Topos Theory

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2007

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1 Presheaves

We start by reviewing the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of contravariant functors from $\mathcal{C}$ to Set. $\mathcal{C}$ is assumed to be a small category throughout. Objects of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ are called *presheaves on* $\mathcal{C}$.

We have the Yoneda embedding $y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$; we write its effect on objects C and arrows f as y_C, y_f respectively. So for $f : C \to D$ we have $y_f : y_C \to y_D$. Recall: $y_C(C') = \mathcal{C}(C', C)$, the set of arrows $C' \to C$ in $\mathcal{C}$; for $\alpha : C'' \to C'$ we have $y_C(\alpha) : y_C(C') \to y_C(C'')$ which is defined by composition with α, so $y_C(\alpha)(g) = g\alpha$ for $g : C' \to C$. For $f : C \to D$ we have $y_f : y_C \to y_D$ which is a natural transformation with components

$$(y_f)_{C'} : y_C(C') \to y_D(C')$$

given by $(y_f)_{C'}(g) = fg$. Note, that the naturality of y_f is just the associativity of composition in $\mathcal{C}$.

Presheaves of the form y_C are called *representable*.

The *Yoneda Lemma* says that there is a 1-1 correspondence between elements of $X(C)$ and arrows in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ from y_C to X, for presheaves X and objects C of $\mathcal{C}$, and this correspondence is natural in both X and C. To every element $x \in X(C)$ corresponds a natural transformation $\mu : y_C \to X$ such that $(\mu)_C(\mathrm{id}_C) = x$; and natural transformations from y_C are completely determined by their effect on id_C. An important consequence of the Yoneda lemma is that the Yoneda embedding is actually an embedding, that is: full and faithful, and injective on objects.

Examples of presheaf categories

1. A first example is the category of presheaves on a monoid (a one-object category) M. Such a presheaf is nothing but a set X together with a right M-*action*, that is: we have a map $X \times M \to X$, written $x, f \mapsto xf$, satisfying $xe = x$ (for the unit e of the monoid), and $(xf)g = x(fg)$. There is only one representable presheaf.

2. If the category C is a poset $(P, \leq)$, for $p \in P$ we have the representable y_p with $y_p(q) = \{*\}$ if $q \leq p$, and $\emptyset$ otherwise. So we can identify the representable y_p with the *downset* $\downarrow(p) = \{q \mid q \leq p\}$.

3. The category of directed graphs and graph morphisms is a presheaf category: it is the category of presheaves on the category with two objects e and v, and two non-identity arrows $\sigma, \tau : v \to e$. For a presheaf X on this category, $X(v)$ can be seen as the set of vertices,

$X(e)$ the set of edges, and $X(\sigma), X(\tau) : X(e) \to X(v)$ as the source and target maps.

4. A *tree* is a partially ordered set T with a least element, such that for any $x \in T$, the set $\downarrow(x) = \{y \in T \mid y \le x\}$ is a finite linearly ordered subset of T. A *morphism* of trees $f : T \to S$ is an order-preserving function wth the property that for any element $x \in T$, the restriction of f to $\downarrow(x)$ is a bijection from $\downarrow(x)$ to $\downarrow(f(x))$. A *forest* is a set of trees; a map of forests $X \to Y$ is a function $\phi : X \to Y$ together with an X-indexed collection $(f_x \mid x \in X)$ of morphisms of trees such that $f_x : x \to \phi(x)$. The category of forests and their maps is just the category of presheaves on ω, the first infinite ordinal.

Recall the definition of the category $y{\downarrow}X$ (an example of a 'comma category' construction): objects are pairs (C, μ) with C an object of $\mathcal{C}$ and $\mu : y_C \to X$ an arrow in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. A morphism $(C, \mu) \to (C', \nu)$ is an arrow $f : C \to C'$ in $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} y_C \arrow[rr, "y_f"] \arrow[dr, "\mu"'] & & y_{C'} \arrow[dl, "\nu"] \\ & X & \end{tikzcd}$$

commutes.

Note that if this is the case and $\mu : y_C \to X$ corresponds to $\xi \in X(C)$ and $\nu : y_{C'} \to X$ corresponds to $\eta \in X(C')$, then $\xi = X(f)(\eta)$.

There is a functor $U_X : y{\downarrow}X \to \mathcal{C}$ (the forgetful functor) which sends (C, μ) to C and f to itself; by composition with y we get a diagram

$$y{\circ}U_X : y{\downarrow}X \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$$

Clearly, there is a natural transformation ρ from $y{\circ}U_X$ to the constant functor Δ_X from $y{\downarrow}X$ to $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ with value X: let $\rho_{(C,\mu)} = \mu : y_C \to X$. So there is a cocone in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ for $y{\circ}U_X$ with vertex X.

Proposition 1.1 *The cocone $\rho : y{\circ}U_X \Rightarrow \Delta_X$ is colimiting.*

Proof. Suppose $\lambda : y{\circ}U_X \Rightarrow \Delta_Y$ is another cocone. Define $\nu : X \to Y$ by $\nu_C(\xi) = (\lambda_{(C,\mu)})_C(\mathrm{id}_C)$, where $\mu : y_C \to X$ corresponds to ξ in the Yoneda Lemma.

Then ν is natural: if $f : C' \to C$ in $\mathcal{C}$ and $\mu' : y_{C'} \to X$ corresponds to $X(f)(\xi)$, the diagram

$$\begin{tikzcd} y_{C'} \arrow[rr, "y_f"] \arrow[dr, "\mu'"'] & & y_C \arrow[dl, "\mu"] \\ & X & \end{tikzcd}$$

commutes, so f is an arrow $(C', \mu') \to (C, \mu)$ in $y{\downarrow}X$. Since λ is a cocone, we have that

$$\begin{tikzcd} y_{C'} \arrow[rr, "y_f"] \arrow[dr, "\lambda_{(C',\mu')}"'] & & y_C \arrow[dl, "\lambda_{(C,\mu)}"] \\ & Y & \end{tikzcd}$$

commutes; so

$$\begin{array}{lcrc} \nu_{C'}(X(f)(\xi)) & = & (\lambda_{(C',\mu')})_{C'}(\mathrm{id}_{C'}) & = \\ (\lambda_{(C,\mu)})_{C'}((y_f)_{C'}(\mathrm{id}_{C'})) & = & (\lambda_{(C,\mu)})_{C'}(f) & = \\ Y(f)((\lambda_{(C,\mu)})_C(\mathrm{id}_C)) & = & Y(f)(\nu_C(\xi)) & \end{array}$$

It is easy to see that $\lambda : y{\circ}U_X \Rightarrow \Delta_Y$ factors through ρ via ν, and that the factorization is unique. ∎

Proposition 1.1 is often referred to by saying that "every presheaf is a colimit of representables".

Let us note that the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is complete and cocomplete, and that limits and colimits are calculated 'pointwise': if I is a small category and $F : I \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is a diagram, then for every object C of $\mathcal{C}$ we have a diagram $F_C : I \to Set$ by $F_C(i) = F(i)(C)$; if X_C is a colimit for this diagram in Set, there is a unique presheaf structure on the collection $(X_C \,|\, C \in \mathcal{C}_0)$ making it into the vertex of a colimit for F. The same holds for limits. Some immediate consequences of this are:

i) an arrow $\mu : X \to Y$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is mono (resp. epi) if and only if every component μ_C is an injective (resp. surjective) function of sets;

ii) the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is regular, and every epimorphism is a regular epi;

iii) the initial object of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the constant presheaf with value $\emptyset$;

iv) X is terminal in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ if and only if every set $X(C)$ is a singleton;

v) for every presheaf X, the functor $(-) \times X : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ preserves colimits.

Furthermore we note the following fact: the Yoneda embedding $\mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the 'free colimit completion' of $\mathcal{C}$. That is: for any functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is a cocomplete category, there is, up to isomorphism, exactly one *colimit preserving* functor $\tilde{F} : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{D}$ such that the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "y"'] & & \mathcal{D} \\ & \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

commutes. $\tilde{F}(X)$ is computed as the colimit in $\mathcal{D}$ of the diagram

$$y{\downarrow}X \stackrel{U_X}{\to} \mathcal{C} \stackrel{F}{\to} \mathcal{D}$$

The functor $\tilde{F}$ is also called the 'left Kan extension of F along y'.

We shall now calculate explicitly some structure of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Exponentials can be calculated using the Yoneda Lemma and proposition 1.1. For Y^X, we need a natural 1-1 correspondence

$$\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(Z, Y^X) \simeq \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(Z \times X, Y)$$

In particular this should hold for representable presheaves y_C; so, by the Yoneda Lemma, we should have a 1-1 correspondence

$$Y^X(C) \simeq \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C \times X, Y)$$

which is natural in C. This leads us to *define* a presheaf Y^X by: $Y^X(C) = \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C \times X, Y)$, and for $f : C' \to C$ we let $Y^X(f) : Y^X(C) \to Y^X(C')$ be defined by composition with $y_f \times \mathrm{id}_X : y_{C'} \times X \to y_C \times X$. Then certainly, Y^X is a well-defined presheaf and for representable presheaves we have the natural bijection $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C, Y^X) \simeq \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C \times X, Y)$ we want. In order to show that it holds for arbitrary presheaves Z we use proposition 1.1. Given Z, we have the diagram $y \circ U_Z : y{\downarrow}Z \to \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of which Z is a colimit. Therefore arrows $Z \to Y^X$ correspond to cocones on $y \circ U_Z$ with vertex Y^X. Since we have our correspondence for representables y_C, such cocones correspond to cocones on the diagram

$$y{\downarrow}Z \stackrel{U_Z}{\to} \mathcal{C} \stackrel{y}{\to} \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \stackrel{(-)\times X}{\to} \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$$

with vertex Y. Because, as already noted, the functor $(-) \times X$ preserves colimits, these correspond to arrows $Z \times X \to Y$, as desired.

It is easy to see that the construction of Y^X gives a functor $(-)^X : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ which is right adjoint to $(-) \times X$, thus establishing that $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is cartesian closed. The *evaluation map* $\mathrm{ev}_{X,Y} : Y^X \times X \to Y$ is given by

$$(\phi, x) \;\mapsto\; \phi_C(\mathrm{id}_C, x)$$

Exercise 1 Show that the map $\mathrm{ev}_{X,Y}$, thus defined, is indeed a natural transformation.

Exercise 2 Prove that $y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ preserves all limits which exist in $\mathcal{C}$. Prove also, that if $\mathcal{C}$ is cartesian closed, y preserves exponents.

Another piece of structure we shall need is that of a *subobject classifier*.

Suppose $\mathcal{E}$ is a category with finite limits. A subobject classifier is a monomorphism $t : T \to \Omega$ with the property that for any monomorphism $m : A \to B$ in $\mathcal{E}$ there is a *unique* arrow $\phi : B \to \Omega$ such that there is a pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "m"'] & T \arrow[d, "t"] \\ B \arrow[r, "\phi"'] & \Omega \end{tikzcd}$$

We say that the unique arrow ϕ *classifies* m or rather, the subobject represented by m (if m and m' represent the same subobject, they have the same classifying arrow). In Set, any two element set $\{a, b\}$ together with a specific choice of one of them, say b (considered as arrow $1 \to \{a, b\}$) acts as a subobject classifier: for $A \subset B$ we have the unique *characteristic function* $\phi_A : B \to \{a, b\}$ defined by $\phi_A(x) = b$ if $x \in A$, and $\phi_A(x) = a$ otherwise.

It is no coincidence that in Set, the domain of $t : T \to \Omega$ is a terminal object: T is always terminal. For, for any object A the arrow $\phi : A \to \Omega$ which classifies the identity on A factors as tn for some $n : A \to T$. On the other hand, if $k : A \to T$ is any arrow, then we have pullback diagrams

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "\mathrm{id}_A"'] & T \arrow[r, "\mathrm{id}_T"] \arrow[d, "\mathrm{id}_T"] & T \arrow[d, "t"] \\ A \arrow[r, "k"'] & T \arrow[r, "t"'] & \Omega \end{tikzcd}$$

so tk classifies id_A. By uniqueness of the classifying map, $tn = tk$; since t is mono, $n = k$. So T is terminal. Henceforth we shall write $1 \xrightarrow{t} \Omega$ for the subobject classifier, or, by abuse of language, just Ω.

Note: if $1 \xrightarrow{t} \Omega$ is a subobject classifier in $\mathcal{E}$ then we have a 1-1 correspondence between arrows $A \xrightarrow{\phi} \Omega$ and *subobjects* of A. This correspondence is natural in the following sense: given $f : B \to A$ and a subobject U of A; by $f^\sharp(U)$ we denote the subobject of B obtained by pulling back U along f. Then if ϕ classifies U, ϕf classifies $f^\sharp(U)$.

First a remark about subobjects in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. A subobject of X can be identified with a *subpresheaf* of X: that is, a presheaf Y such that $Y(C) \subseteq X(C)$ for each C, and $Y(f)$ is the restriction of $X(f)$ to $Y(\mathrm{cod}(f))$. This follows easily from epi-mono factorizations pointwise, and the corresponding fact in Set.

Again, we use the Yoneda Lemma to compute the subobject classifier in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. We need a presheaf Ω such that at least for each representable presheaf y_C, $\Omega(C)$ is in 1-1 correspondence with the set of subobjects (in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$) of y_C. So we define Ω such that $\Omega(C)$ is the set of subpresheaves of y_C; for $f : C' \to C$ we have $\Omega(f)$ defined by the action of pulling back along y_f.

What do subpresheaves of y_C look like? If R is a subpresheaf of y_C then R can be seen as a set of arrows with codomain C such that if $f : C' \to C$ is in R and $g : C'' \to C'$ is arbitrary, then fg is in R (for, $fg = y_C(g)(f)$). Such a set of arrows is called a *sieve* on C.

Under the correspondence between subobjects of y_C and sieves on C, the operation of pulling back a subobject along a map y_f (for $f : C' \to C$) sends a sieve R on C to the sieve $f^*(R)$ on C' defined by

$$f^*(R) \;=\; \{g : D \to C' \mid fg \in R\}$$

So Ω can be defined as follows: $\Omega(C)$ is the set of sieves on C, and $\Omega(f)(R) = f^*(R)$. The map $t : 1 \to \Omega$ sends, for each C, the unique element of $1(C)$ to the maximal sieve on C (i.e., the unique sieve which contains id_C).

Exercise 3 Suppose $\mathcal{C}$ is a preorder $(P, \leq)$. For $p \in P$ we let $\downarrow(p) = \{q \in P \mid q \leq p\}$. Show that sieves on p can be identified with downwards closed subsets of $\downarrow(p)$. If we denote the unique arrow $q \to p$ by qp and U is a downwards closed subset of $\downarrow(p)$, what is $(qp)^*(U)$?

Let us now prove that $t : 1 \to \Omega$, thus defined, is a subobject classifier in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Let Y be a subpresheaf of X. Then for any C and any $x \in X(C)$, the set

$$\phi_C(x) \;=\; \{f : D \to C \mid X(f)(x) \in Y(D)\}$$

is a sieve on C, and defining $\phi : X \to \Omega$ in this way gives a natural transformation: for $f : C' \to C$ we have

$$\begin{array}{rcr} \phi_{C'}(X(f)(x)) & = & \{g : D \to C' \mid X(g)(X(f)(x)) \in Y(D)\} \\ & = & \{g : D \to C' \mid X(gf)(x) \in Y(D)\} \\ & = & \{g : D \to C' \mid fg \in \phi_C(x)\} \\ & = & f^*(\phi_C(x)) \\ & = & \Omega(f)(\phi_C(x)) \end{array}$$

Moreover, if we take the pullback of t along ϕ, we get the subpresheaf of X consisting of (at each object C) of those elements x for which $\mathrm{id}_C \in \phi_C(x)$; that is, we get Y. So ϕ classifies the subpresheaf Y.

On the other hand, if $\phi : X \to \Omega$ is any natural transformation such that pulling back t along ϕ gives Y, then for every $x \in X(C)$ we have that $x \in Y(C)$ if and only if $\mathrm{id}_C \in \phi_C(x)$. But then by naturality we get for any $f : C' \to C$ that

$$X(f)(x) \in Y(C') \;\Leftrightarrow\; \mathrm{id}_{C'} \in f^*(\phi_C(x)) \;\Leftrightarrow\; f \in \phi_C(x)$$

which shows that the classifying map ϕ is unique.

Combining the subobject classifier with the cartesian closed structure, we obtain *power objects*. In a category $\mathcal{E}$ with finite products, we call an object A a *power object* of the object X, if there is a natural 1-1 correspondence

$$\mathcal{E}(Y, A) \simeq \mathrm{Sub}_{\mathcal{E}}(Y \times X)$$

The naturality means that if $f : Y \to A$ and $g : Z \to Y$ are arrows in $\mathcal{E}$ and f corresponds to the subobject U of $Y \times X$, then $fg : Z \to A$ corresponds to the subobject $(g \times \mathrm{id}_X)^\sharp(U)$ of $Z \times X$.

Power objects are unique up to isomorphism; the power object of X, if it exists, is usually denoted $\mathcal{P}(X)$. Note the following consequence of the definition: to the identity map on $\mathcal{P}(X)$ corresponds a subobject of $\mathcal{P}(X) \times X$ which we call the "element relation" $\in_X$; it has the property that whenever $f : Y \to \mathcal{P}(X)$ corresponds to the subobject U of $Y \times X$, then $U = (f \times \mathrm{id}_X)^\sharp(\in_X)$.

Convince yourself that power objects in the category Set are just the familiar power sets.

In a cartesian closed category with subobject classifier Ω, power objects exist: let $\mathcal{P}(X) = \Omega^X$. Clearly, the defining 1-1 correspondence is there.

$$\mathcal{P}(X)(C) \;=\; \mathrm{Sub}(y_C \times X)$$

with action $\mathcal{P}(X)(f)(U) = (y_f \times \mathrm{id}_X)^\sharp(U)$.

Exercise 4 Show that $\mathcal{P}(X)(C) = \mathrm{Sub}(y_C \times X)$ and that, for $f : C' \to C$, $\mathcal{P}(X)(f)(U) = (y_f \times \mathrm{id}_X)^\sharp(U)$. Prove also, that the element relation, as a subpresheaf $\in_X$ of $\mathcal{P}(X) \times X$, is given by

$$(\in_X)(C) = \{(U, x) \in \mathrm{Sub}(y_C \times X) \times X(C) \mid (\mathrm{id}_C, x) \in U(C)\}$$

Definition 1.2 A *topos* is a category with finite limits, which is cartesian closed and has a subobject classifier.

1.1 Recovering the category from its presheaves?

In this short section we shall see to what extent the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ determines $\mathcal{C}$. In other words, suppose $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ and $\mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$ are equivalent categories; what can we say about $\mathcal{C}$ and $\mathcal{D}$?

Definition 1.3 In a regular category an object P is called (regular) *projective* if for every regular epi $f : A \to B$, any arrow $P \to B$ factors through f. Equivalently, every regular epi with codomain P has a section.

Exercise 5 Prove the equivalence claimed in definiton 1.3.

Definition 1.4 An object X is called *indecomposable* if whenever X is a coproduct $\coprod_i U_i$, then for *exactly* one i the object U_i is not initial.

Note, that an initial object is not indecomposable, just as 1 is not a prime number.

In $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, coproducts are *stable*, which means that they are preserved by pullback functors; this is easy to check. Another triviality is that the initial object is *strict*: the only maps into it are isomorphisms.

Proposition 1.5 *In $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, a presheaf X is indecomposable and projective if and only if it is a retract of a representable presheaf: there is a diagram $X \xrightarrow{i} y_C \xrightarrow{r} X$ with $ri = \mathrm{id}_X$.*

Proof. Check yourself that every retract of a projective object is again projective. Similarly, a retract of an indecomposable object is indecomposable: if $X \xrightarrow{i} Y \xrightarrow{r} X$ is such that $ri = \mathrm{id}_X$ and Y is indecomposable, any presentation of X as a coproduct $\coprod_i U_i$ can be pulled back along r to produce, by stability of coproducts, a presentation of Y as coproduct $\coprod_i V_i$ such that

$$\begin{tikzcd} V_i \arrow[r] \arrow[d] & Y \arrow[d, "r"] \\ U_i \arrow[r] & X \end{tikzcd}$$

is a pullback; for exactly one i then, V_i is non-initial; hence since r is epi and the initial object is strict, for exactly one i we have that U_i is non-initial. We see that the property of being projective and indecomposable is inherited by retracts. Moreover, every representable is indecomposable and projective, as we leave for you to check.

Conversely, assume X is indecomposable and projective. By proposition 1.1 and the standard construction of colimits from coproducts and coequalizers, there is an epi $\coprod_i y_{C_i} \to X$ from a coproduct of representables. Since X is projective, this epi has a section ι. Pulling back along ι we get a presentation of X as a coproduct $\coprod_i V_i$ such that

$$\begin{tikzcd} V_i \arrow[r] \arrow[d] & X \arrow[d, "\iota"] \\ y_{C_i} \arrow[r] & \coprod_i y_{C_i} \end{tikzcd}$$

is a pullback diagram. X was assumed indecomposable, so exactly one V_i is non-initial. But this means that X is a retract of y_{C_i}. ∎

If X is a retract of y_C, say $X \xrightarrow{\mu} y_C \xrightarrow{\nu} X$ with $\nu\mu = \mathrm{id}_X$, consider $\mu\nu : y_C \to y_C$. This arrow is *idempotent*: $(\mu\nu)(\mu\nu) = \mu(\nu\mu)\nu = \mu\nu$, and since the Yoneda embedding is full and faithful, $\mu\nu = y_e$ for an idempotent $e : C \to C$ in $\mathcal{C}$.

A category $\mathcal{C}$ is said to be *Cauchy complete* if for every idempotent $e : C \to C$ there is a diagram $D \xrightarrow{i} C \xrightarrow{r} D$ with $ri = \mathrm{id}_D$ and $ir = e$. One also says: "idempotents split". In the situation above (where X is a retract of y_C) we see that X must then be isomorphic to y_D for a retract D of C in $\mathcal{C}$. We conclude:

Theorem 1.6 *If $\mathcal{C}$ is Cauchy complete, $\mathcal{C}$ is equivalent to the full subcategory of* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ *on the indecomposable projectives. Hence if $\mathcal{C}$ and $\mathcal{D}$ are Cauchy complete and* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ *and* $\mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$ *are equivalent, so are $\mathcal{C}$ and $\mathcal{D}$.*

Exercise 6 Show that if $\mathcal{C}$ has equalizers, $\mathcal{C}$ is Cauchy complete.

1.2 The Logic of Presheaves

Definition 1.7 A *Heyting algebra* H is a lattice $(\bot, \top, \vee, \wedge)$ together with a binary operation $\to$ (called *Heyting implication*), which satisfies the following equivalence for all $a, b, c \in H$:

$$a \wedge b \leq c \;\Leftrightarrow\; a \leq b \to c$$

Exercise 7 Prove that every Heyting algebra, as a lattice, is *distributive*: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ hold, for all $x, y, z \in H$.

For a presheaf X we shall write $\mathrm{Sub}(X)$ for the set of subpresheaves of X. So if $\phi : X \to Y$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ we gave $\phi^\sharp : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ by pulling back.

Theorem 1.8 *Every poset* $\mathrm{Sub}(X)$ *is a Heyting algebra. For every* $\phi : Y \to X$*, the map* $\phi^\sharp$ *commutes with the Heyting structure* $(\bot, \top, \wedge, \vee, \to)$*. Moreover,* $\phi^\sharp$ *has both a right and a left adjoint, denoted* $\forall_\phi$ *and* $\exists_\phi$ *respectively.*

Proof. Since limits and colimits are computed pointwise, $\wedge$ and $\vee$ (between subpresheaves) are given by pointwise intersection and union, respectively. The empty subpresheaf is, of course, the bottom element of $\mathrm{Sub}(X)$, and X itself is the top element. Heyting implication is *not* done pointwise, since if A and B are subpresheaves of X, setting

$$(A \to B)(C) \;=\; \{x \in X(C) \,|\, x \in A(C) \Rightarrow x \in B(C)\}$$

does not necessarily define a subpresheaf of X (check this!). Therefore we put

$$(A \to B)(C) \;=\; \{x \in X(C) \,|\, \forall f : C' \to C(X(f)(x) \in A(C') \Rightarrow X(f)(x) \in B(C'))\}$$

Then $(A \to B)$ is a subpresheaf of X. It is easy to verify that if D is another subpresheaf of X then D is a subpresheaf of $(A \to B)$ if and only if $D \cap A$ is a subpresheaf of B.

Let us check that $\phi^\sharp$ preserves Heyting implication (the rest of the structure is left to you):

$$\begin{array}{rcl}
(\phi^\sharp(A) \to \phi^\sharp(B))(C) & = & \{y \in Y(C) \forall f : C' \to C(Y(f)(y) \in \phi^\sharp(A)(C') \Rightarrow \\
 & & \qquad Y(f)(y) \in \phi^\sharp(B)(C'))\} \\
 & = & \{y \in Y(C) \,|\, \forall f : C' \to C(\phi_{C'}(Y(f)(y)) \in A(C') \Rightarrow \\
 & & \qquad \phi_{C'}(Y(f)(y)) \in B(C'))\} \\
 & = & \{y \in Y(C) \,|\, \forall f : C' \to C(X(f)(\phi_C(y)) \in A(C') \Rightarrow \\
 & & \qquad X(f)(\phi_C(y)) \in B(C'))\} \\
 & = & \{y \in Y(C) \,|\, \phi_C(y) \in (A \to B)(C)\} \\
 & = & \phi^\sharp(A \to B)(C)
\end{array}$$

The left adjoint $\exists_\phi(A)$ (where now A is a subpresheaf of Y) is, just as in the case of regular categories, given by the *image* of A under ϕ, and this is done pointwise. So,

$$\exists_\phi(A)(C) \;=\; \{x \in X(C) \,|\, \exists y \in A(C)(x = \phi_C(y))\}$$

Clearly then, if B is a subpresheaf of X, we have $\exists_\phi(A) \leq B$ in $\mathrm{Sub}(X)$ if and only if $A \leq \phi^\sharp(B)$ in $\mathrm{Sub}(Y)$.

The right adjoint $\forall_\phi(A)$ is given by

$$\forall_\phi(A)(C) \;=\; \{x \in X(C) \,|\, \forall f : D \to C \forall y \in Y(D)(\phi_D(y) = x \Rightarrow y \in A(D))\}$$

Check for yourself that then, $B \leq \forall_\phi(A)$ in $\mathrm{Sub}(X)$ if and only if $\phi^\sharp(B) \leq A$ in $\mathrm{Sub}(Y)$. ∎

Exercise 8 Prove that for $\phi : Y \to X$, A a subpresheaf of X and B a subpresheaf of Y,

$$\exists_\phi(\phi^\sharp(A) \wedge B) \;=\; A \wedge \exists_\phi(B)$$

Which property of the map $\phi^\sharp$ do you need?

Exercise 9 Suppose X is a presheaf on $\mathcal{C}$. Let $\mathcal{S}$ be the set of all those $\mathcal{C}_0$-indexed collections of sets $A = (A_C \,|\, c \in \mathcal{C}_0)$ for which A_C is a subset of $X(C)$ for each C. $\mathcal{S}$ is ordered pointwise: $A \leq B$ iff for each C, $A_C \subseteq B_C$.

Let $\iota : \mathrm{Sub}(X) \to \mathcal{S}$ be the inclusion. Show that ι has both a left and a right adjoint.

1.2.1 First-order structures in categories of presheaves

In any regular category which satisfies the properties of theorem 1.8 (such a category is often called a 'Heyting category'), one can extend the interpretation of 'regular logic' in regular categories to full first-order logic. We shall retain as much as possible the notation from chapter 4 of 'Basic Category Theory'.

We have a language $\mathcal{L}$, which consists of a collection of *sorts* $S, T, \ldots$, possibly *constants* c^S of sort S, *function symbols* $f : S_1, \ldots, S_n \to S$, and *relation symbols* $R \subseteq S_1, \ldots, S_n$. The definition of formula is extended with the clauses:

i) If φ and ψ are formulas then $(\varphi \vee \psi)$, $(\varphi \to \psi)$ and $\neg\varphi$ are formulas;

ii) if φ is a formula and x^S a variable of sort S then $\forall x^S \varphi$ is a formula.

For the notations $FV(t)$ and $FV(\varphi)$ we refer to the mentioned chapter 4. Again, an interpretation assigns objects $[\![S]\!]$ to the sorts S, arrows to the function symbols and subobjects to relation symbols. This then leads to the definition of the interpretation of a formula φ as a subobject $[\![\varphi]\!]$ of $[\![FV(\varphi)]\!]$, which is a chosen product of the interpretations of all the sorts of the free variables of φ: if $FV(\varphi) = \{x_1^{S_1}, \ldots, x_n^{S_n}\}$ then $[\![FV(\varphi)]\!] = [\![S_1]\!] \times \cdots \times [\![S_n]\!]$.

The definition of $[\![\varphi]\!]$ of the mentioned chapter 4 is now extended by the clauses:

i) If $[\![\varphi]\!] \to [\![FV(\varphi)]\!]$ and $[\![\psi]\!] \to [\![FV(\psi)]\!]$ are given and

$$\begin{tikzcd} {[\![FV(\varphi \wedge \psi)]\!]} \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & {[\![FV(\varphi)]\!]} \\ & {[\![FV(\psi)]\!]} \end{tikzcd}$$

are the projections, then

$$\begin{array}{rcll} [\![\varphi \vee \psi]\!] & = & (\pi_1)^\sharp([\![\varphi]\!]) \vee (\pi_2)^\sharp([\![\psi]\!]) & \text{in Sub}([\![FV(\varphi \wedge \psi)]\!]) \\ [\![\varphi \to \psi]\!] & = & (\pi_1)^\sharp([\![\varphi]\!]) \to (\pi_2)^\sharp([\![\psi]\!]) & \text{in Sub}([\![FV(\varphi \wedge \psi)]\!]) \\ [\![\neg\varphi]\!] & = & [\![\varphi]\!] \to \bot & \text{in Sub}([\![FV(\varphi)]\!]) \end{array}$$

(Note that $FV(\varphi \wedge \psi) = FV(\varphi \vee \psi) = FV(\varphi \to \psi)$)

ii) if $[\![\varphi]\!] \to [\![FV(\varphi)]\!]$ is given and $\pi : [\![FV(\varphi)]\!] \to [\![FV(\exists x\varphi)]\!]$ is the projection, let $FV'(\varphi) = FV(\varphi \wedge x = x)$ and $\pi' : [\![FV'(\varphi)]\!] \to [\![FV(\varphi)]\!]$ the projection. Then

$$[\![\forall x\varphi]\!] \;=\; \forall_{\pi\pi'}((\pi')^\sharp([\![\varphi]\!]))$$

We shall now write out what this means, concretely, in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. For a formula φ, we have $[\![\varphi]\!]$ as a subobject of $[\![FV(\varphi)]\!]$, hence we have a classifying map $\{\varphi\} : [\![FV(\varphi)]\!] \to \Omega$ with components $\{\varphi\}_C : [\![FV(\varphi)]\!](C) \to \Omega(C)$; for $(a_1, \ldots, a_n) \in [\![FV(\varphi)]\!](C)$, $\{\varphi\}_C(a_1, \ldots, a_n)$ is a sieve on C.

Definition 1.9 For φ a formula with free variables $x_1, \ldots, x_n$, C an object of $\mathcal{C}$ and $(a_1, \ldots, a_n) \in [\![FV(\varphi)]\!](C)$, the notation $C \Vdash \varphi(a_1, \ldots, a_n)$ means that $\mathrm{id}_C \in \{\varphi\}_C(a_1, \ldots, a_n)$.

The pronunciation of "$\Vdash$" is 'forces'.

Notation. For φ a formula with free variables $x_1^{S_1}, \ldots, x_n^{S_n}$, C an object of $\mathcal{C}$ and $(a_1, \ldots, a_n) \in [\![FV(\varphi)]\!](C)$ as above, so $a_i \in [\![S_i]\!](C)$, if $f : C' \to C$ is an arrow in $\mathcal{C}$ we shall write $a_i f$ for $[\![S_i]\!](f)(a_i)$.

Note: with this notation and φ, C, $a_1, \ldots, a_n$, $f : C' \to C$ as above, we have $f \in \{\varphi\}_C(a_1, \ldots, a_n)$ if and only if $C' \Vdash \varphi(a_1 f, \ldots, a_n f)$.

Using the characterization of the Heyting structure of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ given in the proof of theorem 1.8, we can easily write down an inductive definition for the notion $C \Vdash \varphi(a_1, \ldots, a_n)$:

- $C \Vdash (t = s)(a_1, \ldots, a_n)$ if and only if $[\![t]\!]_C(a_1, \ldots, a_n) = [\![s]\!]_C(a_1, \ldots, a_n)$
- $C \Vdash R(t_1, \ldots, t_k)(a_1, \ldots, a_n)$ if and only if

$$([\![t_1]\!]_C(a_1, \ldots, a_n), \ldots, [\![t_k]\!]_C(a_1, \ldots, a_n)) \in [\![R]\!](C)$$

- $C \Vdash (\varphi \wedge \psi)(a_1, \ldots, a_n)$ if and only if

$$C \Vdash \varphi(a_1, \ldots, a_n) \text{ and } C \Vdash \psi(a_1, \ldots, a_n)$$

- $C \Vdash (\varphi \vee \psi)(a_1, \ldots, a_n)$ if and only if

$$C \Vdash \varphi(a_1, \ldots, a_n) \text{ or } C \Vdash \psi(a_1, \ldots, a_n)$$

- $C \Vdash (\varphi \to \psi)(a_1, \ldots, a_n)$ if and only if for every arrow $f : C' \to C$,

$$\text{if } C' \Vdash \varphi(a_1 f, \ldots, a_n f) \text{ then } C' \Vdash \psi(a_1 f, \ldots, a_n f)$$

- $C \Vdash \neg\varphi(a_1, \ldots, a_n)$ if and only if for no arrow $f : C' \to C$, $C' \Vdash \varphi(a_1 f, \ldots, a_n f)$
- $C \Vdash \exists x^S \varphi(a_1, \ldots, a_n)$ if and only if for some $a \in [\![S]\!](C)$, $C \Vdash \varphi(a, a_1, \ldots, a_n)$
- $C \Vdash \forall x^S \varphi(a_1, \ldots, a_n)$ if and only if for every arrow $f : C' \to C$ and every $a \in [\![S]\!](C')$,

$$C' \Vdash \varphi(a, a_1 f, \ldots, a_n f)$$

Exercise 10 Prove: if $C \Vdash \varphi(a_1, \ldots, a_n)$ and $f : C' \to C$ is an arrow, then $C' \Vdash \varphi(a_1 f, \ldots, a_n f)$.

Now let ϕ be a sentence of the language, so $[\![\phi]\!]$ is a subobject of 1 in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Note: a subobject of 1 is 'the same thing' as a collection X of objects of $\mathcal{C}$ such that whenever $C \in X$ and $f : C' \to C$ is arbitrary, then $C' \in X$ also. The following theorem is straightforward.

Theorem 1.10 *For a language $\mathcal{L}$ and interpretation $[\![\cdot]\!]$ of $\mathcal{L}$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, we have that for every $\mathcal{L}$-sentence ϕ, $[\![\phi]\!] = \{C \in \mathcal{C}_0 \,|\, C \Vdash \phi\}$. Hence, ϕ is true for the interpretation in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ if and only if for every C, $C \Vdash \phi$.*

If Γ is a set of $\mathcal{L}$-sentences and ϕ an $\mathcal{L}$-sentence, we write $\Gamma \Vdash \phi$ to mean: in every interpretation in a presheaf category such that every sentence of Γ is true, ϕ is true.

We mention without proof:

Theorem 1.11 (Soundness and Completeness) *If Γ is a set of $\mathcal{L}$-sentences and ϕ an $\mathcal{L}$-sentence, we have $\Gamma \Vdash \phi$ if and only if ϕ is provable from Γ in intuitionistic predicate calculus.*

Intuitionistic predicate calculus is what one gets from classical logic by deleting the rule which infers ϕ from a proof that $\neg\phi$ implies absurdity. In a Gentzen calculus, this means that one restricts attention to those sequents $\Gamma \Rightarrow \Delta$ for which Δ consists of at most one formula.

Exercise 11 Let N denote the constant presheaf with value $\mathbb{N}$.

i) Show that there are maps $0 : 1 \to N$ and $S : N \to N$ which make N into a natural numbers object in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

ii) Accordingly, there is an interpretation of the language of first-order arithmetic in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, where the unique sort is interpreted by N. Prove, that for this interpretation, a sentence in the language of arithmetic is true if and only if it is true classically in the standard model $\mathbb{N}$.

Exercise 12 Prove that for every object C of $\mathcal{C}$, the set $\Omega(C)$ of sieves on C is a Heyting algebra, and that for every map $f : C' \to C$ in $\mathcal{C}$, $\Omega(f) : \Omega(C) \to \Omega(C')$ preserves the Heyting structure. Write out explicitly the Heyting implication $(R \to S)$ of two sieves.

1.3 Two examples and applications

1.3.1 Kripke semantics

Kripke semantics is a special kind of presheaf semantics: $\mathcal{C}$ is taken to be a poset, and the sorts are interpreted by presheaves X such that for every $q \leq p$ the map $X(q \leq p) : X(p) \to X(q)$ is an inclusion of sets. Let us call these presheaves *Kripke presheaves*.

The soundness and completeness theorem 1.11 already holds for Kripke semantics. This raises the question whether the greater generality of presheaves achieves anything new. In this example, we shall see that general presheaves are richer than Kripke models if one considers *intermediate logics*: logics stronger than intuitionistic logic but weaker than classical logic.

In order to warm up, let us look at Kripke models for *propositional logic*. The propositional variables are interpreted as subobjects of 1 in $\mathrm{Set}^{\mathcal{K}^{\mathrm{op}}}$ (for a poset $(\mathcal{K}, \leq)$); that means, as downwards closed subsets of $\mathcal{K}$ (see te remark just before theorem 1.10). Let, for example, $\mathcal{K}$ be the poset:

$$\begin{tikzcd} & 0 & \\ k \arrow[ur] & & l \arrow[ul] \end{tikzcd}$$

and let $[\![\, p \,]\!] = \{k\}$. Then $0 \not\Vdash p$, $0 \not\Vdash \neg p$ (since $k \leq 0$ and $k \Vdash p$) and $0 \not\Vdash \neg\neg p$ (since $l \leq 0$ and $l \Vdash \neg p$). So $p \vee \neg p \vee \neg\neg p$ is not true for this interpretation. Even simpler, if $\mathcal{K} = \{0 \leq 1\}$ and $[\![\, p \,]\!] = \{0\}$, then $1 \not\Vdash p \vee \neg p$. However, if $\mathcal{K}$ is a linear order, then $(p \to q) \vee (q \to p)$ is always true on $\mathcal{K}$, since if $\mathcal{K}$ is linear, then so is the poset of its downwards closed subsets. From this one can conclude that if one adds to intuitionistic propositional logic the axiom scheme

$$(\phi \to \psi) \vee (\psi \to \phi)$$

one gets a logic which is strictly between intuitionistic and classical logic.

Exercise 13 Prove that in fact, $\mathcal{K}$ is linear *if and only if* $(p \to q) \vee (q \to p)$ is always true on $\mathcal{K}$. Prove also, that $\neg p \vee \neg\neg p$ is always true on $\mathcal{K}$ if and only if $\mathcal{K}$ has the following property: whenever two elements have an upper bound, they also have a lower bound.

Not only certain properties of posets can be characterized by the propositional logic they satisfy in the sense of exercise 13, also properties of presheaves.

Exercise 14 Let X be a Kripke presheaf on a poset $\mathcal{K}$. Show that the following axiom scheme of predicate logic:

$$\mathrm{D} \qquad \forall x(A(x) \vee B) \to (\forall x A(x) \vee B)$$

(where A and B may contain additional variables, but the variable x is not allowed to occur in B) is always true in X, if and only if for every $k' \leq k$ in $\mathcal{K}$, the map $X(k) \to X(k')$ is the identity.

Suppose now one considers the logic D-J, which is intuitionistic logic extended with the axiom schemes $\neg\phi \vee \neg\neg\phi$ and the axiom scheme D from exercise 14. One might expect (in view of exercises 13 and 14) that this logic is complete with respect to constant presheaves on posets $\mathcal{K}$ which have the property that whenever two elements have an upper bound, they also have a lower bound. However, this is not the case!

Proposition 1.12 *Suppose X is a constant presheaf on a poset $\mathcal{K}$ which has the property that whenever two elements have an upper bound, they also have a lower bound. Then the following axiom scheme is always true on X:*

$$\begin{array}{c}\forall x[(R \to (S \vee A(x))) \vee (S \to (R \vee A(x)))] \wedge \neg\forall x A(x) \\ \to \\ [(R \to S) \vee (S \to R)]\end{array}$$

Exercise 15 Prove proposition 1.12.

However, the axiom scheme in proposition 1.12 is not a consequence of the logic D-J, which fact can be shown using presheaves. This was also shown by Ghilardi. We give the relevant statements without proof; the interested reader is referred to *Arch.Math.Logic* **29** (1989), 125–136.

Proposition 1.13 *i) The axiom scheme $\neg\phi \vee \neg\neg\phi$ is true in every interpretation in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ if and only if the category $\mathcal{C}$ has the property that every pair of arrows with common codomain fits into a commutative square.*

ii) Let X be a presheaf on a category $\mathcal{C}$. Suppose X has the property that for all $f : C' \to C$ in $\mathcal{C}$, all $n \geq 0$, all $x_1, \ldots, x_n \in X(C)$ and all $y \in X(C')$ there is $f' : C' \to C$ and $x \in X(C)$ such that $xf = y$ and $x_1 f = x_1 f', \ldots, x_n f = x_n f'$. Then for every interpretation on X the axiom scheme D *of exercise 14 is true.*

iii) There exist a category $\mathcal{C}$ satisfying the property of i), and a presheaf X on $\mathcal{C}$ satisfying the property of ii), and an interpretation on X for which an instance of the axiom scheme of proposition 1.12 is not true.

1.3.2 Failure of the Axiom of Choice

In this example, due to M. Fourman and A. Scedrov (*Manuscr. Math.* **38** (1982), 325–332), we explore a bit the higher-order structure of a presheaf category. Recall that the Axiom of Choice says: if X is a set consisting of nonempty sets, there is a function $F : X \to \bigcup X$ such that $F(x) \in x$ for every $x \in X$. This axiom is not provable in Zermelo-Fraenkel set theory, but it is classically totally unproblematic for *finite* X (induction on the cardinality of X).

We exhibit here a category $\mathcal{C}$, a presheaf Y on $\mathcal{C}$, and a subpresheaf X of the power object $\mathcal{P}(Y)$ such that the following statements are true in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$:

$\forall \alpha\beta \in X(\alpha = \beta)$ ("X has at most one element")

$\forall\alpha \in X \exists xy \in Y(x \neq y \wedge \forall z \in Y(z \in \alpha \leftrightarrow z = x \vee z = y))$ ("every element of X has exactly two elements")

There is no arrow $X \to \bigcup X$ (this is stronger than: X has no choice function).

Consider the category $\mathcal{C}$ with two objects and two non-identity arrows:

$$\begin{tikzcd} D \arrow[loop left, "\beta"] \arrow[r, "\alpha"] & E \end{tikzcd}$$

subject to the equations $\beta^2 = \mathrm{id}_D$ and $\alpha\beta = \alpha$.

We calculate the representables y_D and y_E, and the map $y_\alpha : y_D \to y_E$:

$$\begin{array}{ll}
y_D(E) = \emptyset & (y_\alpha)_D(\mathrm{id}_D) = \alpha \\
y_D(D) = \{\mathrm{id}_D, \beta\} & (y_\alpha)_D(\beta) = \alpha \\
y_D(\alpha) \text{ is the empty function} & (y_\alpha)_E \text{ is the empty function} \\
y_D(\beta)(\mathrm{id}_D) = \beta & y_D(\beta)(\beta) = \mathrm{id}_D
\end{array}$$

Since E is terminal in $\mathcal{C}$, y_E is a terminal object in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$:

$$y_E(E) = \{\mathrm{id}_E\},\ y_E(D) = \{\alpha\},\ y_E(\alpha)(\mathrm{id}_E) = \alpha,\ y_E(\beta)(\alpha) = \alpha$$

Now let us calculate the power object $\mathcal{P}(y_D)$. According to the explicit construction of power objects in presheaf categories, we have

$$\begin{array}{rcl}
\mathcal{P}(y_D)(E) & = & \mathrm{Sub}(y_E \times y_D) \\
\mathcal{P}(y_D)(D) & = & \mathrm{Sub}(y_D \times y_D)
\end{array}$$

$(y_E \times y_D)(D)$ is the two-element set $\{(\alpha, \mathrm{id}_D), (\alpha, \beta)\}$ which are permuted by the action of β, and $(y_E \times y_D)(E) = \emptyset$. So we see that $\mathrm{Sub}(y_E \times y_D)$ has two elements: $\emptyset$ (the empty presheaf) and $y_E \times y_D$ itself. $(y_D \times y_D)(D)$ has 4 elements: $(\mathrm{id}_D, \beta), (\beta, \mathrm{id}_D), (\beta, \beta), (\mathrm{id}_D, \mathrm{id}_D)$ and we have: $(\mathrm{id}_D, \beta)\beta = (\beta, \mathrm{id}_D)$ and $(\beta, \beta)\beta = (\mathrm{id}_D, \mathrm{id}_D)$.

So $\mathrm{Sub}(y_D \times y_D)$ has 4 elements: $\emptyset, y_D \times y_D, A, B$ where A and B are such that

$$\begin{array}{ll}
A(E) = \emptyset & A(D) = \{(\mathrm{id}_D, \beta), (\beta, \mathrm{id}_D)\} \\
B(E) = \emptyset & B(D) = \{(\beta, \beta), (\mathrm{id}_D, \mathrm{id}_D)\}
\end{array}$$

Summarizing: we have $\mathcal{P}(y_D)(E) = \{\emptyset, y_E \times y_D\}$, $\mathcal{P}(y_D)(D) = \{\emptyset, y_D \times y_D, A, B\}$. The map $\mathcal{P}(y_D)(\alpha)$ is given by pullback along $y_\alpha \times \mathrm{id}_{y_D}$ and sends therefore $\emptyset$ to $\emptyset$ and $y_E \times y_D$ to $y_D \times y_D$. $\mathcal{P}(y_D)(\beta)$ is by pullback along $y_\beta \times \mathrm{id}_{y_D}$ and sends $\emptyset$ to $\emptyset$, $y_D \times y_D$ to $y_D \times y_D$, and permutes A and B.

Now let X be the subpresheaf of $\mathcal{P}(y_D)$ given by:

$$X(E) = \emptyset \quad X(D) = \{y_D \times y_D\}$$

Then X is a 'set of sets' (a subobject of a power object), and clearly, in X, the sentence $\forall xy(x = y)$ is true. So X 'has at most one element'. We have the element relation $\in_{y_D}$ as a subobject of $\mathcal{P}(y_D) \times y_D$, and its restriction to a subobject of $X \times y_D$. This is the presheaf Z with $Z(E) = \emptyset$ and $Z(D) = \{(y_D \times y_D, \mathrm{id}_D), (y_D \times y_D, \beta)\}$. So we see that the sentence expressing 'every element of X has exactly two elements' is true. The presheaf $\bigcup X$ of 'elements of elements of X' is the presheaf $(\bigcup X)(E) = \emptyset$, $(\bigcup X)(D) = \{\mathrm{id}_D, \beta\}$ as subpresheaf of y_D. Now there cannot be any arrow in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ from X to $\bigcup X$, because, in $X(D)$, the unique element is fixed by the action of β; however, in $(\bigcup X)(D)$ there is no fixed point for the action of β. Hence there is no 'choice function'.

2 Sheaves

In this chapter we shall generalize the notion of a 'sheaf on a topological space' to arbitrary categories.

First, let us recall what a sheaf on a topological space is. Let X be a space with set of opens $\mathcal{O}(X)$, considered as a poset with the inclusion order.

A *presheaf* F on X is simply a presheaf on $\mathcal{O}(X)$. So for two opens $U \subseteq V$ in X we have $F(U \subseteq V) : F(V) \to F(U)$, with the usual conditions. F is called *separated* if for any two elements x, y of $F(U)$ and any open cover $U = \bigcup_i U_i$ of U, if $F(U_i \subseteq U)(x) = F(U_i \subseteq U)(y)$ for all i, then $x = y$.

F is called a *sheaf*, if for every system of elements $x_i \in F(U_i)$, indexed by an open cover $U = \bigcup_i U_i$ of U, such that for every pair i, j we have $F(U_i \cap U_j \subseteq U_i)(x_i) = F(U_i \cap U_j \subseteq U_j)(x_j)$, there is a *unique* element $x \in F(U)$ such that $x_i = F(U_i \subseteq U)(x)$ for each i.

Such a system of elements x_i is called a *compatible family*, and x is called an *amalgamation* of it. The most common examples of sheaves on X are sheaves of partial functions: $F(U)$ is a set of functions $U \to Y$ (for example, continuous functions to a space Y), and $F(U \subseteq V)(f)$ is the restriction of f to U.

Example. Let X be $\mathbb{R}$ with the discrete topology; for $U \subseteq \mathbb{R}$ let $F(U)$ be the set of *injective* functions from U to $\mathbb{N}$. F is separated, but not a sheaf (check!).

In generalizing from $\mathcal{O}(X)$ to an arbitrary category $\mathcal{C}$, we see that what we lack is the notion of a 'cover'. Because $\mathcal{C}$ is in general not a preorder, it will not do to define a 'cover of an object C' as a collection of *objects* (as in the case of $\mathcal{O}(X)$); rather, a cover of C will be a *sieve* on C.

We shall denote the maximal sieve on C by $\max(C)$.

Definition 2.1 Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ specifies, for every object C of $\mathcal{C}$, a family $\mathrm{Cov}(C)$ of 'covering sieves' on C, in such a way that the following conditions are satisfied:

i) $\max(C) \in \mathrm{Cov}(C)$

ii) If $R \in \mathrm{Cov}(C)$ then for every $f : C' \to C$, $f^*(R) \in \mathrm{Cov}(C')$

iii) If R is a sieve on C and S is a covering sieve on C, such that for every arrow $f : C' \to C$ from S we have $f^*(R) \in \mathrm{Cov}(C')$, then $R \in \mathrm{Cov}(C)$

We note an immediate consequence of the definition:

Proposition 2.2 *a) If* $R \in \mathrm{Cov}(C)$*,* S *a sieve on* C *and* $R \subseteq S$*, then* $S \in \mathrm{Cov}(C)$*;*

b) If $R, S \in \mathrm{Cov}(C)$ *then* $R \cap S \in \mathrm{Cov}(C)$

Proof. For a), just observe that for every $f \in R$, $f^*(S) = \max(C')$; apply i) and iii) of 2.1. For b), note that if $f \in R$ then $f^*(S) = f^*(R \cap S)$, and apply ii) and iii). ∎

Definition 2.3 A *universal closure operation* on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ assigns to every presheaf X an operation $\bar{(\cdot)} : \mathrm{Sub}(X) \to \mathrm{Sub}(X)$ such that the following hold:

i) $A \leq \bar{A}$

ii) $\bar{A} = \bar{\bar{A}}$

iii) $A \leq B \Rightarrow \bar{A} \leq \bar{B}$

iv) For $\phi : Y \to X$ and $A \in \mathrm{Sub}(X)$, $\phi^\sharp(\bar{A}) = \overline{\phi^\sharp(A)}$

Definition 2.4 A *Lawvere-Tierney topology* on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is an arrow $J : \Omega \to \Omega$ (where, as usual, Ω denotes the subobject classifier of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$), such that the following hold:

i) $R \subseteq J_C(R)$ for every sieve R on C

ii) $J_C(R \cap S) = J_C(R) \cap J_C(S)$

iii) $J_C(J_C(R)) = J_C(R)$

Theorem 2.5 *The following notions are equivalent (that is, each of them determines the others uniquely):*

1) A Grothendieck topolosy on $\mathcal{C}$

2) A Lawvere-Tierney topology on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$

3) A universal closure operation on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$

4) A full subcategory $\mathcal{E}$ *of* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$*, such that the inclusion* $\mathcal{E} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ *has a left adjoint which preserves finite limits*

Proof. We first prove the equivalence of the first three notions; the equivalence of these with notion 4 requires more work and is relegated to a separate proof.

1) $\Rightarrow$ 2). Given a Grothendieck topology Cov on $\mathcal{C}$, define $J : \Omega \to \Omega$ by

$$J_C(R) = \{h : C' \to C \,|\, h^*(R) \in \mathrm{Cov}(C')\}$$

If $h : C' \to C$ is an element of $J_C(R)$ and $g : C'' \to C'$ arbitrary, then $(hg)^*(R) = g^*(h^*(R)) \in \mathrm{Cov}(C'')$ so $hg \in J_C(R)$. Hence $J_C(R)$ is a sieve on C. Similarly, J is a natural transformation: for $f : C' \to C$ we have

$$\begin{array}{rcr} J_{C'}(f^*(R)) & = & \{h : C'' \to C' \,|\, h^*(f^*(R)) \in \mathrm{Cov}(C'')\} \\ & = & \{h : C'' \to C' \,|\, (fh)^*(R) \in \mathrm{Cov}(C'')\} \\ & = & \{h : C'' \to C' \,|\, fh \in J_C(R)\} \\ & = & f^*(J_C(R)) \end{array}$$

To prove $R \subseteq J_C(R)$ we just apply condition i) of 2.1, since $h^*(R) = \max(C')$ for $h \in R$.

By 2.2 we have $R \cap S \in \mathrm{Cov}(C)$ if and only if both $R \in \mathrm{Cov}(C)$ and $S \in \mathrm{Cov}(C)$, and together with the equation $h^*(R \cap S) = h^*(R) \cap h^*(S)$ this implies that $J_C(R \cap S) = J_C(R) \cap J_C(S)$.

Finally, since J preserves $\cap$, it preserves $\subseteq$ ($A \subseteq B$ iff $A \cap B = A$). We have proved $R \subseteq J_C(R)$, so $J_C(R) \subseteq J_C(J_C(R))$ follows. For the converse, suppose $h \in J_C(J_C(R))$ so $h^*(J_C(R)) \in \mathrm{Cov}(C')$. We need to prove $h^*(R) \in \mathrm{Cov}(C')$. Now for any $g \in h^*(J_C(R))$ we have $(hg)^*(R) \in \mathrm{Cov}(C'')$ so $g^*(h^*(R)) \in \mathrm{Cov}(C'')$. Hence by condition iii) of 2.1, $h^*(R) \in \mathrm{Cov}(C')$. This completes the proof that J is a Lawvere-Tierney topology.

2) $\Rightarrow$ 3). Suppose we are given a Lawvere-Tierney topology J. Define the operation $\bar{(\cdot)} : \mathrm{Sub}(X) \to \mathrm{Sub}(X)$ as follows: if $A \in \mathrm{Sub}(X)$ is classified by $\phi : X \to \Omega$ then $\bar{A}$ is classified by $J\phi$. So

$$\bar{A}(C) = \{x \in X(C) \,|\, J_C(\phi_C(x)) = \max(C)\}$$

Then if $f : Y \to X$ is a map of presheaves and $A \in \mathrm{Sub}(X)$, both subobjects $f^\sharp(\bar{A})$ and $\overline{f^\sharp(A)}$ are classified by $J\phi f : Y \to \Omega$, hence they are equal. This proves iv) of definition 2.3. i) follows from condition i) of 2.4 and ii) from iii) of 2.4; finally, that $\bar{(\cdot)}$ is order-preserving follows from the fact that J_C preserves $\subseteq$.

3) $\Rightarrow$ 1). Given a universal closure operation $\bar{(\cdot)}$ on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, we define $\mathrm{Cov}(C) = \{R \in \Omega(C) \,|\, \bar{R} = y_C \text{ in } \mathrm{Sub}(y_C)\}$. Under the identification of

sieves on C with subobjects of y_C, $f^*(R)$ corresponds to $(y_f)^\sharp(R)$. So from condition iv) in 2.3 it follows that if $R \in \mathrm{Cov}(C)$ and $f : C' \to C$, $f^*(R) \in \mathrm{Cov}(C')$. Condition i) ($\max(C) \in \mathrm{Cov}(C)$) follows from i) of 2.3.

To prove iii) of 2.1, suppose $R \in \Omega(C)$, $S \in \mathrm{Cov}(C)$ and for every $f : C' \to C$ in S we have $f^*(R) \in \mathrm{Cov}(C')$. So $\bar{S} = y_C$ and for all $f \in S$, $y_{C'} = \overline{f^*(R)} = (y_f)^\sharp(\bar{R}) = f^*(\bar{R})$. But that means that for all $f \in S$, $f \in \bar{R}$. So $S \subseteq \bar{R}$; hence by iii) of 2.3, $y_C = \bar{S} = \bar{\bar{R}}$; but $\bar{\bar{R}} = \bar{R}$ by ii) of 2.3, so $y_C = \bar{R}$, so $R \in \mathrm{Cov}(C)$, as desired. ∎

As said in the beginning of this proof, the equivalence of 4) with the other notions requires more work. We start with some definitions.

Definition 2.6 Let Cov be a Grothendieck topology on $\mathcal{C}$, and $\bar{(\cdot)}$ the associated universal closure operation on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

A presheaf F is *separated* for Cov if for each $C \in \mathcal{C}_0$ and $x, y \in F(C)$, if the sieve $\{f : C' \to C \mid F(f)(x) = F(f)(y)\}$ covers C, then $x = y$.

A subpresheaf G of F is *closed* if $\bar{G} = G$ in $\mathrm{Sub}(F)$.

A subpresheaf G of F is *dense* if $\bar{G} = F$ in $\mathrm{Sub}(F)$.

Exercise 16 i) A subpresheaf G of F is closed if and only if for each $x \in F(C)$: if $\{f : C' \to C \mid F(f)(x) \in G(C')\}$ covers C, then $x \in G(C)$

ii) F is separated if and only if the diagonal: $F \to F \times F$ is a closed subobject (this explains the term 'separated': in French, the word 'séparé' is synonymous with 'Hausdorff')

iii) A subpresheaf G of F is dense if and only if for each $x \in F(C)$, the sieve $\{f : C' \to C \mid F(f)(x) \in G(C')\}$ is covering

iv) A sieve R on C, considered as subobject of y_C, is dense if and only if it is covering

Definition 2.7 Let F be a presheaf, C an object of $\mathcal{C}$. A *compatible family* in F at C is a family $(x_f \mid f \in R)$ indexed by a sieve R on C, of elements $x_f \in F(\mathrm{dom}(f))$, such that for $f : C' \to C$ in R and $g : C'' \to C'$ arbitrary, $x_{fg} = F(g)(x_f)$. In other words, a compatible family is an arrow $R \to F$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. An *amalgamation* of such a compatible family is an element x of $F(C)$ such that $x_f = F(f)(x)$ for all $f \in R$. In other words, an amalgamation is an extension of the map $R \to F$ to a map $y_C \to F$.

Exercise 17 F is separated if and only if each compatible family in F, indexed by a covering sieve, has *at most one* amalgamation.

Definition 2.8 F is a *sheaf* if every compatible family in F, indexed by a covering sieve, has *exactly one* amalgamation.

Exercise 18 Suppose G is a subpresheaf of F. If G is a sheaf, then G is closed in $\mathrm{Sub}(F)$. Conversely, every closed subpresheaf of a sheaf is a sheaf.

Example. Let Y be a presheaf. Define a presheaf Z as follows: $Z(C)$ consists of all pairs (R, ϕ) such that $R \in \mathrm{Cov}(C)$ and $\phi : R \to Y$ is an arrow in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. If $f : C' \to C$ then $Z(f)(R, \phi) = (f^*(R), \phi f')$ where f' is such that

$$\begin{tikzcd} f^*(R) \arrow[r, "f'"] \arrow[d] & R \arrow[d] \\ y_{C'} \arrow[r, "y_f"'] & y_C \end{tikzcd}$$

is a pullback.

Suppose we have a compatible family in Z, indexed by a covering sieve S on C. So for each $f \in S$, $f : C' \to C$ there is $R_f \in \mathrm{Cov}(C')$, $\phi_f : R_f \to Y$, such that for $g : C'' \to C'$ we have that $R_{fg} = g^*(R_f)$ and $\phi_{fg} : R_{fg} \to Y$ is $\phi_f g'$ where $g' : R_{fg} \to R_f$ is the pullback of $y_g : y_{C''} \to y_{C'}$.

Then this family has an amalgamation in Z: define $T \in \mathrm{Cov}(C)$ by $T = \{fg \,|\, f \in S, g \in R_f\}$. T is covering since for every $f \in S$ we have $R_f \subseteq f^*(T)$. We can define $\chi : T \to Y$ by $\chi(fg) = \phi_f(g)$. So the presheaf Z satisfies the 'existence' part of the amalgamation condition for a sheaf. It does not in general satisfy the uniqueness part.

Exercise 19 Prove that F is a sheaf if and only if for every presheaf X and every dense subpresheaf A of X, any arrow $A \to F$ has a unique extension to an arrow $X \to F$.

The 'plus' construction. We define, for every presheaf X on $\mathcal{C}$, a presheaf X^+ as follows. $X^+(C)$ is the set of *equivalence classes* of pairs (R, ϕ) with $R \in \mathrm{Cov}(C)$ and $\phi : R \to X$, where $(R, \phi) \sim (S, \psi)$ holds if and only if there is a covering sieve T on C, such that $T \subseteq R \cap S$ and ϕ and ψ agree on T. Since $\mathrm{Cov}(C)$ is closed under intersections, this is evidently an equivalence relation.

The construction $(\cdot)^+$ is a functor $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$: for $f : X \to Y$ define $(f)^+ : X^+ \to Y^+$ by $(f)^+_C(R, \phi) = (R, f\phi)$. This is well-defined on equivalence classes.

We have a natural transformation ζ from the identity on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ to $(\cdot)^+$: $(\zeta_X)_C(x) = (y_C, \phi)$ where $\phi : y_C \to X$ corresponds to x in the Yoneda Lemma ($\phi_C(\mathrm{id}_C) = x$).

The following lemma is 'by definition'.

Lemma 2.9 *Let X be a presheaf on $\mathcal{C}$.*

i) *X is separated if and only if ζ_X is mono.*

ii) *X is a sheaf if and only if ζ_X is an isomorphism.*

Lemma 2.10 *Let X be a presheaf, F a sheaf, $g : X \to F$ an arrow in* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. *Then g factors through $\zeta_X : X \to X^+$ via a unique $\tilde{g} : X^+ \to F$:*

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\zeta_X"'] & & F \\ & X^+ \arrow[ur, dashed, "\tilde{g}"'] & \end{tikzcd}$$

Proof. For $[(R, \phi)] \in X^+(C)$, define $\tilde{g}_C([(R, \phi)])$ to be the unique amalgamation in $F(C)$ of the composite $g\phi : R \to F$. This is well-defined, for if $(R, \phi) \sim (S, \psi)$ then for some covering sieve $T \subseteq R \cap S$ we have that $g\phi$ and $g\psi$ agree on T; hence they have the same amalgamation. Convince yourself that $\tilde{g}$ is natural. By inspection, the diagram commutes, and $\tilde{g}$ is the unique arrow with this property. ∎

Lemma 2.11 *For every presheaf X, X^+ is separated.*

Proof. For, suppose (R, ϕ) and (S, ψ) are representatives of elements of $X^+(C)$ such that, for some covering sieve T of C it holds that for each $f \in T$ there is a cover $T_f \subseteq F^*(R) \cap f^*(S)$ such that $\phi f'$ and $\psi f''$ agree on T_f, where f' and f'' are as in the pullback diagrams

$$\begin{tikzcd} f^*(R) \arrow[r, "f'"] \arrow[d] & R \arrow[d] & f^*(S) \arrow[r, "f''"] \arrow[d] & S \arrow[d] \\ y_{C'} \arrow[r, "f"'] & y_C & y_{C''} \arrow[r, "f"'] & y_C \end{tikzcd}$$

Let $U = \{fg \mid f \in T, g \in T_f\}$. Then U is a covering sieve on C, $U \subseteq R \cap S$, and ϕ and ψ agree on U. Hence $(R, \phi) \sim (S, \psi)$, so they represent the same element of $X^+(C)$. ∎

Lemma 2.12 *If X is separated, X^+ is a sheaf.*

Proof. Suppose we have a compatible family in X^+, indexed by a covering sieve R on C. So for each $f : C' \to C$ in R we have (R_f, ϕ_f), $\phi_f : R \to X$.

In order to find an amalgamation, we define a sieve S and a map $\psi : S \to X$ by:

$$\begin{aligned} S &= \{fg \mid f \in R, g \in R_f\} \\ \psi(fg) &= \phi_f(g) \end{aligned}$$

Certainly S is a covering sieve on C, but it is not a priori clear that ψ is well-defined. For it may be the case that for $f, f' \in R$, $g \in R_f$ and $g' \in R_{f'}$, $fg = f'g'$:

$$\begin{tikzcd} & C' \arrow[dr, "f"] & \\ C'' \arrow[ur, "g"] \arrow[dr, "g'"'] & & C \\ & D \arrow[ur, "f'"'] & \end{tikzcd}$$

We need to show that in this case, $\phi_f(g) = \phi_{f'}(g')$.

The fact that we have a compatible family means that $(g^*(R_f), \phi_f h) \sim ((g')^*(R_{f'}), \phi_{f'} h')$ in the equivalence relation defining $X^+(C'')$, where h and h' are as in the pullback diagrams

$$\begin{tikzcd} g^*(R_f) \arrow[r, "h"] \arrow[d] & R_f \arrow[d] & (g')^*(R_{f'}) \arrow[r, "h'"] \arrow[d] & R_{f'} \arrow[d] \\ y_{C''} \arrow[r, "y_g"'] & y_{C'} & y_{C''} \arrow[r, "y_{g'}"'] & y_D \end{tikzcd}$$

That means that there is a covering sieve T on C'' such that $T \subseteq g^*(R_f) \cap (g')^*(R_{f'})$ on which $\phi_f h$ and $\phi_{f'} h'$ coincide; hence, for all $k \in T$ we have that $X(k)(\phi_f(g)) = X(k)(\phi_{f'}(g'))$. Since X is separated by assumption, $\phi_f(g) = \phi_{f'}(g')$ as desired.

We have obtained an amalgamation. It is unique because X^+ is separated by lemma 2.11. ∎

The functor $\mathbf{a} : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is defined by applying $(\cdot)^+$ twice: $\mathbf{a}(X) = X^{++}$. By lemmas 2.11 and 2.12, $\mathbf{a}(X)$ is always a sheaf. There is a natural transformation η from the identity to $\mathbf{a}$ obtained by the composition $X \xrightarrow{\zeta} X^+ \xrightarrow{\zeta} X^{++} = \mathbf{a}(X)$; by twice applying lemma 2.10 one sees that every arrow from a presheaf X to a sheaf F factors uniquely through η.

This exhibits η as the unit of an adjunction between $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ and its full subcategory of sheaves. If we denote the latter by $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ and regard $\mathbf{a}$ as a functor $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$, then $\mathbf{a}$ is left adjoint to the inclusion functor.

The functor $\mathbf{a}$ is usually called *sheafification*, or the *associated sheaf* functor.

Lemma 2.13 *The sheafification functor preserves finite limits.*

Proof. It is enough to show that $(\cdot)^+$ preserves the terminal object and pullbacks. That $(\cdot)^+$ preserves 1 is obvious (1 is always a sheaf). Suppose

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "v"'] & Y \arrow[d, "f"] \\
Z \arrow[r, "g"'] & V
\end{tikzcd}$$

is a pullback diagram. In order to show that also its $(\cdot)^+$-image is, suppose (R, ϕ) and (S, ψ) represent elements of $Y^+(C)$, $Z^+(C)$, respectively, such that $(R, f\phi) \sim (S, g\psi)$ in the equivalence relation defining $V^+(C)$. Then there is a covering sieve $T \subseteq R \cap S$ such that the square

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
Z \arrow[r, "g"'] & V
\end{tikzcd}$$

commutes. By the pullback property, there is a unique factoring map $\chi : T \to X$; that is, an element (T, χ) of $X^+(C)$ such that $[(R, \phi)] = u^+([(T, \chi)])$ and $[(S, \psi)] = v^+([(T, \chi)])$, which shows that the $(\cdot)^+$-image of the given diagram is a pullback. ∎

We now wrap up to finish the proof of Theorem 2.5: every Grothendieck topology gives a category of sheaves, which is a full subcategory of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ such that the inclusion has a left adjoint $\mathbf{a}$ which preserves finite limits by 2.13.

Conversely, suppose such a full subcategory $\mathcal{E}$ of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is given, with inclusion $i : \mathcal{E} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ and left adjoint $r : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{E}$, r preserving finite limits. This then determines a universal closure operation as follows. Given a subpresheaf A of X, define $\bar{A}$ as given by the pullback:

$$\begin{tikzcd}
\bar{A} \arrow[r] \arrow[d] & ir(A) \arrow[d] \\
X \arrow[r, "\eta"'] & ir(X)
\end{tikzcd}$$

where $\eta : X \to ir(X)$ is the unit of the adjunction $r \dashv i$. Since r preserves finite limits, ir preserves monos, so the map $\bar{A} \to X$ is monic. Writing down the naturality square for η for the inclusion $A \to X$ we see that this inclusion factors through $\bar{A} \to X$, so $A \leq \bar{A}$. It is immediate that $\bar{(\cdot)}$ is order-preserving; and because i is full and faithful so ri is naturally isomorphic to the identity on $\mathcal{E}$, ir is (up to isomorphism) idempotent, from which it is easy to deduce that $\bar{\bar{A}} = \bar{A}$. The final property of a universal closure operation, stability under pullback, follows again from the fact that r, and hence ir, preserves pullbacks. ∎

2.1 Examples of Grothendieck topologies

1. As always, there are the two trivial extremes. The smallest Grothendieck topology (corresponding to the maximal subcategory of sheaves) has $\mathrm{Cov}(C) = \{\max(C)\}$ for all C. The only dense subpresheaves are the maximal ones; every presheaf is a sheaf.

2. The other extreme is the biggest Grothendieck topology: $\mathrm{Cov}(C) = \Omega(C)$. Every subpresheaf is dense; the only sheaf is the terminal object 1.

3. Let X be a topological space with set of opens $\mathcal{O}(X)$, regarded as a category: a poset under the inclusion order. A sieve on an open set U can be identified with a downwards closed collection R of open subsets of U. The standard Grothendieck topology has $R \in \mathrm{Cov}(U)$ iff $\bigcup R = U$. Sheaves for this Grothendieck topology coincide with the familiar sheaves on the space X.

4. The *dense* or $\neg\neg$-topology is defined by:

   $$\mathrm{Cov}(C) \;=\; \{R \in \Omega(C) \,|\, \forall f : C' \to C \exists g : C'' \to C' \, (fg \in R)\}$$

   This topology corresponds to the Lawvere-Tierney topology $J : \Omega \to \Omega$ defined by

   $$J_C(R) \;=\; \{h : C' \to C \,|\, \forall f : C'' \to C' \exists g : C''' \to C'' \, (hfg \in R)\}$$

   This topology has the property that for every sheaf F, the collection of subsheaves of F forms a Boolean algebra

5. For this example we assume that in the category $\mathcal{C}$, every pair of arrows with common codomain fits into a commutative square. Then the

atomic topology takes all *nonempty* sieves as covers. This corresponds to the Lawvere-Tierney topology

$$J_C(R) \;=\; \{h : C' \to C \,|\, \exists f : C'' \to C' \,(hf \in R)\}$$

This topology has the property that for every sheaf F, the collection of subsheaves of F forms an *atomic* Boolean algebra: an *atom* in a Boolean algebra is a minimal non-bottom element. An atomic Boolean algebra is such that for every non-bottom x, there is an atom which is $\leq x$.

6. Let U be a subpresheaf of the terminal presheaf 1. With U we can associate a set of objects $\tilde{U}$ of $\mathcal{C}$ such that whenever $f : C' \to C$ is an arrow and $C \in \tilde{U}$, then $C' \in \tilde{U}$. Namely, $\tilde{U} = \{C \,|\, U(C) \neq \emptyset\}$. To such U corresponds a Grothendieck topology, the *open topology* determined by U, given by

$$\mathrm{Cov}(C) \;=\; \{R \in \Omega(C) \,|\, \forall f : C' \to C \,(C' \in \tilde{U} \Rightarrow f \in R)\}$$

and associated Lawvere-Tierney topology

$$J_C(R) \;=\; \{h : C' \to C \,|\, \forall f : C'' \to C' \,(C'' \in \tilde{U} \Rightarrow hf \in R)\}$$

Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ on the objects in $\tilde{U}$. Then there is an equivalence of categories between $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ and $\mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$.

7. For U and $\tilde{U}$ as in the previous example, there is also the *closed* topology determined by U, given by

$$\mathrm{Cov}(C) \;=\; \{R \in \Omega(C) \,|\, C \in \tilde{U} \text{ or } R = \max(C)\}$$

There is an equivalence between $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ and the category of presheaves on the full subcategory of $\mathcal{C}$ on the objects *not* in $\tilde{U}$.

2.2 Structure of the category of sheaves

In this section we shall see, among other things, that also the category $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ is a topos.

Proposition 2.14 $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ *is closed under arbitrary limits in* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Proof. This is rather immediate from the defining property of sheaves and the way (point-wise) limits are calculated in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Suppose $F : I \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$

is a diagram of sheaves with limiting cone $(X, (\mu_i : X \to F(i)))$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. We show that X is a sheaf.

Suppose $R \in \mathrm{Cov}(C)$ and $\phi : R \to X$ is a map in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Since every $F(i)$ is a sheaf, every composite $\mu_i \phi : R \to F(i)$ has a unique amalgamation $y_i \in F(i)(C)$, and by uniqueness these satisfy, for every map $k : i \to j$ in the index category I, the equality $(F(k))_C(y_i) = y_j$. Since $X(C)$ is the vertex of a limiting cone for the diagram $F(\cdot)(C) : I \to \mathrm{Set}$, there is a unique $x \in X(C)$ such that $(\mu_i)_C(x) = y_i$ for each i. But this means that x is an amalgamation (and the unique such) for $R \xrightarrow{\phi} X$. ∎

Proposition 2.15 *Let X be a presheaf, Y a sheaf. Then Y^X is a sheaf.*

Proof. Suppose $A \to Z$ is a dense subobject, and $A \xrightarrow{\phi} Y^X$ a map. By exercise 19 we have to see that ϕ has a unique extension to a map $Z \to Y^X$. Now ϕ transposes to a map $\tilde{\phi} : A \times X \to Y$. By stability of the closure operation, if $A \to Z$ is dense then so is $A \times X \to Z \times X$. Since Y is a sheaf, $\tilde{\phi}$ has a unique extension to $\psi : Z \times X \to Y$. Transposing back gives $\bar{\psi} : Z \to Y^X$, which is the required extension of ϕ. ∎

Corollary 2.16 *The category* $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ *is cartesian closed.*

Now we turn to the subobject classifier in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. Let $J : \Omega \to \Omega$ be the associated Lawvere-Tierney topology. Sieves on C which are in the image of J_C are called *closed*. This is good terminology, since a closed sieve on C is the same thing as a closed subpresheaf of y_C.

By exercise 18 we know that subsheaves of a sheaf are the closed subpresheaves, and from exercise 16i) we know that a subpresheaf is closed if and only if its classifying map takes values in the image of J. This is a subobject of Ω; let us call it Ω_J. So subobjects in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ admit unique classifying maps into Ω_J; note that the map $1 \xrightarrow{t} \Omega$, which picks out the maximal sieve on any C, factors through Ω_J since every maximal sieve is closed. So $1 \xrightarrow{t} \Omega_J$ is a subobject classifier in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ provided we can show that it is a map between sheaves. It is easy to see (and a special case of 2.14) that 1 is a sheaf. For Ω_J this requires a little argument.

Proposition 2.17 *The presheaf Ω_J is a sheaf.*

Proof. We have seen that the arrow $1 \xrightarrow{t} \Omega_J$ classifies closed subobjects. Therefore, in order to show that Ω_J has the unique-extension property w.r.t. dense inclusions, it is enough to see that whenever X is a dense subpresheaf of Y there is a bijection between the closed subpresheaves of X and the closed subpresheaves of Y.

For a closed subpresheaf A of X let $k(A)$ be the closure of A in $\mathrm{Sub}(Y)$. For a closed subpresheaf B of Y let $l(B) = B \cap X$; this is a closed subpresheaf of X.

Now $kl(B) = k(B \cap X) = \overline{B \cap X} = \bar{B} \cap \bar{X} = \bar{B} = B$ since X is dense and B closed. Conversely, $lk(A) = \bar{A} \cap X$ which is (by stability of closure) the closure of A in X. But A was closed, so this is A. Hence the maps k and l are inverse to each other, which finishes the proof. ∎

Corollary 2.18 *The category* $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ *is a topos.*

Definition 2.19 A pair $(\mathcal{C}, \mathrm{Cov})$ of a small category and a Grothendieck topology on it is called a *site*. For a sheaf on $\mathcal{C}$ for Cov, we also say that it is a *sheaf on the site* $(\mathcal{C}, \mathrm{Cov})$. A *Grothendieck topos* is a category of sheaves on a site.

Not every topos is a Grothendieck topos. For the moment, there is only one simple example to give of a topos that is not Grothendieck: the category of finite sets. It is not a Grothendieck topos, for example because it does not have all small limits.

Exercise 20 The terminal category **1** is a topos. Is it a Grothendieck topos?

Let us say something about power objects and the natural numbers in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$.

For power objects there is not much more to say than this: for a sheaf X, its power object in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ is Ω_J^X; we shall also write $\mathcal{P}_J(X)$. By the Yoneda Lemma we have a natural 1-1 correspondence between $\mathcal{P}_J(X)(C)$ and the set of closed subpresheaves of $y_C \times X$; for $f : C' \to C$ and A a closed subpresheaf of $y_C \times X$, $\mathcal{P}_J(X)(f)(A)$ is given by $(y_f \times \mathrm{id}_X)^\sharp(A)$.

Next, let us discuss natural numbers. We use exercise 11 which says that the constant presheaf with value $\mathbb{N}$ is a natural numbers object in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, and we also use the following result:

Exercise 21 Suppose $\mathcal{E}$ has a natural numbers object and $F : \mathcal{E} \to \mathcal{F}$ is a functor which has a right adjoint and preserves the terminal object. Then F preserves the natural numbers object.

So the natural numbers object in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ is N^{++}, where N is the constant presheaf with value $\mathbb{N}$. In fact, we don't have to apply the 'plus' construction twice, because N is 'almost' separated: clearly, if n, m are two distinct natural numbers and $R \in \mathrm{Cov}(C)$ is such that for all $f \in R$ we have $nf =$

mf, then $R = \emptyset$. So the only way that N can fail to be separated is that for some objects C we have $\emptyset \in \mathrm{Cov}(C)$. Now define the presheaf N' as follows:

$$N'(C) \;=\; \begin{cases} \mathbb{N} & \text{if } \emptyset \notin \mathrm{Cov}(C) \\ \{*\} & \text{if } \emptyset \in \mathrm{Cov}(C) \end{cases}$$

Exercise 22 Prove:

a) N' is separated

b) $\zeta_N : N \to N^+$ factors through N'

c) $N^{++} \simeq (N')^+$

Colimits in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ are calculated as follows: take the colimit in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, then apply the associated sheaf functor. For coproducts of sheaves, we have a simplification comparable to that of N. We write $\bigsqcup$ for the coproduct in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ and $\bigsqcup_J$ for the coproduct in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. So $\bigsqcup_J F_i = \mathbf{a}(\bigsqcup F_i)$, but if we define $\bigsqcup' F_i$ by

$$(\bigsqcup{}' F_i)(C) \;=\; \begin{cases} \bigsqcup F_i(C) & \text{if } \emptyset \notin \mathrm{Cov}(C) \\ \{*\} & \text{if } \emptyset \in \mathrm{Cov}(C) \end{cases}$$

then it is not too hard to show that $\bigsqcup_J F_i \simeq (\bigsqcup' F_i)^+$. Concretely, a compatible family in $\bigsqcup' F_i$ indexed by a covering sieve R on C, i.e./ a map $\phi : R \to \bigsqcup' F_i$, gives for each i a sub-sieve R_i and a map $\phi_i : R_i \to F_i$. The system of subsieves R_i has the property that if $h : C' \to C$ is an element of $R_i \cap R_j$ and $i \neq j$, then $\emptyset \in \mathrm{Cov}(C')$. Of course, such compatible families are still subject to the equivalence relation defining $(\bigsqcup' F_i)^+$.

Exercise 23 Prove:

i) Coproducts are stable in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$

ii) For any sheaf F, $F^{N_J} \simeq \prod_{n \in \mathbb{N}} F$

Images in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$: given a map $\phi : F \to G$ between sheaves, the image of ϕ (as subsheaf of G) is the closure of the image in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of the same map. The arrow ϕ is an epimorphism in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ if and only if for each C and each $x \in G(C)$, the sieve $\{f : C' \to C \mid \exists y \in F(C')(\phi_{C'}(y) = xf)\}$ covers C.

Exercise 24 Prove this characterization of epis in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. Prove also that in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$, an arrow which is both mono and epi is an isomorphism.

Regarding the structure of the lattice of subobjects in $\mathrm{Sh}(\mathcal{C},\mathrm{Cov})$ of a sheaf F, we know that these are the closed subpresheaves, so the fixed points of the closure operation. That the subobjects again form a Heyting algebra is then a consequence of the following exercise.

Exercise 25 Suppose H is a Heyting algebra with operations $\bot,\top,\wedge,\vee,\to$ and let $j : H \to H$ be order-preserving, idempotent, inflationary (that is: $x \leq j(x)$ for all $x \in H$), and such that $j(x \wedge y) = j(x) \wedge j(y)$. Let H_j be the set of fixed points of j. Then H_j is a Heyting algebra with operations:

$$\begin{array}{rr} \top_j = \top & \bot_j = j(\bot) \\ x \wedge_j y = x \wedge y & x \vee_j y = j(x \vee y) \\ & x \to_j y = x \to y \end{array}$$

Exercise 26 If H is a Heyting algebra, show that the map $\neg\neg : x \mapsto (x \to \bot) \to \bot$ satisfies the requirements of the map j in exercise 25. Show also that $H_{\neg\neg}$ is a Boolean algebra.

Exercise 27 Let J be the Lawvere-Tierney topology corresponding to the dense topology (see section 2.1). Show that in the Heyting algebra $\Omega(C)$, J_C is the map $\neg\neg$ of exercise 26.

As for presheaves, we can express the interpretation of first-order languages in $\mathrm{Sh}(\mathcal{C},\mathrm{Cov})$ in terms of a 'forcing' definition. The basic setup is the same; only now, of course, we take sheaves as interpretation of the sorts, and closed subpresheaves (subsheaves) as interpretation of the relation symbols. We then define $[\![\,\varphi\,]\!]$ as a subsheaf of $[\![\,FV(\varphi)\,]\!]$ and let $\{\varphi\} : [\![\,FV(\varphi)\,]\!] \to \Omega_J$ be its classifying map. The notation $C \Vdash_J \varphi(a_1,\ldots,a_n)$ again means that $\{\varphi\}_C(a_1,\ldots,a_n)$ is the maximal sieve on C. This relation then again admits a definition by recursion on the formula φ. The inductive clauses of the definition of $\Vdash_J$ are the same for $\Vdash$ for the cases: atomic formula, $\wedge$, $\to$ and $\forall$, and we put:

- $C \Vdash_J (\varphi \vee \psi)(a_1,\ldots,a_n)$ if and only if the sieve $\{g : C' \to C \,|\, C' \Vdash_J \varphi(a_1g,\ldots,a_ng)$ or $C' \Vdash_J \psi(a_1g,\ldots,a_ng)\}$ covers C;
- $C \Vdash_J \exists x\varphi(x,a_1,\ldots,a_n)$ if and only if the sieve $\{g : C' \to C \,|\, \exists x \in F(C')\, C' \Vdash_J \varphi(x,a_1g,\ldots,a_ng)\}$ covers C (where F is the interpretation of the sort of x).

That this works should be no surprise in view of our characterisation of images in $\mathrm{Sh}(\mathcal{C},\mathrm{Cov})$ and our treatment of the Heyting structure on the subsheaves of a sheaf. We have the following properties of the relation $\Vdash_J$:

Theorem 2.20 *i) If $C \Vdash_J \varphi(a_1, \ldots, a_n)$ then for each arrow $f : C' \to C$, $C' \Vdash_J \varphi(a_1 f, \ldots, a_n f)$;*

ii) if R is a covering sieve on C and for every arrow $f : C' \to C$ in R we have $C' \Vdash_J \varphi(a_1 f, \ldots, a_n f)$, then $C \Vdash_J \varphi(a_1, \ldots, a_n)$.

Exercise 28 Let N_J be the natural numbers object in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. Prove the same result as we had in exercise 11, that is: for the standard interpretation of te language of arithmetic in N_J, a sentence is true if and only it is true in the (classical) standard model of natural numbers.

2.3 Application: a model for the independence of the Axiom of Choice

In this section we treat a model, due to P. Freyd, which shows that in toposes where *classical* logic always holds, the axiom of choice need not be valid. Specifically, we construct a topos $\mathcal{F} = \mathrm{Sh}(F, \mathrm{Cov})$ and in $\mathcal{F}$ a subobject E of $N_J \times \mathcal{P}_J(N_J)$ with the properties:

i) $\mathcal{F}$ is *Boolean*, that is: every subobject lattice is a Boolean algebra;

ii) $\Vdash_J \forall n \exists \alpha ((n, \alpha) \in E)$

iii) $\Vdash \neg \exists f \in \mathcal{P}_J(N_J)^{N_J} \forall n \, ((n, f(n)) \in E)$

So, E is an N_J-indexed collection of nonempty (in a strong sense) subsets of $\mathcal{P}_J(N_J)$, but admits no choice function.

Let $\mathbb{F}$ be the following category: it has objects $\bar{n}$ for each natural number n, and an arrow $f : \bar{m} \to \bar{n}$ is a function $\{0, \ldots, m\} \to \{0, \ldots, n\}$ such that $f(i) = i$ for every i with $0 \leq i \leq n$. It is understood that there are no morphisms $\bar{m} \to \bar{n}$ for $m < n$. Note, that $\bar{0}$ is a terminal object in this category.

On $\mathbb{F}$ we let Cov be the dense topology, so a sieve R on $\bar{m}$ covers $\bar{m}$ if and only if for every arrow $g : \bar{n} \to \bar{m}$ there is an arrow $h : \bar{k} \to \bar{n}$ such that $gh \in R$. We shall work in the topos $\mathcal{F} = \mathrm{Sh}(\mathbb{F}, \mathrm{Cov})$, the *Freyd topos*. Let E_n be the object $\mathbf{a}(y_{\bar{n}})$, the sheafification of the representable presheaf on $\bar{n}$.

Lemma 2.21 Cov *has the following properties:*

a) Every covering sieve is nonempty

b) Every nonempty sieve on $\bar{0}$ is a cover *c)* *Every representable presheaf is separated*

d) $y_{\bar{0}}$ *has only two closed subobjects*

Proof. For a), apply the definition of 'R covers $\bar{m}$' to the identity on $\bar{m}$; it follows that there is an arrow $h : \bar{k} \to \bar{m}$ such that $h \in R$.

For b), suppose S is a sieve on $\bar{0}$ and $\bar{k} \xrightarrow{f} \bar{0}$ is in S. Since $\bar{0}$ is terminal, for any $\bar{m} \xrightarrow{g} \bar{0}$ and any maps $\overline{m+k} \to \bar{k}$, $\overline{m+k} \to \bar{m}$, the square

$$\begin{tikzcd} \overline{m+k} \arrow[r] \arrow[d] & \bar{k} \arrow[d, "f"] \\ \bar{m} \arrow[r, "g"'] & \bar{0} \end{tikzcd}$$

commutes, so for any such g there is an h with $gh \in R$, hence R covers $\bar{0}$.

For c), suppose $g, g' : \bar{k} \to \bar{n}$ are such that for a cover R of $\bar{k}$ we have $gf = g'f$ for all $f \in R$. We need to see that $g = g'$. Pick $i \leq k$. Let $h : \overline{k+1} \to \bar{k}$ be such that $h(k+1) = i$. Since R covers $\bar{k}$ there is $u : \bar{l} \to \overline{k+1}$ such that $hu \in R$. Then $ghu = g'hu$, which means that $g(i) = ghu(k+1) = g'hu(k+1) = g'(i)$. So $g = g'$, as desired.

Finally, d) follows directly from b): suppose R is a closed sieve on $\bar{0}$. If $R \neq \emptyset$, then R is covering by b), hence (being also closed) equal to $\max(\bar{0})$. Hence the only closed sieves are $\emptyset$ and $\max(\bar{0})$. ∎

Proposition 2.22 *The unique map* $E_n \to 1$ *is an epimorphism.*

Proof. By lemma 2.21d), $1 = \mathbf{a}(y_{\bar{0}})$ has only two subobjects and $y_{\bar{n}}$ is nonempty, so the image of $E_n \to 1$ is 1. ∎

Proposition 2.23 *If* $n > m$ *then* $E_n(\bar{m}) = \emptyset$.

Proof. Since $y_{\bar{n}}$ is separated by 2.21c), $E_n = (y_{\bar{n}})^+$, so $E_n(\bar{m})$ is an equivalence class of morphisms $\tau : S \to y_{\bar{n}}$ in $\mathrm{Set}^{\mathbb{F}^{\mathrm{op}}}$, for a cover S of $\bar{m}$. We claim that such τ don't exist.

For, since such S is nonempty (2.21a)), pick $s : \bar{k} \to \bar{m}$ in S and let $f = \tau_{\bar{k}}(s)$, so $f : \bar{k} \to \bar{n}$. Let $g, h : \overline{k+1} \to \bar{k}$ be such that $g(k+1) = n$, and $h(k+1) = s(n) \leq m < n$. Then $sg = sh$ (check!). So

$$fg = \tau_{\bar{k}}(s)g = \tau_{\overline{k+1}}(sg) = \tau_{\overline{k+1}}(sh) = \tau_{\bar{k}}(s)h = fh$$

However, $fg(k+1) = f(n) = n$, whereas $fh(k+1) = f(s(n+1)) = s(n)$. Contradiction. ∎

Corollary 2.24 *The product sheaf $\prod_{n\in\mathbb{N}} E_n$ is empty.*

Proof. For, if $(\prod_n E_n)(\bar{m}) \neq \emptyset$ then by applying the projection $\prod_n E_n \to E_{\overline{m+1}}$ we would have $E_{m+1}(\bar{m}) \neq \emptyset$, contradicting 2.23. ∎

Proposition 2.25 *For each n there is a monomorphism $E_n \to \mathcal{P}_J(N_J)$.*

Proof. Since $E_n = \mathbf{a}(y_{\bar{n}})$ and $\mathcal{P}_J(N_J)$ is a sheaf, it is enough to construct a monomorphism $y_{\bar{n}} \to \mathcal{P}_J(N_J)$, which gives then a unique extension to a map from E_n; since $\mathbf{a}$ preserves monos, the extension will be mono if the given map is.

Fix n for the rest of the proof. Let $(g_k)_{k\in\mathbb{N}}$ be a 1-1 enumeration of all the arrows in $\mathbb{F}$ with codomain $\bar{n}$. For each g_i, let C_i be the smallest closed sieve on $\bar{n}$ containing g_i (i.e., C_i is the $J_{\bar{n}}$-image of the sieve generated by g_i).

$\mathcal{P}_J(N_J)(\bar{m})$ is the set of closed subpresheaves of $y_{\bar{m}} \times N_J$. Elements of $(y_{\bar{m}} \times N_J)(\bar{k})$ are pairs $(h, (S_i)_{i\in\mathbb{N}})$ where $h : \bar{k} \to \bar{m}$ and $(S_i)_i$ is an $\mathbb{N}$-indexed collection of sieves on $\bar{k}$, such that $S_i \cap S_j = \emptyset$ for $i \neq j$, and $\bigcup_i S_i$ covers $\bar{k}$.

Define $\mu_{\bar{m}} : y_{\bar{n}}(\bar{m}) \to \mathcal{P}_J(N_J)(\bar{m})$ as follows. For $f : \bar{m} \to \bar{n}$, $\mu_{\bar{m}}(f)$ is the subpresheaf of $y_{\bar{m}} \times N_J$ given by: $(h, (S_i)_i) \in \mu_{\bar{m}}(f)(\bar{k})$ iff for each i, $S_i \subseteq (fh)^*(C_i)$. It is easily seen that $\mu_{\bar{m}}(f)$ is a closed subpresheaf of $y_{\bar{m}} \times N_J$.

Let us first see that μ is a natural transformation. Suppose $g : \bar{l} \to \bar{m}$. For $h' : \bar{k} \to \bar{l}$ we have:

$$\begin{array}{ll} & (h', (S_i)_i) \in (y_g \times \mathrm{id}_{N_J})^\sharp(\mu_{\bar{m}}(f))(\bar{k}) \\ \text{iff} & (gh', (S_i)_i) \in \mu_{\bar{m}}(f)(\bar{k}) \\ \text{iff} & \forall i(\, S_i \subseteq (fgh')^*(C_i)) \\ \text{iff} & (h', (S_i)_i) \in \mu_{\bar{l}}(fg)(\bar{k}) \end{array}$$

Next, let us prove that μ is mono. Suppose $\mu_{\bar{m}}(f) = \mu_{\bar{m}}(f')$ for $f, f' : \bar{m} \to \bar{n}$. Let j and j' be such that in our enumeration, $f = g_j$ and $f' = g_{j'}$. Now consider the pair $\xi = (\mathrm{id}_{\bar{m}}, (S_i)_i)$, where S_i is the empty sieve if $i \neq j$, and $S_j = \max(\bar{m})$. Then ξ is easily seen to be an element of $\mu_{\bar{m}}(f)(\bar{m})$, so it must also be an element of $\mu_{\bar{m}}(f')(\bar{m})$, which means that $f' \in C_j$. So $C_j \cap C_{j'} \neq \emptyset$. But this means that we must have a commutative square in $\mathbb{F}$:

$$\begin{tikzcd} \bar{l} \arrow[r] \arrow[d] & \bar{m} \arrow[d, "f'"] \\ \bar{m} \arrow[r, "f"'] & \bar{n} \end{tikzcd}$$

It is easy to conclude from this that $f = f'$. ∎

2.4 Application: a model for "every function from reals to reals is continuous"

In 1924, L.E.J. Brouwer published a paper: *Beweis, dass jede volle Funktion gleichmässig stetig ist* (Proof, that every total function is uniformly continuous), *Nederl. Akad. Wetensch. Proc.* **27**, pp.189–193. His lucubrations on intuitionistic mathematics had led him to the conclusion that every function from $\mathbb{R}$ to $\mathbb{R}$ must be continuous. Among present-day researchers of constructive mathematics, this statement is known as *Brouwer's Principle* (although die-hard intuitionists still refer to it as Brouwer's Theorem).

The principle can be made plausible in a number of ways; one is, to look at the reals from a computational point of view. If a computer, which can only deal with finite approximations of reals, computes a function, then for every required precision for $f(x)$ it must be able to approximate x closely enough and from there calculate $f(x)$ within the prescribed precision; this just means that f must be continuous.

In this section we shall show that the principle is *consistent* with higher-order intuitionistic type theory, by exhibiting a topos in which it holds, for the standard real numbers. In order to do this, we have of course to say what the "object of real numbers" in a topos is. That will be done in the course of the construction.

We shall work with a full subcategory $\mathbb{T}$ of the category Top of topological spaces and continuous functions. It doesn't really matter so much what $\mathbb{T}$ exactly is, but we require that:

- $\mathbb{T}$ is closed under finite products and open subspaces
- $\mathbb{T}$ contains the space $\mathbb{R}$ (with the euclidean topology)

We specify a Grothendieck topology on $\mathbb{T}$ by defining, for an object T of $\mathbb{T}$, that a sieve R on T covers T, if the set of open subsets U of T for which the inclusion $U \to T$ is in R, forms an open covering of T. It is easy to verify that this is a Grothendieck topology.

The first thing to note is that for this topology (we call it Cov), every representable presheaf is a sheaf, because it is a presheaf of (continuous) functions: given a compatible family $R \to y_T$ for R a covering sieve on X, this family contains maps $f_U : U \to T$ for every open U contained in a covering of X; and these maps agree on intersections, because we have a sieve. So they have a unique amalgamation to a continuous map $f : X \to T$, i.e. an element of $y_T(X)$.

Also for spaces S not necessarily in the category $\mathbb{T}$ we have sheaves $y_S = \mathrm{Cts}(-, S)$.

Recall that the Yoneda embedding preserves existing exponents in $\mathbb{T}$. This also extends to exponents which exist in Top but are not in $\mathbb{T}$. If T is a locally compact space, then for any space X we have an exponent X^T in Top: it is the set of continuous functions $T \to X$, equipped with the compact-open topology (a subbase for this topology is given by the sets $\mathcal{C}(C, U)$ of those continuous functions that map C into U, for a compact subset C of T and an open subset U of X). Thus, even if X is not an object of $\mathbb{T}$, we still have in $\mathrm{Sh}(\mathbb{T}, \mathrm{Cov})$:

$$y_{X^T} \simeq (y_X)^{(Y_T)}$$

Exercise 29 Prove this fact.

From now on, we shall denote the category $\mathrm{Sh}(\mathbb{T}, \mathrm{Cov})$ by $\mathcal{T}$.
Notation: in this section we shall dispense with all subscripts $(\cdot)_J$, since we shall only work in $\mathcal{T}$. So, N denotes the *sheaf* of natural numbers, $\mathcal{P}(X)$ is the power *sheaf* of X, $\Vdash$ refers to forcing in sheaves, etc.

The natural numbers are given by the constant sheaf N, the $\mathbb{N}$-fold coproduct of copies of 1. The rational numbers are formed as a quotient of $N \times N$ by an equivalence relation which can be defined in a quantifier-free way, and hence is also a constant sheaf; therefore the object of rational numbers Q is the constant sheaf on the classical rational numbers $\mathbb{Q}$, and therefore the $\mathbb{Q}$-fold coproduct of copies of 1.

Proposition 2.26 *In $\mathcal{T}$, N and Q are isomorphic to the representable sheaves $y_{\mathbb{N}}$, $y_{\mathbb{Q}}$ respectively, where $\mathbb{N}$ and $\mathbb{Q}$ are endowed with the discrete topology.*

Proof. We shall do this for N; the proof for Q is similar. An element of $y_{\mathbb{N}}(X)$ is a continuous function from X to the discrete space $\mathbb{N}$; this is the same thing as an open covering $\{U_n \mid n \in \mathbb{N}\}$ of pairwise disjoint sets; which in turn is the same thing as an (equivalence class of an) $\mathbb{N}$-indexed collection $\{R_n \mid n \in \mathbb{N}\}$ of sieves on X such that whenever for $n \neq m$, $f : Y \to X$ is in $R_n \cap R_m$, $Y = \emptyset$; and moreover the sieve $\bigcup_n R_n$ covers X. But that last thing is just an element of $(\coprod_n 1)(X)$. ∎

Under this isomorphism, the order on N and Q corresponds to the pointwise ordering on functions.

Exercise 30 Show that in $\mathcal{T}$, the objects N and Q are linearly ordered, that is: for every space X in $\mathbb{T}$, $X \Vdash \forall rs \in Q\,(r < s \vee r = r \vee s < r)$.

We now construct the *object of Dedekind reals* R_d. Just as in the classical definition, a real number is a Dedekind cut of rational numbers, that is: a pair (L, R) of subsets of Q satisfying:

i) $\forall q \in Q \neg(q \in L \wedge q \in R)$

ii) $\exists q(q \in L) \wedge \exists r(r \in R)$

iii) $\forall qr(q < r \wedge r \in L \to q \in L) \ \wedge \ \forall st(s < t \wedge s \in R \to t \in R)$

iv) $\forall q \in L \exists r(q < r \wedge r \in L) \ \wedge \ \forall s \in R \exists t(t < s \wedge t \in R)$

v) $\forall qr(q < r \to q \in L \vee r \in R)$

Write $\mathrm{Cut}(L, R)$ for the conjunction of these formulas. So the object of reals R_d is the subsheaf of $\mathcal{P}(Q) \times \mathcal{P}(Q)$ given by:

$$R_d(X) \ = \ \{(L, R) \in (\mathcal{P}(Q) \times \mathcal{P}(Q))(X) \mid X \Vdash \mathrm{Cut}(L, R)\}$$

This is always a sheaf, by theorem 2.20ii).

Proposition 2.27 *The sheaf R_d is isomorphic to the representable sheaf $y_{\mathbb{R}}$.*

Proof. Let W be an object of $\mathbb{T}$ and $(L, R) \in R_d(W)$. Then L and R are subsheaves of $y_W \times Q$, which is isomorphic to $y_{W \times \mathbb{Q}}$. So both L and R consist of pairs of maps (α, p) with $\alpha : Y \to W$, $p : Y \to \mathbb{Q}$ continuous. Since L and R are sub*sheaves* we have: if $(\alpha, p) \in L(Y)$ then for any $f : V \to Y$, $(\alpha f, pf) \in L(V)$, and if $(\alpha \restriction V_i, p \restriction V_i) \in L(V_i)$ for an open cover $\{V_i\}_i$ of Y, then $(\alpha, p) \in L(Y)$ (and similar for R, of course).

Now for such $(L, R) \in \mathcal{P}(\mathbb{Q})(W) \times \mathcal{P}(\mathbb{Q})(W)$ we have $(L, R) \in R_d(W)$ if and only if $W \Vdash \mathrm{Cut}(L, R)$. We are now going to spell out what this means, and see that such (L, R) uniquely determine a continuous function $W \to \mathbb{R}$.

i)′ For $\beta : W' \to W$ and $q : W' \to \mathbb{Q}$, not both $(\beta, q) \in L(W')$ and $(\beta, q) \in R(W')$

ii)′ There is an open covering $\{W_i\}$ of W such that for each i there are $W_i \xrightarrow{l_i} \mathbb{Q}$ and $W_i \xrightarrow{r_i} \mathbb{Q}$ with $(W_i \to W, W_i \xrightarrow{l_i} \mathbb{Q}) \in L(W_i)$, and $(W_i \to W, W_i \xrightarrow{r_i} \mathbb{Q}) \in R(W_i)$

iii)′ For any map $\beta : W' \to W$ and any $q, r : W' \to \mathbb{Q}$: if $(\beta, r) \in L(W)$ and $q(x) < r(x)$ for all $x \in W'$, then $(\beta, q) \in L(W')$, and similar for R

iv)′ For any $\beta : W' \to W$ ad $q : W' \to \mathbb{Q}$: if $(\beta, q) \in L(W')$ there is an open covering $\{W'_i\}$ of W', and maps $r_i : W'_i \to \mathbb{Q}$ such that $(\beta \restriction W'_i, r_i) \in L(W'_i)$, and $r_i(x) > q(x)$ for all $x \in W'_i$. And similar for R

v)′ For any $\beta : W' \to W$ and $q, r : W' \to \mathbb{Q}$ satisfying $q(x) < r(x)$ for all $x \in W'$, there is an open covering $\{W'_i\}$ of W' such that for each i, either $(\beta \restriction W'_i, q \restriction W'_i) \in L(W'_i)$ or $(\beta \restriction W'_i, q \restriction W'_i) \in R(W'_i)$.

Let $\hat{q} : W \to \mathbb{Q}$ be the constant function with value q. For every $x \in W$ we define:

$$\begin{array}{rcl} L_x & = & \{q \in \mathbb{Q} \,|\, \exists \text{open } V \subseteq W (x \in V \wedge (V \to W, \hat{q} \restriction V) \in L(V))\} \\ R_x & = & \{q \in \mathbb{Q} \,|\, \exists \text{open } V \subseteq W (x \in V \wedge (V \to W, \hat{q} \restriction V) \in R(V))\} \end{array}$$

Then you should verify that (L_x, R_x) form a Dedekind cut in Set, hence determine a real number $f_{L,R}(x)$.

By definition of L_x and R_x, if q, r are rational numbers then $q < f_{L,R}(x) < r$ holds if and only if $q \in L_x$ and $r \in R_x$; so the preimage of the open interval (q, r) under $f_{L,R}$ is open; that is, $f_{L,R}$ is continuous. We have therefore defined a map $(L, R) \mapsto f_{L,R} : R_d(W) \to y_{\mathbb{R}}(W)$. It is easy to verify that this gives a map of sheaves: $R_d \to y_{\mathbb{R}}$.

For the other direction, if $f : W \to \mathbb{R}$ is continuous, one defines subsheaves L_f, R_f of $y_{W \times \mathbb{Q}}$ as follows: for $\beta : W' \to W, p : W' \to \mathbb{Q}$ put

$$\begin{array}{lll} (\beta, p) \in L_f(W') & \text{iff} & \forall x \in W'(p(x) < f(\beta(x))) \\ (\beta, p) \in R_f(W') & \text{iff} & \forall x \in W'(p(x) > f(\beta(x))) \end{array}$$

We leave it to you to verify that then $W \Vdash \mathrm{Cut}(L_f, R_f)$ and that the two given operations between $y_{\mathbb{R}}(W)$ and $R_d(W)$ are inverse to each other. You should observe that every continuous function $f : W \to \mathbb{Q}$ is locally constant, as $\mathbb{Q}$ is discrete. ∎

Corollary 2.28 *The exonential $R_d^{R_d}$ is isomorphic to $y_{\mathbb{R}^{\mathbb{R}}}$, where $\mathbb{R}^{\mathbb{R}}$ is the set of continuous maps $\mathbb{R} \to \mathbb{R}$ with the compact-open topology.*

Proof. This follows at once from proposition 2.27, the observation that y preserves exponents, and the fact that $\mathbb{R}$ is locally compact. ∎

From the corollary we see at once that arrows $R_d \to R_d$ in $\mathcal{T}$ correspond bijectively to continuous functions $\mathbb{R} \to \mathbb{R}$, but this is not yet quite Brouwer's statement that all functions (defined, possibly, with extra parameters) from R_d to R_d are continuous. So we prove that now.

Theorem 2.29 $\mathcal{T} \Vdash$ *"All functions $R_d \to R_d$ are continuous"*

Proof. . In other words, we have to prove that the sentence

$$\begin{aligned}&\forall f \in R_d^{R_d} \forall x \in R_d \forall \epsilon \in R_d (\epsilon > 0 \to \exists \delta \in R_d (\delta > 0 \wedge \\ &\quad \forall y \in R_d (x - \delta < y < x + \delta \to f(x) - \epsilon < f(y) < f(x) + \epsilon)))\end{aligned}$$

is true in $\mathcal{T}$.

We can work in $y_{\mathbb{R}^{\mathbb{R}}}$ for $R_d^{R_d}$, so $R_d^{R_d}(W) = \mathrm{Cts}(W \times \mathbb{R}, \mathbb{R})$. Take $f \in R_d^{R_d}(W)$ and $a, \epsilon \in R_d(W)$ such that $W \Vdash \epsilon > 0$. So $f : W \times \mathbb{R} \to \mathbb{R}$, and $a, \epsilon : W \to \mathbb{R}$, $\epsilon(x) > 0$ for all $x \in W$. We have to show:

(∗) $W \Vdash \exists \delta \in R_d (\delta > 0 \wedge \forall y \in R_d (a - y < \delta < a + \delta \to$
$f(a) - \epsilon < f(y) < f(a) + \epsilon))$

Now f and ϵ are continuous, so for each $x \in W$ there is an open neighborhood $W_x \subseteq W$ of x, and a $\delta_x > 0$ such that for each $\xi \in W_x$ and $t \in (a(x) - \delta_x, a(x) + \delta_x)$:

(1) $|a(\xi) - a(x)| < \frac{1}{2}\delta_x$

(2) $|f(\xi, t) - f(\xi, a(x))| < \frac{1}{2}\epsilon(\xi)$

We claim:

$$W_x \Vdash \forall y (a - \frac{1}{2}\delta_x < y < a + \frac{1}{2}\delta_x \to f(a) - \epsilon < f(y) < f(a) + \epsilon)$$

Note that this establishes what we want to prove.

To prove the claim, choose $\beta : V \to W_x$, $b : V \to \mathbb{R}$ such that

$$V \Vdash a\beta - \frac{1}{2}\delta_x < b < a\beta + \frac{1}{2}\delta_x$$

Then for all $\zeta \in V$, $|a\beta(\zeta) - b(\zeta)| < \frac{1}{2}\delta_x$, so by (1),

$$|a(x) - b(\zeta)| < \delta_x$$

Therefore we can substitute $\beta\zeta$ for ξ, and $b(\zeta)$ for t in (2) to obtain

$$|f(\beta(\zeta), b(\zeta)) - f(x, a(x))| < \tfrac{1}{2}\epsilon\beta(\zeta)$$
and
$$|f(\beta(\zeta), a\beta(\zeta)) - f(x, a(x))| < \tfrac{1}{2}\epsilon\beta(\zeta)$$

We conclude that $|f(\beta(\zeta)), b(\zeta)) - f(\beta(\zeta), a\beta(\zeta))| < \epsilon\beta(\zeta)$. Hence,

$$V \Vdash (f\beta)(a\beta) - \epsilon\beta < (f\beta)(b) < (f\beta)(a\beta) + \epsilon\beta$$

which proves the claim and we are done. ∎

3 The Effective Topos

In this chapter we shall meet an example of a topos which is not Grothendieck. The construction of this topos was given by J.M.E. Hyland in 1982 (Hyland, *The effective topos*, in: Troelstra and Van Dalen (eds.), The L.E.J. Brouwer Centenary Symposium, North Holland 1982, pp.165–216).

In constructive mathematics, several types of models for various theories had been given with recursion-theoretic methods. The pioneer of this approach had been S.C. Kleene, who had given an interpretation of first-order intuitionistic arithmetic based on recursive functions (S.C. Kleene, *On the Interpretation of Intuitionistic Number Theory*, JSL **10** (1945), pp.109–124), which interpretation became known as "realizability". We shall see that there is a close connection between Hyland's topos and Kleene's work.

Let us define this topos, which we call $\mathcal{E}ff$, straight away. An *object* of $\mathcal{E}ff$ is a pair (X, R) where X is a set and $R : X \times X \to \mathcal{P}(\mathbb{N})$ is a function (we think of elements of $R(x, y)$ as numbers coding information to the effect that "$x = y$"). The function R has to satisfy the condition that there exist partial recursive functions σ and τ (for "symmetry" and "transitivity" respectively), such that the following hold:

- For $x, y \in X$ and $n \in R(x, y)$, $\sigma(n)\downarrow$ and $\sigma(n) \in R(y, x)$;
- For $x, y, z \in X$, $n \in R(x, y)$ and $m \in R(y, z)$, $\tau(n, m)\downarrow$ and $\tau(n, m) \in R(x, z)$.

Notation. We shall use a primitive recursive coding of pairs $\langle \cdot, \cdot \rangle$ and sequences; context will make clear when we regard a number n as a code of a pair, in which case we write n_1, n_2 for its components, so: $n = \langle n_1, n_2 \rangle$. Partial recursive function application may be written as $\varphi_e(x)$, $\{e\}(x)$ (the old, clumsy, Kleene notation), $e(x)$ or simply ex.

In the conditions on R for an object (X, R), we have not required "reflexivity". Elements of $R(x, x)$ (we shall also write $E(x)$ for this set) are thought of as witnesses for the "existence" of x.

Suppose (X, R) and $Y, S)$ are objects of $\mathcal{E}ff$. A "functional relation" from (X, R) to (Y, S) is a function $F : X \times Y \to \mathcal{P}(\mathbb{N})$, such that there exist partial recursive functions $\mathrm{ex}, \mathrm{st}, \mathrm{sv}, \mathrm{tl}$ with the following properties:

- For $n \in R(x', x)$, $m \in S(y, y')$ and $k \in F(x, y)$, $\mathrm{ex}(n, m, k)\downarrow$ and $\mathrm{ex}(n, m, k) \in F(x', y')$ (The function ex testifies that F is "extensional", i.e. respects the equality relations);

- For $n \in F(x,y)$, $\mathrm{st}(n)_1 \in E(x)$ and $\mathrm{st}(n)_2 \in E(y)$ (If F relates x to y, then both these elements exist; F is “strict”);

- For $n \in F(x,y)$ and $m \in F(x,y')$, $\mathrm{sv}(n,m)\downarrow$ and $\mathrm{sv}(n,m) \in S(y,y')$ (“F is single-valued”);

- For $n \in E(x)$, $\mathrm{tl}(n)\downarrow$ and $\mathrm{tl}(n) \in \bigcup_{y\in Y} F(x,y)$ (“F is total”).

Functional relations from (X,R) to (Y,S) form a preorder: $F \leq G$ if for some partial recursive function e, whenever $n \in F(x,y)$ then $e(n) \in G(x,y)$ (for all x,y).

A *morphism* $(X,R) \to (Y,S)$ in $\mathcal{E}ff$ is now an equivalence class of functional relations from (X,R) to (Y,S), under the equivalence relation $F \leq G \wedge G \leq F$.

Exercise 31 Show that for functional relations F and G in fact, one inequality $F \leq G$ implies the other.

We should define how such morphisms can be composed, and what identity morphisms are.

We define: if F represents a morphism $f : (X,R) \to (Y,S)$ and G represents a morphism $g : (Y,S) \to (Z,T)$, then the composition gf is the arrow represented by

$$G \circ F(x,z) = \bigcup_{y\in Y} \{\langle a,b\rangle \mid a \in F(x,y) \wedge b \in G(y,z)\}$$

Exercise 32 Show that this is well-defined. That is: show that $G \circ F$ is indeed a functional relation if F and G are; and moreover show that the equivalence class of $G \circ F$ does not depend on the choice of representatives F and G.

For every object (X,R), R itself is a functional relation from (X,R) to itself.

Exercise 33 Show that the composition as just defined, is associative. Show also, that for each (X,R), the arrow represnted by R is an identity on (X,R).

It is very important to do these two exercises, in order to obtain some understanding of the category $\mathcal{E}ff$.

3.1 Some subcategories and functors

The "constant objects" functor $\nabla : \mathrm{Set} \to \mathcal{E}\!\mathit{ff}$ is defined as follows: $\nabla(X) = (X, R)$ with $R(x, x') = \begin{cases} \mathbb{N} & \text{if } x = x' \\ \emptyset & \text{else} \end{cases}$. For a function $f : X \to Y$, the arrow $\nabla(f)$ is represented by the functional relation $\nabla(f)(x, y) = \begin{cases} \mathbb{N} & \text{if } f(x) = y \\ \emptyset & \text{else} \end{cases}$.

Exercise 34 Check that ∇ is a well-defined functor. Also show the following important fact: ∇ is full and faithful.

The category $\mathbb{P}$ ($\mathbb{P}$ for "projective"; this will be explained later) has as objects pairs (X, α) where X is a set and $\alpha : X \to \mathbb{N}$ is a function. A morphism $(X, \alpha) \to (Y, \beta)$ is a function $f : X \to Y$ such that there exists some partial recursive function e such that for every $x \in X$, $e(\alpha(x))\downarrow$ and $e(\alpha(x)) = \beta(f(x))$. Obviously, $\mathbb{P}$ is a category.

There is a functor $\pi : \mathbb{P} \to \mathcal{E}\!\mathit{ff}$ given on objects by $\pi(X, \alpha) = (X, R)$ where $R(x, y) = \{\alpha(x) \mid x = y\}$.

Exercise 35 Describe yourself the action of π on arrows. Show that π is full and faithful.

A third category we consider is the category $\mathbb{S}$ (for "separated"). Objects are pairs (X, E) where $E(x)$ is a *nonempty* subset of $\mathbb{N}$. A morphism $(X, E) \to (Y, E')$ is a function $f : X \to Y$ such that there is a partial recursive function e with the property that for any $x \in X$ and any $n \in E(x)$, $e(n)\downarrow$ and $e(n) \in E'(f(x))$. The partial function e is said to "track" the function f. Objects of $\mathbb{S}$ are also often called "assemblies".

There is also a full and faithful functor from $\mathbb{S}$ to $\mathcal{E}\!\mathit{ff}$ which sends (X, E) to the object (X, R) where $R(x, x') = \{n \mid n \in E(x) \text{ and } x = x'\}$.

Exercise 36 Do the same as in the previous exercise, for the functor $\mathbb{S} \to \mathcal{E}\!\mathit{ff}$.

Clearly, the category $\mathbb{P}$ can be seen as a subcategory of $\mathbb{S}$: send (X, α) to (X, E_α) where $E_\alpha(x) = \{\alpha(x)\}$. Objects from $\mathbb{P}$ are called "partitioned assemblies".

We also have an embedding from Set into $\mathbb{P}$ which sends a set X to the $\mathbb{P}$-object $(X, 0_X)$ where 0_X is the function on X which is constant zero.

Exercise 37 Show that the following diagram of functors

$$\begin{tikzcd} \mathrm{Set} \arrow[r] \arrow[dr, "\nabla"'] & \mathbb{P} \arrow[r] \arrow[d, "\pi"'] & \mathbb{S} \arrow[dl] \\ & \mathcal{E}\!\mathit{ff} & \end{tikzcd}$$

commutes up to isomorphism.

For any locally small category $\mathcal{E}$ with terminal object 1 there is the *global sections* functor, usually denoted Γ from $\mathcal{E}$ to Set given by $\Gamma(X) = \mathcal{E}(1, X)$.

Exercise 38 a) Let $\{*\}$ be a one-element set, and $R(*, *) = \mathbb{N}$. Show that $(\{*\}, R)$ is a terminal object 1 in $\mathcal{E}ff$.

b) Prove that on $\mathcal{E}ff$, Γ can be described like this: $\Gamma(X, R) = X^+/\sim$, where $X^+ = \{x \in X \mid E(x) \neq \emptyset\}$, and $x \sim x'$ iff $R(x, x') \neq \emptyset$. Define also the effect of Γ on arrows.

c) Prove that Γ is *left* adjoint to $\nabla : \text{Set} \to \mathcal{E}ff$.

Here we see a contrast with the category of sheaves on a site. The 'constant sheaves functor' $\Delta : \text{Set} \to \text{Sh}(\mathcal{C}, \text{Cov})$ is not right adjoint, but *left* adjoint to Γ. Later we shall see, that ∇ does not have a right adjoint.

On the other hand, the embedding $\mathbb{S} \to \mathcal{E}ff$ does have a left adjoint, the 'separated reflection': it sends (X, R) to $(\Gamma(X, R), E)$ where $E([x]) = \bigcup_{x \in X} R(x, x)$.

Exercise 39 Again, define this functor on arrows and show that it is left adjoint to the embedding $\mathbb{S} \to \mathcal{E}ff$.

3.2 Structure of $\mathcal{E}ff$

3.2.1 Finite products

Binary products in $\mathcal{E}ff$ are given as follows: for objects (X, R) and (Y, S), the product $(X, R) \times (Y, S)$ is $(X \times Y, R \times S)$ where $R \times S$ is defined as follows:

$$(R \times S)((x, y), (x', y')) \;=\; R(x, x') \times S(y, y') \;=\; \{n \mid n_1 \in R(x, x') \text{ and } n_2 \in S(y, y')\}$$

Exercise 40 a) Check the universal property for products given above.

b) Check that also the categories $\mathbb{P}$ and $\mathbb{S}$ have products, and that the embeddings $\mathbb{P} \to \mathcal{E}ff$, $\mathbb{S} \to \mathcal{E}ff$ and $\text{Set} \to \mathcal{E}ff$ preserve products.

c) Check also that the functor $\Gamma : \mathcal{E}ff \to \text{Set}$ preserves products.

3.2.2 Exponentials

Exponentials are less straightforward. Given (X, R) and (Y, S), the exponential $(Y, S)^{(X,R)}$ is the object (Φ, T) where Φ is the set of functions $X \times Y \to \mathcal{P}(\mathbb{N})$. In order to define T, we first define, for $F \in \Phi$, the set $E(F)$ of elements which testify that F "exists"; in other words: is a function $(X, R) \to (Y, S)$. We define $E(F)$ as the set of all 4-tuples $n = \langle n_1, n_2, n_3, n_4\rangle$, such that n_1 is an index for a partial recursive function which testifies that F is extensional (see the definition of morphisms in $\mathcal{E}\!f\!f$), and similarly n_2, n_3, n_4 for respectively strict, single-valued and total.

Given $F, G \in \mathcal{P}(\mathbb{N})^{X\times Y}$, write $[F \leq G]$ for the set of indices n such that for all x, y and all $m \in F)(x, y)$, $nm\!\downarrow$ and $nm \in G(x, y)$. Now we put

$$T(F, G) \;=\; \{\langle k, l, m\rangle \mid k \in E(F), l \in E(G), m \in [F \leq G]\}$$

The definition doesn't look symmetrical. However, we have the following exercise.

Exercise 41 Show that given $\langle k, l, m\rangle \in T(F, G)$, we can recursively find from this an element of $[G \leq F]$.

Exercise 42 Check the universal property of $(Y, S)^{(X,R)}$ as just defined.

Exercise 43 In the definition of $(Y, S)^{(X,R)}$, the set Φ can be trimmed down a bit. Check that for *any* object (X, R) of $\mathcal{E}\!f\!f$, (X, R) is isomorphic to (X^+, R) where X^+ consists of those elements $x \in X$ for which $E(x) = R(x, x)$ is nonempty.

Now suppose that (X, E_X) and (Y, E_Y) are assemblies (seen as objects of $\mathcal{E}\!f\!f$ via the embedding $\mathbb{S} \to \mathcal{E}\!f\!f$), and $(Y, E_Y)^{(X,E_X)} = (\Phi, T)$ as defined above. Suppose $F \in \Phi$ such that $E(F) \neq \emptyset$. Then since the equality relation on Y gives the empty set for two distinct elements $y \neq y'$ of Y, we must have (using the totality and single-valuedness of F) that for every $x \in X$ there is exactly one $y \in Y$ such that $F(x, y)$ is nonempty. Moreover, another use of totality, combined with strictness, gives us a partial recursive function τ with the property that for each $x \in X$ and each $n \in E_X(x)$, $\tau(n)\!\downarrow$ and there is a $y \in Y$ such that $\tau(n) \in F(x, y)$. That is, F determines a unique function $f : X \to Y$ and any element of $E(F)$ determines a partial recursive function which tracks f. Hence, the exponential $(Y, E_Y)^{(X,E_X)}$ in $\mathcal{E}\!f\!f$ is isomorphic to the assembly $(\mathcal{T}, E)$ where $\mathcal{T}$ is the set of morphisms $(X, E_X) \to (Y, E_Y)$ in $\mathbb{S}$, and $E(f)$ is the set of indices of partial recursive functions which track f.

Exercise 44 Show that in fact, if (Y, E_Y) is an assembly then for *any* object (X, R) of $\mathcal{E}ff$, the exponential $(Y, E_Y)^{(X,R)}$ is isomorphic to an assembly.

Note that in general, there is more than one partial recursive function which tracks a given morphism in $\mathbb{S}$, and every partial recursive function has infinitely many indices. This means that, except for trivial cases, exponentials of assemblies are almost never objects of $\mathbb{P}$ (of course, if (Y, E_Y) is isomorphic to a set, under the embedding Set $\to \mathbb{S}$, then $(Y, E_Y)^{(X,E_X)}$ is isomorphic to an object of $\mathbb{P}$, in fact it is isomorphic to a set).

Exercise 45 Consider the adjoint pair $\Gamma \dashv \nabla : \mathcal{E}ff \to \mathrm{Set}$.

a) Show that for objects (X, R) of $\mathcal{E}ff$ and A of Set,

$$\Gamma(X, R) \times A \cong \Gamma((X, R) \times \nabla(A))$$

b) Conclude from part a) and the adjunction, that ∇ preserves exponentials.

3.2.3 Natural numbers object

We write N for the object $(\mathbb{N}, R)$ with $R(n, n') = \{n \mid n = n'\}$. Let $1 \xrightarrow{0} N$ be the arrow determined by the function $* \mapsto 0$, and $S : N \to N$ the arrow determined by the function $n \mapsto n + 1$. We claim that the diagram

$$1 \xrightarrow{0} N \xrightarrow{S} N$$

is a natural numbers object in $\mathcal{E}ff$.

To see this, suppose $1 \xrightarrow{a} (X, R) \xrightarrow{f} (X, R)$ is another such diagram, with a represented by $A \in \mathcal{P}(\mathbb{N})^{\{*\}\times X}$ and f by $F \in \mathcal{P}(\mathbb{N})^{X\times X}$. Since A is total and strict we find elements $x_0 \in X$ and $m, k_0 \in \mathbb{N}$ such that $m \in A(*, x_0)$ and $k_0 \in E(x_0)$. Let α and s be partial recursive functions which testify, respectively, that F is total and strict in its second argument, that is: for every $x \in X$ and $n \in E(x)$, there is $y \in X$ with $\alpha(n) \in F(x, y)$; and for every $x, y \in X$ and $n \in F(x, y)$, $\mathsf{s}(n) \in E(y)$. We now define a function

$$h(n) = (\xi(n), \eta(n)) : \mathbb{N} \to \{(x, k) \mid x \in X, k \in E(x)\}$$

as follows: let $h(0) = (x_0, k_0)$. If $h(n) = (x_n, k_n)$ has been defined, pick x_{n+1} such that $\alpha(k_n) \in F(x_n, x_{n+1})$, and let $h(n + 1) = (x_{n+1}, \mathsf{s}(\alpha(k_n)))$. Note that the function $\eta : \mathbb{N} \to \mathbb{N}$ is total recursive. We now have a functional relation $H : \mathbb{N}\times X \to \mathcal{P}(\mathbb{N})$ by $H(n, x) = \{\langle n, a\rangle \mid a \in R(x_n, x)\}$. You should check that H is strict, extensional and single-valued; and using η we find a partial recursive function which testifies that it is total. So H represents an arrow $h : N \to (X, R)$. We leave it to you to check that the diagram

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "a"'] & N \arrow[r, "S"] \arrow[d, "h"'] & N \arrow[d, "h"] \\
& (X,R) \arrow[r, "f"'] & (X,R)
\end{tikzcd}$$

commutes.

3.2.4 Finite Coproducts

Exercise 46 Suppose (X, R) satisfies $R(x, x) = \emptyset$ for all $x \in X$. Show that (X, R) is initial in $\mathcal{E}\!f\!f$.

For the disjoint sum (coproduct) of two objects (X, R) and (Y, S) of $\mathcal{E}\!f\!f$, let $X + Y$ be the disjoint sum in Set, and define $R \oplus S$ on $X + Y$ by: $n \in (R \oplus S)(u, v)$ iff either both $u, v \in X$ and $n = \langle 0, k\rangle$ with $k \in R(u, v)$, or both $u, v \in Y$ and $n = \langle 1, k\rangle$ with $k \in S(u, v)$.

Exercise 47 As usual, check the universal property.

Exercise 48 Show that the categories $\mathbb{P}$ and $\mathbb{S}$ also have coproducts, and that the embeddings into $\mathcal{E}\!f\!f$ preserve them.

Contrary to what you might think, the embedding $\nabla : \mathrm{Set} \to \mathcal{E}\!f\!f$ does *not* preserve coproducts. In fact we have:

Proposition 3.1 *Every arrow* $\nabla(\{0, 1\}) \to 1{+}1$ *is constant, that is: factors through one of the coproduct inclusions* $1 \to 1 + 1$.

Proof. You should check that $1 + 1$ is isomorphic to the object $(\{0, 1\}, R)$ with $R(0, 0) = \{0\}$, $R(1, 1) = \{1\}$ and $R(0, 1) = R(1, 0) = \emptyset$. Suppose $F : \{*\} \times \{0, 1\} \to \mathcal{P}(\mathbb{N})$ represents an arrow in $\mathcal{E}\!f\!f$. If t testifies that F is total and s that F is strict in its second argument, then for some $i \in \{0, 1\}$, $\mathsf{t}(0) \in F(*, i)$ hence $\mathsf{s}(\mathsf{t}(i)) = i$. Then the arrow represented by F factors through $i : 1 \to 1 + 1$. ∎

Proposition 3.1 proves also that ∇ cannot have a right adjoint (why?). Actually, the argument of 3.1 can be used more generally, as the following proposition says.

Proposition 3.2 *For any set* A*, any map* $\nabla(A) \to N$ *is constant.*

Exercise 49 Prove proposition 3.2.

3.2.5 Finite limits

We start with equalizers. Suppose we have two parallel arrows

$$\begin{tikzcd}(X,R) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,S)\end{tikzcd}$$

represented by F and G respectively. Define a map $R' : X \to \mathcal{P}(\mathbb{N})$ by:

$$\begin{array}{rcl} R'(x,x') & = & \{\langle n,m,k\rangle \mid n \in R(x,x') \text{ and for some } y \in Y, \\ & & \quad m \in F(x,y) \text{ and } k \in G(x,y)\} \end{array}$$

The elements m, k witness that "$f(x) = g(x)$".

We have an obvious map $(X,R') \xrightarrow{i} (X,R)$.

Proposition 3.3 *The diagram*

$$\begin{tikzcd}(X,R') \arrow[r, "i"] & (X,R) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,S)\end{tikzcd}$$

is an equalizer in $\mathcal{E}ff$.

Proof. Suppose $(Z,T) \xrightarrow{h} (X,R)$ is represented by H and such that $fh = gh$. That means that we have a partial recursive function φ such that for all $z \in Z$, $y \in Y$ and $n, m \in \mathbb{N}$, if for some $x \in X$ both $n \in H(z,x)$ and $m \in F(x,y)$ then $\varphi(n,m)\downarrow$ and for some $x' \in X$, $\varphi(n,m)_1 \in H(z,x')$ and $\varphi(n,m)_2 \in G(x',y)$.

But we can modify φ a bit, using the single-valuedness of H (from $n \in H(z,x)$ and $\varphi(n,m)_1 \in H(z,x')$ we obtain an element of $R(x,x')$) and the extensionality of H, to construct a partial recursive φ' such that: if $n \in H(x,z)$ and $m \in F(x,y)$ then $\varphi'(n,m)\downarrow$, $\varphi'(n,m)_1 \in H(z,x)$ and $\varphi'(n,m)_2 \in G(x,y)$.

Now using that H is total, we find, for each $k \in T(z,z)$ a pair $\langle a,b\rangle$ such that for some $x \in X$, $a \in H(z,x)$ and $b \in R'(x,x)$. That is, h factors through i.

We need to see that the factorization is unique. But if we had another pair $\langle a',b'\rangle$ such that for $x' \in X$, $a' \in H(z,x')$ and $b' \in R'(x',x)$ then we find from this an element in $R'(x',x)$. We leave it to you to convince yourself that this means that the factorization is indeed unique. ∎

Pullbacks are, in any category, constructed from products and equalizers, so here we content ourselves with giving an expression for the pullback of two arrows with common codomain.

Given

$$
\begin{tikzcd}
 & (X,R) \arrow[d, "f"] \\
(Y,S) \arrow[r, "g"'] & (Z,T)
\end{tikzcd}
$$

with f and g represented by F and G, construct an object $(X \times Y, U)$ as follows:

$$
\begin{array}{rcl}
U((x,y),(x',y')) & = & \{\langle a,b,c,d\rangle \mid a \in R(x,x'), b \in S(y,y') \text{ and for some} \\
 & & z \in Z,\ c \in F(x,z) \text{ and } d \in G(y,z)\}
\end{array}
$$

There are arrows $(X \times Y, U) \stackrel{\pi_1}{\to} (X,R)$ and $(X \times Y, U) \stackrel{\pi_2}{\to} (Y,S)$ represented by $P_1((x,y),x') = \{\langle a,b\rangle \mid a \in U((x,y),(x,y)), b \in R(x,x')\}$ and $P_2((x,y),y') = \{\langle a,b\rangle \mid a \in U((x,y),(x,y)), b \in S(y,y')\}$. Then

$$
\begin{tikzcd}
(X \times Y, U) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & (X,R) \arrow[d, "f"] \\
(Y,S) \arrow[r, "g"'] & (Z,T)
\end{tikzcd}
$$

is a pullback in $\mathcal{E}ff$. You should verify all this!

3.2.6 Monics and the subobject classifier

Before we treat general monomorphisms in $\mathcal{E}ff$, we have a look at the following structure. For a set X and a map $K : X \to \mathcal{P}(\mathbb{N})$ we think of the statement $n \in K(x)$ that "n witnesses, or realizes, thet x has property K". In view of this interpretation we shall also write $n \Vdash x \in K$ (n 'forces' $x \in K$) for this statement. So we think of K as a predicate on the set X. We can now extend this 'forcing' to more complicated predicates formed by logical connectives. Define:

$$
\begin{array}{lll}
n \Vdash x \in K \wedge L & \text{iff} & n_1 \Vdash x \in K \text{ and } n_2 \Vdash x \in L \\
n \Vdash x \in K \vee L & \text{iff} & (n_1 = 0 \text{ and } n_2 \Vdash x \in K) \text{ or } (n_1 = 1 \text{ and } n_2 \Vdash x \in L) \\
n \Vdash x \in K \to L & \text{iff} & \forall m (m \Vdash x \in K \Rightarrow nm{\downarrow} \text{ and } nm \Vdash x \in L) \\
n \Vdash x \in \neg K & \text{iff} & \forall m (m \nVdash x \in K)
\end{array}
$$

We can also define 'constants' $\bot$ and $\top$ by: $n \Vdash x \in \bot$ *never* holds, and $n \Vdash x \in \top$ always holds.

Exercise 50 Check that $n \Vdash x \in \neg K$ if and only if $n \Vdash x \in K \to \bot$.

We can consider the set $\mathcal{P}(\mathbb{N})^X$ of all functions $X \to \mathcal{P}(\mathbb{N})$ as preordered by: $K \leq L$ if for some n, $n \Vdash x \in K \to L$ for every $x \in X$. In this poset, the meet $K \wedge L$ of two elements is (up to isomorphism) the function which assigns to each x the set $\{n \,|\, n \Vdash x \in K \wedge L\}$; and similar for join, etc.

Now consider two sets X and Y. The operation $\mathcal{P}(\mathbb{N})^Y \to \mathcal{P}(\mathbb{N})^{X\times Y}$ given by composition with the projection $X \times Y \to Y$, is order-preserving, as is easy to check. Moreover, this operation has both a right and a left adjoint $\forall x$ and $\exists x$: for $K : X \times Y \to \mathcal{P}(\mathbb{N})$, define

$$\begin{array}{lll} n \Vdash y \in \forall x K & \text{iff} & \text{for all } x \in X,\, n \Vdash (x,y) \in K \\ n \Vdash y \in \exists x K & \text{iff} & \text{for some } x \in X,\, n \Vdash (x,y) \in K \end{array}$$

Exercise 51 Check the adjunctions.

Now consider an object (X,R) of $\mathcal{E}\!f\!f$. The set X is equipped with an 'equality' R, which is transitive and symmetric in the sense that with the $\Vdash$ notation just introduced we have:

$$\begin{array}{l} \text{For some } n,\, n \Vdash \forall x \forall y \forall z (R(x,y) \wedge R(y,z) \to R(x,z)) \\ \text{For some } n,\, n \Vdash \forall x \forall y (R(x,y) \to R(y,x)) \end{array}$$

We shall abbreviate the expression "for some n, $n \Vdash \cdots$" by "$\Vdash \cdots$".

Now given an object (X,R) and $K : X \to \mathcal{P}(\mathbb{N})$ we say that

$$\begin{array}{lll} K \text{ is strict (for } R) & \text{iff} & \Vdash \forall x (x \in K \to E(x)) \\ K \text{ is extensional (for } R) & \text{iff} & \Vdash \forall xy (x \in K \wedge R(x,y) \to y \in K) \end{array}$$

Such a K will be called a *strict relation* for (X,R).

We now return to the category $\mathcal{E}\!f\!f$.

Lemma 3.4 *Let $f : (X,R) \to (Y,S)$ be a morphism in $\mathcal{E}\!f\!f$ represented by $F : X \times Y \to \mathcal{P}(\mathbb{N})$. Then f is a monomorphism if and only if*

$$\Vdash \forall x x' y (F(x,y) \wedge F(x',y) \to R(x,x'))$$

Proof. Let (Z,T) be the object of $\mathcal{E}\!f\!f$ defined by $Z = X \times X$, and $n \in T((a,b),(a',b'))$ if and only if $n = \langle n_1, n_2, n_3 \rangle$ such that $n_1 \Vdash R(a,a')$, $n_2 \Vdash R(b,b')$ and for some $c \in Y$, $n_3 \Vdash F(a,c) \wedge F(b,c)$. Then we have a diagram

$$(Z,T) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} (X,R) \xrightarrow{f} (Y,S)$$

where π_1 is represented by $P_1((a,b),x) = E((a,b)) \wedge R(a,x)$ and π_2 by $P_1((a,b),x) = E((a,b)) \wedge R(b,x)$. You should check that $f\pi_1 = f\pi_2$; actually, we have a pullback diagram

$$\begin{tikzcd}
(Z,T) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & (X,R) \arrow[d, "f"] \\
(X,R) \arrow[r, "f"'] & (Y,S)
\end{tikzcd}$$

in $\mathcal{E}ff$. Now if f is mono, it follows that $\pi_1 = \pi_2$; it is easy to derive the expression in the lemma from this. Conversely, if the expression in the lemma holds, then it follows that $\pi_1 = \pi_2$ and that this is an isomorphism with inverse the map $(X,R) \to (Z,T)$ represented by $(x,(a,b)) \mapsto E(x) \wedge E(a,b) \wedge R(x,a) \wedge R(x,b)$. From this it follows that f is mono. ∎

Suppose (X,R) is an object of $\mathcal{E}ff$ and K is a strict relation for (X,R). Then we can form the object (X,R_K) where

$$R_K(x,x') \;=\; \{\langle a,b\rangle \mid a \in K(x) \wedge b \in R(x,x')\}$$

Exercise 52 Check the following statements:

a) (X,R_K) is a well-defined object of $\mathcal{E}ff$

b) R_K represents an arrow: $(X,R_K) \to (X,R)$

c) This arrow is monic.

Monics of this form are called *standard monos*.

Proposition 3.5 *Every mono is isomorphic to a standard mono.*

Proof. Suppose $F : X \times Y \to \mathcal{P}(\mathbb{N})$ represents a monomorphism $(X,R) \to (Y,S)$. Then the map $K : Y \to \mathcal{P}(\mathbb{N})$ given by $y \mapsto \{n \mid n \Vdash \exists x F(x,y)\}$ is a strict relation for (Y,S), and you should check that F actually induces an isomorphism from (X,R) to (Y,S_K). ∎

Corollary 3.6 *The lattice* $\mathrm{Sub}(Y,S)$ *of subobjects of* (Y,S) *in* $\mathcal{E}ff$ *is isomorphic to the lattice of equivalence classes of strict relations for* (Y,S)*, where we have* $K \sim L$ *iff* $\Vdash \forall y(E(y) \to (K(y) \leftrightarrow L(y)))$.

Now let us consider the object $\Omega = (\mathcal{P}(\mathbb{N}, \Leftrightarrow)$ where $A \Leftrightarrow B = \{n \mid n \Vdash A \leftrightarrow B\}$. Let us spell this out concretely: an $n \in A \Leftrightarrow B$ is a pair $\langle n_1, n_2\rangle$ of indices of partial recursive functions, such that for every $a \in A$, $n_1 a{\downarrow}$ and $n_1 a \in B$, and for every $b \in B$, $n_2 b{\downarrow}$ and $n_2 b \in A$.

Exercise 53 Prove that there is a number n such that $n \in \bigcap_{A\in\mathcal{P}(\mathbb{N})} A \Leftrightarrow (A \Leftrightarrow \mathbb{N}))$.

There is an arrow $t : 1 \to \Omega$, represented by the identity on $\mathcal{P}(\mathbb{N})$. This map is monic (why?).

Now suppose that $K : X \to \mathcal{P}(\mathbb{N})$ is a strict relation for (X, R). Then we have an arrow from (X, R) to $(\mathcal{P}(\mathbb{N}), \Leftrightarrow)$ represented by $(x, A) \mapsto \{n \,|\, n \Vdash E(x) \wedge (A \Leftrightarrow K(x))\}$. Let us call this map χ_K.

Proposition 3.7 *We have a pullback diagram*

$$\begin{tikzcd} (X, R_K) \arrow[r] \arrow[d] & 1 \arrow[d, "t"] \\ (X, R) \arrow[r, "\chi_K"'] & \Omega \end{tikzcd}$$

Moreover, the map χ_K is unique with this property.

The collection of strict relations for (X, R) is a Heyting algebra. The top element is the constant function $\mathbb{N}$, the bottom element is the constant function $\emptyset$, and $\vee$, $\wedge$ and $\to$ are as follows:

$$\begin{array}{lcl} (K \vee L)(x) & = & \{n \,|\, n \Vdash K(x) \vee L(x)\} \\ (K \wedge L)(x) & = & \{n \,|\, n \Vdash K(x) \wedge L(x)\} \\ (K \to L)(x) & = & \{n \,|\, n \Vdash E(x) \wedge (K(x) \to L(x))\} \end{array}$$

The *power object* $\mathcal{P}(X, R)$ of (X, R) in $\mathcal{E}ff$ is the object $(\mathcal{P}(\mathbb{N})^X, T)$ with

$$T(K, L) \;=\; \{n \,|\, n \Vdash \text{“}K \text{ is a strict relation”} \wedge \forall x(E(x) \to (K(x) \leftrightarrow L(x)))\}$$

Just as in the case of presheaves and sheaves before, if $f : (X, R) \to (Y, S)$ is a morphism, the operation $f^\sharp : \mathrm{Sub}(Y, S) \to \mathrm{Sub}(X, R)$ of pulling back along f commutes with the Heyting structure and has both a left and a right adjoint, $\exists_f$ and $\forall_f$ respectively. These maps are given as follows (we assume that f is represented by F:

$$\begin{array}{lcl} \exists_f(K)(y) & = & \{n \,|\, n \Vdash \exists x(K(x) \wedge F(x, y))\} \\ \forall_f(K)(y) & = & \{n \,|\, n \Vdash E(y) \wedge \forall x(F(x, y) \to K(x))\} \end{array}$$

Exercise 54 Suppose $F : X \times Y \to \mathcal{P}(\mathbb{N})$ represents an arrow $f : (X, R) \to (Y, S))$ in $\mathcal{E}ff$.

a) Prove that f is epi if and only if $\Vdash \forall y(E(y) \to \exists x F(x, y))$

b) Suppose f is epi. Define R' on X by

$$R'(x,x') \equiv \{n \mid n \Vdash \exists y(F(x,y) \wedge F(x',y))\}$$

Prove that (X,R') is an object of $\mathcal{E}\!f\!f$ and that $(X,R') \cong (Y,S)$.

c) Prove that in $\mathcal{E}\!f\!f$, every arrow factors as an epi followed by a mono.

d) Prove that in $\mathcal{E}\!f\!f$, every arrow which is both epi and mono is an isomorphism.

3.3 Intermezzo: interpretation of languages and theories in toposes

We start by looking at many-sorted first order logic. We have a collection of basic sorts $X, Y, X_i, \ldots$, and a rule: if $X_1, \ldots, X_n$ are sorts then so is $X_1 \times \cdots \times X_n$. We also have the terminal sort 1. For any sort X we have variables x^X of sort X, sufficiently many of them. Furthermore our language may contain:

- Relation symbols $R \subseteq X_1 \times \cdots \times X_n$ (we think of the relation symbol R together with its "arity", that is the sorts of its arguments);
- Function symbols $F : X_1 \times \cdots \times X_n \to Y$. We may also have constants of sort Y, which may be thought of as function symbols $c : 1 \to Y$. For any list of sorts $X_1, \ldots, X_n$ and any $1 \le i \le n$ we have a function symbol $\pi_i : X_1 \times \cdots \times X_n \to X_i$.

Terms (which are always of a specified sort) are built up from variables, constants and function symbols as usual, with one additional term-forming operation: if $X_1, \ldots, X_n$ is a list of sorts and t_i is a term of sort X_i for $1 \le i \le n$, then we have a term

$$\langle t_1, \ldots, t_n \rangle$$

of sort $X_1 \times \cdots \times X_n$.

When we write $t(x_1, \ldots, x_n)$ for a term t (of sort Y say, and the x_i variables of sort X_i), we always understand this as meaning that the list $x_1, \ldots, x_n$ *contains all variables actually occurring in* t (but the list may be longer).

We shall now say what an *interpretation* $[\![\cdot]\!]$ of such a language in a topos $\mathcal{E}$ is. First, for any basic sort X we have an object $[\![X]\!]$ of $\mathcal{E}$. This then extends to an object $[\![X]\!]$ for any sort X by the clauses:

- $[\![1]\!]$ is a terminal object of $\mathcal{E}$;
- If $X_1, \ldots, X_n$ is a list of sorts such that $[\![X_1]\!], \ldots, [\![X_n]\!]$ have been defined, choose a product

$$[\![X_1]\!] \times \cdots \times [\![X_n]\!]$$

  and put $[\![X_1 \times \cdots \times X_n]\!] = [\![X_1]\!] \times \cdots \times [\![X_n]\!]$.

In the interpretation of sorts there is some choice of terminal object and products, but once the interpretation of the basic sorts has been fixed, the rest is determined up to isomorphism. This situation will recur later on.

Secondly, for any function symbol $F : X_1 \times X_n \to Y$, $[\![F]\!]$ is to be a morphism in $\mathcal{E}$ from $[\![X_1 \times \cdots \times X_n]\!]$ to $[\![Y]\!]$. For the special function symbols π_i we require that they be interpreted as the corresponding projections (which are part of the product structure of $[\![X_1 \times \cdots \times X_n]\!]$).

Thirdly, for a relation symbol $R \subseteq X_1 \times \cdots \times X_n$, $[\![R]\!]$ will be a subobject of $[\![X_1 \times \cdots \times X_n]\!]$.

Suppose we are given an interpretation $[\![\cdot]\!]$ of a language in a topos $\mathcal{E}$. For any term $t(x_1, \ldots, x_n)$ we have then a morphism $[\![t]\!] : [\![X_1 \times \cdots \times X_n]\!]$, defined as follows:

- If t is a variable x^{X_i}, $[\![t]\!]$ is the i-th projection;
- If $t_1, \ldots, t_n$ are terms of sorts $Y_1, \ldots, Y_n$ respectively, and $x_1^{X_1}, \ldots, x_m^{X_m}$ is a list of variables containing all the variables in the t_i, and by induction hypothesis we have

$$[\![t_i]\!] : [\![X_1 \times \cdots \times X_m]\!] \to [\![Y_i]\!]$$

  then if $F : Y_1 \times \cdots \times Y_n \to Z$ is a function symbol, we let $[\![F(t_1, \ldots, t_n)]\!]$ be the composite

$$[\![X_1 \times \cdots \times X_m]\!] \overset{\langle [\![t_i]\!] \rangle_{i=1}^n}{\longrightarrow} [\![Y_1 \times \cdots \times Y_n]\!] \overset{[\![F]\!]}{\to} [\![Z]\!]$$

- If $t_1, \ldots, t_n$, $Y_1, \ldots, Y_n$ and $x_1^{X_1}, \ldots, x_m^{X_m}$ are as in the previous item, then

$$[\![\langle t_1, \ldots, t_n \rangle]\!]$$

  is the unique map from $[\![X_1 \times \cdots \times X_m]\!]$ to $[\![Y_1 \times \cdots \times Y_n]\!]$ such that its composition with the i-th projection is $[\![t_i]\!]$, for each i.

Exercise 55 Write out t where $t = f(f(x,y), g(x,u,c))$ where x, y, u are variables, c a constant and f and g function symbols of appropriate arities.

Next, we extend the interpretation to formulas. Formulas are built up from: atomic formulas (which are of the form $R(t_1, \ldots, t_n)$ for a relation symbol R and $t_1, \ldots, t_n$ of the right sorts, or equalities $t = s$ when t and s have the same sort) by the logical connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$ and the quantifiers $\forall x^X$ and $\exists x^X$. As with terms, by $\varphi(x_1, \ldots, x_n)$ we understand that all *free* variables of φ are in the list $x_1, \ldots, x_n$.

Given $\varphi(x_1^{X_1}, \ldots, x_n^{X_n})$, $[\![\varphi]\!]$ will be a *subobject* of $[\![X_1 \times \cdots \times X_n]\!]$. We start by giving $[\![\varphi]\!]$ for atomic φ.

- Suppose $\varphi = R(t_1, \ldots, t_k)$ where t_i are terms of sort Y_i. By the interpretation of terms we have

$$[\![t_i(x_1, \ldots, x_n)]\!] : [\![X_1 \times \cdots \times X_n]\!] \rightarrow [\![Y_i]\!]$$

  and by definition of interpretation we have a subobject $[\![R]\!]$ of $[\![Y_1 \times \cdots \times Y_k]\!]$. We let $[\![\varphi(x_1, \ldots, x_n)]\!]$ be the subobject of $[\![X_1 \times \cdots \times X_n]\!]$ given by the pullback

$$\begin{tikzcd}
{[\![\varphi]\!]} \arrow[r] \arrow[d] & {[\![R]\!]} \arrow[d] \\
{[\![X_1 \times \cdots \times X_n]\!]} \arrow[r, "{\langle [\![t_1]\!], \ldots, [\![t_k]\!] \rangle}"'] & {[\![Y_1 \times \cdots \times Y_k]\!]}
\end{tikzcd}$$

- Suppose $\varphi = (t = s)(x_1, \ldots, x_n)$. We have $[\![s]\!], [\![t]\!] : [\![X_1 \times \cdots \times X_n]\!] \rightarrow [\![Y]\!]$, where Y is the sort of t and s. We let $[\![t = s]\!]$ be the equalizer of $[\![s]\!]$ and $[\![t]\!]$.

For the propositional connectives $\wedge$, $\vee$, $\neg$ and $\rightarrow$, we need that $\mathrm{Sub}(X)$ is a Heyting algebra. This is the case in any topos $\mathcal{E}$. We state the following facts without proof:

- Every topos is a regular category;
- Every topos has finite colimits, and the initial object is strict;
- In every topos, every operation $\phi^\sharp : \mathrm{Sub}(Y) \rightarrow \mathrm{Sub}(X)$ of pulling back along $\phi : X \rightarrow Y$ has a right adjoint $\forall_\phi$ (note that a left adjoint exists since $\mathcal{E}$ is regular).

We have then, that the meet of two subobjects is given by a simple pullback. The join of two subobjects $A \xrightarrow{m} X$ and $B \xrightarrow{n} X$ is the image of the arrow $[m,n] : A+B \to X$. Implication $A \to B$ (for $A \xrightarrow{m} X$ and $B \xrightarrow{n} X$) is $\forall_m(A \wedge B)$. $\neg A$ is $A \to \bot$ where $\bot$ is the initial subobject $0 \to X$.

As for the adjoints $\exists_\phi$ and $\forall_\phi$, we have the following fact: if

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "\phi'"'] & Y' \end{tikzcd}$$

is a pullback square, then for any subobject A of Y, we have that $\exists_g(\phi^\sharp(A)) = \phi'^\sharp(\exists_f(A))$ as subobjects of X'; for the maps $\exists_g$ and $\exists_f$ are given by images, and these are stable under pullback since $\mathcal{E}$ is regular. By mirroring the pullback diagram we get, for a subobject B of X', that $\exists_\phi(g^\sharp(B)) = f^\sharp(\exists_{\phi'}(B))$. Now taking right adjoints we see that for $A \in \mathrm{Sub}(Y)$, $\forall_g(\phi^\sharp(A)) = \phi'^\sharp(\forall_f(A))$. This fact is called the *Beck-Chevalley condition*.

We now return to the definition of $[\![\, \varphi(x_1,\ldots,x_n) \,]\!]$. Suppose $\varphi(x_1,\ldots,x_n) \equiv \exists y^Y \psi(y,x_1,\ldots,x_n)$ and by inductive hypothesis we have defined ψ as a subobject of $[\![\, Y \times X_1 \times \cdots \times X_n \,]\!]$. Then $[\![\, \varphi \,]\!]$ is $\exists_\pi([\![\, \psi \,]\!])$, where π is the projection from $[\![\, Y \times X_1 \times \cdots \times X_n \,]\!]$ to $[\![\, X_1 \times \cdots \times X_n \,]\!]$. The definition of $[\![\, \forall y^Y \psi \,]\!]$ is quite similar and uses $\forall_\pi$.

The following lemma holds.

Lemma 3.8 *Suppose $\varphi(x^X, x_1, \ldots, x_n)$ is a formula and $t(y)$ is a term of sort X with a variable of sort Y. Suppose that t is substitutable for x in φ. Then we have a pullback diagram*

$$\begin{tikzcd} {[\![\, \varphi[t/x] \,]\!]} \arrow[r] \arrow[d] & {[\![\, \varphi \,]\!]} \arrow[d] \\ {[\![\, Y \times X_1 \times \cdots \times X_n \,]\!]} \arrow[r, "{[\![\, t(y,\vec{x}) \,]\!]}"'] & {[\![\, X_1 \times \cdots \times X_n \,]\!]} \end{tikzcd}$$

The proof is a straightforward induction on φ, where for the quantifier steps one uses the Beck-Chevalley condition.

Summing up, we have defined an interpretation $[\![\, \varphi \,]\!]$ as subobject of $[\![\, X_1 \times \cdots \times X_n \,]\!]$, for any sequence of variables $x_1^{X_1}, \ldots, x_n^{X_n}$ which contains all the free variables of φ; starting from a basic interpretation of the sorts, the relation symbols and the function symbols. We now say that $\varphi(x_1,\ldots,x_n)$ is *true* under this interpretation, if $[\![\, \varphi \,]\!]$ is the maximal subobject of $[\![\, X_1 \times \cdots \times X_n \,]\!]$.

Exercise 56 Prove that this does not depend on the list $x_1, \ldots, x_n$; in fact, if $x_{i_1}, \ldots, x_{i_k}$ is the sublist of all the free variables of φ, then $\varphi(x_1, \ldots, x_n)$ is true if and only if $\varphi(x_{i_1}, \ldots, x_{i_k})$ is true.

We say that the formula is *valid* in $\mathcal{E}$, if φ is true for any interpretation $[\![\cdot]\!]$ of its language in $\mathcal{E}$.

Theorem 3.9 *Every theorem of intuitionistic predicate logic is valid in every topos $\mathcal{E}$.*

The interpretation of first-order logic in a topos $\mathcal{E}$ can be extended to higher-order type theory. We extend the collection of sorts by the following:

- We have a basic sort N (for natural numbers);
- Given two sorts X and Y, we have a sort Y^X of 'functions $Y \to X$';
- For each sort X we have a power sort $\mathcal{P}(X)$.

Together with a few new relation and function symbols, as well as another term forming operation:

- We have a constant 0 of type N and a function symbol $S : N \to N$;
- We have a function symbol $\mathsf{ev}_{X,Y} : Y^X \times X \to Y$ and for any term t of sort Y with *free* variables $x^X, x_1, \ldots, x_n$, we have a term $\lambda x^X.t$ of sort Y^X with free variables $x_1, \ldots, x_n$ (the variable x^X is *bound* in $\lambda x^X.t$);
- We have a relation symbol $\in_X \subseteq X \times \mathcal{P}(X)$;
- For any formula φ with free variables $x^X, x_1, \ldots, x_n$ we have a term $\{x^X : \varphi\}$ of sort $\mathcal{P}(X)$ and with free variables $x_1, \ldots, x_n$ (again, x^X is bound in $\{x^X : \varphi\}$);
- For any term s of sort X with free variables among $x_1, \ldots, x_n$ and any term t of sort X with free variables among $x^X, x_1, \ldots, x_n$ (where it is understood that x does not appear in the list $x_1, \ldots, x_n$) we have a term $\mathsf{Rec}[s; t]$ of sort X and free variables $x_1, \ldots, x_n, y^N$ where y^N is a fresh variable of sort N.

We then extend the interpretation $[\![\cdot]\!]$ in the following, obvious, way:

- $[\![N]\!]$ is a natural numbers object of $\mathcal{E}$, and $[\![0]\!]$ and $[\![S]\!]$ are the structure morphisms for the natural numbers object;

- $[\![Y^X]\!]$ is an exponential $[\![Y]\!]^{[\![X]\!]}$, and $[\![\mathsf{ev}_{X,Y}]\!]$ is to be the evaluation morphism which is part of the structure of an exponential. If t is a term of sort Y with free variables $x^X, x_1^{X_1}, \ldots, x_n^{X_n}$, and inductively t is interpretated as $[\![t]\!] : [\![X \times X_1 \times \cdots \times X_n]\!] \to [\![Y]\!]$, then $[\![\lambda x^X.t]\!] : [\![X_1 \times \cdots \times X_n]\!] \to [\![Y^X]\!]$ is to be the exponential transpose of $[\![t]\!]$;

- $[\![\mathcal{P}(X)]\!]$ is a chosen power object of $[\![X]\!]$, and $[\![\in_X]\!]$ is the canonical 'element relation' which is part of the structure of a power object;

- If φ has been interpreted as a subobject $[\![\varphi]\!]$ of $[\![X \times X_1 \times \cdots \times X_n]\!]$, then $[\![\{x^X : \varphi\}]\!] : [\![X_1 \times \cdots \times X_n]\!] \to [\![\mathcal{P}(X)]\!]$ is the unique map corresponding to $[\![\varphi]\!]$ according to the definition of a power object;

- If $[\![s]\!] : [\![X_1 \times \cdots \times X_n]\!] \to [\![X]\!]$ and $[\![t]\!] : [\![X \times X_1 \times \cdots \times X_n]\!] \to [\![X]\!]$ are interpretations of s and t, then $[\![\mathsf{Rec}[s;t]]\!] : [\![X_1 \times \cdots \times X_n \times N]\!] \to [\![X]\!]$ is the unique map corresponding to $[\![s]\!]$ and $[\![t]\!]$ given by the defining property of a natural numbers object in a cartesian closed category.

We can now write down a number of statements in our language which are easily seen to be valid.

Exercise 57 Convince yourself that the following statements are valid (where all variables and terms are of the appropriate sort, if not indicated):

a) $x = \langle \pi_1(x), \ldots, \pi_n(x) \rangle$

b) $\mathsf{ev}_{X,Y}(\lambda x^X.t, x') = t[x'/x]$

c) $\varphi[x'/x] \leftrightarrow x' \in_X \{x^X : \varphi\}$

d) $$\begin{array}{c} \mathsf{Rec}[s:t][0/y] = s \\ \mathsf{Rec}[s:t][S(y)/y] = t[\mathsf{Rec}[s:t]/y] \end{array}$$

e) $\forall x^X \exists ! y^Y \varphi \to \exists ! f^{Y^X} \forall x^X \varphi[\mathsf{ev}_{X,Y}(f,x)/y]$
Here the 'unique existence' quantifier $\exists!$ is an abbreviation: $\exists ! x\varphi$ is short for $\exists x \forall y(\varphi(y) \leftrightarrow y = x)$. The statement in this item is called the principle of 'unique choice'.

A very general way to describe inductively the truth of formulas under an interpretation, is the so-called *Kripke-Joyal semantics*. Suppose $\varphi(x_1, \ldots, x_n)$ is interpreted as $[\![\varphi]\!] \rightarrowtail [\![X_1 \times \cdots \times X_n]\!]$. For an object E of $\mathcal{E}$ and arrows $\alpha_i : E \to [\![X_i]\!]$ (the arrows α_i are called "generalized elements" of the $[\![X_i]\!]$), we write

$$E \Vdash \varphi(\alpha_1, \ldots, \alpha_n)$$

for the statement that the map $\langle\alpha_1,\ldots,\alpha_n\rangle : E \to [\![X_1\times\cdots\times X_n]\!]$ factors through the subobject $[\![\varphi]\!]$.

With this notation, one can give another inductive description of the truth of formulas. We shall just write α for $\alpha_1,\ldots,\alpha_n$ (this is actually no loss of generality). We have:

- $E \Vdash (t=s)(\alpha)$ iff $[\![t]\!]\circ\alpha = [\![s]\!]\circ\alpha$
- $E \Vdash (\varphi\wedge\psi)(\alpha)$ iff $E\Vdash\varphi(\alpha)$ and $E\Vdash\psi(\alpha)$
- $E \Vdash (\varphi\vee\psi)(\alpha)$ iff there are arrows $\beta_1 : E_1\to E$ and $\beta_2 : E_2\to E$ such that the map $[\beta_1,\beta_2] : E_1+E_2\to E$ is an epimorphism, and both $E_1\Vdash\varphi(\alpha\beta_1)$ and $E_2\Vdash\psi(\alpha\beta_2)$
- $E \Vdash (\varphi\to\psi)(\alpha)$ iff for every arrow $\beta : E'\to E$, if $E'\Vdash\varphi(\alpha\beta)$ then $E'\Vdash\psi(\alpha\beta)$
- $E\Vdash\bot(\alpha)$ iff E is initial in $\mathcal{E}$
- $E\Vdash\exists x^X\varphi(x,\alpha)$ iff there is an epimorphism $\beta : E'\to E$ and a map $\gamma : E'\to[\![X]\!]$ such that $E'\Vdash\varphi(\langle\gamma,\alpha\beta\rangle)$
- $E\Vdash\forall x^X\varphi$ iff for all $\beta : E'\to E$ and all $\gamma : E'\to[\![X]\!]$, $E'\Vdash\varphi(\gamma,\alpha\beta)$

Two important features of this definition are:
Monotonicity: if $E\Vdash\varphi(\alpha)$ then for any $\beta : E'\to E$ we have $E'\Vdash\varphi(\alpha\beta)$.
Local Character: if $\{E_i\stackrel{\beta_i}{\to}E\,|\,i\in I\}$ is an epimorphic family (meaning that $\forall i(f\beta_i=g\beta_i)$ implies $f=g$ for two parallel f,g) and for all i, $E_i\Vdash\varphi(\alpha\beta_i)$, then $E\Vdash\varphi(\alpha)$.

The forcing definition we gave for presheaves and sheaves is actually a form of the Kripke-Joyal semantics. For, by local character it suffices to look at $E\Vdash\varphi(\alpha)$ for representable E (every presheaf is a colimit of representables, hence covered by a sum of representables); and $\alpha : y_C\to[\![X]\!]$ corresponds uniquely to some $x\in[\![X]\!](C)$ by the Yoneda lemma.

In $\mathcal{E}\!f\!f$, we do not have 'representable presheaves' but we have something similar.

Proposition 3.10 *In $\mathcal{E}\!f\!f$, for every object (X,R) there is an object (P,S) of $\mathbb{P}$ and an epimorphism $(P,S)\to(X,R)$.*

Proof. Let $P=\{(x,n)\,|\,n\in E(x)\}$ (recall that $E(x)$ abbreviates $R(x,x)$). Define S by

$$S((x,n),x',n')) \;=\; \{n\,|\,x=x' \text{ and } n=n'\}$$

Let $F : P \times X \to \mathcal{P}(\mathbb{N})$ be defined as

$$F((x,n),x') \;=\; \{\langle n,k\rangle \mid k \in R(x,x')\}$$

The proof that F represents an epimorphism $(P,S) \to (X,R)$ is left to you. ∎

Exercise 58 Fill in the missing details of the proof of proposition 3.10, using exercise 54.

Proposition 3.11 *Every object of $\mathbb{P}$ is projective in $\mathcal{E}ff$.*

Proof. Suppose (P,S) is an object of $\mathbb{P}$ and F represents an epimorphism $(X,R) \to (P,S)$. We have to show that this has a section. Without loss of generality we may assume that for each $p \in P$, $E(p)$ is a singleton. By the characterization of epis in exercise 54 there is a number e such that for every $p \in P$ and every $n \in E(p)$, $en{\downarrow}$ and there is an $x \in X$ such that $en \in F(x,p)$. Using strictness of F we find for such x, $\mathsf{st}(en) \in E(x)$. By the axiom of choice, choose for each p an $x = f(p)$ such that $en \in F(f(p),p)$ and $\mathsf{st}(en) \in E(f(p))$. Define $G : P \times X \to \mathcal{P}(\mathbb{N})$ by

$$G(p,x) \;=\; R(x,f(p))$$

Then G represents a section of F, as is left for you to check. ∎

From proposition 3.10 it follows that if we want to do Kripke-Joyal semantics for $\mathcal{E}ff$, we can restrict the definition to $P \Vdash \cdots$ for partitioned assemblies P. Moreover, from proposition 3.11 it follows that the clauses for disjunction and existential quantification can be simplified to:

- $P \Vdash (\varphi \vee \psi)(\alpha)$ iff P is a coproduct $A + B$ and $\alpha = [\alpha_1, \alpha_2]$, and $A \Vdash \varphi(\alpha_1)$ and $B \Vdash \psi(\alpha_2)$
- $P \Vdash \exists x^X \varphi(\alpha)$ iff there is a map $\beta : P \to [\![\, X \,]\!]$ such that $P \Vdash \varphi(\langle \beta, \alpha\rangle)$

3.4 Elements of the logic of $\mathcal{E}ff$

We continue the study of $\mathcal{E}ff$. Recall the characterization of power objects from section 3.2.6. In every topos, any arrow $f : X \to Y$ gives rise to a map $\exists_f : \mathcal{P}(X) \to \mathcal{P}(Y)$: this is the map which corresponds to the subobject of $\mathcal{P}(X) \times Y$ which is the image of the subobject $\in_X$ of $\mathcal{P}(X) \times X$ under the map $\mathrm{id}_{\mathcal{P}(X)} \times f$. In logical terms, we have:

$$y \in \exists_f(A) \leftrightarrow \exists x(x \in A \wedge f(x) = y)$$

In $\mathcal{E}\!f\!f$, the map $\exists_f$ is represented by

$$(K,L) \mapsto E(K) \wedge \{n \mid n \Vdash \forall y^Y (y \in L \leftrightarrow \exists x^X (F(x,y) \wedge x \in K))\}$$

when F represents f. It is easy to see, and in fact valid in general, that if f is epi, so is $\exists_f$.

We call an object X of $\mathcal{E}\!f\!f$ *uniform* iff there is an epi $\nabla(A) \to X$.

Exercise 59 Prove that the following two assertions are equivalent for an object X of $\mathcal{E}\!f\!f$:

a) X is uniform

b) X is isomorphic to an object (Y,R) which has the property that $\bigcap_{y \in Y} E(y) \neq \emptyset$

The following proposition generalizes 3.2.

Proposition 3.12 *If X is uniform, then the following statement is true in $\mathcal{E}\!f\!f$:*

$$\forall A : \mathcal{P}(X \times N)[\forall x : X \exists n : N((x,n) \in A) \to \exists n : N \forall x : X((x,n) \in A)]$$

Exercise 60 Prove this.

Proposition 3.13 *In $\mathcal{E}\!f\!f$, every power object is uniform.*

Proof. First we prove this for power objects of assemblies. Let (X,R) be an object of $\mathbb{S}$; so we have a function $E : X \to \mathcal{P}(\mathbb{N})$ such that $R(x,y)$ is given by: $R(x,y) = E(x)$ if $x = y$ and $\emptyset$ otherwise. Consider $K : X \to \mathcal{P}(\mathbb{N})$. Then K is always extensional for R; to make it strict, let $K'(x) = \{\langle k,n\rangle \mid k \in K(x), n \in E(x)\}$. Check that this defines a map: $\nabla(\mathcal{P}(\mathbb{N})^X) \to \mathcal{P}(X)$ and that this map is epi.

Now for every X there is an epi $f : P \to X$ for some partitioned assembly P; then, as we have seen, $\exists_f$ defines an epi $\mathcal{P}(P) \to \mathcal{P}(X)$. Since $\mathcal{P}(P)$ is uniform, $\mathcal{P}(X)$ is. ∎

As a special application of propositions 3.12 and 3.13 we have that the following principle of higher-order arithmetic is true in $\mathcal{E}\!f\!f$:

$$\forall F \subseteq \mathcal{P}(N) \times N[\forall U \subseteq N \exists n((U,n) \in F) \to \exists n \forall U((U,n) \in F)]$$

which is called the *Uniformity Principle*.

We now give an inductive definition for truth of a first order formula (relative to an interpretation $[\![\cdot]\!]$ in $\mathcal{E}\!f\!f$). For simplicity we assume φ is of the form $\varphi(a)$; we define the inductive rules for "$n \Vdash \varphi(a)$".

- $n \Vdash \varphi(a) \wedge \psi(a)$ iff $n_1 \Vdash \varphi(a)$ and $n_2 \Vdash \psi(a)$
- $n \Vdash \varphi(a) \vee \psi(a)$ iff $n_1 = 0$ and $n_2 \Vdash \varphi(a)$, or $n_1 \neq 0$ and $n_2 \Vdash \psi(a)$
- $n \Vdash \perp$ never holds
- $n \Vdash \varphi(a) \to \psi(a)$ iff $n_1 \in E(a)$ and for all m, if $m \Vdash \varphi(a)$ then $n_2 m\downarrow$ and $n_2 m \Vdash \psi(a)$
- $n \Vdash \exists x \varphi(x,a)$ iff there is an $x \in X$ such that $n_1 \in E(x)$ and $n_2 \Vdash \varphi(x,a)$ (here we assume that the sort of x is interpreted by the object (X,R), so $E(x) = R(x,x)$)
- $n \Vdash \forall x \varphi(x,a)$ iff $n_1 \in E(a)$ and for all $x \in X$ and all $m \in E(x)$, $n_2 m\downarrow$ and $n_2 m \Vdash \varphi(x,a)$ (with the same assumption on x as in the previous clause)

Then, if the atomic formulas are interpreted by strict relations, the function $a \mapsto \{n \mid n \Vdash \varphi(a)\}$ is also a strict relation.

We shall now have a look at some principles of higher-order arithmetic which are true in $\mathcal{E}ff$. First, two easy remarks.

Proposition 3.14 *i) The standard interpretation of primitive recursive functions in $\mathcal{E}ff$ interprets every primitive recursive function 'as itself' (as map $N^k \to N$).*

ii) $n \Vdash \neg\neg\varphi(a)$ iff $n_1 \in E(a)$ and the set $\{m \mid m \Vdash \varphi(a)\}$ is nonempty.

Proposition 3.15 Markov's Principle *is the axiom*

$$\forall U{:}\mathcal{P}(N)[\forall n(n \in U \vee \neg(n \in U)) \wedge \neg\neg\exists n(n \in U) \to \exists n(n \in U)]$$

Markov's Principle is true in $\mathcal{E}ff$.

Proof. Suppose $U : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is a strict relation, $m \Vdash \forall n(n \in U \vee \neg(n \in U))$ and $k \Vdash \neg\neg\exists n(n \in U)$. Then m codes a total recursive function such that for all n, if $(mn)_1 = 0$ then $(mn)_2 \Vdash n \in U$. By the very existence of k we know that there must be an n with $(mn)_1 = 0$. So if F is the partial recursive function which, with input m, k, ouputs $\langle n, (mn)_2\rangle$ for the first n such that $(mn)_1 = 0$, then $F(m,k)\downarrow$ and $F(m,k) \Vdash \exists n(n \in U)$. Hence any index for F forces the statement in question. ∎

Proposition 3.16 Church's Thesis *is (in the context of constructivism) the name for the statement that every function $N \to N$ is recursive:*

$$\forall f{:}N^N \exists e \forall n \exists m (T(e,n,m) \wedge U(m) = f(n))$$

Church's thesis is true in $\mathcal{E}\!f\!f$.

Proof. In section 3.2.2 we have seen that N^N is the assembly (M, E) where M is the set of morphisms $N \to N$ in $\mathbb{S}$ (or in $\mathcal{E}\!f\!f$), and $E(f)$ is the set of indices of partial recursive functions which track f. But any morphism $N \to N$ in $\mathcal{E}\!f\!f$ is (by the requirement that it has a tracking) total recursive; so N^N is isomorphic to the assembly (Rec, E) where Rec is the set of total recursive functions, and $E(f)$ is the set of indices for f. In order to find an n such that n forces Church's Thesis, we have to find a partial recursive function ϕ such that for every $f \in \mathrm{Rec}$ and every $e \in E(f)$,

$$(\phi(e))_2 \Vdash \forall n \exists m (T(e,n,m) \wedge U(m) = f(n))$$

Let $\phi(e)_1 = e$ and $\phi(e)_2$ such that for each n, $\phi(e)_2 n$ is the least m such that $T(e,n,m)$. Then any index for ψ forces Church's Thesis. ∎

Proposition 3.17 Brouwer's Principle *is the statement that every function from N^N to N is continuous, that is:*

$$\forall \Phi{:}N^{(N^N)} \forall f{:}N^N \exists n \forall g{:}N^N [\forall i \leq n (f(i) = g(i)) \to \Phi(f) = \Phi(g)]$$

Brouwer's Principle is true in $\mathcal{E}\!f\!f$.

Proof. We repeat the analysis according to section 3.2.2 for the object $N^{(N^N)}$. A morphism $N^N \to N$ in $\mathcal{E}\!f\!f$ is a function $\mathrm{Rec} \to \mathbb{N}$ which has a tracking. That means, we have a partial recursive function F such that for every index e of a total recursive function, $F(e)$ is defined, and if e and e' are indices of the same total recursive function, then $F(e) = F(e')$. Such a function F is called an *effective operation* in Recursion Theory. Regarding effective operations, there is the following theorem in Recursion Theory:

Kreisel-Lacombe-Shoenfield Theorem (KLS theorem). There is a partial recursive function φ of two variables, such that for every index f of an effective operation F and every index e of a total recursive function, $\varphi(f,e)\downarrow$ and for every index e' of a total recursive function it holds that whenever $ex = e'x$ for all $x \leq \varphi(f,e)$, then $F(e) = F(e')$.

Using the KLS theorem it should be easy to find an n which forces Brouwer's Principle. ∎

Exercise 61 Use propositions 3.16 and 3.17 in order to prove that the object N^N is not projective in $\mathcal{E}\!f\!f$.

4 Morphisms between toposes

In this section we consider two kinds of morphisms between toposes; the two kinds reflect the situation that a topos is both an *algebraic* (logical) and a *geometric* (topological) object.

Definition 4.1 Let $\mathcal{E}$ and $\mathcal{F}$ be toposes. A *logical functor* $\mathcal{E} \to \mathcal{F}$ is a functor L which preserves the categorical structure in the definition of 'topos', that is:

- finite limits
- exponentials
- subobject classifier

Exercise 62 Show that L also preserves power objects.

In fact, it can be shown that a logical functor preserves much more: it preserves finite colimits, and the natural numbers object (if it exists).

The word 'preserves' in definition 4.1 means of course: up to isomorphism. More specifically, for products we demand that the map

$$L(X \times Y) \xrightarrow{\langle L(\pi_X), L(\pi_Y)\rangle} L(X) \times L(Y)$$

is an isomorphism. Given this, there is also a canonical map

$$L(Y^X) \to L(Y)^{L(X)}$$

(which one?) which we require to be an isomorphism. For the subobject classifier we have: if L preserves finite limits, it preserves monos, so we have a map

$$\theta : L(\Omega_{\mathcal{E}}) \to \Omega_{\mathcal{F}}$$

whic classifies the subobject $1 \cong L(1) \stackrel{L(t)}{\to} L(\Omega_{\mathcal{E}})$; and we require θ to be an isomorphism.

Definition 4.2 Again let $\mathcal{E}, \mathcal{F}$ be toposes. A *geometric morphism* $\mathcal{E} \to \mathcal{F}$ is a pair $f = (f_*, f^*)$ of functors with $f_* : \mathcal{E} \to \mathcal{F}$ and f^* is left adjoint to f_* and f^* preserves finite limits. The functors f^* and f_* are called *inverse image* and *direct image* respectively.

Remarks 1. The inverse image f^* preserves not only finite limits, but, being a left adjoint, also all colimits. Of course, the functors f_* and f^* determine each other up to isomorphism. If $\mathcal{E}$ and $\mathcal{F}$ are Grothendieck toposes, then by the adjoint functor theorem any functor which preserves all colimits has a right adjoint; hence the functor f^* specifies a geometric morphism.
2. It happens that f^* is a logical functor; also f_* can be logical.

Examples

1. Any continuous map $f : X \to Y$ of topological spaces induces a geometric morphism
$$f : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$$
between the toposes of sheaves. This is a special case of 3. below.

2. Any functor $F : \mathcal{C} \to \mathcal{D}$ between small categories gives a geometric morphism
$$f : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$$
The functor f^* is given by composition with $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$: $f^*(Q)(C) = Q(F(C))$. Because limits and colimits are calculated 'pointwise' in presheaf categories, f^* preserves both. In fact, f^* has not only a right adjoint f_*, but also a left adjoint $f_!$.

   The right adjoint f_* is easily computed using the Yoneda lemma: $f_*(P)(D)$ is naturally isomorphic, by the Yoneda lemma, to $\mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}(y_D, f_*(P))$, which is, by the adjunction, $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(f^*(y_D), P)$.

   As for $f_!$, if it exists we must have by the adjunction and the Yoneda lemma, for representables y_C:
$$\begin{array}{rcl} \mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}(f_!(y_C), Q) & \cong & \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C, f^*(Q)) \\ & \cong & f^*(Q)(C) \\ & \cong & Q(F(C)) \\ & \cong & \mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}(y_{F(C)}, Q) \end{array}$$
   We conclude that $f_!(y_C) \cong y_{F(C)}$ naturally. This then determines $f_!$ uniquely up to isomorphism; for every presheaf is a colimit of representables (proposition 1.1) and $f_!$, being a left adjoint, preserves colimits. We see that $f_!$ is in fact the left Kan extension of the composite $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{y_{\mathcal{D}}} \mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$ along $y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. There is a similar notion of 'right Kan extension', and f_* is the right Kan extension of the composite $\mathcal{D} \xrightarrow{y_{\mathcal{D}}} \mathrm{Set}^{\mathcal{D}^{\mathrm{op}}} \xrightarrow{f^*} \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ along $y_{\mathcal{D}}$.

3. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories with finite limits which are equipped with Grothendieck topologies $\mathrm{Cov}_{\mathcal{C}}$ and $\mathrm{Cov}_{\mathcal{D}}$. Let $F : \mathcal{D} \to \mathcal{C}$ be a functor which preserves finite limits *and covers*, which means: if R is a covering sieve on D ($R \in \mathrm{Cov}_{\mathcal{D}}(D)$) then the sieve generated by $\{F(f) \mid f \in R\}$ is an element of $\mathrm{Cov}_{\mathcal{C}}(F(D))$. Then there is a geometric morphism $f : \mathrm{Sh}(\mathcal{C}, \mathrm{Cov}_{\mathcal{C}}) \to \mathrm{Sh}(\mathcal{D}, \mathrm{Cov}_{\mathcal{D}})$.

   For, in this situation the functor $f^* : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$ which is by composition with F^{op} as in the previous example, is easily seen to map sheaves to sheaves, so restricts to a functor $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov}_{\mathcal{C}}) \to \mathrm{Sh}(\mathcal{D}, \mathrm{Cov}_{\mathcal{D}})$, and this functor has a left adjoint which is the composition of (the restriction to sheaves of) $f_!$ and the sheafification functor, and this preserves finite limits.

   Note that example 1) is an instance of this; for a sheaf on a topological space X is a sheaf on the site $(\mathcal{O}(X), \mathrm{Cov})$ where $\mathcal{O}(X)$ is the category (poset) of opens of X and Cov is the ordinary covering relation. A continuous map $f : X \to Y$ gives a map $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ (inverse image!) which commutes with finite intersections (limits) and unions (preserves covers).

4. For every Grothendieck topos $\mathcal{E}$ there is a unique geometric morphism γ to Set. In Set, every set X is the colimit of a diagram of X copies of 1, hence $\gamma^*(X)$ must be the colimit of a diagram of X copies of $\gamma^*(1) \cong 1$ in $\mathcal{E}$. In fact, this functor (which obviously preserves fimite limits) has a right adjoint γ_*: for an object E of $\mathcal{E}$, $\gamma_*(E)$ is the set $\mathcal{E}(1, E)$. The functors γ^* and γ_* are called *constant sheaves functor* and *global sections functor* respectively.

5. The topos $\mathcal{E}ff$ is not a Grothendieck topos. In fact, the situation here is rather the opposite of the case with Grothendieck toposes: the global sections functor $\Gamma : \mathcal{E}ff \to \mathrm{Set}$ is not a right adjoint, but a left adjoint; its right adjoint is $\nabla : \mathrm{Set} \to \mathcal{E}ff$, and we have a geometric morphism $\mathrm{Set} \to \mathcal{E}ff$.

6. For any site $(\mathcal{C}, \mathrm{Cov})$ we have a geometric morphism $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov}) \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$: the direct image functor is the inclusion, and the inverse image functor is sheafification.

7. Suppose $\mathcal{E}$ is a topos and X an object of $\mathcal{E}$. Recall the definition of the "slice category" $\mathcal{E}/X$: objects are maps $Y \xrightarrow{f} X$ with codomain X; for two such maps f, f', an arrow $f \to f'$ is an arrow $g : \mathrm{dom}(f) \to$

dom(f') such that the triangle

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "f"'] & Y' \arrow[d, "f'"] \\ & X \end{tikzcd}$$

commutes. $\mathcal{E}$ can of course be identified with the slice $\mathcal{E}/1$. The various slice categories are connected by functors: for $f : X \to Y$ we have a functor $f^* : \mathcal{E}/Y \to \mathcal{E}/X$ by pull back along f. In particular there is $X^* : \mathcal{E} \to \mathcal{E}/X$.

The *fundamental theorem of topos theory* says that every slice $\mathcal{E}/X$ is again a topos, and that every functor f^* is logical. Moreover, f^* has both adjoints, Σ_f and Π_f. Therefore the pair (Π_f, f^*) defines a geometric morphism $\mathcal{E}/X \to \mathcal{E}/Y$, with an inverse image functor which is logical.

8. Another example where the inverse image functor is logical, is the unique geometric morphism from G-Set to Set, for a group G. Here G-Set is the category of sets with a right G-action, i.e. presheaves on G (the first example of a presheaf category in section 1). This geometric morphism, γ, can be decribed as follows: for a set S, $\gamma^*(S)$ is S with the trivial G-action; for a G-set X, $\gamma_*(X)$ is the set of fixed points of X under G, i.e. $\{x \in X \,|\, \forall g \in G\,(xg = x)\}$. The functor γ^* is logical and has also a left adjoint: $\gamma_!$ sends a G-set X to its set of orbits X/G.

The category of toposes and logical functors has an initial object: the *free topos*. The free topos is constructed syntactically. We refer to tha language outlined in section 3.3: we have a collection of sorts which contains 1, N, is closed under $\times$, $(\cdot)^{(\cdot)}$ and $\mathcal{P}(\cdot)$ and terms as given there. We now consider axioms: such as given in exercise 57 (we only sketch the construction here). This leads to a theory $\mathcal{T}$. For any formula $\varphi(x_1^{X_1}, \ldots, x_n^{X_n})$ (recall our convention that the list $x_1, \ldots, x_n$ contains all free variables of φ) we have an object $(X_1 \times \cdots \times X_n, \varphi)$ which we think of as $\{\vec{x} \in X_1 \times \cdots \times X_n \,|\, \varphi\}$. Given two such objects $(X_1 \times \cdots \times X_n, \varphi)$ and $Y_1 \times \cdots \times Y_m, \psi)$ (where we may assume that the lists of variables $\vec{x}$ and $\vec{y}$ are disjoint), an arrow $(X_1 \times \cdots \times X_n, \varphi) \to (Y_1 \times \cdots \times Y_m, \psi)$ is an equivalence class of formulas $\rho(\vec{x}, \vec{y})$ such that

$$\mathcal{T} \vdash \forall \vec{x}\,(\varphi(\vec{x}) \to \exists! \vec{y} \rho(\vec{x}, \vec{y}))$$
$$\mathcal{T} \vdash \forall \vec{x}\vec{y}\,(\rho(\vec{x}, \vec{y}) \to \varphi(\vec{x} \wedge \psi(\vec{y}))$$

where two such formulas ρ and rho' define the same morphism if

$$\mathcal{T} \vdash \forall \vec{x}\vec{y}(\rho(\vec{x},\vec{y}) \leftrightarrow \rho'(\vec{x},\vec{y}))$$

One can now formally prove that this is a topos, the *free topos* $\mathcal{F}$. Since there is a unique logical functor from $\mathcal{F}$ to any topos $\mathcal{E}$ and logical functors preserve truth of formulas, exactly those formulas are true in $\mathcal{F}$ which are true in every topos.

Instead of considering the theory $\mathcal{T}$, one can also take extra axioms (like, for example, Church's Thesis), and form a theory $\mathcal{T}'$ on which to construct a topos. Or, one can expand the language by extra sorts and terms; for example, if $\mathcal{C}$ is a small category, one could add an extra sort C for every object C of $\mathcal{C}$, and an extra function symbol $f : C \to D$ for every arrow $f : C \to D$ in $\mathcal{C}$. One obtains the *free topos on* $\mathcal{C}$, $\mathcal{F}_{\mathcal{C}}$. There is an embedding $\eta_{\mathcal{C}} : \mathcal{C} \to \mathcal{F}_{\mathcal{C}}$ and for every functor $F : \mathcal{C} \to \mathcal{E}$ where $\mathcal{E}$ is a topos, there is a unique logical functor $\tilde{F} : \mathcal{F}_{\mathcal{C}} \to \mathcal{E}$ such that

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\eta_{\mathcal{C}}"] \arrow[dr, "F"'] & \mathcal{F}_{\mathcal{C}} \arrow[d, "\tilde{F}"] \\ & \mathcal{E} \end{tikzcd}$$

commutes.

5 Literature

A good introduction to the field of topos theory is the book *Sheaves in Geometry and Logic* by S. Mac Lane and I. Moerdijk. Both the connections to geometry and to logic are treated in detail. For a more category-theoretic treatment (and not so many applications), see *Topos Theory* by P.T. Johnstone. Recently, Johnstone has completed the first two volumes of an intended three-volume compendium on topos theory, *Sketches of an Elephant*.

For formal aspects of topos theory and also other category theory, such as the construction of the free topos (but also, the free cartesian closed category on a category) and a good formulation of the language of toposes, see J. Lambek and Ph. Scott, *Introduction to higher-order categorical logic*.

For the effective topos, there is as yet nothing better than the seminal paper by J.M.E. Hyland, *The effective topos*, in: Troelstra and Van Dalen (eds), *The L.E.J.Brouwer Centenary Symposium*, North-Holland 1982, pp.165–216.

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