc}
V & \twoheadrightarrow & U \\
\downarrow & & \downarrow \\
f^*A \times \psi^*(C) & \longrightarrow & \psi^*(C) \\
\downarrow & \lrcorner & \downarrow \\
f^*A & \longrightarrow & 1
\end{array}$$

Moreover, $V \rightarrowtail f^*A \times \psi^*(C)$ is a definable subobject since each $U_a \rightarrowtail \psi^*(C)$ is definable.

More generally, the following diagram depicts what we have termed *a weak ψ-cover* in $\mathscr{F}$ [3].

$$\begin{array}{ccc}
V & \twoheadrightarrow & U \\
\downarrow & & \downarrow \\
\psi^*E & \xrightarrow{\psi^*m} & \psi^*C \\
{\scriptstyle \psi^*x}\downarrow & \lrcorner & \downarrow{\scriptstyle \psi^*y} \\
f^*A & \xrightarrow{f^*l} & f^*B
\end{array}
\qquad\qquad
\begin{array}{ccc}
E & \xrightarrow{m} & C \\
{\scriptstyle x}\downarrow & \lrcorner & \downarrow{\scriptstyle y} \\
e^*A & \xrightarrow{e^*l} & e^*B
\end{array}
\qquad (4)$$

The subobjects $V \rightarrowtail \psi^*E$ and $U \rightarrowtail \psi^*C$ are definable, and the square coming from $\mathscr{E}$ (above right) is a pullback.

A ψ-cover in $\mathscr{F}$ is a diagram

$$\begin{array}{ccc} V & \twoheadrightarrow & U \\ \downarrow & & \downarrow \\ \psi^* E & \xrightarrow{\psi^* m} & \psi^* C \\ {\scriptstyle \psi^* x}\downarrow & & \downarrow{\scriptstyle \psi^* y} \\ f^* A & \xrightarrow{f^* l} & f^* B \end{array} \qquad\qquad \begin{array}{ccc} E & \xrightarrow{m} & C \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ e^* A & \xrightarrow{e^* l} & e^* B \end{array} \tag{5}$$

where the subobjects $V \rightarrowtail \psi^* E$ and $U \rightarrowtail \psi^* C$ are definable, but m is not required to be definable. Thus, weak ψ-covers are ψ-covers.

Let $\mathscr{Z} \rightarrowtail P(\mathbb{H})$ denote the subtopos of sheaves for the topology in $\mathbb{H}$ generated by the weak ψ-covers. The topos of sheaves on $\mathbb{H}$ for the ψ-covers is the image topos of $\mathscr{F} \xrightarrow{q} P(\mathbb{H})$, which is a subtopos of $\mathscr{Z}$ since every weak ψ-cover is a ψ-cover.

We may now factor ψ as follows, refining diagram (2).

$$\begin{array}{ccccc} \mathscr{F} & \xrightarrow{\rho} & \mathscr{X} & \rightarrowtail & \mathscr{Z} \\ & {\scriptstyle \psi}\searrow & \downarrow & \lrcorner & \downarrow \\ & & \mathscr{E}^{H^{\mathrm{op}}} & \rightarrowtail & P(\mathbb{H}) \\ & & \downarrow & \lrcorner & \downarrow \\ & & \mathscr{E} & \rightarrowtail & P(\mathbb{C}) \end{array} \tag{6}$$

2.1. Definition. [3] We shall say that $\mathscr{F} \xrightarrow{\rho} \mathscr{E}$ is *hyperpure* if any weak ρ-cover

$$\begin{array}{ccc} V & \twoheadrightarrow & \rho^* U \\ \downarrow & & \downarrow{\scriptstyle \rho^* u} \\ \rho^* E & \xrightarrow{\rho^* m} & \rho^* C \\ \downarrow & \lrcorner & \downarrow \\ f^* A & \longrightarrow & f^* B \end{array} \tag{7}$$

is given locally by a diagram in $\mathscr{E}$, where m and u are definable. This means that there is a collective epimorphism

$$\begin{array}{ccc} C' & \twoheadrightarrow & C \\ \downarrow & & \downarrow \\ e^* B' & \longrightarrow & e^* B \end{array}$$

in $\mathscr{E}$ such that the pullback of (7) to f^*B' is given by a diagram

$$\begin{array}{ccc} V' & \twoheadrightarrow & U' \\ \downarrow & & \downarrow{\scriptstyle u'} \\ E' & \xrightarrow{m'} & C' \\ \downarrow & \lrcorner & \downarrow \\ e^*A' & \longrightarrow & e^*B' \end{array} \tag{8}$$

over e^*B'.

We also require a uniqueness condition: for any two representations (8) of a given (7), the two witnessing collective epimorphisms have a common refining collective epimorphism such that the pullback of the two representing diagrams (8) to the refinement are equal.

2.2. Remark. Hyperpure geometric morphisms are pure. In fact, the direct image functor of a hyperpure geometric morphism preserves $\mathscr{S}$-coproducts [3], and any such geometric morphism is pure.

We shall say that a geometric morphism $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ over $\mathscr{S}$ is *a complete spread* if ρ in diagram (6) is an equivalence. When $\mathscr{F}$ is a definable dominance, the factorization (6) is the essentially unique factorization of ψ into its hyperpure and complete spreads factors, said to be its *hyperpure, complete spread* factorization [3].

2.3. Proposition. *Every complete spread (whose domain is a definable dominance) is a spread.*

Proof. This follows from the characterization of spreads given in terms of σ in diagram (2). ∎

In particular, we may consider the hyperpure, complete spread factorization of a definable dominance $\mathscr{F} \xrightarrow{f} \mathscr{S}$ ($\mathscr{E} = \mathscr{S}$ in this case).

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\rho} & X \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & \mathscr{S} & \end{array} \tag{9}$$

We call X the *locale of quasicomponents of* $\mathscr{F}$, as its construction clearly justifies this terminology.

2.4. Remark. A point $1 \longrightarrow X$ (should it exist) is a filter (upclosed and closed under finite infima) of definable subobjects of $1_{\mathscr{F}}$ that is inaccesible by joins in $\mathscr{F}$. In topology [13], this agrees with the usual notion of quasicomponent.

2.5. Lemma. *The hyperpure ρ in (9) is a surjection.*

Proof. If we take 1 as a site for $\mathscr{S}$, then $\mathbb{H}$ consists of the definable subobjects of $1_{\mathscr{F}}$. The f-covers (5) and the weak f-covers (4) generate the same topology in $\mathbb{H}$ because every morphism in $\mathscr{S}$ is $\mathscr{S}$-definable. ∎

By Remark 2.2, Prop. 2.3, and Lemma 2.5 we have the following.

2.6. Corollary. *The hyperpure, complete spread factorization of a definable dominance coincides with its pure surjection, spread factorization. In particular, the locale of quasi-components of a definable dominance agrees with its 0-dimensional locale reflection.*

2.7. Remark. A pure surjection whose domain is a definable dominance is hyperpure. Indeed, a pure surjection can have no non-trivial complete spread factor since a complete spread is a spread. Hence, it must be hyperpure. For surjections, hyperpure, pure, and direct image functor preserves $\mathscr{S}$-coproducts are equivalent.

3. V-determined toposes

Let $\mathscr{S}$ denote a topos, called the base topos. Let **Loc** denote the category of locales in $\mathscr{S}$, where for any object A of $\mathscr{S}$, regarded as a discrete locale, we define

$$\mathbf{Loc}^A = \mathbf{Loc}/A\ .$$

Loc has Σ satisfying the BCC. **Loc** also has small hom-objects, as an $\mathscr{S}$-indexed category.

The *interior* of a localic geometric morphism $\mathscr{Y} \longrightarrow \mathscr{E}$ is an object Y of $\mathscr{E}$ such that

$$\begin{array}{ccc} \mathscr{E}/Y & \longrightarrow & \mathscr{Y} \\ & \searrow & \downarrow \\ & & \mathscr{E} \end{array}$$

commutes, and any $\mathscr{E}/Z \longrightarrow \mathscr{Y}$ over $\mathscr{E}$ factors uniquely through $\mathscr{E}/Y$. The interior of a localic geometric morphism always exists. The terminology 'interior' is suggested by the idea that an étale map over a locale is a generalized open part of the (frame of the) locale, so that the largest such is a generalized interior.

If $\mathscr{F}$ is a topos over a base topos $\mathscr{S}$, then there is an $\mathscr{S}$-indexed functor

$$F^* : \mathbf{Loc} \longrightarrow \mathscr{F}$$

such that $F^*(X)$ is the interior of the topos pullback below, left.

$$\begin{array}{ccc} \mathscr{F} \times X & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ \mathscr{F} & \xrightarrow{f} & \mathscr{S} \end{array} \qquad \begin{array}{ccc} \mathscr{F}/F^*(X) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathscr{F} & \xrightarrow{f} & \mathscr{S} \end{array}$$

Note: throughout we often write X in a topos diagram when we mean $Sh(X)$, for a locale X. We have a commutative square of toposes above, right. We refer to the top horizontal in this square as a projection. For any object Y of $\mathscr{F}$, we have natural bijections

$$\begin{array}{c} Y \longrightarrow F^*(X) \text{ in } \mathscr{F} \\ \hline \text{geometric morphisms } \mathscr{F}/Y \longrightarrow X \text{ over } \mathscr{S} \\ \hline \text{locale morphisms } L(Y) \longrightarrow X \ , \end{array}$$

where $L(Y)$ denotes the localic reflection of Y: $O(L(Y)) = Sub_{\mathscr{F}}(Y)$. These bijections are not equivalences of categories, when the 2-cell structure of **Loc** is taken into account: we say locale morphisms

$$m \leq l : W \longrightarrow X$$

if $m^*U \leq l^*U$, for any $U \in O(X)$. Then $\mathscr{F}(Y, F^*X)$ is discrete in this sense, but $\mathbf{Loc}(L(Y), X)$ may not be - for instance, take X to be Sierpinski space. Thus, F^* forgets 2-cells.

These remarks motivate the introduction of the following terminology.

3.1. Definition. We shall say that a locale Z is *locally discrete* if for every locale X the partial ordering in $\mathbf{Loc}(X, Z)$ is discrete. Likewise, a map $Z \xrightarrow{p} B$ is *locally discrete* if for every $X \xrightarrow{q} B$, $\mathbf{Loc}/B(q, p)$ is discrete.

3.2. Example. Spreads and étale maps (of locales) are locally discrete maps. In particular, 0-dimensional locales and discrete locales are locally discrete.

Let **LD** denote the category of locally discrete locales in $\mathscr{S}$. It is easy to verify that **LD** may be likewise regarded as an $\mathscr{S}$-indexed category. As such **LD** has Σ satisfying the BCC, and small hom-objects. **LD** is closed under limits, which are created in **Loc**. **LD** has the following additional properties:

1. If $Y \longrightarrow Z$ is a locally discrete map, and Z is locally discrete, then Y is locally discrete.

2. If Y is locally discrete, then any locale morphism $Y \longrightarrow Z$ is locally discrete.

3. The pullback of a locally discrete map along another locally discrete map is again locally discrete.

4. If Z is locally discrete, then any sublocale $S \rightarrowtail Z$ is also locally discrete.

3.3. Example. Peter Johnstone communicated to us an example of an étale map $Y \longrightarrow X$ into a 0-dimensional locale X, for which Y is not 0-dimensional. X is the subspace $\{0\} \cup \{\frac{1}{n} \mid n \geq 1\}$ of the reals, and $Y = X + X/\sim$, indentifying the two $\frac{1}{n}$'s, for every n. The topology on Y is T_1, but not Hausdorff. The map $Y \longrightarrow X$ identifying the two 0's is étale, X is 0-dimensional, but Y is not. However, according to 1 above, Y is locally discrete. Y is also locally 0-dimensional in the sense that its 0-dimensional open subsets form a base. In general, if $Y \longrightarrow X$ is étale and X is locally 0-dimensional, then so is Y.

3.4. Assumption. In what follows, $\mathbf{V}$ denotes an $\mathscr{S}$-indexed full subcategory of $\mathbf{LD}$, with Σ satisfying the BCC, which is also:

1. closed under open sublocales, and
2. closed under pullbacks

$$\begin{array}{ccc} W & \xrightarrow{q} & Z \\ {\scriptstyle n}\downarrow & \lrcorner & \downarrow{\scriptstyle m} \\ Y & \xrightarrow{p} & X \end{array}$$

in $\mathbf{Loc}$ in which p is étale.

3.5. Example. The categories $\mathbf{LD}$, $\mathbf{Loc}_0$, and $\mathscr{S}$ are all instances of such a $\mathbf{V}$.

We use the same notation F^* when we restrict F^* to $\mathbf{V}$.

3.6. Definition. A topos $\mathscr{F} \xrightarrow{f} \mathscr{S}$ is said to be $\mathbf{V}$*-determined* if there is an $\mathscr{S}$-indexed left adjoint $F_! \dashv F^* : \mathbf{V} \longrightarrow \mathscr{F}$, such that 'the BCC for opens' holds, in the sense that for any open $U \overset{p}{\rightarrowtail} Y$ with Y in $\mathbf{V}$ (hence U in $\mathbf{V}$), the transpose (below, right) of a pullback square (below, left) is again a pullback.

$$\begin{array}{ccc} P & \longrightarrow & F^*U \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle F^*p} \\ E & \xrightarrow{m} & F^*Y \end{array} \qquad \begin{array}{ccc} F_!P & \longrightarrow & U \\ {\scriptstyle F_!q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ F_!E & \xrightarrow{\hat{m}} & Y \end{array}$$

Denote by $\mathbf{T}_{\mathbf{V}}$ the full sub 2-category of $\mathbf{Top}_{\mathscr{S}}$ whose objects are the $\mathbf{V}$-determined toposes.

3.7. Remark. A topos is $\mathscr{S}$-determined iff it is locally connected.

In effect, the BCC for opens means that the transpose locale morphism $\hat{m}$ is defined by the formula: $\hat{m}^* = F_! m^*$, when we interpret m as a geometric morphism $\mathscr{F}/E \longrightarrow Y$. We say that a locale X is $\mathbf{V}$-determined if $Sh(X)$ is $\mathbf{V}$-determined.

3.8. Remark. To say that $F_!$ is $\mathscr{S}$-indexed is the property that if a square, below left, is a pullback in $\mathscr{F}$, then so is the right square in $\mathbf{V}$, where A, B are discrete locales.

$$\begin{array}{ccc} C & \longrightarrow & f^*A \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle f^*p} \\ D & \xrightarrow{m} & f^*B \end{array} \qquad \begin{array}{ccc} F_!C & \longrightarrow & A \\ {\scriptstyle F_!q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ F_!D & \xrightarrow{\hat{m}} & B \end{array}$$

The BCC for opens is thus a strengthening of this property.

3.9. Lemma. *$F_! \dashv F^*$ satisfies the BCC for opens iff for any unit $D \longrightarrow F^*(F_!D)$ and any open $U \overset{p}{\rightarrowtail} F_!(D)$, the transpose of the left-hand pullback is again a pullback.*

$$\begin{array}{ccc} P & \longrightarrow & F^*U \\ \downarrow & \lrcorner & \downarrow{\scriptstyle F^*p} \\ D & \longrightarrow & F^*(F_!D) \end{array} \qquad\qquad \begin{array}{ccc} F_!P & \longrightarrow & U \\ {\scriptstyle F_!q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ F_!D & \xrightarrow{1} & F_!D \end{array}$$

This holds iff the transpose $F_!P \longrightarrow U$ is an isomorphism.

Proof. The condition is clearly necessary. To see that it is sufficient consider an arbitrary open $U \rightarrowtail Y$ and $D \xrightarrow{m} F^*Y$, which factors as the bottom horizontal in the following diagram.

$$\begin{array}{ccccc} P & \longrightarrow & F^*W & \longrightarrow & F^*U \\ \downarrow & \lrcorner & {\scriptstyle F^*q}\downarrow & \lrcorner & \downarrow \\ D & \longrightarrow & F^*(F_!D) & \xrightarrow{F^*\hat{m}} & F^*Y \end{array} \qquad\qquad \begin{array}{ccc} W & \longrightarrow & U \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow \\ F_!D & \xrightarrow{\hat{m}} & Y \end{array}$$

First form the pullback W. The maps $\hat{m}$ and $U \rightarrowtail Y$ are locally discrete, and so is U. The pullback W is locally discrete, and q is open. This pullback remains a pullback under F^*. We are assuming that the transpose of the left-hand square involving P is a pullback. Thus, we have $F_!P \cong W$ over $F_!D$. ∎

3.10. Remark. $F_!(q)$ is an open sublocale in Def. 3.6; however, in general we cannot expect $F_!$ to carry opens to opens. Indeed, take for $\mathscr{F}$ the topologist's sine curve $\mathscr{Y} = Sh(Y)$, which is connected but not locally connected. Let $U \subset Y$ be any open sufficiently small disk centered on the y-axis. Then $Y_!(U) \longrightarrow Y_!(1) = 1$ is not étale because $Y_!(U)$ is not a discrete space.

3.11. Definition. For any object D of a $\mathbf{V}$-determined topos $\mathscr{F}$, there is a geometric morphism that we denote

$$\rho_D : \mathscr{F}/D \longrightarrow \mathscr{F}/F^*(F_!D) \longrightarrow F_!(D)$$

obtained by composing the projection with the unit of $F_! \dashv F^*$.

3.12. Remark. In slightly more practical terms, the adjointness $F_! \dashv F^*$ says that for any locale W in $\mathbf{V}$, every geometric morphism $\mathscr{F}/D \longrightarrow W$ factors uniquely through ρ_D.

$$\begin{array}{ccc} \mathscr{F}/D & & \\ {\scriptstyle \rho_D}\downarrow & \searrow^{\forall} & \\ F_!(D) & \xrightarrow[\exists!]{} & W \end{array}$$

3.13. Lemma. *Let $\mathscr{F}$ be* **V***-determined. Let X denote $F_!(1_{\mathscr{F}})$ and $\rho = \rho_{1_{\mathscr{F}}}$. Then the inverse image functor of $\mathscr{F} \xrightarrow{\rho} X$ may be described as follows: if $Y \xrightarrow{p} X$ is étale* (i.e., an object of *Sh*(X))*, then $\rho^*(p)$ is the pullback*

$$\begin{array}{ccc} \rho^*(p) & \longrightarrow & F^*Y \\ \downarrow & \lrcorner & \downarrow{\scriptstyle F^*p} \\ 1 & \longrightarrow & F^*X \end{array}$$

in $\mathscr{F}$. Moreover, if p is an open $U \rightarrowtail X$, then the transpose $F_!\rho^*(p) \longrightarrow U$ of the top horizontal is an isomorphism of open sublocales of X.

Proof. In the following diagram, the outer rectangle, the middle square, and the right-hand square are all topos pullbacks (the middle one since p is étale).

$$\begin{array}{ccccccc} \mathscr{F}/\rho^*(p) & \longrightarrow & \mathscr{F}/F^*Y & \longrightarrow & \mathscr{F}\times Y & \longrightarrow & Y \\ \downarrow & & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \mathscr{F} & \longrightarrow & \mathscr{F}/F^*X & \longrightarrow & \mathscr{F}\times X & \longrightarrow & X \end{array}$$

Therefore, the left-hand square is a pullback. The second statement follows from the BCC for opens. ∎

3.14. Proposition. *Let $\mathscr{F}$ be* **V***-determined. Then for any D of $\mathscr{F}$, ρ_D is a surjection.*

Proof. The property $F_!\rho^*(U) \cong U$ for opens $U \rightarrowtail X = F_!(1_{\mathscr{F}})$ implies that the locale morphism from the localic reflection of $\mathscr{F}$ to X is a surjection. Hence, ρ is a surjection. ∎

3.15. Remark. It is tempting to require the BCC for all étale maps $Z \longrightarrow Y$ in Def. 3.6, not just opens $U \rightarrowtail Y$. We feel this is too strong since Lemma 3.13 would imply that the ρ_D's are connected, which excludes some examples.

4. **V**-initial geometric morphisms

If $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ is a geometric morphism over $\mathscr{S}$, and Z is a locale in **V**, then there is a geometric morphism $\mathscr{F}/\psi^*(E^*Z) \longrightarrow \mathscr{F} \times Z$, which factors through $F^*(Z)$ by a morphism $\psi^*(E^*Z) \longrightarrow F^*(Z)$ in $\mathscr{F}$ since $F^*(Z)$ is the interior of $\mathscr{F} \times Z$. Thus, there is a natural transformation

$$\psi^*E^* \Rightarrow F^* \,, \qquad (10)$$

which is an isomorphism when restricted to discrete locales. It is also an isomorphism when ψ is étale. Another fact about (10) is the following.

4.1. Lemma. *The naturality square of $\psi^*E^* \Rightarrow F^*$ for an étale map is a pullback.*

Proof. Suppose that $Z \longrightarrow Y$ is an étale map of locales. In the following diagram we wish to show that the left-hand square is a pullback.

$$\begin{array}{ccccccc} \mathscr{F}/\psi^*E^*Z & \longrightarrow & \mathscr{F}/F^*Z & \longrightarrow & \mathscr{F}\times Z & \longrightarrow & \mathscr{E}\times Z \\ \downarrow & & \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \\ \mathscr{F}/\psi^*E^*Y & \longrightarrow & \mathscr{F}/F^*Y & \longrightarrow & \mathscr{F}\times Y & \longrightarrow & \mathscr{E}\times Y \end{array}$$

The right-hand square is clearly a pullback, and the middle square is one if $Z \longrightarrow Y$ is étale. Thus, it suffices to show that the outer rectangle is a pullback. This rectangle is equal to the outer rectangle below.

$$\begin{array}{ccccc} \mathscr{F}/\psi^*E^*Z & \longrightarrow & \mathscr{E}/E^*Z & \longrightarrow & \mathscr{E}\times Z \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \\ \mathscr{F}/\psi^*E^*Y & \longrightarrow & \mathscr{E}/E^*Y & \longrightarrow & \mathscr{E}\times Y \end{array}$$

Both squares in this rectangle are pullbacks, so we are done. ■

4.2. Definition. We shall say that $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ is ***V**-initial* if the transpose $E^* \Rightarrow \psi_* F^*$ of (10) under $\psi^* \dashv \psi_*$ is an isomorphism.

4.3. Remark. The transpose in Def. 4.2 may be explicitly described as follows. We have

$$\mathscr{E}(D, E^*(Z)) = \mathbf{Frm}(O(Z), Sub_{\mathscr{E}}(D)) \ ,$$

and

$$\mathscr{E}(D, \psi_* F^*(Z)) \cong \mathscr{F}(\psi^* D, F^*(Z)) = \mathbf{Frm}(O(Z), Sub_{\mathscr{F}}(\psi^* D)) \ .$$

The restriction of ψ^* to subobjects is a frame morphism $Sub_{\mathscr{E}}(D) \longrightarrow Sub_{\mathscr{F}}(\psi^* D)$ for each D, natural in D, which induces the desired natural transformation.

4.4. Proposition. *The pullback of a **V**-initial geometric morphism along an étale geometric morphism is **V**-initial.*

Proof. This is a straightforward diagram chase, using the fact that (10) is an isomorphism when ψ is étale. ■

4.5. Lemma. *Consider a triangle of geometric morphisms*

$$\begin{array}{ccc} \mathscr{F} & & \\ {\scriptstyle\eta}\downarrow & \searrow^{p} & \\ \mathscr{X} & \underset{\tau}{\rightarrowtail} & \mathscr{Z} \end{array}$$

in which τ is an inclusion.

1. *If p and τ are both **V**-initial, then so is η.*

2. *If p is **V**-initial and $\tau^* Z^* \Rightarrow X^*$ is an isomorphism, then η is **V**-initial.*

Proof. 1. Consider

$$\begin{array}{ccc} Z^* & & \\ \Downarrow & \searrow\!\!\!\!\searrow & \\ \tau_* X^* & \Longrightarrow & \tau_* \eta_* F^* \end{array}$$

If $p = \tau\eta$ is **V**-initial, then the hypotenuse is an isomorphism. If τ is **V**-initial, then the vertical is an isomorphism. Therefore, the horizontal is an isomorphism, and therefore η is **V**-initial since τ is an inclusion.

2. Applying τ^* to the isomorphism $Z^* \cong p_* F^* \cong \tau_* \eta_* F^*$ gives the top horizontal in the following diagram, which is an isomorphism.

$$\begin{array}{ccc} \tau^* Z^* & \Longrightarrow & \tau^* \tau_* \eta_* F^* \\ \Downarrow & & \Downarrow \\ X^* & \Longrightarrow & \eta_* F^* \end{array}$$

The right vertical is the counit of $\tau^* \dashv \tau_*$, which is an isomorphism since τ is an inclusion. The left vertical is an isomorphism by assumption. We conclude the bottom horizontal is an isomorphism, which says that η is **V**-initial. ∎

4.6. Remark. Suppose that $\mathscr{E}$ and $\mathscr{F}$ are 'quasi **V**-determined' toposes, in the sense that the BCC for opens is not required, just the adjointness. Then a geometric morphism $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ is **V**-initial iff the natural transformation

$$\xi : F_! \psi^* \Rightarrow E_!$$

obtained by twice transposing $\psi^* E^* \Rightarrow F^*$ (under $F_! \dashv F^*$ and $E_! \dashv E^*$) is an isomorphism. This holds by transposing to right adjoints. Equivalently, ψ is **V**- initial iff $F_! \psi^* \dashv E^*$.

4.7. Proposition. *The direct image functor of a **V**-initial geometric morphism preserves $\mathscr{S}$-coproducts.*

Proof. This is clear by restricting to discrete locales. ∎

4.8. Proposition. *A topos $\mathscr{F}$ is **V**-determined iff for every object D of $\mathscr{F}$, there is a locale X_D in **V** and a **V**-initial geometric morphism $\mathscr{F}/D \xrightarrow{\rho_D} X_D$* (natural in D), *in which case every locale $F_!(D) = X_D$ is **V**-determined.*

Proof. Suppose that $\mathscr{F}$ is **V**-determined. Then of course $X_D = F_!(D)$. Consider the case $D = 1_{\mathscr{F}}$, $\rho = \rho_1$, and $X = F_!(1_{\mathscr{F}})$. We claim that $F_! \rho^* \dashv X^*$. This will show at once that X is **V**-determined, and that ρ is **V**-initial. The BCC for opens holds for X because it holds for $\mathscr{F}$. If $Y \xrightarrow{p} X$ is étale, and Z is a locale in **V**, we wish to show that morphisms $p \longrightarrow X^* Z$ over X are in bijection with locale morphisms $F_! \rho^*(p) \longrightarrow Z$. We have

$$p \cong \varinjlim_{A} U_a,$$

where $\{U_a \rightarrowtail X\}$ is a diagram of open sublocales of X. We may forget that this diagram is over X, so that

$$Y \cong \varinjlim_A U_a$$

in **Loc**. Locale morphisms $Y \longrightarrow Z$ are thus in bijection with cocones $\{U_a \longrightarrow Z\}$. Since $F_!\rho^*(U_a) \cong U_a$, the colimit of such a cocone regarded in $\mathbf{V}$ is

$$\varinjlim_A F_!\rho^*(U_a) \cong F_!\rho^*(\varinjlim_A U_a) \cong F_!\rho^*(p) .$$

(Note: this also shows that the colimit exists in $\mathbf{V}$. Warning: the colimit in $\mathbf{V}$ need not be isomorphic to Y, in general.) The first isomorphism holds by the BCC for opens and Lemma 3.13. Thus, cocones $\{U_a \longrightarrow Z\}$ are in bijection with morphisms $F_!\rho^*(p) \longrightarrow Z$. This shows $F_!\rho^* \dashv X^*$.

For the converse, we wish to show first that $F_!(D) = X_D$ is left adjoint to F^*. If W is in $\mathbf{V}$, then we have natural bijections

$$\frac{\dfrac{\dfrac{\text{locale maps } X_D \longrightarrow W}{\text{global sections of } X_D^*(W) \cong (\rho_D)_* F_D^*(W)}}{\text{global sections of } F_D^*(W) \text{ in } \mathscr{F}/D}}{\text{maps } D \longrightarrow F^*(W) \text{ in } \mathscr{F}} \ .$$

As for the BCC for opens, by Lemma 3.9 it suffices to show that it holds for units $D \longrightarrow F^*(F_!D)$ only. We shall establish this first for $D = 1_{\mathscr{F}}$ by essentially reversing the argument in the first paragraph. Let $U \overset{p}{\rightarrowtail} F_!(1_{\mathscr{F}}) = X$ be open. The statement $F_!\rho^* \dashv X^*$ implies that for any locale Z in $\mathbf{V}$, locale morphisms $U \longrightarrow Z$ bijectively correspond to locale morphisms $F_!\rho^*(U) \longrightarrow Z$. But $F_!\rho^*(U)$ and U are both in $\mathbf{V}$, so we have $F_!\rho^*(U) \cong U$. This concludes the argument for $D = 1_{\mathscr{F}}$. When D is arbitrary, we localize to $\mathscr{F}/D$ and repeat this argument for the $\mathbf{V}$-initial ρ_D. ∎

5. Extensive topos doctrines

Lawvere [8] has described a 'comprehension scheme' in terms of an 'elementary existential doctrine', motivated by examples from logic, in which covariant and contravariant aspects coexist and interact. For our purposes, we retain just the covariant aspects: we do not start with a fibration that is also an opfibration, but directly with just the latter. In addition, we interpret the categories of 'predicates' or 'properties' of a certain type, as categories of 'extensive quantities' of a certain type, and we do not assume the existence of a terminal 'extensive quantity' for each type.

Let $\mathbf{V}$ be a category of locales satisfying Assumption 3.4.

5.1. Definition. A $\mathbf{V}$-*distribution* on a topos $\mathscr{E}$ is an $\mathscr{S}$-indexed functor $\mu : \mathscr{E} \longrightarrow \mathbf{V}$ with an $\mathscr{S}$-indexed right adjoint $\mu \dashv \mu_*$.

For $\mathscr{E}$ an object of $\mathbf{Top}_{\mathscr{S}}$, let $\mathbf{E_V}(\mathscr{E})$ denote the category of $\mathbf{V}$-distributions and natural transformations on $\mathscr{E}$. A 1-cell $\mathscr{F} \xrightarrow{\varphi} \mathscr{E}$ of $\mathbf{Top}_{\mathscr{S}}$ induces a 'pushforward' functor

$$\mathbf{E_V}(\varphi) : \mathbf{E_V}(\mathscr{E}) \longrightarrow \mathbf{E_V}(\mathscr{F})$$

that associates with a $\mathbf{V}$-distribution $\mathscr{F} \xrightarrow{\nu} \mathbf{V}$ on $\mathscr{F}$ the $\mathbf{V}$-distribution

$$\mathscr{E} \xrightarrow{\varphi^*} \mathscr{F} \xrightarrow{\nu} \mathbf{V}$$

on $\mathscr{E}$, i.e., $\mathbf{E_V}(\varphi)(\nu) = \nu\varphi^*$. We have a 2-functor

$$\mathbf{E_V} : \mathbf{Top}_{\mathscr{S}} \longrightarrow \mathbf{Cat}$$

Given any object $\mathscr{F}$ of $\mathbf{Top}_{\mathscr{S}}$, we may restrict $F^* : \mathbf{Loc} \longrightarrow \mathscr{F}$ to any $\mathscr{S}$-indexed subcategory $\mathbf{V} \hookrightarrow \mathbf{Loc}$, using the same notation

$$F^* : \mathbf{V} \longrightarrow \mathscr{F} \; . \tag{11}$$

5.2. Definition. An *extensive topos doctrine (ETD)* is a pair $(\mathbf{T}, \mathbf{V})$ consisting of

1. a full sub-2-category $\mathbf{T}$ of $\mathbf{Top}_{\mathscr{S}}$, and
2. a full 2-subcategory $\mathbf{V}$ of $\mathbf{LD}$ satisfying Assumption 3.4.

This data subject to the additional condition that for each $\mathscr{F}$ in $\mathbf{T}$, $\mathscr{F}$ is $\mathbf{V}$-determined. An ETD is said to be *replete* if the converse of this condition holds, i.e., if every $\mathbf{V}$-determined $\mathscr{F}$ is an object of $\mathbf{T}$ - equivalently, if $\mathbf{T} = \mathbf{T_V}$.

5.3. Proposition. *If $(\boldsymbol{T}, \boldsymbol{V})$ is an ETD, then for any topos $\mathscr{F}$ in $\boldsymbol{T}$, and each object Y of $\mathscr{F}$, there is a canonical geometric morphism*

$$\rho_Y : \mathscr{F}/Y \longrightarrow F_!(Y)$$

in $\boldsymbol{T}$. Moreover, ρ_Y is natural in Y.

Proof. This is in more generality the same as Def. 3.11. ∎

If $\mathscr{E}$ is an object of $\mathbf{Top}_{\mathscr{S}}$, then $\mathbf{T}/\mathscr{E}$ shall denote the comma 2-category of geometric morphisms over $\mathscr{E}$ whose domain topos is an object of $\mathbf{T}$.

5.4. Lemma. *If $\mathscr{F} \xrightarrow{\gamma} \mathscr{G}$ is a geometric morphism over $\mathscr{S}$, then there exists a canonical natural transformation $\xi_\gamma : F_!\gamma^* \Rightarrow G_!$.*

Proof. As in Remark 4.6, ξ_γ is obtained by twice transposing the canonical $\gamma^* G^* \Rightarrow F^*$. ∎

5.5. Remark. Let $(\mathbf{T}, \mathbf{V})$ be an ETD. For each object $\mathscr{F}$ of $\mathbf{T}$, denote the $\mathbf{V}$-valued distribution $F_! : \mathscr{F} \longrightarrow \mathbf{V}$ on $\mathscr{F}$ by $t_{\mathscr{F}}$. By doing so we are not assuming that this is the terminal $\mathbf{V}$-valued distribution. The notation $t_{\mathscr{F}}$ is meant to suggest an analogy with the case of Lawvere hyperdoctrines [8], where the terminal property of any given type is given as part of the data.

For any $\mathscr{E}$ in $\mathbf{Top}_{\mathscr{S}}$, we have a 2-functor

$$\Lambda_{\mathscr{E},\mathbf{V}} = \Lambda_{\mathscr{E}} : \mathbf{T}/\mathscr{E} \longrightarrow \mathbf{E}_{\mathbf{V}}(\mathscr{E})$$

such that

$$\Lambda_{\mathscr{E}}(\mathscr{F} \xrightarrow{\psi} \mathscr{E}) = \mathbf{E}_{\mathbf{V}}(\psi)(t_{\mathscr{F}}) = F_!\psi^* \ .$$

For a 1-cell (γ, t) of $\mathbf{T}/\mathscr{E}$, i.e.,

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\gamma} & \mathscr{G} \\ {\scriptstyle\psi}\searrow & \overset{t}{\Rightarrow} & \swarrow{\scriptstyle\varphi} \\ & \mathscr{E} & \end{array}$$

where t is a natural isomorphism, $\Lambda_{\mathscr{E}}(\gamma, t)$ is the natural transformation

$$\begin{array}{ccc} F_!\psi^* & \overset{F_!t}{\Longrightarrow} & F_!\gamma^*\varphi^* \\ & {\scriptstyle\Lambda(\gamma,t)}\searrow\!\!\searrow & \Downarrow{\scriptstyle\xi\varphi^*} \\ & & G_!\varphi^* \end{array}$$

where $\xi_\gamma : F_!\gamma^* \Rightarrow G_!$ (Lemma 5.4).

5.6. Definition. An ETD $(\mathbf{T}, \mathbf{V})$ is said to satisfy

1. *the comprehension scheme (CS)* if for each $\mathscr{E}$ of $\mathbf{Top}_{\mathscr{S}}$, $\Lambda_{\mathscr{E}}$ has a fully faithful pseudo right adjoint
$$\{\_\}_{\mathscr{E}} : \mathbf{E}_{\mathbf{V}}(\mathscr{E}) \longrightarrow \mathbf{T}/\mathscr{E} \ .$$
2. *the restricted comprehension scheme (RCS)* if for each $\mathscr{E}$ of $\mathbf{Top}_{\mathscr{S}}$, confining $\Lambda_{\mathscr{E}}$ to its image has a fully faithful pseudo right adjoint
$$\widehat{\{\_\}}_{\mathscr{E}} : \widehat{\mathbf{E}_{\mathbf{V}}(\mathscr{E})} \longrightarrow \mathbf{T}/\mathscr{E}$$

5.7. Definition. Let $(\mathbf{T}, \mathbf{V})$ be an ETD that satisfies the (restricted) comprehension scheme.

1. A geometric morphism $\mathscr{F} \xrightarrow{\gamma} \mathscr{G}$ in $\mathbf{T}$ is said to be ***V**-initial* if the canonical natural transformation $\xi_\gamma : \mathbf{E}_{\mathbf{V}}(\gamma)(t_{\mathscr{F}}) \Rightarrow t_{\mathscr{G}}$ in $\mathbf{E}_{\mathbf{V}}(\mathscr{G})$ is an isomorphism.

2. An object $\mathscr{D} \xrightarrow{\varphi} \mathscr{E}$ of $\mathbf{T}/\mathscr{E}$ is called a ***V**- fibration* if the unit of the (restricted) comprehension 2-adjunction $\Lambda_{\mathscr{E}} \dashv \{\_\}_{\mathscr{E}}$ evaluated at φ is an isomorphism. $\mathbf{V}$-Fib$/\mathscr{E}$ denotes the category of $\mathbf{V}$-fibrations with codomain $\mathscr{E}$.

5.8. Proposition. *If* $(\boldsymbol{T}, \boldsymbol{V})$ *is an ETD that satisfies the comprehension scheme, then the biadjoint pair*

$$\Lambda_{\mathscr{E}} \dashv \{\_\}_{\mathscr{E}} : \boldsymbol{E}_{\boldsymbol{V}}(\mathscr{E}) \longrightarrow \boldsymbol{T}/\mathscr{E}$$

induces an equivalence of categories

$$\boldsymbol{E}_{\boldsymbol{V}}(\mathscr{E}) \;\simeq\; \boldsymbol{V}\text{-}Fib/\mathscr{E}.$$

A similar but less striking statement can be made in the case of an ETD that satisfies the restricted comprehension scheme.

5.9. Proposition. *Let* $(\boldsymbol{T}, \boldsymbol{V})$ *be an ETD that satisfies the (restricted) comprehension scheme. Then any object* $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ *of* $\boldsymbol{T}/\mathscr{E}$ *admits a factorization*

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\eta} & \mathscr{D} \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & \mathscr{E} & \end{array} \tag{12}$$

into a first factor η *given by the unit of the adjointness* $\Lambda_{\mathscr{E}} \dashv \widehat{\{\_\}}_{\mathscr{E}}$*, and a* $\boldsymbol{V}$*-fibration* φ*. Furthermore,* $\Lambda_{\mathscr{E}}(\eta)$ *is an isomorphism.*

5.10. Definition. The factorization (12) arising from the (restricted) comprehension scheme satisfied by an ETD $(\mathbf{T}, \mathbf{V})$ is said to be *comprehensive* in $\mathbf{T}/\mathscr{E}$ relative to $\mathbf{V}$ if the unit of the 2-adjointness $\Lambda_{\mathscr{E}} \dashv \widehat{\{\_\}}_{\mathscr{E}}$ has $\mathbf{V}$-initial components.

6. The support of a V-distribution

We now consider what we shall call the support of a $\mathbf{V}$-distribution μ on a topos $\mathscr{E}$. This construction is always available, although it is not evident that it does not always depend on the site chosen for $\mathscr{E}$.

Let μ be a $\mathbf{V}$-distribution on $\mathscr{E} \simeq Sh(\mathbb{C}, J)$. Let $\mathbb{M}$ be the category in $\mathscr{S}$ with objects (C, U) with $U \in O(\mu(C))$, and morphisms $(C, U) \longrightarrow (D, V)$ given by $C \xrightarrow{m} D$ in $\mathbb{C}$ such that $U \leq \mu(m)^*(V)$. For $U \in O(\mu(C))$, denote by $U \rightarrowtail \mu(C)$ the corresponding open sublocale.

Let $\mathscr{X}$ be the topos of sheaves for the topology on $\mathbb{M}$ generated by the following families, which we call *weak* μ*-covers*: a family

$$\{(C, U_a) \xrightarrow{1_C} (C, U) \mid a \in A\}$$

is a weak μ-cover if $\bigvee U_a = U$ in $O(\mu(C))$. As usual there is a functor $\mathbb{M} \longrightarrow \mathbb{C}$ that induces a geometric morphism $P(\mathbb{M}) \longrightarrow P(\mathbb{C})$, and hence one $\mathscr{X} \longrightarrow P(\mathbb{C})$.

6.1. Definition. Let μ be a **V**-distribution on $\mathscr{E}$. The geometric morphism φ in the topos pullback

$$\begin{array}{ccc} \mathscr{X} & \rightarrowtail & \mathscr{Z} \\ {\scriptstyle\varphi}\downarrow & \lrcorner & \downarrow \\ \mathscr{E} & \rightarrowtail & P(\mathbb{C}) \end{array}$$

is said to be *the support of* μ, denoted $\{\mu\}_{\mathscr{E}}$.

Going hand-in-hand with the support $\varphi = \{\mu\}_{\mathscr{E}}$ are geometric morphisms

$$\sigma_{(C,U)} : \mathscr{X}/(C,U) \longrightarrow U ,$$

for every object (C,U) of $\mathbb{M}$. In fact, these geometric morphisms are an important part of the support associated with μ. The inverse image functor of such a geometric morphism is defined by

$$\sigma^*_{(C,U)} : O(U) \longrightarrow \mathbb{M}/(C,U) \xrightarrow{\text{Yon}} P(\mathbb{M})/(C,U) \longrightarrow \mathscr{X}/(C,U)$$

such that $V \leq U$ (in $O(\mu(C))$ goes to the $(C,V) \xrightarrow{1_C} (C,U)$ in $\mathbb{M}/(C,U)$. Moreover, if t is the top element of $O(\mu(C))$, then

$$\sigma_{(C,t)} = \sigma_C : \mathscr{X}/\varphi^*C \longrightarrow \mu(C) . \tag{13}$$

For any (C,U), the following is a pullback in $\mathscr{X}$.

$$\begin{array}{ccc} \sigma^*_C(C,U) & \longrightarrow & X^*U \\ \downarrow & \lrcorner & \downarrow \\ \varphi^*C & \longrightarrow & X^*(\mu(C)) \end{array}$$

where the bottom morphism corresponds to σ_C. Note: we have not assumed that the locale U is a member of **V**; however, this is an important consideration that enters the picture in the next two sections.

Next observe that the σ_C's induce a natural transformation

$$\mu_* \Rightarrow \varphi_* X^* . \tag{14}$$

Indeed, for any W in **V**, and C in $\mathbb{C}$, we pass from $C \longrightarrow \mu_*(W)$ to $\mu(C) \longrightarrow W$ to $\mathscr{X}/\varphi^*C \longrightarrow W$ by composing with σ_C. In turn this corresponds to a morphism $\varphi^*C \longrightarrow X^*W$ in $\mathscr{X}$, and hence to one $C \longrightarrow \varphi_*X^*(W)$ in $\mathscr{E}$. This gives a morphism $\mu_*(W) \longrightarrow \varphi_*X^*(W)$ in $\mathscr{E}$, which is a component of (14).

The natural transformation (14) is the adjoint transpose of the counit of the pseudoadjointness studied in § 7, evaluated at μ.

7. A comprehensive factorization

If $\mathscr{F}$ is **V**-determined, then for any K in $\mathscr{F}$ we have a factorization of $\mathscr{F}/K \longrightarrow \mathscr{S}$ into a **V**-initial geometric morphism and a locale X_K in **V**

$$\mathscr{F}/K \xrightarrow{\rho_K} X_K \longrightarrow \mathscr{S} \ ,$$

where $X_K = F_!(K)$. The **V**-initial ρ_K is a surjection in this case.

Consider a topos $\mathscr{E} \simeq Sh(\mathbb{C}, J)$. Suppose that $\mathscr{F}$ is **V**-determined and $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ is a geometric morphism. Let $\mu = \Lambda_{\mathscr{E}}(\psi) = F_!\psi^*$. As in Def. 6.1, we may consider the category $\mathbb{M}$, and the support associated with μ. In this case there is a functor

$$P : \mathbb{M} \longrightarrow \mathscr{F}/\psi^*C \xrightarrow{\Sigma_{\psi^*C}} \mathscr{F}$$

such that $P(C,U) = \Sigma_{\psi^*C}\rho^*_{\psi^*C}(U)$. P is filtering, so P corresponds to a geometric morphism $\mathscr{F} \xrightarrow{p} P(\mathbb{M})$, which makes the following diagram commute.

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{p} & P(\mathbb{M}) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \mathscr{E} & \rightarrowtail & P(\mathbb{C}) \end{array}$$

7.1. Lemma. *For any object (C,U) of $\mathbb{M}$, $P(C,U) = p^*(C,U)$ is the following pullback.*

$$\begin{array}{ccc} p^*(C,U) & \longrightarrow & F^*U \\ \downarrow & \lrcorner & \downarrow \\ \psi^*C & \longrightarrow & F^*(X_{\psi^*C}) \end{array}$$

*The bottom horizontal is a unit of $F_! \dashv F^*$, where X_{ψ^*C} denotes the locale $F_!(\psi^*C)$ in **V**. For every (C,U), we have $F_!p^*(C,U) \cong U$.*

Proof. By Lemma 3.13 for ρ_{ψ^*C}, the left vertical in the diagram is precisely $\rho^*_{\psi^*C}(U)$. The second statement follows from the BCC for opens. ∎

7.2. Lemma. *Let (C,U) be any object of $\mathbb{M}$. Let W be a locale in **V**. Then geometric morphisms $\mathscr{F}/p^*(C,U) \longrightarrow W$ are in natural bijection with locale morphisms $U \longrightarrow W$. Equivalently, $p_*(F^*W)$ is the presheaf on $\mathbb{M}$:*

$$p_*(F^*W)(C,U) = \boldsymbol{V}(U,W) \ .$$

Proof. We have natural bijections

$$\frac{\dfrac{\mathscr{F}/p^*(C,U) \longrightarrow W}{p^*(C,U) \longrightarrow F^*W \ (\text{in } \mathscr{F})}}{U \cong F_!p^*(C,U) \longrightarrow W.}$$

∎

7.3. Lemma. *$\mathscr{F} \xrightarrow{p} P(\mathbb{M})$ factors through the subtopos $\mathscr{Z}$ for the weak μ-covers, by a geometric morphism we denote q. Hence, we have the following diagram.*

$$\begin{array}{ccccc} & & \overset{q}{\frown} & & \\ \mathscr{F} & \xrightarrow{\eta} & \mathscr{X} & \rightarrowtail & \mathscr{Z} \\ & \searrow^{\psi} & \downarrow{\scriptstyle\varphi} & \lrcorner & \downarrow \\ & & & & P(\mathbb{M}) \\ & & & & \downarrow{\scriptstyle\gamma} \\ & & \mathscr{E} & \rightarrowtail & P(\mathbb{C}) \end{array} \tag{15}$$

Moreover, for any (C, U),

$$\begin{array}{ccc} \mathscr{F}/p^*(C,U) & \xrightarrow{\eta_{(C,U)}} & \mathscr{X}/(C,U) \\ {\scriptstyle\rho_{p^*(C,U)}}\downarrow & \swarrow_{\sigma_{(C,U)}} & \\ U & & \end{array} \tag{16}$$

commutes. (Note: $U \cong F_! p^*(C,U) = X_{p^*(C,U)}$, so $\rho_{p^*(C,U)}$ does indeed land in U.)

Proof. Clearly p^* inverts the weak μ-covers. ∎

Let $\mathscr{X}/(C,U) \xrightarrow{g} W$ be a geometric morphism into a locale W in $\mathbf{V}$. Since $U \cong F_! p^*(C,U) = X_{p^*(C,U)}$, there is an essentially unique locale morphism $U \xrightarrow{\widehat{g}} W$ such that

$$\begin{array}{ccc} \mathscr{F}/p^*(C,U) & \xrightarrow{\eta_{(C,U)}} & \mathscr{X}/(C,U) \\ {\scriptstyle\rho_{p^*(C,U)}}\downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{\widehat{g}} & W \end{array} \tag{17}$$

commutes (Remark 3.12).

7.4. Lemma. *For any (C, U), we have a pullback*

$$\begin{array}{ccc} \eta^*_{(C,U)} g^* V & \longrightarrow & F^* \widehat{g}^* V \\ \downarrow & \lrcorner & \downarrow \\ p^*(C,U) & \longrightarrow & F^* U \end{array}$$

in $\mathscr{F}$. Consequently, by the BCC for opens, $\widehat{g}^* V \cong F_!(\eta^*_{(C,U)} g^* V)$.

Proof. This follows from (17), and Lemma 3.13. ∎

7.5. Lemma. *For any g and $\widehat{g}$ as in (17), we have $g \cong \widehat{g}\,\sigma_{(C,U)}$.*

Proof. We claim that

$$g \leq \widehat{g}\,\sigma_{(C,U)}\ . \tag{18}$$

The lemma follows from this since W is in $\mathbf{V}$, hence W is locally discrete. Let V be an open of W. Then $\sigma^*_{(C,U)}(\widehat{g}^*V)$ is the (associated sheaf) of

$$(C, \widehat{g}^*V) \rightarrowtail (C, U)\ .$$

Let i denote the inclusion $\mathscr{X} \rightarrowtail P(\mathbb{M})$. From a given element $(D, U') \xrightarrow{m} i_*g^*V$

$$\begin{array}{ccc} (D,U') & \xrightarrow{m} & i_*g^*V \\ & \searrow^{\widetilde{m}} & \downarrow \\ & & (C,U) \end{array}$$

we obtain a morphism $p^*(D, U') \longrightarrow F^*\widehat{g}^*V$ by first applying $p^*_{(C,U)} \cong \eta^*_{(C,U)}i^*$ to m, and then composing with the top horizontal in Lemma 7.4. The transpose under $F_! \dashv F^*$ of this morphism is a morphism $U' \longrightarrow \widehat{g}^*V$ such that $\widetilde{m}$ factors through $(C, \widehat{g}^*V)$. Thus, we have subobjects

$$\begin{array}{ccc} i_*g^*V & \rightarrowtail & (C,\widehat{g}^*V) \\ & \searrow \quad \swarrow & \\ & (C,U) & \end{array}$$

in $P(\mathbb{M})$. We apply i^* to this, so that for every open $V \rightarrowtail W$, we have a subobject $g^*V \rightarrowtail \sigma^*_{(C,U)}\widehat{g}^*V$ in $\mathscr{X}$, establishing (18). ∎

7.6. Proposition. *The geometric morphism η is $\boldsymbol{V}$-initial, and $\mathscr{X}$ is $\boldsymbol{V}$-determined.*

Proof. We must show that $F_!\eta^* \dashv X^*$. We first show that for any W in $\mathbf{V}$, and an object (C, U) of $\mathbb{M}$, we have a natural bijection

$$\frac{\dfrac{\text{locale morphisms } F_!\eta^*(C,U) \cong U \longrightarrow W}{\text{geometric morphisms } \mathscr{X}/(C,U) \longrightarrow W}}{\text{morphisms } (C,U) \longrightarrow X^*W \text{ in } \mathscr{X}\,.}$$

We pass from a locale morphism $U \longrightarrow W$ to a geometric morphism

$$\mathscr{X}/(C,U) \longrightarrow W$$

by composing with $\sigma_{(C,U)}$. Composing this with $\eta_{(C,U)}$ returns us to the given $U \longrightarrow W$. On the other hand, if we start with a geometric morphism

$$\mathscr{X}/(C,U) \xrightarrow{g} W\ ,$$

giving $U \xrightarrow{\widehat{g}} W$ as in (17), then by Lemma 7.5, composing with $\sigma_{(C,U)}$ returns us to g. This establishes the above bijection. If $X_!$ denotes $F_!\eta^*$, then $X_! \dashv X^*$ follows from the above bijection by a straightforward colimit argument. Finally, the BCC for opens holds for $X_! \dashv X^*$ since it holds for $F_! \dashv F^*$. ∎

7.7. Theorem. *For any topos $\mathscr{E}$ of $\boldsymbol{Top}_{\mathscr{S}}$, $\Lambda_{\mathscr{E}} : \boldsymbol{T_V}/\mathscr{E} \longrightarrow \widehat{\boldsymbol{E_V}(\mathscr{E})}$ has a full and faithful pseudo right adjoint*

$$\widehat{\{\_\}}_{\mathscr{E}} : \widehat{\boldsymbol{E_V}(\mathscr{E})} \longrightarrow \boldsymbol{T_V}/\mathscr{E}\ .$$

8. Well-supported V-distributions

In this section we attempt to learn more about when a given **V**-distribution μ on a topos $\mathscr{E}$ may be resolved by a **V**-determined topos, in the sense that there is a **V**-determined topos $\mathscr{F}$ and a geometric morphism $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ such that $\mu \cong F_!\psi^*$.

8.1. Definition. We shall say that a **V**-distribution μ on a topos $\mathscr{E}$ is *well-supported* if for every C, σ_C (13) is **V**-initial (Def. 4.2).

8.2. Proposition. *Let μ be a $\boldsymbol{V}$-distribution on $\mathscr{E}$, with support $\mathscr{X} \xrightarrow{\varphi} \mathscr{E}$. Then the following are equivalent:*

1. *μ is well-supported;*
2. *(14) is an isomorphism;*
3. *$\mathscr{X}$ is $\boldsymbol{V}$-determined, and $X_!(C,U) = U$;*
4. *μ may be resolved by a $\boldsymbol{V}$-determined topos.*

Proof. 1 and 2 are equivalent because the hypothesis that μ is well-supported amounts to the assertion that composition with a σ_C induces a bijection between locale morphisms $\mu(C) \longrightarrow W$ and geometric morphisms $\mathscr{X}/\varphi^*C \longrightarrow W$, for any locale W in **V**.

$1 \Rightarrow 3$. First observe that every $\sigma_{(C,U)}$ is **V**-initial since

$$\begin{array}{ccc} \mathscr{X}/(C,U) & \rightarrowtail & \mathscr{X}/\varphi^*C \\ {\scriptstyle \sigma_{(C,U)}}\downarrow & \lrcorner & \downarrow{\scriptstyle \sigma_C} \\ U & \rightarrowtail & \mu(C) \end{array}$$

is a topos pullback, so that Prop. 4.4 applies. This gives a bijection between $U \longrightarrow W$ and $\mathscr{X}/(C,U) \longrightarrow W$, for any locale W in **V**. Then the colimit extension of $X_!(C,U) = U$ is left adjoint to X^* by the usual colimit argument, as in the proof of Prop. 7.6. The BCC for opens also holds for $X_! \dashv X^*$.

$3 \Rightarrow 4$ is clear.

$4 \Rightarrow 1$. A distribution $F_!\psi^*$ is well-supported, since σ_C for its support $\mathscr{X}$ coincides with $\rho_{(C,t)}$ for $\mathscr{X}$:

$$\sigma_C = \rho_{(C,t)} : \mathscr{X}/\varphi^* C \longrightarrow F_!\psi^* C = \mu(C)\ .$$

We have seen that $\mathscr{X}$ is **V**-determined, so its ρ's are **V**-initial. ∎

We conclude this section by briefly investigating some special necessary structure that well-supported **V**-distributions have. We begin by reminding the reader that an $\mathscr{S}$-indexed functor $\mathscr{E} \xrightarrow{\mu} \mathbf{V}$ is a functor

$$\mu^A : \mathscr{E}/e^* A \longrightarrow \mathbf{V}/A$$

for every A of $\mathscr{S}$, commuting with the pullback functors.

If an $\mathscr{S}$-indexed functor μ preserves Σ, then $\mu^A(D \longrightarrow e^* A)$ can be written $\mu(D) \longrightarrow A$. Then since μ is $\mathscr{S}$-indexed, we see that for any morphism $B \xrightarrow{p} A$ of $\mathscr{S}$ and any pullback pullback square below left, the transposed one is also a pullback.

$$\begin{array}{ccc} P & \longrightarrow & e^*(B) \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle e^*(p)} \\ D & \xrightarrow{m} & e^*(A) \end{array} \qquad \begin{array}{ccc} \mu(P) & \longrightarrow & B \\ {\scriptstyle \mu(q)}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \mu(D) & \xrightarrow{\hat{m}} & A \end{array} \tag{19}$$

8.3. PROPOSITION. *Suppose that* $\mathscr{E} \xrightarrow{\mu} \boldsymbol{V}$ *is an* $\mathscr{S}$*-indexed functor that preserves* Σ*. Then there is a unique* $\mathscr{S}$*-indexed natural transformation*

$$t : \mu e^* \Rightarrow id\ .$$

We have $\hat{m} = t_A \cdot \mu(m)$ *in* (19)*, and a component morphism* t_A *is part of a pullback*

$$\begin{array}{ccc} \mu(e^* A) & \xrightarrow{t_A} & A \\ \downarrow & \lrcorner & \downarrow \\ \mu(1) & \xrightarrow{t_1 = !} & 1 \end{array}$$

in $\boldsymbol{V}$*. In other words,* $\mu(e^* A) \cong \mu(1) \times A$*.*

We shall say that a natural transformation $\mu E^* \Rightarrow id$ is an extension of the unique $\mu e^* \Rightarrow id$ if they agree on the discrete locales.

8.4. DEFINITION. *A special* $\boldsymbol{V}$*-distribution* μ on a topos $\mathscr{E}$ is a pair $\langle \mu, t \rangle$ consisting of a **V**-distribution μ together with an extension

$$t : \mu E^* \Rightarrow id \ \ \text{(equivalently } E^* \Rightarrow \mu_*\text{)}$$

of $\mu e^* \Rightarrow id$, such that the BCC for opens holds: this means that if a square (below, left) is a pullback, then so is the one on the right, where p is an open sublocale.

$$\begin{array}{ccc} P & \longrightarrow & E^*U \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle E^*(p)} \\ D & \xrightarrow{m} & E^*(Y) \end{array} \qquad \begin{array}{ccc} \mu(P) & \longrightarrow & U \\ {\scriptstyle \mu(q)}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \mu(D) & \xrightarrow{\hat{m}} & Y \end{array}$$

The locale morphism $\hat{m}$ equals $t_Y \cdot \mu(m)$.

A morphism $\langle \mu, t\rangle \longrightarrow \langle \nu, s\rangle$ of such pairs is a natural transformation $\mu \Rightarrow \nu$ that commutes with t and s. Special **V**-distributions on a topos $\mathscr{E}$ and their morphisms form a subcategory of all **V**-distributions.

8.5. Remark. If μ is a special **V**-distribution, and Y is in **V**, then the naturality square of t for an open sublocale $U \rightarrowtail Y$ is a pullback.

$$\begin{array}{ccc} \mu(E^*U) & \xrightarrow{t_U} & U \\ \downarrow & & \downarrow \\ \mu(E^*Y) & \xrightarrow{t_Y} & Y \end{array}$$

In other words, the frame morphism ${t_Y}^*$ satisfies ${t_Y}^*U = \mu(E^*U)$.

8.6. Proposition. *Suppose that $\mathscr{F}$ is **V**-determined. Let $\mathscr{F} \xrightarrow{\psi} \mathscr{E}$ be a geometric morphism over $\mathscr{S}$. Let*

$$\varepsilon : F_!\psi^*E^* \Rightarrow F_!F^* \Rightarrow id$$

*denote the composite of (10) with the counit of $F_! \dashv F^*$. Then $\langle F_!\psi^*, \varepsilon\rangle$ is a special **V**-distribution on $\mathscr{E}$.*

Proof. We have $F_!\psi^* \dashv \psi_*F^*$. Clearly ε extends the canonical $(F_!\psi^*)e^* \cong F_!f^* \Rightarrow id$. We verify the BCC for opens.

$$\begin{array}{ccc} P & \longrightarrow & E^*U \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle E^*(p)} \\ D & \xrightarrow{m} & E^*(Y) \end{array} \qquad \begin{array}{ccc} \psi^*(P) & \longrightarrow & F^*U \\ {\scriptstyle \psi^*(q)}\downarrow & \lrcorner & \downarrow{\scriptstyle F^*(p)} \\ \psi^*(D) & \xrightarrow{\psi^*(m)} & F^*(Y) \end{array} \qquad \begin{array}{ccc} F_!\psi^*(P) & \longrightarrow & U \\ {\scriptstyle F_!\psi^*(q)}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ F_!\psi^*(D) & \xrightarrow{\hat{m}} & Y \end{array}$$

If the far-left square is a pullback in $\mathscr{E}$, then by Lemma 4.1, and since ψ^* is left exact, so is the center one. Then the right square is pullback by the BCC for opens for $F_! \dashv F^*$. ∎

8.7. Remark. Proposition 8.6 shows that well-supported **V**-distributions are special. The converse is an interesting question that we shall leave open. Another way to interpret the fact that well-supported **V**-distributions are special is to observe that in any case we always have natural transformations

$$\begin{array}{c} E^* \\ \Downarrow \\ \mu_* \Rightarrow \varphi_* X^* \end{array}$$

The horizontal one is (14), which is an isomorphism if μ is well-supported. In this case, we evidently have a natural transformation $E^* \Rightarrow \mu_*$ that extends the unique $e^* \Rightarrow \mu_*$, so that μ is special.

9. Michael coverings are comprehensive

The general theory of an extensive topos doctrine immediately gives the following instance of a comprehensive factorization already established in [2] with a different proof.

9.1. Theorem. *The pure, complete spread factorization of a geometric morphism with a locally connected domain is comprehensive.*

Proof. If **L** denotes the full sub-2-category of $\mathbf{Top}_{\mathscr{S}}$ whose objects are locally connected, then clearly $(\mathbf{T}, \mathbf{V}) = (\mathbf{L}, \mathscr{S})$ is a replete ETD that satisfies the (unrestricted) comprehension scheme. The pure, complete spread (with locally connected domain) factorization [2, 4] is comprehensive because pure geometric morphisms are precisely the $\mathscr{S}$-initial ones (between locally connected toposes). $\Lambda_{\mathscr{E}}(\varphi) = f_! \cdot \varphi^*$ in the context of an ETD agrees with that of Lawvere's hyperdoctrines and eeds [8] because the connected components functor $\mathscr{F} \xrightarrow{f_!} \mathscr{S}$ is known a priori to be the terminal Lawvere distribution on a locally connected topos $\mathscr{F}$.

It is our aim to show that Michael coverings are comprehensive, in the sense that the hyperpure, complete spread factorization of [3] is comprehensive. This is a new result even in the classical case [11].

The general theory applies here with the example of $\mathbf{V} = \mathbf{Loc}_0$, the category of 0-dimensional locales.

9.2. Definition. A topos $f : \mathscr{F} \longrightarrow \mathscr{S}$ is called *quasi locally connected* if it is $\mathbf{Loc}_0$-determined. Denote by **Q** the full sub 2-category of $\mathbf{Top}_{\mathscr{S}}$ whose objects are the quasi locally connected toposes.

9.3. Corollary. *A topos $\mathscr{F}$ in $\mathbf{Top}_{\mathscr{S}}$ is quasi locally connected iff for every object D of $\mathscr{F}$, there is a 0-dimensional locale X_D and a $\mathbf{Loc}_0$-initial geometric morphism $\mathscr{F}/D \xrightarrow{\rho_D} X_D$* (natural in D), *in which case every locale $F_!(D) = X_D$ is quasi locally connected.*

Proof. This follows from Proposition 4.8 for $\mathbf{V} = \mathbf{Loc}_0$.

In order to apply Corollary 9.3 we need to compare the hyperpure geometric morphisms with the $\mathbf{Loc}_0$-initial ones.

9.4. Proposition. *A hyperpure geometric morphism between definable dominances is* $\mathbf{Loc}_0$*-initial.*

Proof. We may use Remark 4.6. Assume that $\mathscr{F} \xrightarrow{\psi} \mathscr{G}$ is hyperpure. Let U be an object of $\mathscr{G}$. Consider the hyperpure, (complete) spread factorizations.

$$\begin{array}{ccc} \mathscr{F}/\psi^*U & \xrightarrow{\rho_{\psi^*U}} & F_!(\psi^*U) \\ & {\scriptstyle f_U}\searrow & \downarrow \\ & & \mathscr{S} \end{array} \qquad \begin{array}{ccc} \mathscr{G}/U & \xrightarrow{\rho'_U} & G_!U \\ & {\scriptstyle g_U}\searrow & \downarrow \\ & & \mathscr{S} \end{array}$$

If ψ is hyperpure, then so is ψ/U, and therefore so is the composite $\rho'_U \cdot \psi/U$, both facts by known properties [3]. Hence, $\xi_U : F_!(\psi^*U) \longrightarrow G_!(U)$ must be an equivalence by the uniqueness of the hyperpure, complete spread factorization of f_U [3].

$$\begin{array}{ccc} \mathscr{F}/\psi^*U & \xrightarrow{\psi/U} & \mathscr{G}/U \\ {\scriptstyle \rho_{\psi^*U}}\downarrow & & \downarrow{\scriptstyle \rho'_U} \\ F_!(\psi^*U) & \longrightarrow & G_!(U) \\ \downarrow & & \downarrow \\ \mathscr{S} & = & \mathscr{S} \end{array}$$

Thus, ξ_U is an isomorphism of 0-dimensional locales. ∎

9.5. Proposition. *A definable dominance is quasi locally connected.*

Proof. We have shown that hyperpure geometric morphisms are $\mathbf{Loc}_0$-initial, so Corollary 9.3 applies. ∎

We end this paper with the result that motivated it, namely that Michael coverings are comprehensive.

9.6. Theorem. *The factorization of a geometric morphism with a definable dominance domain into a hyperpure geometric morphism followed by a complete spread is comprehensive. Furthermore, this factorization is the one associated with the (restricted) comprehension scheme satisfied by the ETD* $(\boldsymbol{Q}, \mathbf{Loc}_0)$.

Proof. We need only observe that the comprehensive factorization for objects of $\mathbf{Q}/\mathscr{E}$ coincides with the hyperpure, complete spread factorization of [3]. This is because the constructions of the support of a $\mathbf{Loc}_0$-distribution of the form $\mu = F_!\psi^*$ are the same in both cases. However, it should be noted that the site $\mathbb{H}$ used in [3] is not exactly the same as the site $\mathbb{M}$ used here. It is not hard to see that the resulting topos $\mathscr{X}$ is the same in both cases. ∎

Final remarks.

1. We have introduced here a notion of quasi locally connected topos and investigated its relevance for the hyperpure, complete spread factorization [3]. In particular, we have shown that both factorizations are comprehensive.

2. Whereas over a Boolean topos the two weakenings of the notion of a locally connected topos, given respectively by definable dominance (geometric aspect) and by quasi locally connected (logical aspect), are equivalent and add nothing to the notion of a bounded $\mathscr{S}$-topos, we believe that working constructively reveals new information.

3. It may be possible to characterize the notion of a quasi locally connected topos in terms of sites in a manner analogous to the well-known site characterization of local connectedness [1]. This would give a further insight into this notion.

4. Applications of the present investigation may stem naturally from the various aspects of complete spreads and distributions that we have treated in our book [4]. The machinery is in place for dealing with a notion of the fundamental groupoid of a quasi locally connected topos, as are applications to knot theory and branched coverings in the non-locally connected case. We anticipate no lack of interesting and challenging questions lying ahead in this program.

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