Classification of finite rotation groups

If an object is positioned in $\mathbb{R}^3$ with its centre of gravity at the origin then its rotational symmetry group is a subgroup of $SO_3(\mathbb{R})$.
In this section we will classify all possible *finite* subgroups of $SO_3(\mathbb{R})$.

Let us start with a 2-dimensional result.

Theorem Let G be a finite subgroup of $O_2(\mathbb{R})$. Then G is isomorphic to precisely one of the following groups:

$$C_n \ (n \geq 1), \qquad D_{2n} \ (n \geq 2).$$

Proof: Let G be a finite (non-trivial) subgroup of $O_2(\mathbb{R})$.
Suppose first that $G \leq SO_2(\mathbb{R})$. Then every element of G is a rotation in the plane. Write r_θ for the rotation anticlockwise by θ (where $0 \leq \theta < 2\pi$) around the origin $(0,0)$. Choose $r_\phi \in G$ with ϕ as small as possible (ok as G is finite) and $\phi > 0$ (ok as G is non-trivial). We claim that every other rotation in G is of the form

$$r_{m\phi} = (r_\phi)^m$$

for some m. Let $r_\theta \in G$ then $\theta = m\phi + \psi$ where $0 \leq \psi < \phi$ and $m \in \mathbb{N}$. Now

$$r_\theta = r_{m\phi+\psi} = (r_\phi)^m r_\psi$$

So $r_\psi = (r_\phi)^{-m} r_\theta \in G$ and $0 \leq \psi < \phi$. But as ϕ was the smallest non-zero angle we must have $\psi = 0$ and $\theta = m\phi$ as required. Therefore, G is generated by r_ϕ and so G is cyclic.

Now suppose that G contains a reflection s (in particular $s^2 = e$). Set

$$H = G \cap SO_2(\mathbb{R}).$$

Then H is a subgroup of $SO_2(\mathbb{R})$ and by the first case H is cyclic. So we have

$$H = \{e, r, r^2, \ldots, r^{n-1}\}$$

for some positive integer n. Take any reflection $s' \in G$. Then $s's$ is a rotation, so $s's = r^i$ for some i. Thus we get $s' = r^i s^{-1} = r^i s$. This shows that

$$G = \{e, r, r^2, \ldots, r^{n-1}, s, rs, r^2 s, \ldots, r^{n-1}s\}$$

and satisfies

$$r^n = I, s^2 = I, sr = r^{n-1}s$$

Hence we get that $G \cong D_{2n}$. □

Now we turn to the classification of finite 3-dimensional rotation groups.

Theorem Let G be a finite subgroup of $SO_3(\mathbb{R})$. Then G is isomorphic to precisely one of the following groups:

C_n, $(n \geq 1)$: rotational symmetry group of an n-pyramid
D_{2n}, $(n \geq 2)$: rotational symmetry group of an n-prism
A_4: rotational symmetry group of a regular tetrahedron
S_4: rotational symmetry group of a cube (or a regular octahedron)
A_5: rotational symmetry group of a regular dodecahedron (or a regular icosahedron).

Proof: Let G be a finite subgroup of $SO_3(\mathbb{R})$. Each element of G (other than e) represents a rotation in $\mathbb{R}^3$ around an axis passing through the origin. Take the unit sphere centered at the origin $(0,0,0)$. Then each rotation gives two poles on the unit sphere which are the intersection of the axis of rotation with the unit sphere. Let X denote the set of all poles of all the elements in $G \setminus \{e\}$. We claim that G acts on the set X. To see this, let $g \in G$ and let $x \in X$. Say that x is a pole for $h \in G$ (i.e. $h(x) = x$). Then we have

$$(ghg^{-1})(g(x)) = gh(g^{-1}g)(x) = gh(x) = g(x).$$

So we have that $g(x)$ is a pole for ghg^{-1} and so $g(x) \in X$.

Now the idea of the proof is to apply Burnside Counting theorem to the action of G on X and show that X has to be a particularly 'nice' configuration of points on the sphere.

Let N be the number of orbits of G in X. Choose a representative from each orbit $x_1, x_2, \ldots, x_N$. Now the identity e fixes every pole and each $g \neq I$ fixes exactly two poles. So using Burnside Counting theorem we get

$$\begin{aligned} N &= \frac{1}{|G|}\left(|X| + (|G|-1)2\right) \\ &= \frac{1}{|G|}\left(2(|G|-1) + \sum_{i=1}^{N} |Orb_G(x_i)|\right). \end{aligned}$$

Rearranging and using the Orbit-Stabilizer theorem we get

$$\begin{aligned} 2(1 - \frac{1}{|G|}) &= N - \frac{1}{|G|}\sum_{i=1}^{N} |Orb_G(x_i)| \\ &= N - \sum_{i=1}^{N} \frac{|Orb_G(x_i)|}{|G|} \\ &= N - \sum_{i=1}^{N} \frac{1}{|G_{x_i}|} \\ &= \sum_{i=1}^{N} (1 - \frac{1}{|G_{x_i}|}). \end{aligned}$$

Now assuming that $G \neq \{I\}$ we have

$$1 \leq 2(1 - \frac{1}{|G|}) < 2.$$

And each $|G_{x_i}| \geq 2$ as it contains at least e and one rotation so we have

$$\frac{1}{2} \leq 1 - \frac{1}{|G_{x_i}|} < 1$$

for $1 \leq i \leq N$. This implies that $2 \leq N < 4$ and hence $N = 2$ or 3.

If $N = 2$ then we get

$$|Orb_G(x_1)| + |Orb_G(x_2)| = 2,$$

each orbit contains one pole, and we have two poles in total. Thus each rotation has the same axis. The plane passing through the origin and perpendicular to this axis is preserved by G. So G is isomorphic to a subgroup of $SO_2(\mathbb{R})$. Using the previous Theorem we see that $G \cong C_n$ for some n.

If $N = 3$ the situation is more complicated. Write $x = x_1, y = x_2, z = x_3$. Then we get

$$1 + \frac{2}{|G|} = \frac{1}{|G_x|} + \frac{1}{|G_y|} + \frac{1}{|G_z|} > 1.$$

So we have four possible cases:

(a) $\frac{1}{|G_x|} = \frac{1}{2}$, $\frac{1}{|G_y|} = \frac{1}{2}$, $\frac{1}{|G_z|} = \frac{1}{n}$ for $n \geq 2$.
(b) $\frac{1}{|G_x|} = \frac{1}{2}$, $\frac{1}{|G_y|} = \frac{1}{3}$, $\frac{1}{|G_z|} = \frac{1}{3}$.
(c) $\frac{1}{|G_x|} = \frac{1}{2}$, $\frac{1}{|G_y|} = \frac{1}{3}$, $\frac{1}{|G_z|} = \frac{1}{4}$.
(d) $\frac{1}{|G_x|} = \frac{1}{2}$, $\frac{1}{|G_y|} = \frac{1}{3}$, $\frac{1}{|G_z|} = \frac{1}{5}$.

We will consider each of these in turn.

Case (a):
If $|G_x| = |G_y| = |G_z| = 2$ then we get $|G| = 4$. We have already seen that, up to isomorphism, there are only two groups of order 4, namely C_4 and D_4.
If $|G_x| = |G_y| = 2$ and $|G_z| = n \geq 3$ then we get $|G| = 2n$. Consider G_z the subgroup of all rotations with axis passing through z and $-z$. This group is cyclic of order n, so

$$G_z = \{e, g, g^2, \dots, g^{n-1}\}$$

for some $g \in G$. We claim that $x, g(x), g^2(x), \dots, g^{n-1}(x)$ are all distinct. To see this suppose that $g^i(x) = g^j(x)$ for some $i > j$. Then $g^{i-j}(x) = x$. But z and $-z$ are the only points fixed by G_z and $x \neq -z$ (as $|G_x| = 2$ and $|G_z| = |G_{-z}| = n \geq 3$).
Now we have

$$|x - g(x)| = |g(x) - g^2(x)| = \dots = |g^{n-1} - x|$$

and $|z-x| = |z-g^i(x)|$ for all $i = 1, 2, \dots, n-1$. This means that the points $x, g(x), \dots, g^{n-1}(x)$ all lie in the same plane and form a regular n-gon P.

Now we have that $|Orb_G(x)| = \frac{|G|}{|G_x|} = n$, and so $Orb_G(x) = \{x, g(x), \dots, g^{n-1}(x)\}$. Thus G maps P to P and we get a homomorphism

$$\phi : G \longrightarrow G'$$

where G' denotes the 3-dimensional rotational symmetries of P. Now every non-trivial rotation in G has only two fixed points in X and so doesn't fix P. This means that $Ker\phi = \{e\}$. Now as $|G| = 2n = |G'| = |D_{2n}|$, we see that ϕ is an isomorphism and $G \cong G' \cong D_{2n}$.

Case (b): $|G_x| = 2$, $|G_y| = |G_z| = 3$.
Then we have that $|G| = 12$ and $|Orb_G(z)| = 4$. Let $u \in Orb_G(z)$ with $|z - u| < 2$ (this is always possible as all poles lie on the unit sphere and $|Orb_G(z)| > 2$). So $u \neq -z$. As $|G_z| = 3$ we have that $G_z \cong C_3$. Choose $g \in G_z$ with $\langle g \rangle = G_z$. Then $u, g(u), g^2(u)$ are all distinct (same argument as in Case (a)). As g preserves distances, they form an equilateral triangle and are all equidistant from z. Now the orbit $Orb_G(z) = \{z, u, g(u), g^2(u)\}$ is preserved under the action of G. For $h \in G_u$ we have $h(u) = u$ and h permutes $z, g(u), g^2(u)$. As h preserves distances we see that the distances from u to z, $g(u)$ and $g^2(u)$ are all equal. Hence we have that $\{z, u, g(u), g^2(u)\}$ form a regular tetrahedron T and we have a homomorphism

$$\phi : G \longrightarrow G'$$

where G' is the rotational symmetry group of T. No rotation (other than e) fixes T, so $Ker\,\phi = \{e\}$ and ϕ is one-to-one. Now as $|G| = |G'| = 12$ we have that ϕ is an isomorphism and $G \cong G' \cong A_4$.

Case (c): $|G_x| = 2$, $|G_y| = 3$ and $|G_z| = 4$.
Here we get that $|G| = 24$ and $|Orb_G(z)| = 6$. Now choose $u \in Orb_G(z)$ with $u \neq z, -z$. As $G_z \cong C_4$ we have $G_z = \{I, g, g^2, g^3\}$ for some $g \in G$. We can show as before that $u, g(u), g^2(u), g^3(u)$ form a square equidistant from z. As $-z \notin Orb_G(x)$ or $Orb_G(y)$ (otherwise $|G_{-z}| = |G_x|$ or $|G_y|$) we have

$$Orb_G(z) = \{z, -z, u, g(u), g^2(u), g^3(u)\}.$$

Now, $-u \in Orb_G(z)$ (as $|G_{-u}| = |G_u| = |G_z|$) and $-u \neq z, -z$ (as $u \neq z, -z$). Also we have that $|g(u) - u| = |g^3(u) - u| < 2$ as $u, g(u), g^2(u), g^3(u)$ form a square. Thus $-u = g^2(u)$. This shows that $z, -z, u, g(u), g^2(u), g^3(u)$ form the vertices of a regular octahedron. Let G' be the rotational symmetry group of this regular octahedron. Then we get a homomorphism

$$\phi : G \longrightarrow G'$$

with $\mathrm{Ker}\,\phi = \{e\}$. So ϕ is one-to-one and as $|G| = |G'| = 24$ we have that ϕ is an isomorphism and we have

$$G \cong G' \cong S_4.$$

Case (d): $|G_x| = 2$, $|G_y| = 3$ and $|G_z| = 5$. Then we get that $|G| = 60$ and $|Orb_G(z)| = 12$. As $G_z \cong C_5$ we can find $g \in G$ with $G_z = \{e, g, g^2, g^3, g^4\}$. It can be shown that we can pick $u \in Orb_G(z)$ with $u \neq z, -z$, and $v \in Orb_G(z)$ with $v \neq z, -z, u, g(u), g^2(u), g^3(u), g^4(u)$. Moreover one can check that

$$\{z, -z, u, g(u), g^2(u), g^3(u), g^4(u), v, g(v), g^2(v), g^3(v), g^4(v)\}$$

form a regular icosahedron. Using the same argument as before, if we denote by G' the rotational symmetry group of this regular icosahedron, then we get a one-to-one homomorphism from G to G' and as $|G| = |G'| = 60$ this is in fact an isomorphism. Thus we get that

$$G \cong G' \cong A_5.$$

$\square$