Solutions Manual Elementary Linear Algebra

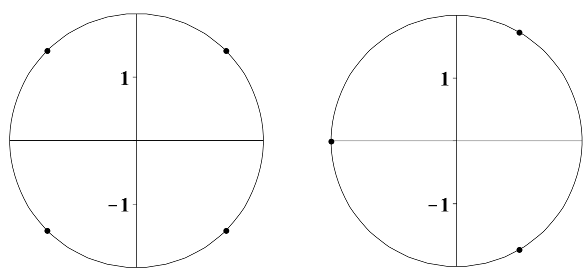
B.1 Exercises 1.8

1. Let $z = 5 + i9$. Find z^{-1}.
   $(5 + i9)^{-1} = \frac{5}{106} - \frac{9}{106}i$

2. Let $z = 2 + i7$ and let $w = 3 - i8$. Find $zw, z + w, z^2$, and w/z.
   $62 + 5i, 5 - i, -45 + 28i$, and $-\frac{50}{53} - \frac{37}{53}i$.

3. Give the complete solution to $x^4 + 16 = 0$.
   $x^4 + 16 = 0$, Solution is: $(1 - i)\sqrt{2}, -(1 + i)\sqrt{2}, -(1 - i)\sqrt{2}, (1 + i)\sqrt{2}$.

4. Graph the complex cube roots of 8 in the complex plane. Do the same for the four fourth roots of 16.
   The cube roots are the solutions to $z^3 + 8 = 0$, Solution is: $i\sqrt{3} + 1, 1 - i\sqrt{3}, -2$
   The fourth roots are the solutions to $z^4 + 16 = 0$, Solution is:
   $$(1 - i)\sqrt{2}, -(1 + i)\sqrt{2}, -(1 - i)\sqrt{2}, (1 + i)$$
   $\sqrt{2}$. When you graph these, you will have three equally spaced points on the circle of radius 2 for the cube roots and you will have four equally spaced points on the circle of radius 2 for the fourth roots. Here are pictures which should result.



5. If z is a complex number, show there exists ω a complex number with $|\omega| = 1$ and $\omega z = |z|$.
   If $z = 0$, let $\omega = 1$. If $z \neq 0$, let $\omega = \dfrac{\overline{z}}{|z|}$

6. De Moivre's theorem says $[r(\cos t + i \sin t)]^n = r^n (\cos nt + i \sin nt)$ for n a positive integer. Does this formula continue to hold for all integers, n, even negative integers? Explain.
   Yes, it holds for all integers. First of all, it clearly holds if $n = 0$. Suppose now that n is a negative integer. Then $-n > 0$ and so
   $$[r(\cos t + i \sin t)]^n = \frac{1}{[r(\cos t + i \sin t)]^{-n}} = \frac{1}{r^{-n}(\cos(-nt) + i \sin(-nt))}$$
   $$\begin{aligned} &= \frac{r^n}{(\cos(nt) - i \sin(nt))} = \frac{r^n (\cos(nt) + i \sin(nt))}{(\cos(nt) - i \sin(nt))(\cos(nt) + i \sin(nt))} \\ &= r^n (\cos(nt) + i \sin(nt)) \end{aligned}$$
   because $(\cos(nt) - i \sin(nt))(\cos(nt) + i \sin(nt)) = 1$.

7. You already know formulas for $\cos(x+y)$ and $\sin(x+y)$ and these were used to prove De Moivre's theorem. Now using De Moivre's theorem, derive a formula for $\sin(5x)$ and one for $\cos(5x)$. **Hint:** Use Problem **??** on Page **??** and if you like, you might use Pascal's triangle to construct the binomial coefficients.

   $\sin(5x) = 5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x$

   $\cos(5x) = \cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x$

8. If z and w are two complex numbers and the polar form of z involves the angle θ while the polar form of w involves the angle ϕ, show that in the polar form for zw the angle involved is $\theta+\phi$. Also, show that in the polar form of a complex number, z, $r = |z|$.

   You have $z = |z|(\cos\theta + i\sin\theta)$ and $w = |w|(\cos\phi + i\sin\phi)$. Then when you multiply these, you get

$$\begin{aligned} & |z|\,|w|\,(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ = {} & |z|\,|w|\,(\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \cos\phi\sin\theta)) \\ = {} & |z|\,|w|\,(\cos(\theta+\phi) + i\sin(\theta+\phi)) \end{aligned}$$

9. Factor x^3+8 as a product of linear factors.

   $x^3+8=0$, Solution is: $i\sqrt{3}+1, 1-i\sqrt{3}, -2$ and so this polynomial equals

$$(x+2)\left(x-\left(i\sqrt{3}+1\right)\right)\left(x-\left(1-i\sqrt{3}\right)\right)$$

10. Write x^3+27 in the form $(x+3)\left(x^2+ax+b\right)$ where x^2+ax+b cannot be factored any more using only real numbers.

    $x^3+27 = (x+3)\left(x^2-3x+9\right)$

11. Completely factor x^4+16 as a product of linear factors.

    $x^4+16=0$, Solution is: $(1-i)\sqrt{2}, -(1+i)\sqrt{2}, -(1-i)\sqrt{2}, (1+i)\sqrt{2}$. These are just the fourth roots of -16. Then to factor, this you get

$$\begin{gathered}\left(x-\left((1-i)\sqrt{2}\right)\right)\left(x-\left(-(1+i)\sqrt{2}\right)\right)\cdot \\ \left(x-\left(-(1-i)\sqrt{2}\right)\right)\left(x-\left((1+i)\sqrt{2}\right)\right)\end{gathered}$$

12. Factor x^4+16 as the product of two quadratic polynomials each of which cannot be factored further without using complex numbers.

    $x^4+16 = \left(x^2-2\sqrt{2}x+4\right)\left(x^2+2\sqrt{2}x+4\right)$. You can use the information in the preceding problem. Note that $(x-z)(x-\overline{z})$ has real coefficients.

13. If z, w are complex numbers prove $\overline{zw} = \overline{z}\,\overline{w}$ and then show by induction that $\overline{z_1\cdots z_m} = \overline{z_1}\cdots\overline{z_m}$. Also verify that $\overline{\sum_{k=1}^m z_k} = \sum_{k=1}^m \overline{z_k}$. In words this says the conjugate of a product equals the product of the conjugates and the conjugate of a sum equals the sum of the conjugates.

    $\overline{(a+ib)(c+id)} = \overline{ac-bd+i(ad+bc)} = (ac-bd) - i(ad+bc)$

    $(a-ib)(c-id) = ac-bd-i(ad+bc)$ which is the same thing. Thus it holds for a product of two complex numbers. Now suppose you have that it is true for the product of n complex numbers. Then

$$\overline{z_1\cdots z_{n+1}} = \overline{z_1\cdots z_n}\;\overline{z_{n+1}}$$

    and now, by induction this equals

$$\overline{z_1}\cdots\overline{z_n}\;\overline{z_{n+1}}$$

As to sums, this is even easier.

$$\overline{\sum_{j=1}^{n}(x_j+iy_j)}=\overline{\sum_{j=1}^{n}x_j+i\sum_{j=1}^{n}y_j}$$

$$=\sum_{j=1}^{n}x_j-i\sum_{j=1}^{n}y_j=\sum_{j=1}^{n}x_j-iy_j=\sum_{j=1}^{n}\overline{(x_j+iy_j)}.$$

14. Suppose $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ where all the a_k are real numbers. Suppose also that $p(z)=0$ for some $z\in\mathbb{C}$. Show it follows that $p(\overline{z})=0$ also.

    You just use the above problem. If $p(z)=0$, then you have

$$\begin{aligned}\overline{p(z)}=0&=\overline{a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0}\\&=\overline{a_nz^n}+\overline{a_{n-1}z^{n-1}}+\cdots+\overline{a_1z}+\overline{a_0}\\&=\overline{a_n}\ \overline{z}^n+\overline{a_{n-1}}\ \overline{z}^{n-1}+\cdots+\overline{a_1}\ \overline{z}+\overline{a_0}\\&=a_n\overline{z}^n+a_{n-1}\overline{z}^{n-1}+\cdots+a_1\overline{z}+a_0\\&=p(\overline{z})\end{aligned}$$

15. Show that $1+i, 2+i$ are the only two zeros to

$$p(x)=x^2-(3+2i)x+(1+3i)$$

    so the zeros do not necessarily come in conjugate pairs if the coefficients are not real.

    $(x-(1+i))(x-(2+i))=x^2-(3+2i)x+1+3i$

16. I claim that $1=-1$. Here is why.

$$-1=i^2=\sqrt{-1}\sqrt{-1}=\sqrt{(-1)^2}=\sqrt{1}=1.$$

    This is clearly a remarkable result but is there something wrong with it? If so, what is wrong?

    Something is wrong. There is no single $\sqrt{-1}$.

17. De Moivre's theorem is really a grand thing. I plan to use it now for rational exponents, not just integers.

$$1=1^{(1/4)}=(\cos 2\pi+i\sin 2\pi)^{1/4}=\cos(\pi/2)+i\sin(\pi/2)=i.$$

    Therefore, squaring both sides it follows $1=-1$ as in the previous problem. What does this tell you about De Moivre's theorem? Is there a profound difference between raising numbers to integer powers and raising numbers to non integer powers?

    It doesn't work. This is because there are four fourth roots of 1.

18. Review Problem 6 at this point. Now here is another question: If n is an integer, is it always true that $(\cos\theta-i\sin\theta)^n=\cos(n\theta)-i\sin(n\theta)$? Explain.

    Yes, this is true.

$$\begin{aligned}(\cos\theta-i\sin\theta)^n&=(\cos(-\theta)+i\sin(-\theta))^n\\&=\cos(-n\theta)+i\sin(-n\theta)\\&=\cos(n\theta)-i\sin(n\theta)\end{aligned}$$

19. Suppose you have any polynomial in $\cos\theta$ and $\sin\theta$. By this I mean an expression of the form $\sum_{\alpha=0}^{m}\sum_{\beta=0}^{n}a_{\alpha\beta}\cos^{\alpha}\theta\sin^{\beta}\theta$ where $a_{\alpha\beta}\in\mathbb{C}$. Can this always be written in the form $\sum_{\gamma=-(n+m)}^{m+n}b_{\gamma}\cos\gamma\theta+\sum_{\tau=-(n+m)}^{n+m}c_{\tau}\sin\tau\theta$? Explain.

Yes it can. It follows from the identities for the sine and cosine of the sum and difference of angles that

$$\begin{aligned}\sin a\sin b &= \frac{1}{2}\left(\cos\left(a-b\right)-\cos\left(a+b\right)\right)\\ \cos a\cos b &= \frac{1}{2}\left(\cos\left(a+b\right)+\cos\left(a-b\right)\right)\\ \sin a\cos b &= \frac{1}{2}\left(\sin\left(a+b\right)+\sin\left(a-b\right)\right)\end{aligned}$$

Now $\cos\theta=1\cos\theta+0\sin\theta$ and $\sin\theta=0\cos\theta+1\sin\theta$. Suppose that whenever $k\leq n$,

$$\cos^{k}\left(\theta\right)=\sum_{j=-k}^{k}a_{j}\cos\left(j\theta\right)+b_{j}\sin\left(j\theta\right)$$

for some numbers a_j,b_j. Then

$$\cos^{n+1}\left(\theta\right)=\sum_{j=-n}^{n}a_{j}\cos\left(\theta\right)\cos\left(j\theta\right)+b_{j}\cos\left(\theta\right)\sin\left(j\theta\right)$$

Now use the above identities to write all products as sums of sines and cosines of $(j-1)\theta, j\theta, (j+1)\theta$. Then adjusting the constants, it follows

$$\cos^{n+1}\left(\theta\right)=\sum_{j=-n+1}^{n+1}a_{j}'\cos\left(\theta\right)\cos\left(j\theta\right)+b_{j}'\cos\left(\theta\right)\sin\left(j\theta\right)$$

You can do something similar with $\sin^{n}\left(\theta\right)$ and with products of the form

$$\cos^{\alpha}\theta\sin^{\beta}\theta.$$

20. Suppose $p\left(x\right)=a_{n}x^{n}+a_{n-1}x^{n-1}+\cdots+a_{1}x+a_{0}$ is a polynomial and it has n zeros,

$$z_1, z_2, \cdots, z_n$$

listed according to multiplicity. (z is a root of multiplicity m if the polynomial $f\left(x\right)=\left(x-z\right)^{m}$ divides $p\left(x\right)$ but $\left(x-z\right)f\left(x\right)$ does not.) Show that

$$p\left(x\right)=a_{n}\left(x-z_{1}\right)\left(x-z_{2}\right)\cdots\left(x-z_{n}\right).$$

$p\left(x\right)=\left(x-z_{1}\right)q\left(x\right)+r\left(x\right)$ where $r\left(x\right)$ is a nonzero constant or equal to 0. However, $r\left(z_{1}\right)=0$ and so $r\left(x\right)=0$. Now do to $q\left(x\right)$ what was done to $p\left(x\right)$ and continue until the degree of the resulting $q\left(x\right)$ equals 0. Then you have the above factorization.

21. Give the solutions to the following quadratic equations having real coefficients.

   (a) $x^{2}-2x+2=0$, Solution is: $1+i, 1-i$
   (b) $3x^{2}+x+3=0$, Solution is: $\frac{1}{6}i\sqrt{35}-\frac{1}{6}, -\frac{1}{6}i\sqrt{35}-\frac{1}{6}$
   (c) $x^{2}-6x+13=0$, Solution is: $3+2i, 3-2i$
   (d) $x^{2}+4x+9=0$, Solution is: $i\sqrt{5}-2, -i\sqrt{5}-2$

(e) $4x^2 + 4x + 5 = 0$, Solution is: $-\frac{1}{2} + i, -\frac{1}{2} - i$

22. Give the solutions to the following quadratic equations having complex coefficients. Note how the solutions do not come in conjugate pairs as they do when the equation has real coefficients.

(a) $x^2 + 2x + 1 + i = 0$, Solution is : $x = -1 + \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}$, $x = -1 - \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}$

(b) $4x^2 + 4ix - 5 = 0$, Solution is : $x = 1 - \frac{1}{2}i$, $x = -1 - \frac{1}{2}i$

(c) $4x^2 + (4 + 4i)x + 1 + 2i = 0$, Solution is : $x = -\frac{1}{2}$, $x = -\frac{1}{2} - i$

(d) $x^2 - 4ix - 5 = 0$, Solution is : $x = -1 + 2i$, $x = 1 + 2i$

(e) $3x^2 + (1 - i)x + 3i = 0$, Solution is : $x = -\frac{1}{6} + \frac{1}{6}\sqrt{19} + \left(\frac{1}{6} - \frac{1}{6}\sqrt{19}\right) i$, $x = -\frac{1}{6} - \frac{1}{6}\sqrt{19} + \left(\frac{1}{6} + \frac{1}{6}\sqrt{19}\right) i$

23. Prove the fundamental theorem of algebra for quadratic polynomials having coefficients in $\mathbb{C}$.

This is pretty easy because you can simply write the quadratic formula. Finding the square roots of complex numbers is easy from the above presentation. Hence, every quadratic polynomial has two roots in $\mathbb{C}$. Note that the two square roots in the quadratic formula are on opposite sides of the unit circle so one is -1 times the other.

B.2 Exercises 2.6

1. Verify all the properties 2.3-2.10.

You just do these. Here is another example. Letting $(\mathbf{v})_j$ denote v_j where $\mathbf{v} = (v_1, \cdots, v_n)$,

$$(\alpha(\mathbf{v} + \mathbf{w}))_j = \alpha(v_j + w_j) = \alpha v_j + \alpha w_j = (\alpha\mathbf{v})_j + (\alpha\mathbf{w})_j = (\alpha\mathbf{v} + \alpha\mathbf{w})_j$$

since j is arbitrary, it follows that $\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$

2. Compute $5(1, 2 + 3i, 3, -2) + 6(2 - i, 1, -2, 7)$.

$5\begin{pmatrix} 1 & 2+3i & 3 & -2 \end{pmatrix} + 6\begin{pmatrix} 2-i & 1 & -2 & 7 \end{pmatrix} = \begin{pmatrix} 17-6i & 16+15i & 3 & 32 \end{pmatrix}$

3. Draw a picture of the points in $\mathbb{R}^2$ which are determined by the following ordered pairs.

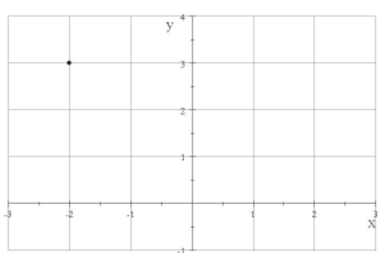
(a) $(1, 2)$

(b) $(-2, -2)$

(c) $(-2, 3)$

(d) $(2, -5)$

This is left for you. However, consider $(-2, 3)$



4. Does it make sense to write $(1, 2) + (2, 3, 1)$? Explain.

It makes absolutely no sense at all.

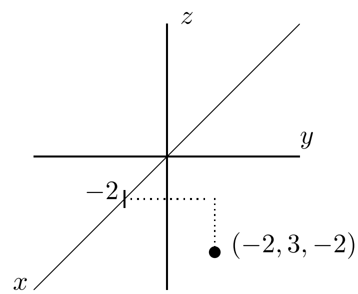
5. Draw a picture of the points in $\mathbb{R}^3$ which are determined by the following ordered triples.

   (a) $(1, 2, 0)$

   (b) $(-2, -2, 1)$

   (c) $(-2, 3, -2)$

   This is harder to do and have it look good. However, here is a picture of the last one.



B.3 Exercises 2.8

1. The wind blows from West to East at a speed of 50 miles per hour and an airplane which travels at 300 miles per hour in still air is heading North West. What is the velocity of the airplane relative to the ground? What is the component of this velocity in the direction North?

   The velocity is the sum of two vectors. $50\mathbf{i} + \frac{300}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = \left(50 + \frac{300}{\sqrt{2}}\right)\mathbf{i} + \frac{300}{\sqrt{2}}\mathbf{j}$. The component in the direction of North is then $\frac{300}{\sqrt{2}} = 150\sqrt{2}$ and the velocity relative to the ground is

$$\left(50 + \frac{300}{\sqrt{2}}\right)\mathbf{i} + \frac{300}{\sqrt{2}}\mathbf{j}$$

2. In the situation of Problem 1 how many degrees to the West of North should the airplane head in order to fly exactly North. What will be the speed of the airplane relative to the ground?

   The speed of the plane is 300. Let the direction vector be (a, b) where this is a unit vector. Then you need to have

$$300a + 50 = 0$$

   Thus $a = -1/6$. Then $b = \frac{1}{6}\sqrt{35}$. Then you would have the velocity of the airplane as

$$300\left(-\frac{1}{6}, \frac{1}{6}\sqrt{35}\right) + (50, 0) = \begin{pmatrix} 0 & 50\sqrt{35} \end{pmatrix}$$

   Hence its speed relative to the ground is

$$50\sqrt{35} = 295.8$$

   The direction vector is $\left(-\frac{1}{6}, \frac{1}{6}\sqrt{35}\right)$ and the cosine of the angle is then equal to $\sqrt{35}/6 = 0.98601$.

   Then you look this up in a table or something. You find it is .167 radians. Hence it is

$$\frac{.167}{\pi} = \frac{\theta}{180}$$

   Which corresponds to $\theta = 9.56$ degrees.

3. In the situation of 2 suppose the airplane uses 34 gallons of fuel every hour at that air speed and that it needs to fly North a distance of 600 miles. Will the airplane have enough fuel to arrive at its destination given that it has 63 gallons of fuel?

   From the above, it goes 295.8 miles every hour. Thus it will take $\frac{600}{295.8} = 2.0284$ hours to get where it is going. This will require $2.0284 \times 34 = 68.966$ gallons of gas. Therefore, it will not make it. This will be the case even if the people in the plane are optimistic and have a good attitude.

4. An airplane is flying due north at 150 miles per hour. A wind is pushing the airplane due east at 40 miles per hour. After 1 hour, the plane starts flying 30° East of North. Assuming the plane starts at $(0,0)$, where is it after 2 hours? Let North be the direction of the positive y axis and let East be the direction of the positive x axis.

   Velocity of plane for the first hour: $(0,150) + (40,0) = \begin{pmatrix} 40 & 150 \end{pmatrix}$. After one hour it is at $(40,150)$. Next the velocity of the plane is $150\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) + (40,0)$ in miles per hour. After two hours it is then at
   $(40,150) + 150\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) + (40,0) = \begin{pmatrix} 155 & 75\sqrt{3}+150 \end{pmatrix} = \begin{pmatrix} 155.0 & 279.9 \end{pmatrix}$

5. City A is located at the origin while city B is located at $(300,500)$ where distances are in miles. An airplane flies at 250 miles per hour in still air. This airplane wants to fly from city A to city B but the wind is blowing in the direction of the positive y axis at a speed of 50 miles per hour. Find a unit vector such that if the plane heads in this direction, it will end up at city B having flown the shortest possible distance. How long will it take to get there?

   Wind: $(0,50)$. Direction it needs to travel: $(3,5)\frac{1}{\sqrt{34}}$. Then you need $250(a,b) + (0,50)$ to have this direction where (a,b) is an appropriate unit vector. Thus you need

   $$\begin{aligned} a^2 + b^2 &= 1 \\ \frac{250b+50}{250a} &= \frac{5}{3} \end{aligned}$$

   Thus $a = \frac{3}{5}, b = \frac{4}{5}$. Then the velocity of the plane relative to the ground is $\begin{pmatrix} 150 & 250 \end{pmatrix}$. The speed of the plane relative to the ground is then

   $$\sqrt{(150)^2 + (250)^2} = 291.55.$$

   It has to go a distance of $\sqrt{(300)^2 + (500)^2} = 583.10$ miles. Therefore, it takes

   $$\frac{583.1}{291.55} = 2 \text{ hours}$$

6. A certain river is one half mile wide with a current flowing at 2 miles per hour from East to West. A man swims directly toward the opposite shore from the South bank of the river at a speed of 3 miles per hour. How far down the river does he find himself when he has swam across? How far does he end up swimming?

   water: $(-2,0)$ swimmer: $(0,3)$. Speed relative to earth. $(-2,3)$. It takes him 1/6 of an hour to get across. Therefore, he ends up travelling $\frac{1}{6}\sqrt{4+9} = \frac{1}{6}\sqrt{13}$ miles. He ends up 1/3 mile down stream.

7. A certain river is one half mile wide with a current flowing at 2 miles per hour from East to West. A man can swim at 3 miles per hour in still water. In what direction should he swim in order to travel directly across the river? What would the answer to this problem be if the river flowed at 3 miles per hour and the man could swim only at the rate of 2 miles per hour?

Man: $3(a,b)$ Water: $(-2,0)$. Then you need $3a=2$ and so $a=2/3$ and hence $b=\sqrt{5}/3$. The vector is then $\left(\frac{2}{3},\frac{\sqrt{5}}{3}\right)$. In the second case, he could not do it. You would need to have a unit vector (a,b) such that $2a=3$. This is not possible, not even if you try real hard.

8. Three forces are applied to a point which does not move. Two of the forces are $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$ Newtons and $\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ Newtons. Find the third force.

   $(2,1,3)+(1,-3,2)+(x,y,z)=(0,0,0)$. Thus the third force is $(-3,2,-5)$.

9. The total force acting on an object is to be $2\mathbf{i}+\mathbf{j}+\mathbf{k}$ Newtons. A force of $-\mathbf{i}+\mathbf{j}+\mathbf{k}$ Newtons is being applied. What other force should be applied to achieve the desired total force?

   $(2,1,1)=(-1,1,1)+(x,y,z)$. The third force is to be $(3,0,0)$.

10. A bird flies from its nest 5 km. in the direction 60° north of east where it stops to rest on a tree. It then flies 10 km. in the direction due southeast and lands atop a telephone pole. Place an xy coordinate system so that the origin is the bird's nest, and the positive x axis points east and the positive y axis points north. Find the displacement vector from the nest to the telephone pole.

    First stopping place: $\left(5\frac{\sqrt{3}}{2},5\frac{1}{2}\right)$

    Next stopping place. $\left(5\frac{\sqrt{3}}{2},5\frac{1}{2}\right)+10\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)=\left(\begin{array}{cc} 5\sqrt{2}+\frac{5}{2}\sqrt{3} & \frac{11}{2}-5\sqrt{2}\end{array}\right)$

11. A car is stuck in the mud. There is a cable stretched tightly from this car to a tree which is 20 feet long. A person grasps the cable in the middle and pulls with a force of 100 pounds perpendicular to the stretched cable. The center of the cable moves two feet and remains still. What is the tension in the cable? The tension in the cable is the force exerted on this point by the part of the cable nearer the car as well as the force exerted on this point by the part of the cable nearer the tree.

    $2T\left(\frac{2}{\sqrt{104}}\right)=100$, Solution is: $T=50\sqrt{26}$.

B.4 Exercises 3.3

1. Use formula 3.11 to verify the Cauchy Schwartz inequality and to show that equality occurs if and only if one of the vectors is a scalar multiple of the other.

   This formula says that $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}|\,|\mathbf{v}|\cos\theta$ where θ is the included angle between the two vectors. Thus
   $$|\mathbf{u}\cdot\mathbf{v}|=|\mathbf{u}|\,|\mathbf{v}|\,|\cos\theta|\leq|\mathbf{u}|\,|\mathbf{v}|$$
   and equality holds if and only if $\theta=0$ or π. This means that the two vectors either point in the same direction or opposite directions. Hence one is a multiple of the other.

2. For $\mathbf{u},\mathbf{v}$ vectors in $\mathbb{R}^3$, define the product, $\mathbf{u}*\mathbf{v}\equiv u_1v_1+2u_2v_2+3u_3v_3$. Show the axioms for a dot product all hold for this funny product. Prove
   $$|\mathbf{u}*\mathbf{v}|\leq(\mathbf{u}*\mathbf{u})^{1/2}(\mathbf{v}*\mathbf{v})^{1/2}.$$
   Hint: Do not try to do this with methods from trigonometry.

   This follows from the Cauchy Schwarz inequality and the proof of Theorem 3.2.15 which only used the properties of the dot product. This new product has the same properties the Cauchy Schwarz inequality holds for it as well.

3. Find the angle between the vectors $3\mathbf{i}-\mathbf{j}-\mathbf{k}$ and $\mathbf{i}+4\mathbf{j}+2\mathbf{k}$.

   $\frac{(3,-1,-1)\cdot(1,4,2)}{\sqrt{9+1+1}\sqrt{1+16+4}}=-\frac{1}{77}\sqrt{11}\sqrt{21}=-0.197\,39=\cos\theta$

   $-0.197\,39=\cos\theta$, Thus $\theta=1.769\,5$ radians.

4. Find the angle between the vectors $\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $\mathbf{i}+2\mathbf{j}-7\mathbf{k}$.

   $\frac{-8}{\sqrt{1+4+1}\sqrt{1+4+49}}=-0.444\,44=\cos\theta$

   $-0.444\,44=\cos\theta$, $\theta=2.031\,3$ radians.

5. Find $\text{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{v}=(1,0,-2)$ and $\mathbf{u}=(1,2,3)$.

   $\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}=\frac{-5}{14}(1,2,3)=\begin{pmatrix}-\frac{5}{14} & -\frac{5}{7} & -\frac{15}{14}\end{pmatrix}$

6. Find $\text{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{v}=(1,2,-2)$ and $\mathbf{u}=(1,0,3)$.

   $\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}=\frac{-5}{10}(1,0,3)=\begin{pmatrix}-\frac{1}{2} & 0 & -\frac{3}{2}\end{pmatrix}$

7. Find $\text{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{v}=(1,2,-2,1)$ and $\mathbf{u}=(1,2,3,0)$.

   $\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}=\frac{(1,2,-2,1)\cdot(1,2,3,0)}{1+4+9}(1,2,3,0)=\begin{pmatrix}-\frac{1}{14} & -\frac{1}{7} & -\frac{3}{14} & 0\end{pmatrix}$

8. Does it make sense to speak of $\text{proj}_{\mathbf{0}}(\mathbf{v})$?

   No, it does not. The 0 vector has no direction and the formula doesn't make sense either.

9. If $\mathbf{F}$ is a force and $\mathbf{D}$ is a vector, show $\text{proj}_{\mathbf{D}}(\mathbf{F})=(|\mathbf{F}|\cos\theta)\,\mathbf{u}$ where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{D}$, $\mathbf{u}=\mathbf{D}/|\mathbf{D}|$ and θ is the included angle between the two vectors, $\mathbf{F}$ and $\mathbf{D}$. $|\mathbf{F}|\cos\theta$ is sometimes called the component of the force, $\mathbf{F}$ in the direction $\mathbf{D}$.

   $\text{proj}_{\mathbf{D}}(\mathbf{F})\equiv\frac{\mathbf{F}\cdot\mathbf{D}}{|\mathbf{D}|}\frac{\mathbf{D}}{|\mathbf{D}|}=(|\mathbf{F}|\cos\theta)\frac{\mathbf{D}}{|\mathbf{D}|}=(|\mathbf{F}|\cos\theta)\,\mathbf{u}$

10. Prove the Cauchy Schwarz inequality in $\mathbb{R}^n$ as follows. For $\mathbf{u},\mathbf{v}$ vectors, consider

$$(\mathbf{u}-\text{proj}_{\mathbf{v}}\,\mathbf{u})\cdot(\mathbf{u}-\text{proj}_{\mathbf{v}}\,\mathbf{u})\geq 0$$

   Now simplify using the axioms of the dot product and then put in the formula for the projection. Of course this expression equals 0 and you get equality in the Cauchy Schwarz inequality if and only if $\mathbf{u}=\text{proj}_{\mathbf{v}}\,\mathbf{u}$. What is the geometric meaning of $\mathbf{u}=\text{proj}_{\mathbf{v}}\,\mathbf{u}$?

$$\left(\mathbf{u}-\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right)\cdot\left(\mathbf{u}-\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right)=|\mathbf{u}|^2-2(\mathbf{u}\cdot\mathbf{v})^2\frac{1}{|\mathbf{v}|^2}+(\mathbf{u}\cdot\mathbf{v})^2\frac{1}{|\mathbf{v}|^2}\geq 0$$

   And so

$$|\mathbf{u}|^2|\mathbf{v}|^2\geq(\mathbf{u}\cdot\mathbf{v})^2$$

   You get equality exactly when $\mathbf{u}=\text{proj}_{\mathbf{v}}\,\mathbf{u}=\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$, in other words, when $\mathbf{u}$ is a multiple of $\mathbf{v}$.

11. A boy drags a sled for 100 feet along the ground by pulling on a rope which is 20 degrees from the horizontal with a force of 40 pounds. How much work does this force do?

    $40\cos\left(\frac{20}{180}\pi\right)100=3758.8$

12. A girl drags a sled for 200 feet along the ground by pulling on a rope which is 30 degrees from the horizontal with a force of 20 pounds. How much work does this force do?

    $20\cos\left(\frac{\pi}{6}\right)200=3464.1$

13. A large dog drags a sled for 300 feet along the ground by pulling on a rope which is 45 degrees from the horizontal with a force of 20 pounds. How much work does this force do?

    $20\left(\cos\frac{\pi}{4}\right)300=4242.6$

14. How much work in Newton meters does it take to slide a crate 20 meters along a loading dock by pulling on it with a 200 Newton force at an angle of 30° from the horizontal?

    $200\left(\cos\left(\frac{\pi}{6}\right)\right)20=3464.1$

15. An object moves 10 meters in the direction of $\mathbf{j}$. There are two forces acting on this object, $\mathbf{F}_1 = \mathbf{i}+\mathbf{j}+2\mathbf{k}$, and $\mathbf{F}_2 = -5\mathbf{i}+2\mathbf{j}-6\mathbf{k}$. Find the total work done on the object by the two forces. **Hint:** You can take the work done by the resultant of the two forces or you can add the work done by each force. Why?

    $(-4,3,-4)\cdot(0,1,0)\times 10 = 30$

    You can consider the resultant of the two forces because of the properties of the dot product. See the next problem for the explanation.

16. An object moves 10 meters in the direction of $\mathbf{j}+\mathbf{i}$. There are two forces acting on this object, $\mathbf{F}_1 = \mathbf{i}+2\mathbf{j}+2\mathbf{k}$, and $\mathbf{F}_2 = 5\mathbf{i}+2\mathbf{j}-6\mathbf{k}$. Find the total work done on the object by the two forces. **Hint:** You can take the work done by the resultant of the two forces or you can add the work done by each force. Why?

    $\mathbf{F}_1\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)10+\mathbf{F}_2\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)10 = (\mathbf{F}_1+\mathbf{F}_2)\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)10$

    $= (6,4,-4)\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)10 = 50\sqrt{2}$

17. An object moves 20 meters in the direction of $\mathbf{k}+\mathbf{j}$. There are two forces acting on this object, $\mathbf{F}_1 = \mathbf{i}+\mathbf{j}+2\mathbf{k}$, and $\mathbf{F}_2 = \mathbf{i}+2\mathbf{j}-6\mathbf{k}$. Find the total work done on the object by the two forces. **Hint:** You can take the work done by the resultant of the two forces or you can add the work done by each force.

    $(2,3,-4)\cdot\left(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)20 = -10\sqrt{2}$

18. If $\mathbf{a},\mathbf{b}$, and $\mathbf{c}$ are vectors. Show that $(\mathbf{b}+\mathbf{c})_\perp = \mathbf{b}_\perp + \mathbf{c}_\perp$ where $\mathbf{b}_\perp = \mathbf{b}-\operatorname{proj}_{\mathbf{a}}(\mathbf{b})$.

    $\mathbf{b}-\operatorname{proj}_{\mathbf{a}}(\mathbf{b})+\mathbf{c}-\operatorname{proj}_{\mathbf{a}}(\mathbf{c}) = \mathbf{b}+\mathbf{c}-(\operatorname{proj}_{\mathbf{a}}(\mathbf{b})+\operatorname{proj}_{\mathbf{a}}(\mathbf{c})) = \mathbf{b}+\mathbf{c}-\operatorname{proj}_{\mathbf{a}}(\mathbf{b}+\mathbf{c})$

    because $\operatorname{proj}_{\mathbf{a}}(\mathbf{b})+\operatorname{proj}_{\mathbf{a}}(\mathbf{c}) =$

    $$\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}+\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} = \frac{(\mathbf{c}+\mathbf{b})\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} = \operatorname{proj}_{\mathbf{a}}(\mathbf{b}+\mathbf{c})$$

19. Find $(1,2,3,4)\cdot(2,0,1,3)$.

    $(1,2,3,4)\cdot(2,0,1,3) = 17$

20. Show that $(\mathbf{a}\cdot\mathbf{b}) = \frac{1}{4}\left[|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2\right]$.

    $$\begin{aligned} & \frac{1}{4}\left[|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2\right] \\ = & \frac{1}{4}(\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}+2\mathbf{a}\cdot\mathbf{b}-(\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b})) \\ = & \mathbf{a}\cdot\mathbf{b} \end{aligned}$$

21. Prove from the axioms of the dot product the parallelogram identity, $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$.

    Start with the left side.

    $$|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 =$$
    $$|\mathbf{a}|^2+|\mathbf{b}|^2+2\mathbf{a}\cdot\mathbf{b}+|\mathbf{a}|^2+|\mathbf{b}|^2-2\mathbf{a}\cdot\mathbf{b}$$
    $$= 2|\mathbf{a}|^2+2|\mathbf{b}|^2$$

B.5 Exercises 3.6

1. Show that if $\mathbf{a} \times \mathbf{u} = \mathbf{0}$ for all unit vectors, $\mathbf{u}$, then $\mathbf{a} = \mathbf{0}$.

   If $\mathbf{a} \neq \mathbf{0}$, then the condition says that $|\mathbf{a} \times \mathbf{u}| = |\mathbf{a}| \sin\theta = 0$ for all angles θ. Hence $\mathbf{a} = \mathbf{0}$ after all.

2. Find the area of the triangle determined by the three points, $(1, 2, 3), (4, 2, 0)$ and $(-3, 2, 1)$.

   $(3, 0, -3) \times (-4, 0, -2) = \begin{pmatrix} 0 & 18 & 0 \end{pmatrix}$. So the area is 9.

3. Find the area of the triangle determined by the three points, $(1, 0, 3), (4, 1, 0)$ and $(-3, 1, 1)$.

   $(3, 1, -3) \times (-4, 1, -2) = \begin{pmatrix} 1 & 18 & 7 \end{pmatrix}$

   $$\frac{1}{2}\sqrt{1 + (18)^2 + 49} = \frac{1}{2}\sqrt{374}$$

4. Find the area of the triangle determined by the three points, $(1, 2, 3), (2, 3, 4)$ and $(3, 4, 5)$. Did something interesting happen here? What does it mean geometrically?

   $(1, 1, 1) \times (2, 2, 2) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$ The area is 0. It means the three points are on the same line.

5. Find the area of the parallelogram determined by the vectors, $(1, 2, 3)$ and $(3, -2, 1)$.

   $(1, 2, 3) \times (3, -2, 1) = \begin{pmatrix} 8 & 8 & -8 \end{pmatrix}$. The area is $8\sqrt{3}$

6. Find the area of the parallelogram determined by the vectors, $(1, 0, 3)$ and $(4, -2, 1)$.

   $(1, 0, 3) \times (4, -2, 1) = \begin{pmatrix} 6 & 11 & -2 \end{pmatrix}$. Area is $\sqrt{36 + 121 + 4} = \sqrt{161}$

7. Find the volume of the parallelepiped determined by the vectors, $\mathbf{i}-7\mathbf{j}-5\mathbf{k}$, $\mathbf{i}-2\mathbf{j}-6\mathbf{k}$,$3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.

   $\begin{vmatrix} 1 & -7 & -5 \\ 1 & -2 & -6 \\ 3 & 2 & 3 \end{vmatrix} = 113$

8. Suppose $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are three vectors whose components are all integers. Can you conclude the volume of the parallelepiped determined from these three vectors will always be an integer?

   Yes. It will involve the sum of product of integers and so it will be an integer.

9. What does it mean geometrically if the box product of three vectors gives zero?

   It means that if you place them so that they all have their tails at the same point, the three will lie in the same plane.

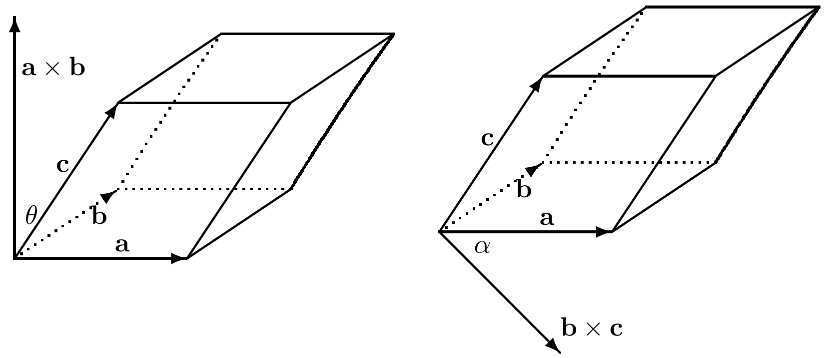
10. Using Problem 9, find an equation of a plane containing the two two position vectors, $\mathbf{a}$ and $\mathbf{b}$ and the point $\mathbf{0}$. **Hint:** If (x, y, z) is a point on this plane the volume of the parallelepiped determined by (x, y, z) and the vectors $\mathbf{a}, \mathbf{b}$ equals 0.

    $\mathbf{x}\cdot (\mathbf{a} \times \mathbf{b}) = 0$

11. Using the notion of the box product yielding either plus or minus the volume of the parallelepiped determined by the given three vectors, show that

    $$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

    In other words, the dot and the cross can be switched as long as the order of the vectors remains the same. **Hint:** There are two ways to do this, by the coordinate description of the dot and cross product and by geometric reasoning. It is better if you use geometric reasoning.



In the picture, you have the same parallelepiped. The one on the left involves $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ which is equal to $|\mathbf{a} \times \mathbf{b}|\,|\mathbf{c}| \cos\theta$ which is the area of the base determined by $\mathbf{a}, \mathbf{b}$ times the altitude which equals $|\mathbf{c}| \cos\theta$. In the second picture, the altitude is measured from the base determined by $\mathbf{c}, \mathbf{b}$. This altitude is $|\mathbf{a}| \cos\alpha$ and the area of the new base is $|\mathbf{c} \times \mathbf{d}|$. Hence the volume of the parallelepiped, determined in this other way is $|\mathbf{b} \times \mathbf{c}|\,|\mathbf{a}| \cos\alpha$ which equals $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$. Thus both $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ yield the same thing and it is the volume of the parallelepiped. If you switch two of the vectors, then the sign would change but in this case, both expressions would give -1 times the volume of the parallelepiped.

12. Is $\mathbf{a}\times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$? What is the meaning of $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$? Explain. **Hint:** Try $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$.

    $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i}$. However, $\mathbf{i}\times (\mathbf{j} \times \mathbf{j}) = \mathbf{0}$ and so the cross product is not associative.

13. Discover a vector identity for $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ and one for $\mathbf{u}\times (\mathbf{v} \times \mathbf{w})$.

$$\begin{aligned} ((\mathbf{u} \times \mathbf{v}) \times \mathbf{w})_i &= \varepsilon_{ijk} (\mathbf{u} \times \mathbf{v})_j w_k = \varepsilon_{ijk}\varepsilon_{jrs} u_r v_s w_k \\ &= -\varepsilon_{jik}\varepsilon_{jrs} u_r v_s w_k = -(\delta_{ir}\delta_{ks} - \delta_{kr}\delta_{is}) u_r v_s w_k \\ &= u_k v_i w_k - u_i v_k w_k = ((\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{v} \cdot \mathbf{w}) \mathbf{u})_i \end{aligned}$$

Hence

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{v} \cdot \mathbf{w}) \mathbf{u}$$

You can do the same thing for the other one or you could do the following.

$$\begin{aligned} \mathbf{u}\times (\mathbf{v} \times \mathbf{w}) &= -(\mathbf{v} \times \mathbf{w}) \times \mathbf{u} = -[(\mathbf{v} \cdot \mathbf{u}) \mathbf{w} - (\mathbf{w} \cdot \mathbf{u}) \mathbf{v}] \\ &= (\mathbf{w} \cdot \mathbf{u}) \mathbf{v} - (\mathbf{v} \cdot \mathbf{u}) \mathbf{w} \end{aligned}$$

14. Discover a vector identity for $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{z} \times \mathbf{w})$.

    It is easiest to use the identity just discovered, although you could do it directly with the permutation symbol and reduction identity. The expression equals

$$\mathbf{u}\cdot (\mathbf{z} \times \mathbf{w}) \mathbf{v} - \mathbf{v}\cdot (\mathbf{z} \times \mathbf{w}) \mathbf{u} = [\mathbf{u}, \mathbf{z}, \mathbf{w}] \mathbf{v} - [\mathbf{v}, \mathbf{z}, \mathbf{w}] \mathbf{u}$$

    Here $[\mathbf{u}, \mathbf{z}, \mathbf{w}]$ denotes the box product.

15. Simplify $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{v} \times \mathbf{w}) \times (\mathbf{w} \times \mathbf{z})$.

    Consider the cross product term. From the above,

$$\begin{aligned} (\mathbf{v} \times \mathbf{w}) \times (\mathbf{w} \times \mathbf{z}) &= [\mathbf{v}, \mathbf{w}, \mathbf{z}] \mathbf{w} - [\mathbf{w}, \mathbf{w}, \mathbf{z}] \mathbf{v} \\ &= [\mathbf{v}, \mathbf{w}, \mathbf{z}] \mathbf{w} \end{aligned}$$

Thus it reduces to

$$(\mathbf{u}\times\mathbf{v})\cdot[\mathbf{v},\mathbf{w},\mathbf{z}]\,\mathbf{w}=[\mathbf{v},\mathbf{w},\mathbf{z}]\,[\mathbf{u},\mathbf{v},\mathbf{w}]$$

16. Simplify $|\mathbf{u}\times\mathbf{v}|^2+(\mathbf{u}\cdot\mathbf{v})^2-|\mathbf{u}|^2\,|\mathbf{v}|^2$.

$$\begin{aligned}|\mathbf{u}\times\mathbf{v}|^2 &= \varepsilon_{ijk}u_jv_k\varepsilon_{irs}u_rv_s=(\delta_{jr}\delta_{ks}-\delta_{kr}\delta_{js})\,u_rv_su_jv_k\\ &= u_jv_ku_jv_k-u_kv_ju_jv_k=|\mathbf{u}|^2\,|\mathbf{v}|^2-(\mathbf{u}\cdot\mathbf{v})^2\end{aligned}$$

It follows that the expression reduces to 0. You can also do the following.

$$\begin{aligned}|\mathbf{u}\times\mathbf{v}|^2 &= |\mathbf{u}|^2\,|\mathbf{v}|^2\sin^2\theta\\ &= |\mathbf{u}|^2\,|\mathbf{v}|^2\left(1-\cos^2\theta\right)\\ &= |\mathbf{u}|^2\,|\mathbf{v}|^2-|\mathbf{u}|^2\,|\mathbf{v}|^2\cos^2\theta\\ &= |\mathbf{u}|^2\,|\mathbf{v}|^2-(\mathbf{u}\cdot\mathbf{v})^2\end{aligned}$$

which implies the expression equals 0.

17. For $\mathbf{u},\mathbf{v},\mathbf{w}$ functions of t, show the product rules

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})' &= \mathbf{u}'\times\mathbf{v}+\mathbf{u}\times\mathbf{v}'\\ (\mathbf{u}\cdot\mathbf{v})' &= \mathbf{u}'\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{v}'\end{aligned}$$

You can do this more elegantly by repeating the proof of the product rule given in calculus and using the properties of the two products. However, I will show it using the summation convention and permutation symbol.

$$\begin{aligned}\left((\mathbf{u}\times\mathbf{v})'\right)_i &\equiv ((\mathbf{u}\times\mathbf{v})_i)'=(\varepsilon_{ijk}u_jv_k)'\\ &= \varepsilon_{ijk}u_j'v_k+\varepsilon_{ijk}u_kv_k'=(\mathbf{u}'\times\mathbf{v}+\mathbf{u}\times\mathbf{v}')_i\end{aligned}$$

and so $(\mathbf{u}\times\mathbf{v})'=\mathbf{u}'\times\mathbf{v}+\mathbf{u}\times\mathbf{v}'$. Similar but easier reasoning shows the next one.

18. If $\mathbf{u}$ is a function of t, and the magnitude $|\mathbf{u}(t)|$ is a constant, show from the above problem that the velocity $\mathbf{u}'$ is perpendicular to $\mathbf{u}$.

$\mathbf{u}\cdot\mathbf{u}=c$ where c is a constant. Differentiate both sides.

$$\mathbf{u}'\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{u}'=0$$

Hence $\mathbf{u}'\cdot\mathbf{u}=0$. For example, if you move about on the surface of a sphere, then your velocity will be perpendicular to the position vector from the center of the sphere.

19. When you have a rotating rigid body with angular velocity vector $\mathbf{\Omega}$, then the velocity vector $\mathbf{v}\equiv\mathbf{u}'$ is given by

$$\mathbf{v}=\mathbf{\Omega}\times\mathbf{u}$$

where $\mathbf{u}$ is a position vector. The acceleration is the derivative of the velocity. Show that if $\mathbf{\Omega}$ is a constant vector, then the acceleration vector $\mathbf{a}=\mathbf{v}'$ is given by the formula

$$\mathbf{a}=\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{u}).$$

Now simplify the expression. It turns out this is centripetal acceleration.

$$\mathbf{a}=\mathbf{v}'=\mathbf{\Omega}\times\mathbf{u}'=\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{u})$$

Of course you can simplify the right side. It equals

$$\mathbf{a} = (\boldsymbol{\Omega} \cdot \mathbf{u})\,\boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega})\,\mathbf{u}$$

Also, its magnitude is $|\boldsymbol{\Omega}|\,|\boldsymbol{\Omega} \times \mathbf{u}| = |\boldsymbol{\Omega}|^2\,|\mathbf{u}| \sin\theta$ and is perpendicular to $\boldsymbol{\Omega}$.

20. Verify directly that the coordinate description of the cross product, $\mathbf{a} \times \mathbf{b}$ has the property that it is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Then show by direct computation that this coordinate description satisfies

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= |\mathbf{a}|^2\,|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= |\mathbf{a}|^2\,|\mathbf{b}|^2 \left(1 - \cos^2(\theta)\right) \end{aligned}$$

where θ is the angle included between the two vectors. Explain why $|\mathbf{a} \times \mathbf{b}|$ has the correct magnitude. All that is missing is the material about the right hand rule. Verify directly from the coordinate description of the cross product that the right thing happens with regards to the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Next verify that the distributive law holds for the coordinate description of the cross product. This gives another way to approach the cross product. First define it in terms of coordinates and then get the geometric properties from this. However, this approach does not yield the right hand rule property very easily.

From the coordinate description,

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = \varepsilon_{ijk} a_j b_k a_i = -\varepsilon_{jik} a_j b_k a_i = -\varepsilon_{jik} b_k a_i a_j = -\mathbf{a} \times \mathbf{b} \cdot \mathbf{a}$$

and so $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$. Similarly, $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{b}$. The Problem 16 above shows that

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2 \left(1 - \cos^2\theta\right) = |\mathbf{a}|^2\,|\mathbf{b}|^2 \sin^2\theta$$

and so $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, the area of the parallelogram determined by $\mathbf{a}, \mathbf{b}$. Only the right hand rule is a little problematic. However, you can see right away from the component definition that the right hand rule holds for each of the standard unit vectors. Thus $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ etc.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}$$

B.6 Exercises 4.3

1. Find the point, (x_1, y_1) which lies on both lines, $x + 3y = 1$ and $4x - y = 3$.

   $\begin{array}{l} x + 3y = 1 \\ 4x - y = 3 \end{array}$, Solution is: $\left[x = \frac{10}{13}, y = \frac{1}{13}\right]$

2. Solve Problem 1 graphically. That is, graph each line and see where they intersect.

3. Find the point of intersection of the two lines $3x + y = 3$ and $x + 2y = 1$.

   $\begin{array}{l} 3x + y = 3 \\ x + 2y = 1 \end{array}$, Solution is: $[x = 1, y = 0]$

4. Solve Problem 3 graphically. That is, graph each line and see where they intersect.

5. Do the three lines, $x+2y = 1, 2x-y = 1$, and $4x+3y = 3$ have a common point of intersection? If so, find the point and if not, tell why they don't have such a common point of intersection.

   $\begin{array}{l} x + 2y = 1 \\ 2x - y = 1 \\ 4x + 3y = 3 \end{array}$, Solution is: $\left[x = \frac{3}{5}, y = \frac{1}{5}\right]$

6. Do the three planes, $x+y-3z=2$, $2x+y+z=1$, and $3x+2y-2z=0$ have a common point of intersection? If so, find one and if not, tell why there is no such point.

   No solution exists. You can see this by writing the augmented matrix and doing row operations. $\begin{pmatrix} 1 & 1 & -3 & 2 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & -2 & 0 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Thus one of the equations says $0=1$ in an equivalent system of equations.

7. You have a system of k equations in two variables, $k \geq 2$. Explain the geometric significance of

   (a) No solution.

   (b) A unique solution.

   (c) An infinite number of solutions.

8. Here is an augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right)$$

   It appears that the solution exists but is not unique.

9. Here is an augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array}\right)$$

   It appears that there is a unique solution.

10. Here is an augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & 0 & * & 0 & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right)$$

11. Here is an augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & 0 & * & \blacksquare \end{array}\right)$$

   There might be a solution. If so, there are infinitely many.

12. Suppose a system of equations has fewer equations than variables. Must such a system be consistent? If so, explain why and if not, give an example which is not consistent.

    No. Consider $x+y+z=2$ and $x+y+z=1$.

13. If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, tell why not.

    These can have a solution. For example, $x+y=1, 2x+2y=2, 3x+3y=3$ even has an infinite set of solutions.

14. Find h such that
$$\left(\begin{array}{cc|c} 2 & h & 4 \\ 3 & 6 & 7 \end{array}\right)$$
is the augmented matrix of an inconsistent matrix.

    $h=4$

15. Find h such that
$$\left(\begin{array}{cc|c} 1 & h & 3 \\ 2 & 4 & 6 \end{array}\right)$$
is the augmented matrix of a consistent matrix.

    Any h will work.

16. Find h such that
$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & h & 12 \end{array}\right)$$
is the augmented matrix of a consistent matrix.

    Any h will work.

17. Choose h and k such that the augmented matrix shown has one solution. Then choose h and k such that the system has no solutions. Finally, choose h and k such that the system has infinitely many solutions.
$$\left(\begin{array}{cc|c} 1 & h & 2 \\ 2 & 4 & k \end{array}\right).$$

    If $h \neq 2$ there will be a unique solution for any k. If $h=2$ and $k \neq 4$, there are no solutions. If $h=2$ and $k=4$, then there are infinitely many solutions.

18. Choose h and k such that the augmented matrix shown has one solution. Then choose h and k such that the system has no solutions. Finally, choose h and k such that the system has infinitely many solutions.
$$\left(\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & h & k \end{array}\right).$$

    If $h \neq 4$, then there is exactly one solution. If $h=4$ and $k \neq 4$, then there are no solutions. If $h=4$ and $k=4$, then there are infinitely many solutions.

19. Determine if the system is consistent. If so, is the solution unique?
$$\begin{aligned} x+2y+z-w &= 2 \\ x-y+z+w &= 1 \\ 2x+y-z &= 1 \\ 4x+2y+z &= 5 \end{aligned}$$

    There is no solution. The system is inconsistent. You can see this from the augmented matrix.
$$\begin{pmatrix} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 0 & 1 \\ 4 & 2 & 1 & 0 & 5 \end{pmatrix}, \text{ row echelon form: } \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

20. Determine if the system is consistent. If so, is the solution unique?

$$\begin{gathered} x + 2y + z - w = 2 \\ x - y + z + w = 0 \\ 2x + y - z = 1 \\ 4x + 2y + z = 3 \end{gathered}$$

Solution is: $\left[w = \frac{3}{2}y - 1, x = \frac{2}{3} - \frac{1}{2}y, z = \frac{1}{3}\right]$

21. Find the general solution of the system whose augmented matrix is

$$\left(\begin{array}{cccccc} 1 & 2 & 0 & | & 2 \\ 1 & 3 & 4 & | & 2 \\ 1 & 0 & 2 & | & 1 \end{array} \right).$$

22. Find the general solution of the system whose augmented matrix is

$$\left(\begin{array}{cccccc} 1 & 2 & 0 & | & 2 \\ 2 & 0 & 1 & | & 1 \\ 3 & 2 & 1 & | & 3 \end{array} \right).$$

The row reduced echelon form is $\left(\begin{array}{cccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{array} \right)$. Therefore, the solution is of the form $z = t, y = \frac{3}{4} + t\left(\frac{1}{4}\right), x = \frac{1}{2} - \frac{1}{2}t$ where $t \in \mathbb{R}$.

23. Find the general solution of the system whose augmented matrix is

$$\left(\begin{array}{ccccc} 1 & 1 & 0 & | & 1 \\ 1 & 0 & 4 & | & 2 \end{array} \right).$$

The row reduced echelon form is $\left(\begin{array}{cccc} 1 & 0 & 4 & 2 \\ 0 & 1 & -4 & -1 \end{array} \right)$ and so the solution is $z = t, y = 4t, x = 2 - 4t$.

24. Find the general solution of the system whose augmented matrix is

$$\left(\begin{array}{ccccccc} 1 & 0 & 2 & 1 & 1 & | & 2 \\ 0 & 1 & 0 & 1 & 2 & | & 1 \\ 1 & 2 & 0 & 0 & 1 & | & 3 \\ 1 & 0 & 1 & 0 & 2 & | & 2 \end{array} \right).$$

The row reduced echelon form is $\left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 9 & 3 \\ 0 & 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & -7 & -1 \\ 0 & 0 & 0 & 1 & 6 & 1 \end{array} \right)$ and so

$x_5 = t, x_4 = 1 - 6t, x_3 = -1 + 7t, x_2 = 4t, x_1 = 3 - 9t$

25. Find the general solution of the system whose augmented matrix is

$$\left(\begin{array}{ccccccc} 1 & 0 & 2 & 1 & 1 & | & 2 \\ 0 & 1 & 0 & 1 & 2 & | & 1 \\ 0 & 2 & 0 & 0 & 1 & | & 3 \\ 1 & -1 & 2 & 2 & 2 & | & 0 \end{array} \right).$$

The row reduced echelon form is $\begin{pmatrix} 1 & 0 & 2 & 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Therefore, let $x_5 = t, x_3 = s$.

Then the other variables are given by $x_4 = -\frac{1}{2} - \frac{3}{2}t, x_2 = \frac{3}{2} - t\frac{1}{2},, x_1 = \frac{5}{2} + \frac{1}{2}t - 2s$.

26. Give the complete solution to the system of equations, $7x + 14y + 15z = 22$, $2x + 4y + 3z = 5$, and $3x + 6y + 10z = 13$.

    Solution is: $[x = 1 - 2t, z = 1, y = t]$

27. Give the complete solution to the system of equations, $3x - y + 4z = 6$, $y + 8z = 0$, and $-2x + y = -4$.

    Solution is: $[x = 2 - 4t, y = -8t, z = t]$

28. Give the complete solution to the system of equations, $9x-2y+4z = -17$, $13x-3y+6z = -25$, and $-2x - z = 3$.

    Solution is: $[x = -1, y = 2, z = -1]$

29. Give the complete solution to the system of equations, $65x+84y+16z = 546$, $81x+105y+20z = 682$, and $84x + 110y + 21z = 713$.

    Solution is: $[x = 2, y = 4, z = 5]$

30. Give the complete solution to the system of equations, $8x + 2y + 3z = -3, 8x + 3y + 3z = -1$, and $4x + y + 3z = -9$.

    Solution is: $[x = 1, y = 2, z = -5]$

31. Give the complete solution to the system of equations, $-8x+2y+5z = 18, -8x+3y+5z = 13$, and $-4x + y + 5z = 19$.

    Solution is: $[x = -1, y = -5, z = 4]$

32. Give the complete solution to the system of equations, $3x - y - 2z = 3$, $y - 4z = 0$, and $-2x + y = -2$.

    Solution is: $[x = 2t + 1, y = 4t, z = t]$

33. Give the complete solution to the system of equations, $-9x + 15y = 66, -11x + 18y = 79$,$-x + y = 4$, and $z = 3$.

    Solution is: $[x = 1, y = 5, z = 3]$

34. Give the complete solution to the system of equations, $-19x+8y = -108$, $-71x+30y = -404$, $-2x + y = -12$, $4x + z = 14$.

    Solution is: $[x = 4, y = -4, z = -2]$

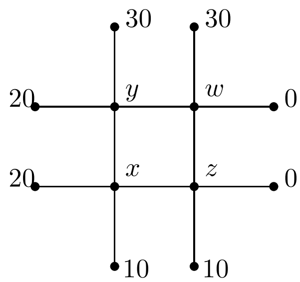
35. Consider the system $-5x+2y-z = 0$ and $-5x-2y-z = 0$. Both equations equal zero and so $-5x + 2y - z = -5x - 2y - z$ which is equivalent to $y = 0$. Thus x and z can equal anything. But when $x = 1$, $z = -4$, and $y = 0$ are plugged in to the equations, it doesn't work. Why?

    These are not legitimate row operations. They do not preserve the solution set of the system.

36. Four times the weight of Gaston is 150 pounds more than the weight of Ichabod. Four times the weight of Ichabod is 660 pounds less than seventeen times the weight of Gaston. Four times the weight of Gaston plus the weight of Siegfried equals 290 pounds. Brunhilde would balance all three of the others. Find the weights of the four sisters.

    $$\begin{array}{c} 4g - I = 150 \\ 4I - 17g = -660 \\ 4g + s = 290 \\ g + I + s - b = 0 \end{array}, \text{ Solution is : } \{g = 60, I = 90, b = 200, s = 50\}$$

37. The steady state temperature, u in a plate solves Laplace's equation, $\Delta u = 0$. One way to approximate the solution which is often used is to divide the plate into a square mesh and require the temperature at each node to equal the average of the temperature at the four adjacent nodes. This procedure is justified by the mean value property of harmonic functions. In the following picture, the numbers represent the observed temperature at the indicated nodes. Your task is to find the temperature at the interior nodes, indicated by x, y, z, and w. One of the equations is $z = \frac{1}{4}(10 + 0 + w + x)$.



You need $\begin{array}{l} \frac{1}{4}(20 + 30 + w + x) - y = 0 \\ \frac{1}{4}(y + 30 + 0 + z) - w = 0 \\ \frac{1}{4}(20 + y + z + 10) - x = 0 \\ \frac{1}{4}(x + w + 0 + 10) - z = 0 \end{array}$, Solution is: $[w = 15, x = 15, y = 20, z = 10]$.

B.7 Exercises 5.2

1. Here are some matrices:

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 7 \end{pmatrix}, B = \begin{pmatrix} 3 & -1 & 2 \\ -3 & 2 & 1 \end{pmatrix}, \\ C &= \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, D = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}, E = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \end{aligned}$$

Find if possible $-3A, 3B - A, AC, CB, AE, EA$. If it is not possible explain why.
$\begin{pmatrix} -3 & -6 & -9 \\ -6 & -3 & -21 \end{pmatrix}$, $\begin{pmatrix} 8 & -5 & 3 \\ -11 & 5 & -4 \end{pmatrix}$,Not possible, $\begin{pmatrix} -3 & 3 & 4 \\ 6 & -1 & 7 \end{pmatrix}$, Not possible, Not possible.

2. Here are some matrices:

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{pmatrix}, B = \begin{pmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{pmatrix}, \\ C &= \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}, D = \begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix}, E = \begin{pmatrix} 1 \\ 3 \end{pmatrix}. \end{aligned}$$

Find if possible $-3A, 3B - A, AC, CA, AE, EA, BE, DE$. If it is not possible explain why.

3. Here are some matrices:

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{pmatrix}, B = \begin{pmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{pmatrix}, \\ C &= \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}, D = \begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix}, E = \begin{pmatrix} 1 \\ 3 \end{pmatrix}. \end{aligned}$$

Find if possible $-3A^T, 3B - A^T, AC, CA, AE, E^T B, BE, DE, EE^T, E^T E$. If it is not possible explain why.

$\begin{pmatrix} -3 & -9 & -3 \\ -6 & -6 & 3 \end{pmatrix}, \begin{pmatrix} 5 & -18 & 5 \\ -11 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 11 & 2 \\ 13 & 6 \\ -4 & 2 \end{pmatrix}$,

Not possible, $\begin{pmatrix} 7 \\ 9 \\ -2 \end{pmatrix}, \begin{pmatrix} -7 & 1 & 5 \end{pmatrix}$, Not possible, $\begin{pmatrix} 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}, 10$

4. Here are some matrices:

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{pmatrix}, B = \begin{pmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{pmatrix}, \\ C &= \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}, D = \begin{pmatrix} -1 \\ 4 \end{pmatrix}, E = \begin{pmatrix} 1 \\ 3 \end{pmatrix}. \end{aligned}$$

Find the following if possible and explain why it is not possible if this is the case. $AD, DA, D^T B, D^T BE, E^T D, DE^T$.

$\begin{pmatrix} 7 \\ 5 \\ -5 \end{pmatrix}$, Not possible, $\begin{pmatrix} -14 & 13 & 2 \end{pmatrix}$, Not possible, 11, $\begin{pmatrix} -1 & -3 \\ 4 & 12 \end{pmatrix}$

5. Let $A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & -2 \\ 2 & 1 & -2 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ -3 & -1 & 0 \end{pmatrix}$. Find if possible.

   (a) $AB = \begin{pmatrix} 3 & 0 & -4 \\ -4 & 1 & 6 \\ 5 & 1 & -6 \end{pmatrix}$
   (b) $BA = \begin{pmatrix} 1 & -2 \\ -2 & -3 \end{pmatrix}$
   (c) $AC =$Not possible.
   (d) $CA = \begin{pmatrix} -4 & -6 \\ -5 & -3 \\ -1 & -2 \end{pmatrix}$
   (e) $CB =$Not possible.
   (f) $BC = \begin{pmatrix} 8 & 1 & -3 \\ 7 & 6 & -6 \end{pmatrix}$

6. Suppose A and B are square matrices of the same size. Which of the following are correct?

   (a) $(A - B)^2 = A^2 - 2AB + B^2$
   (b) $(AB)^2 = A^2 B^2$
   (c) $(A + B)^2 = A^2 + 2AB + B^2$
   (d) $(A + B)^2 = A^2 + AB + BA + B^2$
   (e) $A^2 B^2 = A(AB)B$
   (f) $(A + B)^3 = A^3 + 3A^2 B + 3AB^2 + B^3$
   (g) $(A + B)(A - B) = A^2 - B^2$
   Part d,e.

7. Let $A = \begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix}$. Find $\boxed{\underline{\textbf{all}}}$ 2×2 matrices, B such that $AB = 0$.

   $\begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} -x-z & -w-y \\ 3x+3z & 3w+3y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and so you have to solve

   Solution is: $[w = -y, x = -z]$ So the matrices are of the form $\begin{pmatrix} x & y \\ -x & -y \end{pmatrix}$.

8. Let $\mathbf{x} = (-1,-1,1)$ and $\mathbf{y} = (0,1,2)$. Find $\mathbf{x}^T\mathbf{y}$ and $\mathbf{x}\mathbf{y}^T$ if possible.

   $\begin{pmatrix} 0 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix}, 1$

9. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & k \end{pmatrix}$. Is it possible to choose k such that $AB = BA$? If so, what should k equal?

   $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & k \end{pmatrix} = \begin{pmatrix} 7 & 2k+2 \\ 15 & 4k+6 \end{pmatrix}$

   $\begin{pmatrix} 1 & 2 \\ 3 & k \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 3k+3 & 4k+6 \end{pmatrix}$

   Thus you must have

   $\begin{array}{l} 3k+3 = 15 \\ 2k+2 = 10 \end{array}$, Solution is: $[k = 4]$

10. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 1 & k \end{pmatrix}$. Is it possible to choose k such that $AB = BA$? If so, what should k equal?

    $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & k \end{pmatrix} = \begin{pmatrix} 3 & 2k+2 \\ 7 & 4k+6 \end{pmatrix}$

    $\begin{pmatrix} 1 & 2 \\ 1 & k \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 3k+1 & 4k+2 \end{pmatrix}$

    However, $7 \neq 3$ and so there is no possible choice of k which will make these matrices commute.

11. In 5.1 - 5.8 describe $-A$ and 0.

    To get $-A$, just replace every entry of A with its additive inverse. The 0 matrix is the one which has all zeros in it.

12. Let A be an $n \times n$ matrix. Show A equals the sum of a symmetric and a skew symmetric matrix. (M is skew symmetric if $M = -M^T$. M is symmetric if $M^T = M$.) **Hint:** Show that $\frac{1}{2}\left(A^T + A\right)$ is symmetric and then consider using this as one of the matrices.

    $A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$.

13. Show every skew symmetric matrix has all zeros down the main diagonal. The main diagonal consists of every entry of the matrix which is of the form a_{ii}. It runs from the upper left down to the lower right.

    If $A = -A^T$, then $a_{ii} = -a_{ii}$ and so each $a_{ii} = 0$.

14. Suppose M is a 3×3 skew symmetric matrix. Show there exists a vector $\mathbf{\Omega}$ such that for all $\mathbf{u} \in \mathbb{R}^3$

    $$M\mathbf{u} = \mathbf{\Omega} \times \mathbf{u}$$

Hint: Explain why, since M is skew symmetric it is of the form

$$M=\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

where the ω_i are numbers. Then consider $\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k}$.

Since M is skew symmetric, it is of the form mentioned above. Now it just remains to verify that $\mathbf{\Omega} \equiv \omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k}$ is of the right form. But $\begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = i\omega_2u_3 - i\omega_3u_2 - j\omega_1u_3 + j\omega_3u_1 + k\omega_1u_2 - k\omega_2u_1$. In terms of matrices, this is $\begin{pmatrix} \omega_2u_3-\omega_3u_2 \\ -\omega_1u_3+\omega_3u_1 \\ \omega_1u_2-\omega_2u_1 \end{pmatrix}$. If you multiply by the matrix, you get

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}=\begin{pmatrix} \omega_2u_3-\omega_3u_2 \\ \omega_3u_1-\omega_1u_3 \\ \omega_1u_2-\omega_2u_1 \end{pmatrix}$$

which is the same thing.

15. Using only the properties 5.1 - 5.8 show $-A$ is unique.

    Suppose B also works. Then

    $$-A=-A+(A+B)=(-A+A)+B=0+B=B$$

16. Using only the properties 5.1 - 5.8 show 0 is unique.

    Suppose $0'$ is another one. Then $0'=0'+0=0$.

17. Using only the properties 5.1 - 5.8 show $0A=0$. Here the 0 on the left is the scalar 0 and the 0 on the right is the zero for $m\times n$ matrices.

    $0A=(0+0)\,A=0A+0A$. Now add $-(0A)$ to both sides. Then $0=0A$.

18. Using only the properties 5.1 - 5.8 and previous problems show $(-1)\,A=-A$.

    $A+(-1)\,A=(1+(-1))\,A=0A=0$. Therefore, from the uniqueness of the additive inverse proved in the above Problem 15, it follows that $-A=(-1)\,A$.

19. Prove 5.17.

    $\left((\alpha A+\beta B)^T\right)_{ij}=(\alpha A+\beta B)_{ji}=\alpha A_{ji}+\beta B_{ji}$

    $=\alpha\left(A^T\right)_{ij}+(\beta B)^T_{ij}=\left(\alpha A^T+\beta B^T\right)_{ij}$

20. Prove that $I_mA=A$ where A is an $m\times n$ matrix.

    $(I_mA)_{ij}\equiv\sum_j\delta_{ik}A_{kj}=A_{ij}$.

21. Give an example of matrices, A,B,C such that $B\neq C$, $A\neq 0$, and yet $AB=AC$.

    $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

    $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

22. Suppose $AB = AC$ and A is an invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not. What if A were a non invertible $n \times n$ matrix?

    Yes. Multiply on the left by A^{-1}.

23. Find your own examples:

    (a) 2×2 matrices, A and B such that $A \neq 0, B \neq 0$ with $AB \neq BA$.
    $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$
    $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

    (b) 2×2 matrices, A and B such that $A \neq 0, B \neq 0$, but $AB = 0$.
    $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

    (c) 2×2 matrices, A, D, and C such that $A \neq 0, C \neq D$, but $AC = AD$.
    $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

    $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

24. Explain why if $AB = AC$ and A^{-1} exists, then $B = C$.

    Because you can multiply on the left by A^{-1}.

25. Give an example of a matrix, A such that $A^2 = I$ and yet $A \neq I$ and $A \neq -I$.
    $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

26. Give an example of matrices, A, B such that neither A nor B equals zero and yet $AB = 0$.
    $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

27. Give another example other than the one given in this section of two square matrices, A and B such that $AB \neq BA$.
    $\begin{pmatrix} 3 & -7 \\ 5 & 11 \end{pmatrix}\begin{pmatrix} 7 & -8 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 7 & -24 \\ 57 & -40 \end{pmatrix}$
    $\begin{pmatrix} 7 & -8 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 3 & -7 \\ 5 & 11 \end{pmatrix} = \begin{pmatrix} -19 & -137 \\ 6 & -14 \end{pmatrix}$

28. Let
    $$A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}.$$
    Find A^{-1} if possible. If A^{-1} does not exist, determine why.
    $\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{pmatrix}$

29. Let
    $$A = \begin{pmatrix} 0 & 1 \\ 5 & 3 \end{pmatrix}.$$
    Find A^{-1} if possible. If A^{-1} does not exist, determine why.
    $\begin{pmatrix} 0 & 1 \\ 5 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{3}{5} & \frac{1}{5} \\ 1 & 0 \end{pmatrix}$

30. Let
$$A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}.$$
Find A^{-1} if possible. If A^{-1} does not exist, determine why.
$\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{pmatrix}$

31. Let
$$A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}.$$
Find A^{-1} if possible. If A^{-1} does not exist, determine why.
$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}^{-1}$ does not exist. The row reduced echelon form of this matrix is $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$.

32. Let A be a 2×2 matrix which has an inverse. Say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find a formula for A^{-1} in terms of a, b, c, d.
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$

33. Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{pmatrix}.$$
Find A^{-1} if possible. If A^{-1} does not exist, determine why.
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 4 & -5 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{pmatrix}$

34. Let
$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}.$$
Find A^{-1} if possible. If A^{-1} does not exist, determine why.
$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix}$

35. Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & 10 \end{pmatrix}.$$
Find A^{-1} if possible. If A^{-1} does not exist, determine why.
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & 10 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix}$. There is no inverse.

36. Let
$$A = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, determine why.

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 3 & \frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} \\ -1 & 0 & 0 & 1 \\ -2 & -\frac{3}{4} & \frac{1}{4} & \frac{9}{4} \end{pmatrix}$$

37. Write $\begin{pmatrix} x_1 - x_2 + 2x_3 \\ 2x_3 + x_1 \\ 3x_3 \\ 3x_4 + 3x_2 + x_1 \end{pmatrix}$ in the form $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ where A is an appropriate matrix.

$$A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 3 & 0 & 3 \end{pmatrix}$$

38. Write $\begin{pmatrix} x_1 + 3x_2 + 2x_3 \\ 2x_3 + x_1 \\ 6x_3 \\ x_4 + 3x_2 + x_1 \end{pmatrix}$ in the form $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ where A is an appropriate matrix.

$$A = \begin{pmatrix} 1 & 3 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 6 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix}$$

39. Write $\begin{pmatrix} x_1 + x_2 + x_3 \\ 2x_3 + x_1 + x_2 \\ x_3 - x_1 \\ 3x_4 + x_1 \end{pmatrix}$ in the form $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ where A is an appropriate matrix.

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$$

40. Using the inverse of the matrix, find the solution to the systems

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}.$$

Now give the solution in terms of a, b, and c to

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix}$. Therefore, the solutions desired are respectively $\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$

$$\begin{pmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix},$$

$$\begin{pmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \text{ which equal}$$

$$\begin{pmatrix} 7 \\ -\frac{4}{3} \\ -2 \end{pmatrix}, \begin{pmatrix} -4 \\ \frac{1}{3} \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{2}{3} \\ 0 \end{pmatrix}, \begin{pmatrix} -12 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 3c-2a \\ \frac{1}{3}b-\frac{2}{3}c \\ a-c \end{pmatrix}$$

41. Using the inverse of the matrix, find the solution to the systems

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}.$$

Now give the solution in terms of a, b, and c to

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This one is just like the one above it.

42. Using the inverse of the matrix, find the solution to the system

$$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 3 & \frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} \\ -1 & 0 & 0 & 1 \\ -2 & -\frac{3}{4} & \frac{1}{4} & \frac{9}{4} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 3 & \frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} \\ -1 & 0 & 0 & 1 \\ -2 & -\frac{3}{4} & \frac{1}{4} & \frac{9}{4} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix}$. Thus the solution is $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} =$ $\begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+2b+2d \\ a+b+2c \\ 2a+b-3c+2d \\ a+2b+c+2d \end{pmatrix}$

43. Show that if A is an $n \times n$ invertible matrix and $\mathbf{x}$ is a $n \times 1$ matrix such that $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b}$ an $n \times 1$ matrix, then $\mathbf{x} = A^{-1}\mathbf{b}$.

    Multiply on the left of both sides by A^{-1}.

44. Prove that if A^{-1} exists and $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$.

    Multiply on both sides on the left by A^{-1}. Thus

$$\mathbf{0} = A^{-1}\mathbf{0} = A^{-1}(A\mathbf{x}) = \left(A^{-1}A\right)\mathbf{x} = I\mathbf{x} = \mathbf{x}$$

45. Show that if A^{-1} exists for an $n\times n$ matrix, then it is unique. That is, if $BA=I$ and $AB=I$, then $B=A^{-1}$.

    $A^{-1}=A^{-1}I=A^{-1}(AB)=\left(A^{-1}A\right)B=IB=B.$

46. Show that if A is an invertible $n\times n$ matrix, then so is A^T and $\left(A^T\right)^{-1}=\left(A^{-1}\right)^T$.

    You just need to show that $\left(A^{-1}\right)^T$ acts like the inverse of A^T because from uniqueness in the above problem, this will imply it is the inverse. But from properties of the transpose,

    $$\begin{aligned} A^T\left(A^{-1}\right)^T &= \left(A^{-1}A\right)^T=I^T=I \\ \left(A^{-1}\right)^T A^T &= \left(AA^{-1}\right)^T=I^T=I \end{aligned}$$

    Hence $\left(A^{-1}\right)^T=\left(A^T\right)^{-1}$ and this last matrix exists.

47. Show $(AB)^{-1}=B^{-1}A^{-1}$ by verifying that $AB\left(B^{-1}A^{-1}\right)=I$ and $B^{-1}A^{-1}(AB)=I$. **Hint:** Use Problem 45.

    $(AB)B^{-1}A^{-1}=A\left(BB^{-1}\right)A^{-1}=AA^{-1}=I$

    $B^{-1}A^{-1}(AB)=B^{-1}\left(A^{-1}A\right)B=B^{-1}IB=B^{-1}B=I$

48. Show that $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$ by verifying that $(ABC)\left(C^{-1}B^{-1}A^{-1}\right)=I$ and $\left(C^{-1}B^{-1}A^{-1}\right)(ABC)=I$. **Hint:** Use Problem 45.

    This is just like the above problem.

49. If A is invertible, show $\left(A^2\right)^{-1}=\left(A^{-1}\right)^2$. **Hint:** Use Problem 45.

    $A^2\left(A^{-1}\right)^2=AAA^{-1}A^{-1}=AIA^{-1}=AA^{-1}=I$

    $\left(A^{-1}\right)^2A^2=A^{-1}A^{-1}AA=A^{-1}IA=A^{-1}A=I$

50. If A is invertible, show $\left(A^{-1}\right)^{-1}=A$. **Hint:** Use Problem 45.

    $A^{-1}A=AA^{-1}=I$ and so by uniqueness, $\left(A^{-1}\right)^{-1}=A$.

51. Let A and be a real $m\times n$ matrix and let $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{y}\in\mathbb{R}^m$. Show $(A\mathbf{x},\mathbf{y})_{\mathbb{R}^m}=\left(\mathbf{x},A^T\mathbf{y}\right)_{\mathbb{R}^n}$ where $(\cdot,\cdot)_{\mathbb{R}^k}$ denotes the dot product in $\mathbb{R}^k$. In the notation above, $A\mathbf{x}\cdot\mathbf{y}=\mathbf{x}\cdot A^T\mathbf{y}$. Use the definition of matrix multiplication to do this.

    $A\mathbf{x}\cdot\mathbf{y}=\sum_k(A\mathbf{x})_k y_k=\sum_k\sum_i A_{ki}x_iy_k=\sum_i\sum_k A^T_{ik}x_iy_k=\left(\mathbf{x},A^T\mathbf{y}\right)$

52. Use the result of Problem 51 to verify directly that $(AB)^T=B^TA^T$ without making any reference to subscripts.

    $(AB\mathbf{x},\mathbf{y})=\left(B\mathbf{x},A^T\mathbf{y}\right)=\left(\mathbf{x},B^TA^T\mathbf{y}\right)$

    $(AB\mathbf{x},\mathbf{y})=\left(\mathbf{x},(AB)^T\mathbf{y}\right)$

    Since this is true for all $\mathbf{x}$, it follows that, in particular, it holds for

    $$\mathbf{x}=B^TA^T\mathbf{y}-(AB)^T\mathbf{y}$$

    and so from the axioms of the dot product,

    $$\left(B^TA^T\mathbf{y}-(AB)^T\mathbf{y},B^TA^T\mathbf{y}-(AB)^T\mathbf{y}\right)=0$$

    and so $B^TA^T\mathbf{y}-(AB)^T\mathbf{y}=\mathbf{0}$. However, this is true for all $\mathbf{y}$ and so $B^TA^T-(AB)^T=0$.

53. A matrix A is called a projection if $A^2 = A$. Here is a matrix.

$$\begin{pmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$$

Show that this is a projection. Show that a vector in the column space of a projection is left unchanged by multiplication by A.

$$\begin{pmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$$

A typical vector in the column space of A is $A\mathbf{x}$. Therefore,

$$A\left(A\mathbf{x}\right) = A^2\mathbf{x} = A\mathbf{x}$$

B.8 Exercises 6.3

1. Find the determinants of the following matrices.

   (a) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 0 & 9 & 8 \end{pmatrix}$ (The answer is 31.)

   (b) $\begin{pmatrix} 4 & 3 & 2 \\ 1 & 7 & 8 \\ 3 & -9 & 3 \end{pmatrix}$ (The answer is 375.)

   (c) $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 3 \\ 4 & 1 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix}$, (The answer is -2.)

2. Find the following determinant by expanding along the first row and second column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 6$$

3. Find the following determinant by expanding along the first column and third row.

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 2$$

4. Find the following determinant by expanding along the second row and first column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 6$$

5. Compute the determinant by cofactor expansion. Pick the easiest row or column to use.

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix} = -4$$

6. Find the determinant using row operations.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -4 & 1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -4 & 1 & 2 \end{vmatrix} = -6$$

7. Find the determinant using row operations.

$$\begin{vmatrix} 2 & 1 & 3 \\ 2 & 4 & 2 \\ 1 & 4 & -5 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 2 & 4 & 2 \\ 1 & 4 & -5 \end{vmatrix} = -32$$

8. Find the determinant using row operations.

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 3 \\ -1 & 0 & 3 & 1 \\ 2 & 3 & 2 & -2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 3 \\ -1 & 0 & 3 & 1 \\ 2 & 3 & 2 & -2 \end{vmatrix} = 63$$

9. Find the determinant using row operations.

$$\left|\begin{array}{cccc} 1 & 4 & 1 & 2 \\ 3 & 2 & -2 & 3 \\ -1 & 0 & 3 & 3 \\ 2 & 1 & 2 & -2 \end{array}\right|$$

$$\left|\begin{array}{cccc} 1 & 4 & 1 & 2 \\ 3 & 2 & -2 & 3 \\ -1 & 0 & 3 & 3 \\ 2 & 1 & 2 & -2 \end{array}\right| = 211$$

10. Verify an example of each property of determinants found in Theorems 6.1.23 - 6.1.25 for 2×2 matrices.

This is routine. Here is the verification of the formula for products.

$\det \left(\left(\begin{array}{cc} x & y \\ z & w \end{array} \right) \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \right) = \det \left(\begin{array}{cc} ax + cy & bx + dy \\ cw + az & dw + bz \end{array} \right)$

$= adwx - bcwx - adyz + bcyz$

$\det \left(\begin{array}{cc} x & y \\ z & w \end{array} \right) \det \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) = (ad - bc)(wx - yz) = adwx - bcwx - adyz + bcyz$

which is the same thing.

11. An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right), \left(\begin{array}{cc} a & c \\ b & d \end{array} \right)$$

It does not change the determinant. This was just taking the transpose.

12. An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right), \left(\begin{array}{cc} c & d \\ a & b \end{array} \right)$$

In this case two rows were switched and so the resulting determinant is -1 times the first.

13. An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right), \left(\begin{array}{cc} a & b \\ a + c & b + d \end{array} \right)$$

The determinant is unchanged. It was just the first row added to the second.

14. An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right), \left(\begin{array}{cc} a & b \\ 2c & 2d \end{array} \right)$$

The second row was multiplied by 2 so the determinant of the result is 2 times the original determinant.

15. An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

In this case the two columns were switched so the determinant of the second is -1 times the determinant of the first.

16. Let A be an $r \times r$ matrix and suppose there are $r-1$ rows (columns) such that all rows (columns) are linear combinations of these $r-1$ rows (columns). Show $\det(A) = 0$.

If the determinant is non zero, then this will be unchanged with row operations applied to the matrix. However, by assumption, you can obtain a row of zeros by doing row operations. Thus the determinant must have been zero after all.

17. Show $\det(aA) = a^n \det(A)$ where here A is an $n \times n$ matrix and a is a scalar.

$\det(aA) = \det(aIA) = \det(aI)\det(A) = a^n \det(A)$. The matrix which has a down the main diagonal has determinant equal to a^n.

18. Illustrate with an example of 2×2 matrices that the determinant of a product equals the product of the determinants.

$\det\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix}\right) = -8$

$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\det\begin{pmatrix} -1 & 2 \\ -5 & 6 \end{pmatrix} = -2(4) = -8$

19. Is it true that $\det(A+B) = \det(A) + \det(B)$? If this is so, explain why it is so and if it is not so, give a counter example.

This is not true at all. Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

20. An $n \times n$ matrix is called **nilpotent** if for some positive integer, k it follows $A^k = 0$. If A is a nilpotent matrix and k is the smallest possible integer such that $A^k = 0$, what are the possible values of $\det(A)$?

It must be 0 because $0 = \det(0) = \det\left(A^k\right) = (\det(A))^k$.

21. A matrix is said to be **orthogonal** if $A^T A = I$. Thus the inverse of an orthogonal matrix is just its transpose. What are the possible values of $\det(A)$ if A is an orthogonal matrix?

You would need $\det\left(AA^T\right) = \det(A)\det\left(A^T\right) = \det(A)^2 = 1$ and so $\det(A) = 1$, or -1.

22. Fill in the missing entries to make the matrix orthogonal as in Problem 21.

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{12}}{6} \\ \frac{1}{\sqrt{2}} & \_ & \_ \\ \_ & \frac{\sqrt{6}}{3} & \_ \end{pmatrix}.$$

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{12}}{6} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{12}}{6} \\ 0 & \frac{\sqrt{6}}{3} & -\frac{\sqrt{12}}{6} \end{pmatrix}$$

23. Let A and B be two $n \times n$ matrices. $A \sim B$ (A is **similar** to B) means there exists an invertible matrix, S such that $A = S^{-1}BS$. Show that if $A \sim B$, then $B \sim A$. Show also that $A \sim A$ and that if $A \sim B$ and $B \sim C$, then $A \sim C$.

    If $A = S^{-1}BS$, then $SAS^{-1} = B$ and so if $A \sim B$, then $B \sim A$. It is obvious that $A \sim A$ because you can let $S = I$. Say $A \sim B$ and $B \sim C$. Then $A = P^{-1}BP$ and $B = Q^{-1}CQ$. Therefore,
$$A = P^{-1}Q^{-1}CQP = (QP)^{-1} C (QP)$$
    and so $A \sim C$.

24. In the context of Problem 23 show that if $A \sim B$, then $\det(A) = \det(B)$.

    $\det(A) = \det\left(S^{-1}BS\right) = \det\left(S^{-1}\right)\det(B)\det(S) = \det(B)\det\left(S^{-1}S\right) = \det(B)$.

25. Two $n \times n$ matrices, A and B, are similar if $B = S^{-1}AS$ for some invertible $n \times n$ matrix, S. Show that if two matrices are similar, they have the same characteristic polynomials. The characteristic polynomial of an $n \times n$ matrix, M is the polynomial, $\det(\lambda I - M)$.

    Say $M = S^{-1}NS$. Then
$$\begin{aligned}\det(\lambda I - M) &= \det\left(\lambda I - S^{-1}NS\right) \\ &= \det\left(\lambda S^{-1}S - S^{-1}NS\right) = \det\left(S^{-1}(\lambda I - N)S\right)\end{aligned}$$
$$= \det(\lambda I - N)$$
    by the above problem.

26. Tell whether the statement is true or false.

    (a) If A is a 3×3 matrix with a zero determinant, then one column must be a multiple of some other column.
    False. Consider $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 5 & 4 \\ 0 & 3 & 3 \end{pmatrix}$

    (b) If any two columns of a square matrix are equal, then the determinant of the matrix equals zero.
    True

    (c) For A and B two $n \times n$ matrices, $\det(A + B) = \det(A) + \det(B)$.
    False

    (d) For A an $n \times n$ matrix, $\det(3A) = 3\det(A)$
    False

    (e) If A^{-1} exists then $\det\left(A^{-1}\right) = \det(A)^{-1}$.
    True

    (f) If B is obtained by multiplying a single row of A by 4 then $\det(B) = 4\det(A)$.
    True

    (g) For A an $n \times n$ matrix, $\det(-A) = (-1)^n \det(A)$.
    True

    (h) If A is a real $n \times n$ matrix, then $\det\left(A^T A\right) \geq 0$.
    True

    (i) Cramer's rule is useful for finding solutions to systems of linear equations in which there is an infinite set of solutions.
    False

(j) If $A^k = 0$ for some positive integer, k, then $\det(A) = 0$.
True

(k) If $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$, then $\det(A) = 0$.
True

27. Use Cramer's rule to find the solution to

$$\begin{aligned} x + 2y &= 1 \\ 2x - y &= 2 \end{aligned}$$

Solution is: $[x = 1, y = 0]$

28. Use Cramer's rule to find the solution to

$$\begin{aligned} x + 2y + z &= 1 \\ 2x - y - z &= 2 \\ x + z &= 1 \end{aligned}$$

Solution is: $[x = 1, y = 0, z = 0]$. For example,

$$y = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 1 & 0 & 1 \end{vmatrix}} = 0$$

29. Here is a matrix,

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

$\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix} = -13$ and so it has an inverse. This inverse is

$$\frac{1}{-13} \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \end{pmatrix}^T$$

$$= \frac{1}{-13} \begin{pmatrix} -1 & 3 & -6 \\ 3 & -9 & 5 \\ -4 & -1 & 2 \end{pmatrix}^T = \begin{pmatrix} \frac{1}{13} & -\frac{3}{13} & \frac{4}{13} \\ -\frac{3}{13} & \frac{9}{13} & \frac{1}{13} \\ \frac{6}{13} & -\frac{5}{13} & -\frac{2}{13} \end{pmatrix}$$

30. Here is a matrix,

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

$\det\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix} = 7$ so it has an inverse. This inverse is $\frac{1}{7}\begin{pmatrix} 1 & 3 & -6 \\ -2 & 1 & 5 \\ 2 & -1 & 2 \end{pmatrix}^T$

$= \begin{pmatrix} \frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} & -\frac{1}{7} \\ -\frac{6}{7} & \frac{5}{7} & \frac{2}{7} \end{pmatrix}$

31. Here is a matrix,

$$\begin{pmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

$\det\begin{pmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 3$ so it has an inverse which is $\begin{pmatrix} 1 & 0 & -3 \\ -\frac{2}{3} & \frac{1}{3} & \frac{5}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \end{pmatrix}$

32. Here is a matrix,

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 6 & 7 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

33. Here is a matrix,

$$\begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Determine whether the matrix has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

$\det\begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} = 2$ and so it has an inverse. The inverse turns out to equal

$\begin{pmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & -\frac{9}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$

34. Use the formula for the inverse in terms of the cofactor matrix to find if possible the inverses of the matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 4 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

If the inverse does not exist, explain why.

$\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1, \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 4 & 1 & 1 \end{vmatrix} = -15, \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 0$. Thus the first two have inverses and the third does not. The inverses of the first two are respectively

$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{15} & -\frac{1}{15} & \frac{4}{15} \\ -\frac{4}{15} & \frac{11}{15} & \frac{1}{15} \\ \frac{8}{15} & -\frac{7}{15} & -\frac{2}{15} \end{pmatrix}$.

35. Here is a matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

No. It has a nonzero determinant for all t.

36. Here is a matrix,

$$\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{pmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

$\det \begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{pmatrix} = t^3 + 2$ and so it has no inverse when $t = -\sqrt[3]{2}$.

37. Here is a matrix,

$$\begin{pmatrix} e^t & \cosh t & \sinh t \\ e^t & \sinh t & \cosh t \\ e^t & \cosh t & \sinh t \end{pmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

$\det \begin{pmatrix} e^t & \cosh t & \sinh t \\ e^t & \sinh t & \cosh t \\ e^t & \cosh t & \sinh t \end{pmatrix} = 0$ and so this matrix fails to have a nonzero determinant at any value of t.

38. Show that if $\det (A) \neq 0$ for A an $n \times n$ matrix, it follows that if $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$.

If $\det (A) \neq 0$, then A^{-1} exists and so you could multiply on both sides on the left by A^{-1} and obtain that $\mathbf{x} = \mathbf{0}$.

39. Suppose A, B are $n \times n$ matrices and that $AB = I$. Show that then $BA = I$. **Hint:** You might do something like this: First explain why $\det (A), \det (B)$ are both nonzero. Then $(AB) A = A$ and then show $BA (BA - I) = 0$. From this use what is given to conclude $A (BA - I) = 0$. Then use Problem 38.

You have $1 = \det (A) \det (B)$. Hence both A and B have inverses. Letting $\mathbf{x}$ be given,

$$A (BA - I) \mathbf{x} = (AB) A\mathbf{x} - A\mathbf{x} = A\mathbf{x} - A\mathbf{x} = \mathbf{0}$$

and so it follows from the above problem that $(BA - I) \mathbf{x} = \mathbf{0}$. Since $\mathbf{x}$ is arbitrary, it follows that $BA = I$.

40. Use the formula for the inverse in terms of the cofactor matrix to find the inverse of the matrix,

$$A = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & e^t \cos t - e^t \sin t & e^t \cos t + e^t \sin t \end{pmatrix}.$$

$\det\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t\cos t & e^t\sin t \\ 0 & e^t\cos t - e^t\sin t & e^t\cos t + e^t\sin t \end{pmatrix} = e^{3t}$. Hence the inverse is

$$e^{-3t}\begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t}\cos t + e^{2t}\sin t & -\left(e^{2t}\cos t - e^{2t}\sin\right)t \\ 0 & -e^{2t}\sin t & e^{2t}\cos(t) \end{pmatrix}^T$$

$$= \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t}(\cos t + \sin t) & -(\sin t)e^{-t} \\ 0 & -e^{-t}(\cos t - \sin t) & (\cos t)e^{-t} \end{pmatrix}$$

41. Find the inverse if it exists of the matrix,

$$\begin{pmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{pmatrix}.$$

$$\begin{pmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2}e^{-t} & 0 & \frac{1}{2}e^{-t} \\ \frac{1}{2}\cos t + \frac{1}{2}\sin t & -\sin t & \frac{1}{2}\sin t - \frac{1}{2}\cos t \\ \frac{1}{2}\sin t - \frac{1}{2}\cos t & \cos t & -\frac{1}{2}\cos t - \frac{1}{2}\sin t \end{pmatrix}$$

42. Here is a matrix,

$$\begin{pmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{pmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

$\det\begin{pmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{pmatrix} = 5e^{-t} \neq 0$ and so this matrix is always invertible.

43. Suppose A is an upper triangular matrix. Show that A^{-1} exists if and only if all elements of the main diagonal are non zero. Is it true that A^{-1} will also be upper triangular? Explain. Is everything the same for lower triangular matrices?

The given condition is what it takes for the determinant to be non zero. Recall that the determinant of an upper triangular matrix is just the product of the entries on the main diagonal.

44. If A, B, and C are each $n \times n$ matrices and ABC is invertible, why are each of A, B, and C invertible.

This is obvious because $\det(ABC) = \det(A)\det(B)\det(C)$ and if this product is nonzero, then each determinant in the product is nonzero and so each of these matrices is invertible.

45. Let $F(t) = \det\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$. Verify

$$F'(t) = \det\begin{pmatrix} a'(t) & b'(t) \\ c(t) & d(t) \end{pmatrix} + \det\begin{pmatrix} a(t) & b(t) \\ c'(t) & d'(t) \end{pmatrix}.$$

Now suppose

$$F(t) = \det\begin{pmatrix} a(t) & b(t) & c(t) \\ d(t) & e(t) & f(t) \\ g(t) & h(t) & i(t) \end{pmatrix}.$$

Use Laplace expansion and the first part to verify $F'(t) =$

$$\det\begin{pmatrix} a'(t) & b'(t) & c'(t) \\ d(t) & e(t) & f(t) \\ g(t) & h(t) & i(t) \end{pmatrix} + \det\begin{pmatrix} a(t) & b(t) & c(t) \\ d'(t) & e'(t) & f'(t) \\ g(t) & h(t) & i(t) \end{pmatrix}$$
$$+\det\begin{pmatrix} a(t) & b(t) & c(t) \\ d(t) & e(t) & f(t) \\ g'(t) & h'(t) & i'(t) \end{pmatrix}.$$

Conjecture a general result valid for $n \times n$ matrices and explain why it will be true. Can a similar thing be done with the columns?

Yes, a similar thing can be done with the columns because the determinant of the transpose is the same as the determinant of the matrix.

46. Let $Ly = y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_1(x)\, y' + a_0(x)\, y$ where the a_i are given continuous functions defined on a closed interval, (a,b) and y is some function which has n derivatives so it makes sense to write Ly. Suppose $Ly_k = 0$ for $k = 1, 2, \cdots, n$. The **Wronskian** of these functions, y_i is defined as

$$W(y_1, \cdots, y_n)(x) \equiv \det\begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}$$

Show that for $W(x) = W(y_1, \cdots, y_n)(x)$ to save space,

$$W'(x) = \det\begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n)}(x) & \cdots & y_n^{(n)}(x) \end{pmatrix}.$$

Now use the differential equation, $Ly = 0$ which is satisfied by each of these functions, y_i and properties of determinants presented above to verify that $W' + a_{n-1}(x)\, W = 0$. Give an explicit solution of this linear differential equation, **Abel's formula,** and use your answer to verify that the Wronskian of these solutions to the equation, $Ly = 0$ either vanishes identically on (a,b) or never. **Hint:** To solve the differential equation, let $A'(x) = a_{n-1}(x)$ and multiply both sides of the differential equation by $e^{A(x)}$ and then argue the left side is the derivative of something.

The last formula above follows because $W'(x)$ equals the sum of determinants of matrices which have two equal rows except for the last one in the sum which is the displayed expression. Now let

$$m_i(x) = -\left(a_{n-1}(x)\, y_i^{(n-1)} + \cdots + a_1(x)\, y_i' + a_0(x)\, y_i\right)$$

Since each y_i is a solution to $Ly = 0$, it follows that $y_i^{(n)}(t) = m_i(t)$. Now from the properties of determinants, being linear in each row,

$$\begin{aligned} W'(x) &= -a_{n-1}(x) \det\begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix} \\ &= -a_{n-1}(x)\, W'(x) \end{aligned}$$

Now let $A'(x) = a_{n-1}(x)$. Then

$$\frac{d}{dx}\left(e^{A(x)}W(x)\right) = 0$$

and so $W(x) = Ce^{-A(x)}$. Thus the Wronskian either vanishes for all x or for no x.

47. Find the following determinants.

(a) $\det \begin{pmatrix} 2 & 2+2i & 3-3i \\ 2-2i & 5 & 1-7i \\ 3+3i & 1+7i & 16 \end{pmatrix} = 10$

(b) $\det \begin{pmatrix} 10 & 2+6i & 8-6i \\ 2-6i & 9 & 1-7i \\ 8+6i & 1+7i & 17 \end{pmatrix} = 250$

B.9 Exercises 8.6

1. Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be vectors in $\mathbb{R}^n$. The parallelepiped determined by these vectors $P(\mathbf{u}_1, \cdots, \mathbf{u}_n)$ is defined as

$$P(\mathbf{u}_1, \cdots, \mathbf{u}_n) \equiv \left\{ \sum_{k=1}^{n} t_k \mathbf{u}_k : t_k \in [0,1] \text{ for all } k \right\}.$$

Now let A be an $n \times n$ matrix. Show that

$$\{A\mathbf{x} : \mathbf{x} \in P(\mathbf{u}_1, \cdots, \mathbf{u}_n)\}$$

is also a parallelepiped.

By definition, $A(P(\mathbf{u}_1, \cdots, \mathbf{u}_n)) = \{\sum_{k=1}^{n} t_k A\mathbf{u}_k : t_k \in [0,1] \text{ for all } k\}$ which equals

$$P(A\mathbf{u}_1, \cdots, A\mathbf{u}_n)$$

which is a parallelepiped.

2. In the context of Problem 1, draw $P(\mathbf{e}_1, \mathbf{e}_2)$ where $\mathbf{e}_1, \mathbf{e}_2$ are the standard basis vectors for $\mathbb{R}^2$. Thus $\mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1)$. Now suppose

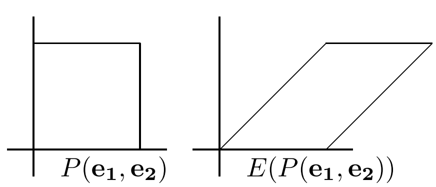
$$E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

where E is the elementary matrix which takes the third row and adds to the first. Draw

$$\{E\mathbf{x} : \mathbf{x} \in P(\mathbf{e}_1, \mathbf{e}_2)\}.$$

In other words, draw the result of doing E to the vectors in $P(\mathbf{e}_1, \mathbf{e}_2)$. Next draw the results of doing the other elementary matrices to $P(\mathbf{e}_1, \mathbf{e}_2)$.

For E the given elementary matrix, the result is as follows.



It is called a shear. Note that it does not change the area.

In case$E = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$, the given square $P(\mathbf{e}_1, \mathbf{e}_2)$ becomes a rectangle. If $\alpha > 0$, the square is stretched by a factor of α in the y direction. If $\alpha < 0$ the rectangle is reflected across the x axis and in addition is stretched by a factor of $|\alpha|$ in the y direction. When $E = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ the result is similar only it features changes in the x direction. These elementary matrices do change the area unless $|\alpha| = 1$ when the area is unchanged. The permutation $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ simply switches $\mathbf{e}_1$ and $\mathbf{e}_2$ so the result appears to be a square just like the one you began with.

3. In the context of Problem 1, either draw or describe the result of doing elementary matrices to $P(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Describe geometrically the conclusion of Corollary 8.5.12.

   It is the same sort of thing. The elementary matrix either switches the $\mathbf{e}_i$ about or it produces a shear or a magnification in one direction.

4. Determine which matrices are in row reduced echelon form.

   (a) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 7 \end{pmatrix}$
   This one is not.

   (b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
   This one is.

   (c) $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$
   This one is.

5. Row reduce the following matrices to obtain the row reduced echelon form. List the pivot columns in the original matrix.

   (a) $\begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 2 & 2 \\ 1 & 1 & 0 & 3 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 3 & 0 & 0 \\ 3 & 2 & 1 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 2 & 1 & 3 \\ -3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

6. Find the rank of the following matrices. If the rank is r, identify r columns **in the original matrix** which have the property that every other column may be written as a linear combination of these. Also find a basis for the row and column spaces of the matrices.

   (a) $\begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. The rank is 3.

(b) $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. The rank is 3

(c) $\begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 2 & 2 \\ 0 & 3 & 2 & 12 & 1 & 6 & 8 \\ 0 & 1 & 1 & 5 & 0 & 2 & 3 \\ 0 & 2 & 1 & 7 & 0 & 3 & 4 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The rank is 3.

(d) $\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & 2 & 6 & 0 & 5 & 4 \\ 0 & 1 & 1 & 2 & 0 & 2 & 2 \\ 0 & 2 & 1 & 4 & 0 & 3 & 2 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The rank is 2.

(e) $\begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 3 & 2 & 6 & 1 & 5 & 1 \\ 0 & 1 & 1 & 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 4 & 0 & 3 & 1 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. The rank is 4.

7. Suppose A is an $m \times n$ matrix. Explain why the rank of A is always no larger than $\min(m, n)$.

   It is because you cannot have more than $\min(m, n)$ nonzero rows in the row reduced echelon form. Recall that the number of pivot columns is the same as the number of nonzero rows from the description of this row reduced echelon form.

8. Let H denote $\operatorname{span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right)$. Find the dimension of H and determine a basis.

   $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. A basis is

$$\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}$$

9. Let H denote $\operatorname{span}\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$. Find the dimension of H and determine a basis.

   $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. A basis is

$$\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}\right\}.$$

10. Let H denote $\operatorname{span}\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$. Find the dimension of H and determine a basis.

$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 4 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. A basis is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

11. Let $M = \left\{ \mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : u_3 = u_1 = 0 \right\}$. Is M a subspace? Explain.

    Yes. This is a subspace. It is closed with respect to vector addition and scalar multiplication.

12. Let $M = \left\{ \mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : u_3 \geq u_1 \right\}$. Is M a subspace? Explain.

    This is not a subspace. $(0, 0, 1, 0)$ is in it. But $(0, 0, -1, 0)$ is not.

13. Let $\mathbf{w} \in \mathbb{R}^4$ and let $M = \left\{ \mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : \mathbf{w} \cdot \mathbf{u} = 0 \right\}$. Is M a subspace? Explain.

    Yes, this is a subspace.

14. Let $M = \left\{ \mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : u_i \geq 0 \text{ for each } i = 1, 2, 3, 4 \right\}$. Is M a subspace? Explain.

    This is clearly not a subspace. $(1, 1, 1, 1)$ is in it. However, $(-1)(1, 1, 1, 1)$ is not.

15. Let $\mathbf{w}, \mathbf{w}_1$ be given vectors in $\mathbb{R}^4$ and define

    $$M = \left\{ \mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : \mathbf{w} \cdot \mathbf{u} = 0 \text{ and } \mathbf{w}_1 \cdot \mathbf{u} = 0 \right\}.$$

    Is M a subspace? Explain.

    Sure. This is a subspace. It is obviously closed with respect to vector addition and scalar multiplication.

16. Let $M = \left\{ \mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : |u_1| \leq 4 \right\}$. Is M a subspace? Explain.

    This is not a subspace. $(1, 1, 1, 1)$ is in it, but $20(1, 1, 1, 1)$ is clearly not.

17. Let $M = \left\{ \mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : \sin(u_1) = 1 \right\}$. Is M a subspace? Explain.

    Not a subspace.

18. Study the definition of span. Explain what is meant by the span of a set of vectors. Include pictures.

    The span of vectors is a long flat thing. In three dimensions, it would be a plane which goes through the origin.

19. Suppose $\{\mathbf{x}_1, \cdots, \mathbf{x}_k\}$ is a set of vectors from $\mathbb{F}^n$. Show that $\mathbf{0}$ is in $\operatorname{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k)$.

    $\sum_{i=1}^{k} 0\mathbf{x}_k = \mathbf{0}$.

20. Study the definition of linear independence. Explain in your own words what is meant by linear independence and linear dependence. Illustrate with pictures.

21. Use Corollary 8.4.17 to prove the following theorem: If A, B are $n \times n$ matrices and if $AB = I$, then $BA = I$ and $B = A^{-1}$. **Hint:** First note that if $AB = I$, then it must be the case that A is onto. Explain why this requires $\operatorname{span}(\text{columns of } A) = \mathbb{F}^n$. Now explain why, using the corollary that this requires A to be one to one. Next explain why $A(BA - I) = 0$ and why the fact that A is one to one implies $BA = I$.

    If $AB = I$, then B must be one to one. Otherwise there exists $\mathbf{x} \neq \mathbf{0}$ such that $B\mathbf{x} = \mathbf{0}$. But then you would have

    $$\mathbf{x} = I\mathbf{x} = AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$$

In particular, the columns of B are linearly independent. Therefore, B is also onto. Also,

$$(BA-I)\,B\mathbf{x} = B\,(AB)\,\mathbf{x} - B\mathbf{x} = \mathbf{0}$$

Since B is onto, it follows that $BA-I$ maps every vector to 0 and so this matrix is 0. Thus $BA = I$.

22. Here are three vectors. Determine whether they are linearly independent or linearly dependent.

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Thus the vectors are linearly independent.

23. Here are three vectors. Determine whether they are linearly independent or linearly dependent.

$$\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 1 & 2 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. These vectors are not linearly independent. They are linearly dependent. In fact -1 times the first added to 2 times the second is the third.

24. Here are three vectors. Determine whether they are linearly independent or linearly dependent.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$\begin{pmatrix} 1 & 4 & 3 \\ 2 & 5 & 1 \\ 3 & 1 & 0 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. These vectors are linearly independent.

25. Here are four vectors. Determine whether they span $\mathbb{R}^3$. Are these vectors linearly independent?

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

These cannot be linearly independent because there are 4 of them. You can have at most three. However, it might be that they span $\mathbb{R}^3$.
$\begin{pmatrix} 1 & 4 & 3 & 2 \\ 2 & 3 & 1 & 4 \\ 3 & 3 & 0 & 6 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The dimension of the span of these vectors is 2 so they do not span $\mathbb{R}^3$.

26. Here are four vectors. Determine whether they span $\mathbb{R}^3$. Are these vectors linearly independent?

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

There are too many vectors to be linearly independent.
$\begin{pmatrix} 1 & 4 & 3 & 2 \\ 2 & 3 & 2 & 4 \\ 3 & 3 & 0 & 6 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. The dimension of the span of these vectors is 3 and so these vectors do span $\mathbb{R}^3$.

27. Determine whether the following vectors are a basis for $\mathbb{R}^3$. If they are, explain why they are and if they are not, give a reason and tell whether they span $\mathbb{R}^3$.

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

These vectors are not linearly independent and so they are not a basis. The remaining question is whether they span.
$\begin{pmatrix} 1 & 4 & 1 & 2 \\ 0 & 3 & 2 & 4 \\ 3 & 3 & 0 & 0 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$. The dimension of the span of these vectors is 3 and so they do span $\mathbb{R}^3$.

28. Determine whether the following vectors are a basis for $\mathbb{R}^3$. If they are, explain why they are and if they are not, give a reason and tell whether they span $\mathbb{R}^3$.

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

They will be a basis if and only if they are linearly independent.
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 3 & 0 & 0 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The vectors are linearly independent and so they are a basis for $\mathbb{R}^3$.

29. Determine whether the following vectors are a basis for $\mathbb{R}^3$. If they are, explain why they are and if they are not, give a reason and tell whether they span $\mathbb{R}^3$.

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

These vectors cannot be a basis because there are too many of them. Also one is 0. However, from the above problem, the first three are linearly independent and so they do span $\mathbb{R}^3$.

30. Determine whether the following vectors are a basis for $\mathbb{R}^3$. If they are, explain why they are and if they are not, give a reason and tell whether they span $\mathbb{R}^3$.

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

They are clearly linearly dependent because on of them is the zero vector. There are also too many of them for them to be linearly independent. The only question is whether they span $\mathbb{R}^3$. However, the third is the sum of the first two and so the dimension of the span of the vectors is only 2. Hence they do not span $\mathbb{R}^3$ either.

31. Consider the vectors of the form

$$\left\{ \begin{pmatrix} 2t+3s \\ s-t \\ t+s \end{pmatrix} : s,t \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

Yes it is. It is the span of the vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$. Since these two vectors are a linearly independent set, the given subspace has dimension 2.

32. Consider the vectors of the form

$$\left\{ \begin{pmatrix} 2t+3s+u \\ s-t \\ t+s \\ u \end{pmatrix} : s,t,u \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^4$? If so, explain why, give a basis for the subspace and find its dimension.

This is the span of the vectors which form the columns of the following matrix.

$\begin{pmatrix} 3 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Thus the vectors are linearly independent. Hence they form a basis and the subspace has dimension 3.

33. Consider the vectors of the form

$$\left\{ \begin{pmatrix} 2t+u \\ t+3u \\ t+s+v \\ u \end{pmatrix} : s,t,u,v \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^4$? If so, explain why, give a basis for the subspace and find its dimension.

It is a subspace and it equals the span of the vectors which form the columns of the following matrix.

$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. It follows that the dimension of this subspace equals 3. A basis is

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

34. If you have 5 vectors in $\mathbb{F}^5$ and the vectors are linearly independent, can it always be concluded they span $\mathbb{F}^5$? Explain.

Yes. If not, there would exist a vector not in the span. But then you could add in this vector and obtain a linearly independent set of vectors with more vectors than a basis.

35. If you have 6 vectors in $\mathbb{F}^5$, is it possible they are linearly independent? Explain.

    No. They can't be.

36. Suppose A is an $m \times n$ matrix and $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$ is a linearly independent set of vectors in $A(\mathbb{F}^n) \subseteq \mathbb{F}^m$. Now suppose $A(\mathbf{z}_i) = \mathbf{w}_i$. Show $\{\mathbf{z}_1, \cdots, \mathbf{z}_k\}$ is also independent.

    Say $\sum_{i=1}^{k} c_i \mathbf{z}_i = \mathbf{0}$. Then you can do A to it.

    $$\sum_{i=1}^{k} c_i A \mathbf{z}_i = \sum_{i=1}^{k} c_i \mathbf{w}_i = \mathbf{0}$$

    and so, by linear independence of the $\mathbf{w}_i$, it follows that each $c_i = 0$.

37. Suppose V, W are subspaces of $\mathbb{F}^n$. Show $V \cap W$ defined to be all vectors which are in both V and W is a subspace also.

    This is obvious. If $\mathbf{x}, \mathbf{y} \in V \cap W$, then for scalars α, β, the linear combination $\alpha \mathbf{x} + \beta \mathbf{y}$ must be in both V and W since they are both subspaces.

38. Suppose V and W both have dimension equal to 7 and they are subspaces of $\mathbb{F}^{10}$. What are the possibilities for the dimension of $V \cap W$? **Hint:** Remember that a linear independent set can be extended to form a basis.

    See the next problem.

39. Suppose V has dimension p and W has dimension q and they are each contained in a subspace, U which has dimension equal to n where $n > \max(p, q)$. What are the possibilities for the dimension of $V \cap W$? **Hint:** Remember that a linear independent set can be extended to form a basis.

    Let $\{x_1, \cdots, x_k\}$ be a basis for $V \cap W$. Then there is a basis for V and W which are respectively

    $$\{x_1, \cdots, x_k, y_{k+1}, \cdots, y_p\}, \ \{x_1, \cdots, x_k, z_{k+1}, \cdots, z_q\}$$

    It follows that you must have $k + p - k + q - k \leq n$ and so you must have

    $$p + q - n \leq k$$

40. If $\mathbf{b} \neq \mathbf{0}$, can the solution set of $A\mathbf{x} = \mathbf{b}$ be a plane through the origin? Explain.

    No. It can't. It does not contain $\mathbf{0}$.

41. Suppose a system of equations has fewer equations than variables and you have found a solution to this system of equations. Is it possible that your solution is the only one? Explain.

    No. There must then be infinitely many solutions. If the system is $A\mathbf{x} = \mathbf{b}$, then there are infinitely many solutions to $A\mathbf{x} = \mathbf{0}$ and so the solutions to $A\mathbf{x} = \mathbf{b}$ are a particular solution to $A\mathbf{x} = \mathbf{b}$ added to the solutions to $A\mathbf{x} = \mathbf{0}$ of which there are infinitely many.

42. Suppose a system of linear equations has a 2×4 augmented matrix and the last column is a pivot column. Could the system of linear equations be consistent? Explain.

    No. This would lead to $0 = 1$.

43. Suppose the coefficient matrix of a system of n equations with n variables has the property that every column is a pivot column. Does it follow that the system of equations must have a solution? If so, must the solution be unique? Explain.

    Yes. It has a unique solution.

44. Suppose there is a unique solution to a system of linear equations. What must be true of the pivot columns in the augmented matrix.

    The last one must not be a pivot column and the ones to the left must each be pivot columns.

45. State whether each of the following sets of data are possible for the matrix equation $A\mathbf{x} = \mathbf{b}$. If possible, describe the solution set. That is, tell whether there exists a unique solution no solution or infinitely many solutions.

    (a) A is a 5×6 matrix, $\text{rank}(A) = 4$ and $\text{rank}(A|\mathbf{b}) = 4$. **Hint:** This says $\mathbf{b}$ is in the span of four of the columns. Thus the columns are not independent.

    Infinite solution set.

    (b) A is a 3×4 matrix, $\text{rank}(A) = 3$ and $\text{rank}(A|\mathbf{b}) = 2$.

    This surely can't happen. If you add in another column, the rank does not get smaller.

    (c) A is a 4×2 matrix, $\text{rank}(A) = 4$ and $\text{rank}(A|\mathbf{b}) = 4$. **Hint:** This says $\mathbf{b}$ is in the span of the columns and the columns must be independent.

    You can't have the rank equal 4 if you only have two columns.

    (d) A is a 5×5 matrix, $\text{rank}(A) = 4$ and $\text{rank}(A|\mathbf{b}) = 5$. **Hint:** This says $\mathbf{b}$ is not in the span of the columns.

    In this case, there is no solution to the system of equations represented by the augmented matrix.

    (e) A is a 4×2 matrix, $\text{rank}(A) = 2$ and $\text{rank}(A|\mathbf{b}) = 2$.

    In this case, there is a unique solution since the columns of A are independent.

46. Suppose A is an $m \times n$ matrix in which $m \leq n$. Suppose also that the rank of A equals m. Show that A maps $\mathbb{F}^n$ onto $\mathbb{F}^m$. **Hint:** The vectors $\mathbf{e}_1, \cdots, \mathbf{e}_m$ occur as columns in the row reduced echelon form for A.

    This says that the columns of A have a subset of m vectors which are linearly independent. Therefore, this set of vectors is a basis for $\mathbb{F}^m$. It follows that the span of the columns is all of $\mathbb{F}^m$. Thus A is onto.

47. Suppose A is an $m \times n$ matrix in which $m \geq n$. Suppose also that the rank of A equals n. Show that A is one to one. **Hint:** If not, there exists a vector, $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$, and this implies at least one column of A is a linear combination of the others. Show this would require the column rank to be less than n.

    The columns are independent. Therefore, A is one to one.

48. Explain why an $n \times n$ matrix, A is both one to one and onto if and only if its rank is n.

    The rank is n is the same as saying the columns are independent which is the same as saying A is one to one which is the same as saying the columns are a basis. Thus the span of the columns of A is all of $\mathbb{F}^n$ and so A is onto. If A is onto, then the columns must be linearly independent since otherwise the span of these columns would have dimension less than n and so the dimension of $\mathbb{F}^n$ would be less than n.

49. Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Show that

$$\dim(\ker(AB)) \leq \dim(\ker(A)) + \dim(\ker(B)).$$

    Hint: Consider the subspace, $B(\mathbb{F}^p) \cap \ker(A)$ and suppose a basis for this subspace is $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$. Now suppose $\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$ is a basis for $\ker(B)$. Let $\{\mathbf{z}_1, \cdots, \mathbf{z}_k\}$ be such that $B\mathbf{z}_i = \mathbf{w}_i$ and argue that

$$\ker(AB) \subseteq \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_r, \mathbf{z}_1, \cdots, \mathbf{z}_k).$$

Here is how you do this. Suppose $AB\mathbf{x} = \mathbf{0}$. Then $B\mathbf{x} \in \ker(A) \cap B(\mathbb{F}^p)$ and so $B\mathbf{x} = \sum_{i=1}^{k} B\mathbf{z}_i$ showing that

$$\mathbf{x} - \sum_{i=1}^{k} \mathbf{z}_i \in \ker(B).$$

Consider $B(\mathbb{F}^p) \cap \ker(A)$ and let a basis be $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$. Then each $\mathbf{w}_i$ is of the form $B\mathbf{z}_i = \mathbf{w}_i$. Therefore, $\{\mathbf{z}_1, \cdots, \mathbf{z}_k\}$ is linearly independent and $AB\mathbf{z}_i = 0$. Now let $\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$ be a basis for $\ker(B)$. If $AB\mathbf{x} = \mathbf{0}$, then $B\mathbf{x} \in \ker(A) \cap B(\mathbb{F}^p)$ and so $B\mathbf{x} = \sum_{i=1}^{k} c_i B\mathbf{z}_i$ which implies

$$\mathbf{x} - \sum_{i=1}^{k} c_i \mathbf{z}_i \in \ker(B)$$

and so it is of the form

$$\mathbf{x} - \sum_{i=1}^{k} c_i \mathbf{z}_i = \sum_{j=1}^{r} d_j \mathbf{u}_j$$

It follows that if $AB\mathbf{x} = \mathbf{0}$ so that $\mathbf{x} \in \ker(AB)$, then

$$\mathbf{x} \in \operatorname{span}(\mathbf{z}_1, \cdots, \mathbf{z}_k, \mathbf{u}_1, \cdots, \mathbf{u}_r).$$

Therefore,

$$\begin{aligned} \dim(\ker(AB)) &\leq k + r = \dim(B(\mathbb{F}^p) \cap \ker(A)) + \dim(\ker(B)) \\ &\leq \dim(\ker(A)) + \dim(\ker(B)) \end{aligned}$$

50. Explain why $A\mathbf{x} = \mathbf{0}$ always has a solution even when A^{-1} does not exist.

    Just let $\mathbf{x} = \mathbf{0}$. Then this solves the equation.

    (a) What can you conclude about A if the solution is unique?

    You can conclude that the columns of A are linearly independent and so A^{-1} exists.

    (b) What can you conclude about A if the solution is not unique?

    You can conclude that the columns of A are dependent and so A^{-1} does not exist. Thus A is not one to one and A is not onto.

51. Suppose $\det(A - \lambda I) = 0$. Show using Theorem 9.2.9 there exists $\mathbf{x} \neq \mathbf{0}$ such that $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

    If $\det(A - \lambda I) = 0$ then $(A - \lambda I)^{-1}$ does not exist and so the columns are not independent which means that for some $\mathbf{x} \neq \mathbf{0}, (A - \lambda I)\mathbf{x} = \mathbf{0}$.

52. Let A be an $n \times n$ matrix and let $\mathbf{x}$ be a nonzero vector such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. When this occurs, the vector $\mathbf{x}$ is called an **eigenvector** and the scalar, λ is called an **eigenvalue**. It turns out that not every number is an eigenvalue. Only certain ones are. Why? **Hint:** Show that if $A\mathbf{x} = \lambda\mathbf{x}$, then $(A - \lambda I)\mathbf{x} = \mathbf{0}$. Explain why this shows that $(A - \lambda I)$ is not one to one and not onto. Now use Theorem 9.2.9 to argue $\det(A - \lambda I) = 0$. What sort of equation is this? How many solutions does it have?

    If A is $n \times n$, then the equation is a polynomial equation of degree n.

53. Let $m < n$ and let A be an $m \times n$ matrix. Show that A is **not** one to one. **Hint:** Consider the $n \times n$ matrix, A_1 which is of the form

$$A_1 \equiv \begin{pmatrix} A \\ 0 \end{pmatrix}$$

where the 0 denotes an $(n-m)\times n$ matrix of zeros. Thus $\det A_1 = 0$ and so A_1 is not one to one. Now observe that $A_1\mathbf{x}$ is the vector,

$$A_1\mathbf{x} = \begin{pmatrix} A\mathbf{x} \\ \mathbf{0} \end{pmatrix}$$

which equals zero if and only if $A\mathbf{x} = \mathbf{0}$. Do this using the Fredholm alternative.

Since A_1 is not one to one, it follows there exists $\mathbf{x} \neq \mathbf{0}$ such that $A_1\mathbf{x} = \mathbf{0}$. Hence $A\mathbf{x} = \mathbf{0}$ although $\mathbf{x} \neq \mathbf{0}$ so it follows that A is not one to one. From another point of view, if A were one to one, then $\ker(A)^{\perp} = \mathbb{R}^n$ and so by the Fredholm alternative, A^T would be onto $\mathbb{R}^n$. However, A^T has only m columns so this cannot take place.

54. Let A be an $m\times n$ real matrix and let $\mathbf{b}\in\mathbb{R}^m$. Show there exists a solution, $\mathbf{x}$ to the system

$$A^TA\mathbf{x} = A^T\mathbf{b}$$

Next show that if $\mathbf{x},\mathbf{x}_1$ are two solutions, then $A\mathbf{x} = A\mathbf{x}_1$. **Hint:** First show that $\left(A^TA\right)^T = A^TA$. Next show if $\mathbf{x}\in\ker\left(A^TA\right)$, then $A\mathbf{x} = \mathbf{0}$. Finally apply the Fredholm alternative. This will give existence of a solution.

That $\left(A^TA\right)^T = A^TA$ follows from the properties of the transpose. Therefore,

$$\left(\ker\left(\left(A^TA\right)^T\right)\right)^{\perp} = \left(\ker\left(A^TA\right)\right)^{\perp}$$

Suppose $A^TA\mathbf{x} = \mathbf{0}$. Then $\left(A^TA\mathbf{x},\mathbf{x}\right) = (A\mathbf{x},A\mathbf{x})$ and so $A\mathbf{x} = \mathbf{0}$. Therefore,

$$\left(A^T\mathbf{b},\mathbf{x}\right) = (\mathbf{b},A\mathbf{x}) = (\mathbf{b},\mathbf{0}) = 0$$

It follows that $A^T\mathbf{b}\in\left(\ker\left(\left(A^TA\right)^T\right)\right)^{\perp}$ and so there exists a solution $\mathbf{x}$ to the equation

$$A^TA\mathbf{x} = A^T\mathbf{b}$$

by the Fredholm alternative.

55. Show that in the context of Problem 54 that if $\mathbf{x}$ is the solution there, then $|\mathbf{b}-A\mathbf{x}| \leq |\mathbf{b}-A\mathbf{y}|$ for every $\mathbf{y}$. Thus $A\mathbf{x}$ is the point of $A(\mathbb{R}^n)$ which is closest to $\mathbf{b}$ of every point in $A(\mathbb{R}^n)$.

$$\begin{aligned}
|\mathbf{b}-A\mathbf{y}|^2 &= |\mathbf{b}-A\mathbf{x}+A\mathbf{x}-A\mathbf{y}|^2 \\
&= |\mathbf{b}-A\mathbf{x}|^2 + |A\mathbf{x}-A\mathbf{y}|^2 + 2(\mathbf{b}-A\mathbf{x},A(\mathbf{x}-\mathbf{y})) \\
&= |\mathbf{b}-A\mathbf{x}|^2 + |A\mathbf{x}-A\mathbf{y}|^2 + 2\left(A^T\mathbf{b}-A^TA\mathbf{x},(\mathbf{x}-\mathbf{y})\right) \\
&= |\mathbf{b}-A\mathbf{x}|^2 + |A\mathbf{x}-A\mathbf{y}|^2
\end{aligned}$$

and so, $A\mathbf{x}$ is closest to $\mathbf{b}$ out of all vectors $A\mathbf{y}$.

56. Let A be an $n\times n$ matrix and consider the matrices $\left\{I,A,A^2,\cdots,A^{n^2}\right\}$. Explain why there exist scalars, c_i not all zero such that

$$\sum_{i=1}^{n^2} c_iA^i = 0.$$

Then argue there exists a polynomial, $p(\lambda)$ of the form

$$\lambda^m + d_{m-1}\lambda^{m-1} + \cdots + d_1\lambda + d_0$$

such that $p(A) = 0$ and if $q(\lambda)$ is another polynomial such that $q(A) = 0$, then $q(\lambda)$ is of the form $p(\lambda) l(\lambda)$ for some polynomial, $l(\lambda)$. This extra special polynomial, $p(\lambda)$ is called the **minimal polynomial**. **Hint:** You might consider an $n \times n$ matrix as a vector in $\mathbb{F}^{n^2}$.

The dimension of $\mathbb{F}^{n^2}$ is n^2. Therefore, there exist scalars c_k such that

$$\sum_{k=0}^{n^2} c_k A^k = 0$$

Let $p(\lambda)$ be the monic polynomial having smallest degree such that $p(A) = 0$. If $q(A) = 0$ then from the Euclidean algorithm,

$$q(\lambda) = p(\lambda) l(\lambda) + r(\lambda)$$

where the degree of $r(\lambda)$ is less than the degree of $p(\lambda)$ or else $r(\lambda)$ equals 0. However, if it is not zero, you could plug in A and obtain

$$0 = q(A) = 0 + r(A)$$

and this would contradict the definition of $p(\lambda)$ as being the polynomial having smallest degree which sends A to 0. Hence $q(\lambda) = p(\lambda) l(\lambda)$.

B.10 Exercises 9.3

1. Study the definition of a linear transformation. State it from memory.
2. Show the map $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ where A is an $m \times n$ matrix and $\mathbf{x}$ is an $m \times 1$ column vector is a linear transformation.

   This is obvious from the properties of matrix multiplication.
3. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/3$.

   $\begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$
4. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/4$.

   $\begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$
5. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $-\pi/3$.

   $\begin{pmatrix} \cos\left(-\frac{\pi}{3}\right) & -\sin\left(-\frac{\pi}{3}\right) \\ \sin\left(-\frac{\pi}{3}\right) & \cos\left(-\frac{\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$
6. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $2\pi/3$.

   $\begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$
7. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/12$. **Hint:** Note that $\pi/12 = \pi/3 - \pi/4$.

$$\begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix} \begin{pmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{4}\sqrt{2}\sqrt{3}+\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2}-\frac{1}{4}\sqrt{2}\sqrt{3} \\ \frac{1}{4}\sqrt{2}\sqrt{3}-\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2}\sqrt{3}+\frac{1}{4}\sqrt{2} \end{pmatrix}$$

8. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $2\pi/3$ and then reflects across the x axis.
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$$

9. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/3$ and then reflects across the x axis.
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$$

10. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/4$ and then reflects across the x axis.
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{pmatrix}$$

11. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/6$ and then reflects across the x axis followed by a reflection across the y axis.
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}$$

12. Find the matrix for the linear transformation which reflects every vector in $\mathbb{R}^2$ across the x axis and then rotates every vector through an angle of $\pi/4$.
$$\begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{pmatrix}$$

13. Find the matrix for the linear transformation which reflects every vector in $\mathbb{R}^2$ across the y axis and then rotates every vector through an angle of $\pi/4$.
$$\begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$$

14. Find the matrix for the linear transformation which reflects every vector in $\mathbb{R}^2$ across the x axis and then rotates every vector through an angle of $\pi/6$.
$$\begin{pmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{pmatrix}$$

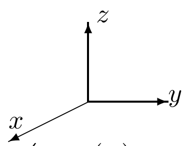
15. Find the matrix for the linear transformation which reflects every vector in $\mathbb{R}^2$ across the y axis and then rotates every vector through an angle of $\pi/6$.
$$\begin{pmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}$$

16. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $5\pi/12$. **Hint:** Note that $5\pi/12 = 2\pi/3 - \pi/4$.
$$\begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} \begin{pmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{4}\sqrt{2}\sqrt{3}-\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2}\sqrt{3}-\frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2}\sqrt{3}+\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2}\sqrt{3}-\frac{1}{4}\sqrt{2} \end{pmatrix}$$

Note that it doesn't matter about the order in this case.

17. Find the matrix of the linear transformation which rotates every vector in $\mathbb{R}^3$ counter clockwise about the z axis when viewed from the positive z axis through an angle of 30° and then reflects through the xy plane.



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) & 0 \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

18. Find the matrix for $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{u} = (1,-2,3)^T$.

   Recall that $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{(\mathbf{v},\mathbf{u})}{|\mathbf{u}|^2}\mathbf{u}$ and so the desired matrix has i^{th} column equal to $\mathrm{proj}_{\mathbf{u}}(\mathbf{e}_i)$.

   Therefore, the matrix desired is $\frac{1}{14}\begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{pmatrix}$

19. Find the matrix for $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{u} = (1,5,3)^T$.

   As in the above, the matrix is $\frac{1}{35}\begin{pmatrix} 1 & 5 & 3 \\ 5 & 25 & 15 \\ 3 & 15 & 9 \end{pmatrix}$

20. Find the matrix for $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ where $\mathbf{u} = (1,0,3)^T$.

   $\frac{1}{10}\begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 9 \end{pmatrix}$

21. Show that the function $T_{\mathbf{u}}$ defined by $T_{\mathbf{u}}(\mathbf{v}) \equiv \mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ is also a linear transformation.

$$\begin{aligned} T_{\mathbf{u}}(a\mathbf{v}+b\mathbf{w}) &= a\mathbf{v}+b\mathbf{w}-\frac{(a\mathbf{v}+b\mathbf{w}\cdot\mathbf{u})}{|\mathbf{u}|^2}\mathbf{u} \\ &= a\mathbf{v}-a\frac{(\mathbf{v}\cdot\mathbf{u})}{|\mathbf{u}|^2}\mathbf{u}+b\mathbf{w}-b\frac{(\mathbf{w}\cdot\mathbf{u})}{|\mathbf{u}|^2}\mathbf{u} \\ &= aT_{\mathbf{u}}(\mathbf{v})+bT_{\mathbf{u}}(\mathbf{w}) \end{aligned}$$

22. Show that $(\mathbf{v}-\mathrm{proj}_{\mathbf{u}}(\mathbf{v}),\mathbf{u}) = 0$ and conclude every vector in $\mathbb{R}^n$ can be written as the sum of two vectors, one which is perpendicular and one which is parallel to the given vector.

   $(\mathbf{v}-\mathrm{proj}_{\mathbf{u}}(\mathbf{v}),\mathbf{u}) = (\mathbf{v},\mathbf{u}) - \left(\frac{(\mathbf{v}\cdot\mathbf{u})}{|\mathbf{u}|^2}\mathbf{u},\mathbf{u}\right) = (\mathbf{v},\mathbf{u}) - (\mathbf{v},\mathbf{u}) = 0.$

   Therefore, $\mathbf{v} = \mathbf{v}-\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) + \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$. The first is perpendicular to $\mathbf{u}$ and the second is a multiple of $\mathbf{u}$ so it is parallel to $\mathbf{u}$.

23. Here are some descriptions of functions mapping $\mathbb{R}^n$ to $\mathbb{R}^n$.

   (a) T multiplies the j^{th} component of $\mathbf{x}$ by a nonzero number b.

(b) T replaces the i^{th} component of $\mathbf{x}$ with b times the j^{th} component added to the i^{th} component.

(c) T switches two components.

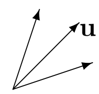
Show these functions are linear and describe their matrices.

Each of these is an elementary matrix. The first is the elementary matrix which multiplies the j^{th} diagonal entry of the identity matrix by b. The second is the elementary matrix which takes b times the j^{th} row and adds to the i^{th} row and the third is just the elementary matrix which switches the i^{th} and the j^{th} rows where the two components are in the i^{th} and j^{th} positions.

24. In Problem 23, sketch the effects of the linear transformations on the unit square in $\mathbb{R}^2$. Give a geometric description of an arbitrary invertible matrix in terms of products of matrices of these special matrices in Problem 23.

This picture was done earlier. Now if A is an arbitrary $n \times n$ matrix, then a product of these elementary matrices $E_1 \cdots E_p$ has the property that $E_1 \cdots E_p A = I$. Hence A is the product of the inverse elementary matrices in the opposite order. Each of these is of the form in the above problem.

25. Let $\mathbf{u} = (a, b)$ be a unit vector in $\mathbb{R}^2$. Find the matrix which reflects all vectors across this vector.



Hint: You might want to notice that $(a, b) = (\cos\theta, \sin\theta)$ for some θ. First rotate through $-\theta$. Next reflect through the x axis which is easy. Finally rotate through θ.

$$\begin{aligned} &\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \\ = &\begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \sin^2\theta - \cos^2\theta \end{pmatrix} \end{aligned}$$

Now to write in terms of (a, b), note that $a/\sqrt{a^2+b^2} = \cos\theta, b/\sqrt{a^2+b^2} = \sin\theta$. Now plug this in to the above. The result is

$$\begin{pmatrix} \frac{a^2-b^2}{a^2+b^2} & 2\frac{ab}{a^2+b^2} \\ 2\frac{ab}{a^2+b^2} & \frac{b^2-a^2}{a^2+b^2} \end{pmatrix} = \frac{1}{a^2+b^2} \begin{pmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{pmatrix}$$

Since this is a unit vector, $a^2 + b^2 = 1$ and so you get

$$\begin{pmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{pmatrix}$$

26. Let $\mathbf{u}$ be a unit vector. Show the linear transformation of the matrix $I - 2\mathbf{u}\mathbf{u}^T$ preserves all distances and satisfies

$$\left(I - 2\mathbf{u}\mathbf{u}^T\right)^T \left(I - 2\mathbf{u}\mathbf{u}^T\right) = I.$$

This matrix is called a Householder reflection. More generally, any matrix Q which satisfies $Q^T Q = QQ^T$ is called an orthogonal matrix. Show the linear transformation determined by an orthogonal matrix always preserves the length of a vector in $\mathbb{R}^n$. **Hint:** First either recall, depending on whether you have done Problem 51 on Page 97, or show that for any matrix A,

$$\langle A\mathbf{x}, \mathbf{y}\rangle = \left\langle \mathbf{x}, A^T\mathbf{y}\right\rangle$$

$$\begin{aligned}\left(I-2\mathbf{u}\mathbf{u}^T\right)^T\left(I-2\mathbf{u}\mathbf{u}^T\right) &= \left(I-2\mathbf{u}\mathbf{u}^T\right)\left(I-2\mathbf{u}\mathbf{u}^T\right)\\ &= I-2\mathbf{u}\mathbf{u}^T-2\mathbf{u}\mathbf{u}^T+4\mathbf{u}\overbrace{\mathbf{u}^T\mathbf{u}}^{=1}\mathbf{u}^T=I\end{aligned}$$

Now, why does this matrix preserve distance? For short, call it Q and note that $Q^TQ=I$. Then

$$|\mathbf{x}|^2=\left(Q^TQ\mathbf{x},\mathbf{x}\right)=(Q\mathbf{x},Q\mathbf{x})=|Q\mathbf{x}|^2$$

and so Q preserves distances.

27. Suppose $|\mathbf{x}|=|\mathbf{y}|$ for $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$. The problem is to find an orthogonal transformation Q, (see Problem 26) which has the property that $Q\mathbf{x}=\mathbf{y}$ and $Q\mathbf{y}=\mathbf{x}$. Show

$$Q\equiv I-2\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}(\mathbf{x}-\mathbf{y})^T$$

does what is desired.

From the above problem this preserves distances and $Q^T=Q$. Now do it to $\mathbf{x}$.

$$Q(\mathbf{x}-\mathbf{y})=\mathbf{x}-\mathbf{y}-2\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}(\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y})=\mathbf{y}-\mathbf{x}$$

$$\begin{aligned}Q(\mathbf{x}+\mathbf{y}) &= \mathbf{x}+\mathbf{y}-2\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}(\mathbf{x}-\mathbf{y})^T(\mathbf{x}+\mathbf{y})\\ &= \mathbf{x}+\mathbf{y}-2\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}\left(|\mathbf{x}|^2-|\mathbf{y}|^2\right)=\mathbf{x}+\mathbf{y}\end{aligned}$$

and so

$$\begin{aligned}Q\mathbf{x}-Q\mathbf{y} &= \mathbf{y}-\mathbf{x}\\ Q\mathbf{x}+Q\mathbf{y} &= \mathbf{x}+\mathbf{y}\end{aligned}$$

Hence, adding these yields $Q\mathbf{x}=\mathbf{y}$ and then subtracting them gives $Q\mathbf{y}=\mathbf{x}$.

28. Let $\mathbf{a}$ be a fixed vector. The function $T_{\mathbf{a}}$ defined by $T_{\mathbf{a}}\mathbf{v}=\mathbf{a}+\mathbf{v}$ has the effect of translating all vectors by adding $\mathbf{a}$. Show this is not a linear transformation. Explain why it is not possible to realize $T_{\mathbf{a}}$ in $\mathbb{R}^3$ by multiplying by a 3×3 matrix.

    Linear transformations take $\mathbf{0}$ to $\mathbf{0}$. Also $T_{\mathbf{a}}(\mathbf{u}+\mathbf{v})\neq T_{\mathbf{a}}\mathbf{u}+T_{\mathbf{a}}\mathbf{v}$.

29. In spite of Problem 28 we can represent both translations and rotations by matrix multiplication at the expense of using higher dimensions. This is done by the homogeneous coordinates. I will illustrate in $\mathbb{R}^3$ where most interest in this is found. For each vector $\mathbf{v}=(v_1,v_2,v_3)^T$, consider the vector in $\mathbb{R}^4$ $(v_1,v_2,v_3,1)^T$. What happens when you do

$$\begin{pmatrix}1&0&0&a_1\\0&1&0&a_2\\0&0&1&a_3\\0&0&0&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\\1\end{pmatrix}?$$

Describe how to consider both rotations and translations all at once by forming appropriate 4×4 matrices.

That product above is of the form

$$\begin{pmatrix} a_1+v_1 \\ a_2+v_2 \\ a_3+v_3 \\ 1 \end{pmatrix}$$

If you just discard the one at the bottom, you have found $\mathbf{a}+\mathbf{v}$. Then to do both rotations and translations, you would look at matrices of the form

$$\begin{pmatrix} R & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$$

where R is a rotation matrix and for translation by $\mathbf{a}$, you use

$$\begin{pmatrix} I & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}$$

To obtain a rotation followed by a translation by $\mathbf{a}$, you would just multiply these two matrices. to get

$$\begin{pmatrix} I & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} R & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}=\begin{pmatrix} R & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix}$$

If you did this to the vector $\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}$, you would get $\begin{pmatrix} R\mathbf{x}+\mathbf{a} \\ 1 \end{pmatrix}$. Now discard the 1 and you have what you want.

30. Write the solution set of the following system as the span of vectors and find a basis for the solution space of the following system.

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 3 & -4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution is: $\begin{pmatrix} -3\hat{t} \\ -\hat{t} \\ \hat{t} \end{pmatrix}, \hat{t}_3 \in \mathbb{R}$. A basis for the solution space is $\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$

31. Using Problem 30 find the general solution to the following linear system.

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 3 & -4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

Note that this has the same matrix as the above problem.

Solution is: $\begin{pmatrix} -3\hat{t}_3 \\ -\hat{t}_3 \\ \hat{t}_3 \end{pmatrix}+\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \hat{t}_3 \in \mathbb{R}$

32. Write the solution set of the following system as the span of vectors and find a basis for the solution space of the following system.

$$\begin{pmatrix} 0 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution is: $\begin{pmatrix} 3\hat{t} \\ 2\hat{t} \\ \hat{t} \end{pmatrix}$, A basis is $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

33. Using Problem 32 find the general solution to the following linear system.

$$\begin{pmatrix} 0 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Solution is: $\begin{pmatrix} 3\hat{t} \\ 2\hat{t} \\ \hat{t} \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$, $\hat{t} \in \mathbb{R}$

34. Write the solution set of the following system as the span of vectors and find a basis for the solution space of the following system.

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & -2 & 0 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution is: $\begin{pmatrix} -4\hat{t} \\ -2\hat{t} \\ \hat{t} \end{pmatrix}$. A basis is $\begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix}$

35. Using Problem 34 find the general solution to the following linear system.

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & -2 & 0 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

$\begin{pmatrix} 1 & -1 & 2 \\ 1 & -2 & 0 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, Solution is: $\begin{pmatrix} -4\hat{t} \\ -2\hat{t} \\ \hat{t} \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, $\hat{t} \in \mathbb{R}$.

36. Write the solution set of the following system as the span of vectors and find a basis for the solution space of the following system.

$$\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & -2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution is: $\begin{pmatrix} -\hat{t} \\ 2\hat{t} \\ \hat{t} \end{pmatrix}$, $\hat{t} \in \mathbb{R}$.

37. Using Problem 36 find the general solution to the following linear system.

$$\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & -2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Solution is: $\begin{pmatrix} -\hat{t} \\ 2\hat{t} \\ \hat{t} \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

38. Write the solution set of the following system as the span of vectors and find a basis for the solution space of the following system.

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & -1 & 3 & 2 \\ 3 & 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution is: $\begin{pmatrix} 0 \\ -\hat{t} \\ -\hat{t} \\ \hat{t} \end{pmatrix}, \hat{t} \in \mathbb{R}$

39. Using Problem 38 find the general solution to the following linear system.

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & -1 & 3 & 2 \\ 3 & 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}.$$

Solution is: $\begin{pmatrix} 0 \\ -\hat{t} \\ -\hat{t} \\ \hat{t} \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \end{pmatrix}$

40. Write the solution set of the following system as the span of vectors and find a basis for the solution space of the following system.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} -s-t \\ s \\ s \\ t \end{pmatrix}, s, t \in \mathbb{R}$. A basis is

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

41. Using Problem 40 find the general solution to the following linear system.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \\ 0 \end{pmatrix}.$$

Solution is: $\begin{pmatrix} -\hat{t} \\ \hat{t} \\ \hat{t} \\ 0 \end{pmatrix} + \begin{pmatrix} -8 \\ 5 \\ 0 \\ 5 \end{pmatrix}$

42. Give an example of a 3×2 matrix with the property that the linear transformation determined by this matrix is one to one but not onto.
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

43. Write the solution set of the following system as the span of vectors and find a basis for the solution space of the following system.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 1 & 2 \\ 3 & 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution is: $\begin{pmatrix} -\frac{1}{2}s - \frac{1}{2}t \\ \frac{1}{2}s - \frac{1}{2}t \\ s \\ t \end{pmatrix}$ for $s, t \in \mathbb{R}$. A basis is

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

44. Using Problem 43 find the general solution to the following linear system.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 1 & 2 \\ 3 & 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}.$$

Solution is: $\begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}s - \frac{1}{2}t \\ \frac{1}{2}s - \frac{1}{2}t \\ s \\ t \end{pmatrix}$

45. Write the solution set of the following system as the span of vectors and find a basis for the solution space of the following system.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution is: $\begin{pmatrix} -\hat{t} \\ \hat{t} \\ \hat{t} \\ 0 \end{pmatrix}$, a basis is $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

46. Using Problem 45 find the general solution to the following linear system.

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \\ 1 \end{pmatrix}.$$

Solution is: $\begin{pmatrix} -\hat{t} \\ \hat{t} \\ \hat{t} \\ 0 \end{pmatrix} + \begin{pmatrix} -9 \\ 5 \\ 0 \\ 6 \end{pmatrix}, t \in \mathbb{R}$.

47. Find $\ker(A)$ for
$$A = \begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 2 & 1 & 1 & 2 \\ 1 & 4 & 4 & 3 & 3 \\ 0 & 2 & 1 & 1 & 2 \end{pmatrix}.$$
Recall $\ker(A)$ is just the set of solutions to $A\mathbf{x} = \mathbf{0}$. It is the solution space to the system $A\mathbf{x} = \mathbf{0}$.

Solution is: $\begin{pmatrix} w - v - 2u \\ -\frac{1}{2}u - \frac{1}{2}v - w \\ u \\ v \\ w \end{pmatrix}, u, v, w \in \mathbb{F}$. A basis is
$$\left\{ \begin{pmatrix} -2 \\ -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

48. Using Problem 47, find the general solution to the following linear system.
$$\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 2 & 1 & 1 & 2 \\ 1 & 4 & 4 & 3 & 3 \\ 0 & 2 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \\ 18 \\ 7 \end{pmatrix}$$
The augmented matrix is
$\begin{pmatrix} 1 & 2 & 3 & 2 & 1 & 11 \\ 0 & 2 & 1 & 1 & 2 & 7 \\ 1 & 4 & 4 & 3 & 3 & 18 \\ 0 & 2 & 1 & 1 & 2 & 7 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 2 & 1 & -1 & 4 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{7}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. It follows that a particular solution is $\begin{pmatrix} 4 \\ 7/2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Then the general solution is
$$s \begin{pmatrix} -2 \\ -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 7/2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

49. Find the general solution to the following linear system.
$$\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 2 & 1 & 1 & 2 \\ 1 & 4 & 4 & 3 & 3 \\ 0 & 2 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 13 \\ 7 \end{pmatrix}$$

50. Suppose $A\mathbf{x}=\mathbf{b}$ has a solution. Explain why the solution is unique precisely when $A\mathbf{x}=\mathbf{0}$ has only the trivial (zero) solution.

    If not, then there would be a infintely many solutions to $A\mathbf{x}=\mathbf{0}$ and each of these added to a solution to $A\mathbf{x}=\mathbf{b}$ would be a solution to $A\mathbf{x}=\mathbf{b}$.

51. Show that if A is an $m\times n$ matrix, then $\ker(A)$ is a subspace.

    If $\mathbf{x},\mathbf{y}\in\ker(A)$ then
    $$A\left(a\mathbf{x}+b\mathbf{y}\right)=aA\mathbf{x}+bA\mathbf{y}=a\mathbf{0}+b\mathbf{0}=\mathbf{0}$$
    and so $\ker(A)$ is closed under linear combinations. Hence it is a subspace.

52. Verify the linear transformation determined by the matrix of 9.2 maps $\mathbb{R}^3$ onto $\mathbb{R}^2$ but the linear transformation determined by this matrix is not one to one.

    This matrix was $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}$. It cannot be one to one because the columns are linearly dependent. It is onto because the columns clearly span $\mathbb{R}^2$ since the rank of the matrix is 2. To see this take the row reduced echelon form.

B.11 Exercises 10.8

1. Find an LU factorization of $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

2. Find an LU factorization of $\begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 1 \\ 5 & 0 & 1 & 3 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 1 \\ 5 & 0 & 1 & 3 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & -10 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -24 & -17 \end{pmatrix}$$

3. Find an LU factorization of the matrix, $\begin{pmatrix} 1 & -2 & -5 & 0 \\ -2 & 5 & 11 & 3 \\ 3 & -6 & -15 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & -2 & -5 & 0 \\ -2 & 5 & 11 & 3 \\ 3 & -6 & -15 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & -5 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. Find an LU factorization of the matrix, $\begin{pmatrix} 1 & -1 & -3 & -1 \\ -1 & 2 & 4 & 3 \\ 2 & -3 & -7 & -3 \end{pmatrix}$.

$$\begin{pmatrix} 1 & -1 & -3 & -1 \\ -1 & 2 & 4 & 3 \\ 2 & -3 & -7 & -3 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5. Find an LU factorization of the matrix, $\begin{pmatrix} 1 & -3 & -4 & -3 \\ -3 & 10 & 10 & 10 \\ 1 & -6 & 2 & -5 \end{pmatrix}$.

$$\begin{pmatrix} 1 & -3 & -4 & -3 \\ -3 & 10 & 10 & 10 \\ 1 & -6 & 2 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & -4 & -3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

6. Find an LU factorization of the matrix, $\begin{pmatrix} 1 & 3 & 1 & -1 \\ 3 & 10 & 8 & -1 \\ 2 & 5 & -3 & -3 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 3 & 1 & -1 \\ 3 & 10 & 8 & -1 \\ 2 & 5 & -3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 & -1 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7. Find an LU factorization of the matrix, $\begin{pmatrix} 3 & -2 & 1 \\ 9 & -8 & 6 \\ -6 & 2 & 2 \\ 3 & 2 & -7 \end{pmatrix}$.

$$\begin{pmatrix} 3 & -2 & 1 \\ 9 & -8 & 6 \\ -6 & 2 & 2 \\ 3 & 2 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & -2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

8. Find an LU factorization of the matrix, $\begin{pmatrix} -3 & -1 & 3 \\ 9 & 9 & -12 \\ 3 & 19 & -16 \\ 12 & 40 & -26 \end{pmatrix}$.

9. Find an LU factorization of the matrix, $\begin{pmatrix} -1 & -3 & -1 \\ 1 & 3 & 0 \\ 3 & 9 & 0 \\ 4 & 12 & 16 \end{pmatrix}$.

$$\begin{pmatrix} -1 & -3 & -1 \\ 1 & 3 & 0 \\ 3 & 9 & 0 \\ 4 & 12 & 16 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & -4 & 1 \end{pmatrix} \begin{pmatrix} -1 & -3 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

10. Find the LU factorization of the coefficient matrix using Dolittle's method and use it to solve the system of equations.

$$x + 2y = 5$$
$$2x + 3y = 6$$

An LU factorization of the coefficient matrix is

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

First solve

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

which gives $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$. Then solve

$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$$

which says that $y = 4$ and $x = -3$.

11. Find the LU factorization of the coefficient matrix using Dolittle's method and use it to solve the system of equations.
$$\begin{gathered} x + 2y + z = 1 \\ y + 3z = 2 \\ 2x + 3y = 6 \end{gathered}$$
An LU factorization of the coefficient matrix is
$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$
First solve
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$$
which yields $u = 1, v = 2, w = 6$. Next solve
$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$$
This yields $z = 6, y = -16, x = 27$.

12. Find the LU factorization of the coefficient matrix using Dolittle's method and use it to solve the system of equations.
$$\begin{gathered} x + 2y + 3z = 5 \\ 2x + 3y + z = 6 \\ x - y + z = 2 \end{gathered}$$

13. Find the LU factorization of the coefficient matrix using Dolittle's method and use it to solve the system of equations.
$$\begin{gathered} x + 2y + 3z = 5 \\ 2x + 3y + z = 6 \\ 3x + 5y + 4z = 11 \end{gathered}$$
An LU factorization of the coefficient matrix is
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$
First solve
$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 11 \end{pmatrix}$$
Solution is: $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix}$. Next solve
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix}$$

Solution is: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7t-3 \\ 4-5t \\ t \end{pmatrix}, t \in \mathbb{R}$.

14. Is there only one LU factorization for a given matrix? **Hint:** Consider the equation

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Sometimes there is more than one LU factorization as is the case in this example. The above equation clearly gives an LU factorization. However, it appears that

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

also. Therefore, this is another.

15. Find a PLU factorization of $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

16. Find a PLU factorization of $\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 4 & 1 \\ 1 & 2 & 1 & 3 & 2 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 4 & 1 \\ 1 & 2 & 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

17. Find a PLU factorization of $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Here are steps for doing this. First use the top row to zero out the entries in the first column which are below the top row. This yields

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & -4 & -2 \end{pmatrix}$$

Obviously there will be no way to obtain an LU factorization because a switch of rows must be done. Switch the second and last row in the original matrix. This will yield one which does have an LU factorization.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Then the original matrix is

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

18. Find a PLU factorization of $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ and use it to solve the systems

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$

First solve

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

This is not too hard because it is the same as solving

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

Thus $t = 1, v = 0, u = 0, w = -1$. Next solve

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

which clearly has no solution. Thus the original problem has no solution either. Note that the augmented matrix has the following row reduced echelon form.

$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Thus there is no solution.

(b) $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

There might not be any solution from part a. Thus suppose there is. First solve

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$\begin{pmatrix} t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} a \\ c-a \\ b-2a \\ 3a-2b-c+d \end{pmatrix}$. Next solve

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ c-a \\ b-2a \\ 3a-2b-c+d \end{pmatrix}$$

Note that there will be no solution unless $3a-2b-c+d=0$, but if this condition holds, then the solution to the problem is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2b-4a+c \\ \frac{3}{2}a-\frac{1}{2}b-\frac{1}{2}c \\ 2a-b \end{pmatrix}$$

To check this, note that

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2b-4a+c \\ \frac{3}{2}a-\frac{1}{2}b-\frac{1}{2}c \\ 2a-b \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ 2b-3a+c \end{pmatrix}$$

where the bottom entry on the right equals d if there is any solution.

19. Find a PLU factorization of $\begin{pmatrix} 0 & 2 & 1 & 2 \\ 2 & 1 & -2 & 0 \\ 2 & 3 & -1 & 2 \end{pmatrix}$ and use it to solve the systems

$$\begin{pmatrix} 0 & 2 & 1 & 2 \\ 2 & 1 & -2 & 0 \\ 2 & 3 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) $\begin{pmatrix} 0 & 2 & 1 & 2 \\ 2 & 1 & -2 & 0 \\ 2 & 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

First solve

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

This is the same as solving

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

and the solution is clearly $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Next solve

$$\begin{pmatrix} 2 & 1 & -2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

This is also fairly easy and gives

$\begin{pmatrix} 2 & 1 & -2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} \frac{5}{4}s + \frac{1}{2}t + \frac{1}{4} \\ \frac{1}{2} - t - \frac{1}{2}s \\ s \\ t \end{pmatrix}$, $s, t \in \mathbb{R}$.

Checking this,

$$\begin{pmatrix} 0 & 2 & 1 & 2 \\ 2 & 1 & -2 & 0 \\ 2 & 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{5}{4}s + \frac{1}{2}t + \frac{1}{4} \\ \frac{1}{2} - t - \frac{1}{2}s \\ s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

so this appears to have worked.

(b) $\begin{pmatrix} 0 & 2 & 1 & 2 \\ 2 & 1 & -2 & 0 \\ 2 & 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

First solve

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

which yields $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$. Next solve

$$\begin{pmatrix} 2 & 1 & -2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{5}{4}s + \frac{1}{2}t \\ 1 - t - \frac{1}{2}s \\ s \\ t \end{pmatrix}, t, s \in \mathbb{R}$$

Check: $\begin{pmatrix} 0 & 2 & 1 & 2 \\ 2 & 1 & -2 & 0 \\ 2 & 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} \frac{5}{4}s + \frac{1}{2}t \\ 1 - t - \frac{1}{2}s \\ s \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

20. Find a QR factorization for the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & -2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & -2 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{11}\sqrt{11} & \frac{13}{66}\sqrt{2}\sqrt{11} & -\frac{1}{6}\sqrt{2} \\ \frac{3}{11}\sqrt{11} & -\frac{5}{66}\sqrt{2}\sqrt{11} & -\frac{1}{6}\sqrt{2} \\ \frac{1}{11}\sqrt{11} & \frac{1}{33}\sqrt{2}\sqrt{11} & \frac{2}{3}\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{11} & -\frac{4}{11}\sqrt{11} & \frac{6}{11}\sqrt{11} \\ 0 & \frac{6}{11}\sqrt{2}\sqrt{11} & \frac{2}{11}\sqrt{2}\sqrt{11} \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

21. Find a QR factorization for the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{11}\sqrt{11} & \frac{1}{11}\sqrt{10}\sqrt{11} & 0 \\ \frac{3}{11}\sqrt{11} & -\frac{3}{110}\sqrt{10}\sqrt{11} & -\frac{1}{10}\sqrt{2}\sqrt{5} \\ \frac{1}{11}\sqrt{11} & -\frac{1}{110}\sqrt{10}\sqrt{11} & \frac{3}{10}\sqrt{2}\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{11} & \frac{2}{11}\sqrt{11} & \frac{6}{11}\sqrt{11} & \frac{4}{11}\sqrt{11} \\ 0 & \frac{2}{11}\sqrt{10}\sqrt{11} & \frac{1}{22}\sqrt{10}\sqrt{11} & -\frac{2}{55}\sqrt{10}\sqrt{11} \\ 0 & 0 & \frac{1}{2}\sqrt{2}\sqrt{5} & \frac{1}{5}\sqrt{2}\sqrt{5} \end{pmatrix}$$

22. If you had a QR factorization, $A = QR$, describe how you could use it to solve the equation $A\mathbf{x} = \mathbf{b}$. This is not usually the way people solve this equation. However, the QR factorization is of great importance in certain other problems, especially in finding eigenvalues and eigenvectors.

    You would have $QR\mathbf{x} = \mathbf{b}$ and so then you would have $R\mathbf{x} = Q^T\mathbf{b}$. Now R is upper triangular and so the solution of this problem is fairly simple.

B.12 Exercises 11.6

1. Maximize and minimize $z = x_1 - 2x_2 + x_3$ subject to the constraints $x_1 + x_2 + x_3 \leq 10$, $x_1 + x_2 + x_3 \geq 2$, and $x_1 + 2x_2 + x_3 \leq 7$ if possible. All variables are nonnegative.

   The constraints lead to the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 10 \\ 1 & 1 & 1 & 0 & -1 & 0 & 2 \\ 1 & 2 & 1 & 0 & 0 & 1 & 7 \end{pmatrix}$$

   The obvious solution is not feasible. Do a row operation.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 & 1 & 0 & 8 \\ 0 & 1 & 0 & -1 & 0 & 1 & -3 \end{pmatrix}$$

   An obvious solution is still not feasible. Do another operation couple of row operations.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 8 \\ 0 & 1 & 0 & 0 & 1 & 1 & 5 \end{pmatrix}$$

At this point, you can spot an obvious feasible solution. Now assemble the simplex tableau.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 5 \\ -1 & 2 & -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Now preserve the simple columns.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 5 \\ 0 & 2 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

First lets work on minimizing this. There is a $+2$. The ratios are then $5, 2$ so the pivot is the 1 on the top of the second column. The next tableau is

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 8 \\ -1 & 0 & -1 & 0 & 2 & 1 & 0 & 3 \\ -2 & 0 & -2 & 0 & 1 & 0 & 1 & -4 \end{pmatrix}$$

There is a 1 on the bottom. The ratios of interest for that column are $3/2, 8$, and so the pivot is the 2 in that column. Then the next tableau is

$$\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{7}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & -\frac{1}{2} & 0 & \frac{13}{2} \\ -1 & 0 & -1 & 0 & 2 & 1 & 0 & 3 \\ -\frac{3}{2} & 0 & -\frac{3}{2} & 0 & 0 & -\frac{1}{2} & 1 & -\frac{11}{2} \end{pmatrix}$$

Now you stop because there are no more positive numbers to the left of 1 on the bottom row. The minimum is $-11/2$ and it occurs when $x_1 = x_3 = x_6 = 0$ and $x_2 = 7/2, x_4 = 13/2, x_6 = -11/2$.

Next consider maximization. The simplex tableau was

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 5 \\ -1 & 2 & -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

This time you work on getting rid of the negative entries. Consider the -1 in the first column. There is only one ratio to consider so 1 is the pivot.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 5 \\ 0 & 3 & 0 & 0 & -1 & 0 & 1 & 2 \end{pmatrix}$$

There remains a -1. The ratios are 5 and 8 so the next pivot is the 1 in the third row and column 5.

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 1 & 0 & 7 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 5 \\ 0 & 4 & 0 & 0 & 0 & 1 & 1 & 7 \end{pmatrix}$$

Then no more negatives remain so the maximum is 7 and it occurs when $x_1 = 7, x_2 = 0, x_3 = 0, x_4 = 3, x_5 = 5, x_6 = 0$.

2. Maximize and minimize the following if possible. All variables are nonnegative.

   (a) $z = x_1 - 2x_2$ subject to the constraints $x_1 + x_2 + x_3 \leq 10$, $x_1 + x_2 + x_3 \geq 1$, and $x_1 + 2x_2 + x_3 \leq 7$
   an augmented matrix for the constraints is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 10 \\ 1 & 1 & 1 & 0 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 1 & 7 \end{pmatrix}$$

The obvious solution is not feasible. Do some row operations.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 9 \\ 1 & 1 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 2 & 1 & 5 \end{pmatrix}$$

Now the obvious solution is feasible. Then include the objective function.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 9 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 2 & 1 & 0 & 5 \\ -1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

First preserve the simple columns.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 9 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 2 & 1 & 0 & 5 \\ -3 & 0 & -2 & 0 & 2 & 0 & 1 & -2 \end{pmatrix}$$

Lets try to maximize first. Begin with the first column. The only pivot is the 1. Use it.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 9 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 6 \\ 0 & 3 & 1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}$$

There is still a -1 on the bottom row to the left of the 1. The ratios are 9 and 6 so the new pivot is the 1 on the third row.

$$\begin{pmatrix} 0 & -1 & 0 & 1 & 0 & -1 & 0 & 3 \\ 1 & 2 & 1 & 0 & 0 & 1 & 0 & 7 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 6 \\ 0 & 4 & 1 & 0 & 0 & 1 & 1 & 7 \end{pmatrix}$$

Then the maximum is 7 when $x_1 = 7$ and $x_1, x_3 = 0$.
Next consider the minimum.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 9 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 2 & 1 & 0 & 5 \\ -3 & 0 & -2 & 0 & 2 & 0 & 1 & -2 \end{pmatrix}$$

There is a positive 2 in the bottom row left of 1. The pivot in that column is the 2.

$$\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & -\frac{1}{2} & 0 & \frac{13}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{7}{2} \\ -1 & 0 & -1 & 0 & 2 & 1 & 0 & 5 \\ -2 & 0 & -1 & 0 & 0 & -1 & 1 & -7 \end{pmatrix}$$

The minimum is -7 and it happens when $x_1 = 0, x_2 = 7/2, x_3 = 0$.

(b) $z = x_1 - 2x_2 - 3x_3$ subject to the constraints $x_1 + x_2 + x_3 \leq 8,\ x_1 + x_2 + 3x_3 \geq 1$, and $x_1 + x_2 + x_3 \leq 7$

This time, lets use artificial variables to find an initial simplex tableau. Thus you add in an artificial variable and then do a minimization procedure.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 8 \\ 1 & 1 & 3 & 0 & -1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

First preserve the seventh column as a simple column by a row operation.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 8 \\ 1 & 1 & 3 & 0 & -1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 7 \\ 1 & 1 & 3 & 0 & -1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Now use the third column.

$$\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & 0 & 1 & \frac{1}{3} & 0 & -\frac{1}{3} & 0 & \frac{23}{3} \\ 1 & 1 & 3 & 0 & -1 & 0 & 1 & 0 & 1 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 1 & -\frac{1}{3} & 0 & \frac{20}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

It follows that a basic solution is feasible if

$$\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & 0 & 1 & \frac{1}{3} & 0 & \frac{23}{3} \\ 1 & 1 & 3 & 0 & -1 & 0 & 1 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 1 & \frac{20}{3} \end{pmatrix}$$

Now assemble the simplex tableau

$$\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & 0 & 1 & \frac{1}{3} & 0 & 0 & \frac{23}{3} \\ 1 & 1 & 3 & 0 & -1 & 0 & 0 & 1 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 1 & 0 & \frac{20}{3} \\ -1 & 2 & 3 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Preserve the simple columns by doing row operations.

$$\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & 0 & 1 & \frac{1}{3} & 0 & 0 & \frac{23}{3} \\ 1 & 1 & 3 & 0 & -1 & 0 & 0 & 1 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 1 & 0 & \frac{20}{3} \\ -2 & 1 & 0 & 0 & 1 & 0 & 1 & -1 \end{pmatrix}$$

Lets do minimization first. Work with the second column.

$$\begin{pmatrix} 0 & 0 & -2 & 1 & 1 & 0 & 0 & 7 \\ 1 & 1 & 3 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 6 \\ -3 & 0 & -3 & 0 & 2 & 0 & 1 & -2 \end{pmatrix}$$

Recall how you have to pick the pivot correctly. There is still a positive number in the bottom row left of the 1. Work with that column.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 6 \\ -3 & 0 & 1 & 0 & 0 & -2 & 1 & -14 \end{pmatrix}$$

There is still a positive number to the left of 1 on the bottom row.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 7 \\ 2 & 2 & 0 & 0 & 1 & 3 & 0 & 20 \\ -4 & -1 & 0 & 0 & 0 & -3 & 1 & -21 \end{pmatrix}$$

It follows that the minimum is -21 and it occurs when $x_1 = x_2 = 0, x_3 = 7$.
Next consider the maximum. The simplex tableau was

$$\begin{pmatrix} \frac{2}{3} & \frac{2}{3} & 0 & 1 & \frac{1}{3} & 0 & 0 & \frac{23}{3} \\ 1 & 1 & 3 & 0 & -1 & 0 & 0 & 1 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 & \frac{1}{3} & 1 & 0 & \frac{20}{3} \\ -2 & 1 & 0 & 0 & 1 & 0 & 1 & -1 \end{pmatrix}$$

Use the first column.

$$\begin{pmatrix} 0 & 0 & -2 & 1 & 1 & 0 & 0 & 7 \\ 1 & 1 & 3 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 6 \\ 0 & 3 & 6 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}$$

There is still a negative on the bottom row to the left of 1.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 6 \\ 0 & 3 & 4 & 0 & 0 & 1 & 1 & 7 \end{pmatrix}$$

There are no more negatives on the bottom row left of 1 so stop. The maximum is 7 and it occurs when $x_1 = 7, x_2 = 0, x_3 = 0$.

(c) $z = 2x_1 + x_2$ subject to the constraints $x_1 - x_2 + x_3 \leq 10$, $x_1 + x_2 + x_3 \geq 1$, and $x_1 + 2x_2 + x_3 \leq 7$.
The augmented matrix for the constraints is

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 & 10 \\ 1 & 1 & 1 & 0 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 1 & 7 \end{pmatrix}$$

The basic solution is not feasible because of that -1. Lets do a row operation to change this. I used the 1 in the second column as a pivot and zeroed out what was above and below it. Now it seems that the basic solution is feasible.

$$\begin{pmatrix} 2 & 0 & 2 & 1 & -1 & 0 & 11 \\ 1 & 1 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 2 & 1 & 5 \end{pmatrix}$$

Assemble the simplex tableau.

$$\begin{pmatrix} 2 & 0 & 2 & 1 & -1 & 0 & 0 & 11 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 2 & 1 & 0 & 5 \\ -2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Then do a row operation to preserve the simple columns.

$$\begin{pmatrix} 2 & 0 & 2 & 1 & -1 & 0 & 0 & 11 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 2 & 1 & 0 & 5 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}$$

Lets do minimization first. Work with the third column because there is a positive entry on the bottom.

$$\begin{pmatrix} 0 & -2 & 0 & 1 & 1 & 0 & 0 & 9 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 6 \\ -2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

It follows that the minimum is 0 and it occurs when $x_1 = x_2 = 0, x_3 = 1$.

Now lets maximize.

$$\begin{pmatrix} 2 & 0 & 2 & 1 & -1 & 0 & 0 & 11 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 2 & 1 & 0 & 5 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}$$

Lets begin with the first column.

$$\begin{pmatrix} 0 & -2 & 0 & 1 & 1 & 0 & 0 & 9 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 6 \\ 0 & 1 & 2 & 0 & -2 & 0 & 1 & 2 \end{pmatrix}$$

There is still a -2 to the left of 1 in the bottom row.

$$\begin{pmatrix} 0 & -3 & 0 & 1 & 0 & -1 & 0 & 3 \\ 1 & 2 & 1 & 0 & 0 & 1 & 0 & 7 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 6 \\ 0 & 3 & 2 & 0 & 0 & 2 & 1 & 14 \end{pmatrix}$$

There are no negatives left so the maximum is 14 and it happens when $x_1 = 7, x_2 = x_3 = 0$.

(d) $z = x_1 + 2x_2$ subject to the constraints $x_1 - x_2 + x_3 \leq 10$, $x_1 + x_2 + x_3 \geq 1$, and $x_1 + 2x_2 + x_3 \leq 7$.

The augmented matrix for the constraints is

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 & 10 \\ 1 & 1 & 1 & 0 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 1 & 7 \end{pmatrix}$$

Of course the obvious or basic solution is not feasible. Do a row operation involving a pivot in the second row to try and fix this.

$$\begin{pmatrix} 2 & 0 & 2 & 1 & -1 & 0 & 11 \\ 1 & 1 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 2 & 1 & 5 \end{pmatrix}$$

Now all is well. Begin to assemble the simplex tableau.

$$\begin{pmatrix} 2 & 0 & 2 & 1 & -1 & 0 & 0 & 11 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 2 & 1 & 0 & 5 \\ -1 & -2 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Do a row operation to preserve the simple columns.

$$\begin{pmatrix} 2 & 0 & 2 & 1 & -1 & 0 & 0 & 11 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 2 & 1 & 0 & 5 \\ 1 & 0 & 2 & 0 & -2 & 0 & 1 & 2 \end{pmatrix}$$

Next lets maximize. There is only one negative number in the bottom left of 1.

$$\begin{pmatrix} \frac{3}{2} & 0 & \frac{3}{2} & 1 & 0 & \frac{1}{2} & 0 & \frac{27}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{7}{2} \\ -1 & 0 & -1 & 0 & 2 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 7 \end{pmatrix}$$

Thus the maximum is 7 and it happens when $x_2 = 7/2, x_3 = x_1 = 0$.
Next lets find the minimum.

$$\begin{pmatrix} 2 & 0 & 2 & 1 & -1 & 0 & 0 & 11 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 2 & 1 & 0 & 5 \\ 1 & 0 & 2 & 0 & -2 & 0 & 1 & 2 \end{pmatrix}$$

Start with the column which has a 2.

$$\begin{pmatrix} 0 & -2 & 0 & 1 & 1 & 0 & 0 & 9 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 6 \\ -1 & -2 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

There are no more positive numbers so the minimum is 0 when $x_1 = x_2 = 0, x_3 = 1$.

3. Consider contradictory constraints, $x_1 + x_2 \geq 12$ and $x_1 + 2x_2 \leq 5$. You know these two contradict but show they contradict using the simplex algorithm.

   You can do this by using artificial variables, x_5. Thus

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 1 & 0 & 12 \\ 1 & 2 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

   Do a row operation to preserve the simple columns.

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 1 & 0 & 12 \\ 1 & 2 & 0 & 1 & 0 & 0 & 5 \\ 1 & 1 & -1 & 0 & 0 & 1 & 12 \end{pmatrix}$$

   Next start with the 1 in the first column.

$$\begin{pmatrix} 0 & -1 & -1 & -1 & 1 & 0 & 7 \\ 1 & 2 & 0 & 1 & 0 & 0 & 5 \\ 0 & -1 & -1 & -1 & 0 & 1 & 7 \end{pmatrix}$$

   Thus the minimum value of $z = x_5$ is 7 but, for there to be a feasible solution, you would need to have this minimum value be 0.

4. Find a solution to the following inequalities for $x, y \geq 0$ if it is possible to do so. If it is not possible, prove it is not possible.

   (a) $\begin{array}{l} 6x + 3y \geq 4 \\ 8x + 4y \leq 5 \end{array}$

   Use an artificial variable. Let $x_1 = x, x_2 = y$ and slack variables x_3, x_4 with artificial variable x_5. Then minimize x_5 as described earlier.

$$\begin{pmatrix} 6 & 3 & -1 & 0 & 1 & 0 & 4 \\ 8 & 4 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

Keep the simple columns.

$$\begin{pmatrix} 6 & 3 & -1 & 0 & 1 & 0 & 4 \\ 8 & 4 & 0 & 1 & 0 & 0 & 5 \\ 6 & 3 & -1 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Now proceed to minimize.

$$\begin{pmatrix} 0 & 0 & -1 & -\frac{3}{4} & 1 & 0 & \frac{1}{4} \\ 8 & 4 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & -1 & -\frac{3}{4} & 0 & 1 & \frac{1}{4} \end{pmatrix}$$

It appears that the minimum value of x_5 is $1/4$ and so there is no solution to these inequalities with $x_1, x_2 \geq 0$.

(b) $\begin{array}{c} 6x_1 + 4x_3 \leq 11 \\ 5x_1 + 4x_2 + 4x_3 \geq 8 \\ 6x_1 + 6x_2 + 5x_3 \leq 11 \end{array}$

The augmented matrix is

$$\begin{pmatrix} 6 & 0 & 4 & 1 & 0 & 0 & 11 \\ 5 & 4 & 4 & 0 & -1 & 0 & 8 \\ 6 & 6 & 5 & 0 & 0 & 1 & 11 \end{pmatrix}$$

It is not clear whether there is a solution which has all variables nonnegative. However, if you do a row operation using 5 in the first column as a pivot, you get

$$\begin{pmatrix} 0 & -\frac{24}{5} & -\frac{4}{5} & 1 & \frac{6}{5} & 0 & \frac{7}{5} \\ 5 & 4 & 4 & 0 & -1 & 0 & 8 \\ 0 & \frac{6}{5} & \frac{1}{5} & 0 & \frac{6}{5} & 1 & \frac{7}{5} \end{pmatrix}$$

and so a solution is $x_1 = 8/5, x_2 = x_3 = 0$.

(c) $\begin{array}{c} 6x_1 + 4x_3 \leq 11 \\ 5x_1 + 4x_2 + 4x_3 \geq 9 \\ 6x_1 + 6x_2 + 5x_3 \leq 9 \end{array}$

The augmented matrix is

$$\begin{pmatrix} 6 & 0 & 4 & 1 & 0 & 0 & 11 \\ 5 & 4 & 4 & 0 & -1 & 0 & 9 \\ 6 & 6 & 5 & 0 & 0 & 1 & 9 \end{pmatrix}$$

Lets include an artificial variable and seek to minimize x_7.

$$\begin{pmatrix} 6 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 11 \\ 5 & 4 & 4 & 0 & -1 & 0 & 1 & 0 & 9 \\ 6 & 6 & 5 & 0 & 0 & 1 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

Preserving the simple columns,

$$\begin{pmatrix} 6 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 11 \\ 5 & 4 & 4 & 0 & -1 & 0 & 1 & 0 & 9 \\ 6 & 6 & 5 & 0 & 0 & 1 & 0 & 0 & 9 \\ 5 & 4 & 4 & 0 & -1 & 0 & 0 & 1 & 9 \end{pmatrix}$$

Use the first column.

$$\begin{pmatrix} 0 & -6 & -1 & 1 & 0 & -1 & 0 & 0 & 2 \\ 0 & -1 & -\frac{1}{6} & 0 & -1 & -\frac{5}{6} & 1 & 0 & \frac{3}{2} \\ 6 & 6 & 5 & 0 & 0 & 1 & 0 & 0 & 9 \\ 0 & -1 & -\frac{1}{6} & 0 & -1 & -\frac{5}{6} & 0 & 1 & \frac{3}{2} \end{pmatrix}$$

It appears that the minimum value for x_7 is $3/2$ and so there will be no solution to these inequalities for which all the variables are nonnegative.

(d) $\begin{array}{c} x_1 - x_2 + x_3 \leq 2 \\ x_1 + 2x_2 \geq 4 \\ 3x_1 + 2x_3 \leq 7 \end{array}$

The augmented matrix is

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 & -1 & 0 & 4 \\ 3 & 0 & 2 & 0 & 0 & 1 & 7 \end{pmatrix}$$

Lets add in an artificial variable and set things up to minimize this artificial variable.

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 & -1 & 0 & 1 & 0 & 4 \\ 3 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

Then

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 & -1 & 0 & 1 & 0 & 4 \\ 3 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 7 \\ 1 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 4 \end{pmatrix}$$

Work with second column.

$$\begin{pmatrix} \frac{3}{2} & 0 & 1 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 4 \\ 1 & 2 & 0 & 0 & -1 & 0 & 1 & 0 & 4 \\ 3 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

It appears the minimum value of x_7 is 0 and so this will mean there is a solution when $x_2 = 2, x_3 = 0, x_1 = 0$.

(e) $\begin{array}{c} 5x_1 - 2x_2 + 4x_3 \leq 1 \\ 6x_1 - 3x_2 + 5x_3 \geq 2 \\ 5x_1 - 2x_2 + 4x_3 \leq 5 \end{array}$

The augmented matrix is

$$\begin{pmatrix} 5 & -2 & 4 & 1 & 0 & 0 & 1 \\ 6 & -3 & 5 & 0 & -1 & 0 & 2 \\ 5 & -2 & 4 & 0 & 0 & 1 & 5 \end{pmatrix}$$

lets introduce an artificial variable x_7 and then minimize x_7.

$$\begin{pmatrix} 5 & -2 & 4 & 1 & 0 & 0 & 0 & 0 & 1 \\ 6 & -3 & 5 & 0 & -1 & 0 & 1 & 0 & 2 \\ 5 & -2 & 4 & 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

Then, preserving the simple columns,

$$\begin{pmatrix} 5 & -2 & 4 & 1 & 0 & 0 & 0 & 0 & 1 \\ 6 & -3 & 5 & 0 & -1 & 0 & 1 & 0 & 2 \\ 5 & -2 & 4 & 0 & 0 & 1 & 0 & 0 & 5 \\ 6 & -3 & 5 & 0 & -1 & 0 & 0 & 1 & 2 \end{pmatrix}$$

work with the first column.

$$\left(\begin{array}{ccccccccc} 5 & -2 & 4 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{3}{5} & \frac{1}{5} & -\frac{6}{5} & -1 & 0 & 1 & 0 & \frac{4}{5} \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 4 \\ 0 & -\frac{3}{5} & \frac{1}{5} & -\frac{6}{5} & -1 & 0 & 0 & 1 & \frac{4}{5} \end{array}\right)$$

There is still a positive entry.

$$\left(\begin{array}{ccccccccc} 5 & -2 & 4 & 1 & 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{4} & -\frac{1}{2} & 0 & -\frac{5}{4} & -1 & 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 4 \\ -\frac{1}{4} & -\frac{1}{2} & 0 & -\frac{5}{4} & -1 & 0 & 0 & 1 & \frac{3}{4} \end{array}\right)$$

It appears that there is no solution to this system of inequalities because the minimum value of x_7 is not 0.

5. Minimize $z = x_1 + x_2$ subject to $x_1 + x_2 \geq 2$, $x_1 + 3x_2 \leq 20$, $x_1 + x_2 \leq 18$. Change to a maximization problem and solve as follows: Let $y_i = M - x_i$. Formulate in terms of y_1, y_2.

   You could find the maximum of $2M - x_1 - x_2$ for the given constraints and this would happen when $x_1 + x_2$ is as small as possible. Thus you would maximize $y_1 + y_2$ subject to the constraints

$$\begin{array}{rcl} M - y_1 + M - y_2 & \geq & 2 \\ M - y_1 + 3(M - y_2) & \leq & 20 \\ M - y_1 + M - y_2 & \leq & 18 \end{array}$$

   To simplify, this would be

$$\begin{array}{rcl} 2M - 2 & \geq & y_1 + y_2 \\ 4M - 20 & \leq & y_1 + 3y_2 \\ 2M - 18 & \leq & y_1 + y_2 \end{array}$$

   You could simply regard M as large enough that $y_i \geq 0$ and use the techniques just developed. The augmented matrix for the constraints is then

$$\left(\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 2M-2 \\ 1 & 3 & 0 & -1 & 0 & 4M-20 \\ 1 & 1 & 0 & 0 & -1 & 2M-18 \end{array}\right)$$

   Here M is large. Use the 3 as a pivot to zero out above and below it.

$$\left(\begin{array}{cccccc} \frac{2}{3} & 0 & 1 & \frac{1}{3} & 0 & \frac{2}{3}M + \frac{14}{3} \\ 1 & 3 & 0 & -1 & 0 & 4M-20 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & -1 & \frac{2}{3}M - \frac{34}{3} \end{array}\right)$$

   Then it is still the case that the basic solution is not feasible. Lets use the bottom row and pick the 2/3 as a pivot.

$$\left(\begin{array}{cccccc} 0 & 0 & 1 & 0 & 1 & 16 \\ 0 & 3 & 0 & -\frac{3}{2} & \frac{3}{2} & 3M-3 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & -1 & \frac{2}{3}M - \frac{34}{3} \end{array}\right)$$

   Now it appears that the basic solution is feasible provided M is large. Then assemble the simplex tableau.

$$\left(\begin{array}{ccccccc} 0 & 0 & 1 & 0 & 1 & 0 & 16 \\ 0 & 3 & 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 3M-3 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & -1 & 0 & \frac{2}{3}M - \frac{34}{3} \\ -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Do row operations to preserve the simple columns.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 16 \\ 0 & 3 & 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 3M-3 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & -1 & 0 & \frac{2}{3}M-\frac{34}{3} \\ 0 & 0 & 0 & 0 & -1 & 1 & 2M-18 \end{pmatrix}$$

There is a negative number to the left of the 1 on the bottom row and we want to maximize so work with this column. Assume M is very large. Then the pivot should be the top entry in this column.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 16 \\ 0 & 3 & -\frac{3}{2} & -\frac{3}{2} & 0 & 0 & 3M-27 \\ \frac{2}{3} & 0 & 1 & \frac{1}{3} & 0 & 0 & \frac{2}{3}M+\frac{14}{3} \\ 0 & 0 & 1 & 0 & 0 & 1 & 2M-2 \end{pmatrix}$$

It follows that the maximum of $y_1 + y_2$ is $2M - 2$ and it happens when $y_1 = M + 7, y_2 = M - 9, y_3 = 0$. Thus the minimum of $x_1 + x_2$ is

$$M - y_1 + M - y_2 = M - (M + 7) + M - (M - 9) = 2$$

B.13 Exercises 12.4

1. State the eigenvalue problem from an algebraic perspective.

2. State the eigenvalue problem from a geometric perspective.

3. If A is the matrix of a linear transformation which rotates all vectors in $\mathbb{R}^2$ through $30°$, explain why A cannot have any real eigenvalues.

   If it did have $\lambda \in \mathbb{R}$ as an eigenvalue, then there would exist a vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for λ a real number. Therefore, $A\mathbf{x}$ and $\mathbf{x}$ would need to be parallel. However, this doesn't happen because A rotates the vectors.

4. If A is an $n \times n$ matrix and c is a nonzero constant, compare the eigenvalues of A and cA.

   Say $A\mathbf{x} = \lambda\mathbf{x}$. Then $cA\mathbf{x} = c\lambda\mathbf{x}$ and so the eigenvalues of cA are just $c\lambda$ where λ is an eigenvalue of A.

5. If A is an invertible $n \times n$ matrix, compare the eigenvalues of A and A^{-1}. More generally, for m an arbitrary integer, compare the eigenvalues of A and A^m.

   $A^m\mathbf{x} = \lambda^m\mathbf{x}$ for any integer. In the case of -1, $A^{-1}\lambda\mathbf{x} = AA^{-1}\mathbf{x} = \mathbf{x}$ so $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$. Thus the eigenvalues of A^{-1} are just λ^{-1} where λ is an eigenvalue of A.

6. Let A, B be invertible $n \times n$ matrices which commute. That is, $AB = BA$. Suppose $\mathbf{x}$ is an eigenvector of B. Show that then $A\mathbf{x}$ must also be an eigenvector for B.

   $BA\mathbf{x} = AB\mathbf{x} = A\lambda\mathbf{x} = \lambda A\mathbf{x}$. Here is is assumed that $B\mathbf{x} = \lambda\mathbf{x}$.

7. Suppose A is an $n \times n$ matrix and it satisfies $A^m = A$ for some m a positive integer larger than 1. Show that if λ is an eigenvalue of A then $|\lambda|$ equals either 0 or 1.

   Let $\mathbf{x}$ be the eigenvector. Then $A^m\mathbf{x} = \lambda^m\mathbf{x}$,$A^m\mathbf{x} = A\mathbf{x} = \lambda\mathbf{x}$ and so

   $$\lambda^m = \lambda$$

   Hence if $\lambda \neq 0$, then

   $$\lambda^{m-1} = 1$$

   and so $|\lambda| = 1$.

8. Show that if $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \lambda\mathbf{y}$, then whenever a, b are scalars,

$$A(a\mathbf{x} + b\mathbf{y}) = \lambda(a\mathbf{x} + b\mathbf{y}).$$

Does this imply that $a\mathbf{x} + b\mathbf{y}$ is an eigenvector? Explain.

The formula is obvious from properties of matrix multiplications. However, this vector might not be an eigenvector because it might equal $\mathbf{0}$ and

$$\boxed{\textbf{Eigenvectors are never } 0}$$

9. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -1 & -1 & 7 \\ -1 & 0 & 4 \\ -1 & -1 & 5 \end{pmatrix}.$$

Determine whether the matrix is defective.

$\begin{pmatrix} -1 & -1 & 7 \\ -1 & 0 & 4 \\ -1 & -1 & 5 \end{pmatrix}$, eigenvectors: $\left\{ \begin{matrix} 3 \\ 1 \\ 1 \end{matrix} \right\} \leftrightarrow 1, \left\{ \begin{matrix} 2 \\ 1 \\ 1 \end{matrix} \right\} \leftrightarrow 2$. This is a defective matrix.

10. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -3 & -7 & 19 \\ -2 & -1 & 8 \\ -2 & -3 & 10 \end{pmatrix}.$$

Determine whether the matrix is defective.

$\begin{pmatrix} -3 & -7 & 19 \\ -2 & -1 & 8 \\ -2 & -3 & 10 \end{pmatrix}$, eigenvectors: $\left\{ \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\} \leftrightarrow 1, \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\} \leftrightarrow 2, \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\} \leftrightarrow 3$

This matrix has distinct eigenvalues so it is not defective.

11. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -7 & -12 & 30 \\ -3 & -7 & 15 \\ -3 & -6 & 14 \end{pmatrix}.$$

Determine whether the matrix is defective.

$\begin{pmatrix} -7 & -12 & 30 \\ -3 & -7 & 15 \\ -3 & -6 & 14 \end{pmatrix}$, eigenvectors: $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \right\} \leftrightarrow -1, \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\} \leftrightarrow 2$

This matrix is not defective because, even though $\lambda = 1$ is a repeated eigenvalue, it has a 2 dimensional eigenspace.

12. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 7 & -2 & 0 \\ 8 & -1 & 0 \\ -2 & 4 & 6 \end{pmatrix}.$$

Determine whether the matrix is defective.

$\begin{pmatrix} 7 & -2 & 0 \\ 8 & -1 & 0 \\ -2 & 4 & 6 \end{pmatrix}$, eigenvectors: $\left\{ \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{pmatrix} \right\} \leftrightarrow 3, \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \leftrightarrow 6$

This matrix is defective.

13. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 3 & -2 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 4 \end{pmatrix}.$$

Determine whether the matrix is defective.

$\begin{pmatrix} 3 & -2 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 4 \end{pmatrix}$, eigenvectors: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \right\} \leftrightarrow 3, \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \leftrightarrow 6$

This matrix is not defective.

14. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 6 & 8 & -23 \\ 4 & 5 & -16 \\ 3 & 4 & -12 \end{pmatrix}$$

Determine whether the matrix is defective.

15. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 5 & 2 & -5 \\ 12 & 3 & -10 \\ 12 & 4 & -11 \end{pmatrix}.$$

Determine whether the matrix is defective.

$\begin{pmatrix} 5 & 2 & -5 \\ 12 & 3 & -10 \\ 12 & 4 & -11 \end{pmatrix}$, eigenvectors: $\left\{ \begin{pmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{5}{6} \\ 0 \\ 1 \end{pmatrix} \right\} \leftrightarrow -1$

This matrix is defective. In this case, there is only one eigenvalue, -1 of multiplicity 3 but the dimension of the eigenspace is only 2.

16. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 20 & 9 & -18 \\ 6 & 5 & -6 \\ 30 & 14 & -27 \end{pmatrix}.$$

Determine whether the matrix is defective.

$\begin{pmatrix} 20 & 9 & -18 \\ 6 & 5 & -6 \\ 30 & 14 & -27 \end{pmatrix}$, eigenvectors:

$\left\{ \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \\ 1 \end{pmatrix} \right\} \leftrightarrow -1, \left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} \right\} \leftrightarrow 2, \left\{ \begin{pmatrix} \frac{9}{13} \\ \frac{3}{13} \\ 1 \end{pmatrix} \right\} \leftrightarrow -3$

Not defective.

17. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 26 & -17 \\ 4 & -4 & 4 \\ -9 & -18 & 9 \end{pmatrix}.$$

Determine whether the matrix is defective.

18. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 3 & -1 & -2 \\ 11 & 3 & -9 \\ 8 & 0 & -6 \end{pmatrix}.$$

Determine whether the matrix is defective.

$\begin{pmatrix} 3 & -1 & -2 \\ 11 & 3 & -9 \\ 8 & 0 & -6 \end{pmatrix}$, eigenvectors: $\left\{ \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \\ 1 \end{pmatrix} \right\} \leftrightarrow 0$ This one is defective.

19. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} -2 & 1 & 2 \\ -11 & -2 & 9 \\ -8 & 0 & 7 \end{pmatrix}.$$

Determine whether the matrix is defective.

$\begin{pmatrix} -2 & 1 & 2 \\ -11 & -2 & 9 \\ -8 & 0 & 7 \end{pmatrix}$, eigenvectors: $\left\{ \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \\ 1 \end{pmatrix} \right\} \leftrightarrow 1$

This is defective.

20. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 3 & -2 \\ 2 & 2 & -1 \end{pmatrix}.$$

Determine whether the matrix is defective.

$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 3 & -2 \\ 2 & 2 & -1 \end{pmatrix}$, eigenvectors: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \leftrightarrow 1, \left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix} \right\} \leftrightarrow 2$

This is non defective.

21. Find the complex eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 4 & -2 & -2 \\ 0 & 2 & -2 \\ 2 & 0 & 2 \end{pmatrix}.$$

$\begin{pmatrix} 4 & -2 & -2 \\ 0 & 2 & -2 \\ 2 & 0 & 2 \end{pmatrix}$, eigenvectors:

$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} \leftrightarrow 4, \left\{ \begin{pmatrix} -i \\ -i \\ 1 \end{pmatrix} \right\} \leftrightarrow 2-2i, \left\{ \begin{pmatrix} i \\ i \\ 1 \end{pmatrix} \right\} \leftrightarrow 2+2i$

22. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 9 & 6 & -3 \\ 0 & 6 & 0 \\ -3 & -6 & 9 \end{pmatrix}.$$

Determine whether the matrix is defective.

$\begin{pmatrix} 9 & 6 & -3 \\ 0 & 6 & 0 \\ -3 & -6 & 9 \end{pmatrix}$, eigenvectors: $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \leftrightarrow 6, \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \leftrightarrow 12$

This is nondefective.

23. Find the complex eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 4 & -2 & -2 \\ 0 & 2 & -2 \\ 2 & 0 & 2 \end{pmatrix}$. Determine whether the matrix is defective.

$\begin{pmatrix} 4 & -2 & -2 \\ 0 & 2 & -2 \\ 2 & 0 & 2 \end{pmatrix}$, eigenvectors:

$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} \leftrightarrow 4, \left\{ \begin{pmatrix} -i \\ -i \\ 1 \end{pmatrix} \right\} \leftrightarrow 2-2i, \left\{ \begin{pmatrix} i \\ i \\ 1 \end{pmatrix} \right\} \leftrightarrow 2+2i$

24. Find the complex eigenvalues and eigenvectors of the matrix $\begin{pmatrix} -4 & 2 & 0 \\ 2 & -4 & 0 \\ -2 & 2 & -2 \end{pmatrix}$. Determine whether the matrix is defective.

$\begin{pmatrix} -4 & 2 & 0 \\ 2 & -4 & 0 \\ -2 & 2 & -2 \end{pmatrix}$, eigenvectors:

$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \leftrightarrow -2, \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} \leftrightarrow -6$

This is not defective.

25. Find the complex eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & 1 & -6 \\ 7 & -5 & -6 \\ -1 & 7 & 2 \end{pmatrix}$. Determine whether the matrix is defective.

$\begin{pmatrix} 1 & 1 & -6 \\ 7 & -5 & -6 \\ -1 & 7 & 2 \end{pmatrix}$, eigenvectors:

$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} \leftrightarrow -6, \left\{ \begin{pmatrix} -i \\ -i \\ 1 \end{pmatrix} \right\} \leftrightarrow 2-6i, \left\{ \begin{pmatrix} i \\ i \\ 1 \end{pmatrix} \right\} \leftrightarrow 2+6i$

This is not defective.

26. Find the complex eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 4 & 2 & 0 \\ -2 & 4 & 0 \\ -2 & 2 & 6 \end{pmatrix}$. Determine whether the matrix is defective.

$\begin{pmatrix} 4 & 2 & 0 \\ -2 & 4 & 0 \\ -2 & 2 & 6 \end{pmatrix}$, eigenvectors:

$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \leftrightarrow 6, \left\{ \begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix} \right\} \leftrightarrow 4-2i, \left\{ \begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix} \right\} \leftrightarrow 4+2i$

This is not defective.

27. Let A be a real 3×3 matrix which has a complex eigenvalue of the form $a + ib$ where $b \neq 0$. Could A be defective? Explain. Either give a proof or an example.

    The characteristic polynomial is of degree three and it has real coefficients. Therefore, there is a real root and two distinct complex roots. It follows that A cannot be defective because it has three distinct eigenvalues.

28. Let T be the linear transformation which reflects vectors about the x axis. Find a matrix for T and then find its eigenvalues and eigenvectors.

    The matrix of T is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

    $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, eigenvectors: $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \leftrightarrow -1, \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \leftrightarrow 1$

29. Let T be the linear transformation which rotates all vectors in $\mathbb{R}^2$ counterclockwise through an angle of $\pi/2$. Find a matrix of T and then find eigenvalues and eigenvectors.

    $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

    $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, eigenvectors: $\left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\} \leftrightarrow -i, \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\} \leftrightarrow i$

30. Let A be the 2×2 matrix of the linear transformation which rotates all vectors in $\mathbb{R}^2$ through an angle of θ. For which values of θ does A have a real eigenvalue?

    When you think of this geometrically, it is clear that the only two values of θ are 0 and π or these added to integer multiples of 2π.

31. Let T be the linear transformation which reflects all vectors in $\mathbb{R}^3$ through the xy plane. Find a matrix for T and then obtain its eigenvalues and eigenvectors.

    $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

    $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, eigenvectors: $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \leftrightarrow -1, \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \leftrightarrow 1$

32. Find the principle direction for stretching for the matrix,

$$\begin{pmatrix} \frac{13}{9} & \frac{2}{15}\sqrt{5} & \frac{8}{45}\sqrt{5} \\ \frac{2}{15}\sqrt{5} & \frac{6}{5} & \frac{4}{15} \\ \frac{8}{45}\sqrt{5} & \frac{4}{15} & \frac{61}{45} \end{pmatrix}.$$

    The eigenvalues are 2 and 1.

    Corresponding to $\lambda = 2$, you have the eigenvector $\begin{pmatrix} \frac{1}{2}\sqrt{5} \\ \frac{3}{4} \\ 1 \end{pmatrix}$. This is the principal direction for stretching.

33. Find the principle directions for the matrix,

$$\begin{pmatrix} \frac{5}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{5}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Recall these directions are just the eigenvectors of the matrix.

$$\begin{pmatrix} \frac{5}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{5}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ eigenvectors: } \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \leftrightarrow 1, \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \leftrightarrow 2, \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\} \leftrightarrow 3$$

34. Suppose the migration matrix for three locations is

$$\begin{pmatrix} .5 & 0 & .3 \\ .3 & .8 & 0 \\ .2 & .2 & .7 \end{pmatrix}.$$

Find a comparison for the populations in the three locations after a long time.

$$\left(\begin{pmatrix} 1/2 & 0 & 3/10 \\ 3/10 & 4/5 & 0 \\ 1/5 & 1/5 & 7/10 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} -\frac{1}{2} & 0 & \frac{3}{10} \\ \frac{3}{10} & -\frac{1}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & -\frac{3}{10} \end{pmatrix}$$

Now find a nonzero vector which is sends to **0.**

$$\begin{pmatrix} -\frac{1}{2} & 0 & \frac{3}{10} & 0 \\ \frac{3}{10} & -\frac{1}{5} & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & -\frac{3}{10} & 0 \end{pmatrix}, \text{ row echelon form:}$$

$$\begin{pmatrix} 1 & 0 & -\frac{3}{5} & 0 \\ 0 & 1 & -\frac{9}{10} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, the eigenvectors are of the form

$$t \begin{pmatrix} 6 \\ 9 \\ 1 \end{pmatrix}$$

you will have the following percentages in the three locations.

$$\begin{pmatrix} 37.5 \\ 56.25 \\ 6.25 \end{pmatrix}$$

35. Suppose the migration matrix for three locations is

$$\begin{pmatrix} .1 & .1 & .3 \\ .3 & .7 & 0 \\ .6 & .2 & .7 \end{pmatrix}.$$

Find a comparison for the populations in the three locations after a long time.

It is like the above problem. You need to find the nonzero vectors which the matrix $\begin{pmatrix} -9/10 & 1/10 & 3/10 \\ 3/10 & -3/10 & 0 \\ 3/5 & 1/5 & -3/10 \end{pmatrix}$ sends to 0. These vectors are of the form

$$t \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix}, t \in \mathbb{R}$$

and so in terms of percentages in the various locations,

$$\begin{pmatrix} 21.429 \\ 21.429 \\ 57.143 \end{pmatrix}$$

36. You own a trailer rental company in a large city and you have four locations, one in the South East, one in the North East, one in the North West, and one in the South West. Denote these locations by SE,NE,NW, and SW respectively. Suppose you observe that in a typical day, .8 of the trailers starting in SE stay in SE, .1 of the trailers in NE go to SE, .1 of the trailers in NW end up in SE, .2 of the trailers in SW end up in SE, .1 of the trailers in SE end up in NE,.7 of the trailers in NE end up in NE,.2 of the trailers in NW end up in NE,.1 of the trailers in SW end up in NE, .1 of the trailers in SE end up in NW, .1 of the trailers in NE end up in NW, .6 of the trailers in NW end up in NW, .2 of the trailers in SW end up in NW, 0 of the trailers in SE end up in SW, .1 of the trailers in NE end up in SW, .1 of the trailers in NW end up in SW, .5 of the trailers in SW end up in SW. You begin with 20 trailers in each location. Approximately how many will you have in each location after a long time? Will any location ever run out of trailers?

   A table for the above information is

	SE	NE	NW	SW
SE	4/5	1/10	1/10	1/5
NE	1/10	7/10	1/5	1/10
NW	1/10	1/10	3/5	1/5
SW	0	1/10	1/10	1/2

   Then you need to find the vectors which the following matrix sends to 0.

$$\begin{pmatrix} -1/5 & 1/10 & 1/10 & 1/5 \\ 1/10 & -3/10 & 1/5 & 1/10 \\ 1/10 & 1/10 & -2/5 & 1/5 \\ 0 & 1/10 & 1/10 & -1/2 \end{pmatrix}$$

   Thus write the augmented matrix and fine row reduced echelon form.

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{7}{2} & 0 \\ 0 & 1 & 0 & -\frac{29}{10} & 0 \\ 0 & 0 & 1 & -\frac{21}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   It follows that the eigenvectors are of the form

$$t \begin{pmatrix} 7/2 \\ 29/10 \\ 21/10 \\ 1 \end{pmatrix}$$

   You need to choose t such that the sum of the entries of this vector equals the total number of trailors. This number is 80. Hence

   $\frac{19}{2}t = 80$, Solution is: $\frac{160}{19}$. Therefore, you would have

$$\begin{pmatrix} 29.474 \\ 24.421 \\ 17.684 \\ 8.4211 \end{pmatrix}$$

   approximately the following numbers in the various locations.

37. Let A be the $n \times n$, $n > 1$, matrix of the linear transformation which comes from the projection $\mathbf{v} \to \operatorname{proj}_{\mathbf{w}}(\mathbf{v})$. Show that A cannot be invertible. Also show that A has an eigenvalue equal to 1 and that for λ an eigenvalue, $|\lambda| \leq 1$.

    Obviously A cannot be onto because the range of A has dimension 1 and the dimension of this space should be 3 if the matrix is onto. Therefore, A cannot be invertible. Its row reduced echelon form cannot be I since if it were, A would be onto. $A\mathbf{w} = \mathbf{w}$ so it has an eigenvalue equal to 1. Now suppose $A\mathbf{x} = \lambda\mathbf{x}$. Thus, from the Cauchy Schwarz inequality,

$$|\mathbf{x}| = \frac{|\mathbf{x}|\,|\mathbf{w}|}{|\mathbf{w}|^2}\,|\mathbf{w}| \geq \frac{|(\mathbf{x},\mathbf{w})|}{|\mathbf{w}|^2}\,|\mathbf{w}| = |\lambda|\,|\mathbf{x}|$$

    and so $|\lambda| \leq 1$.

38. Let $\mathbf{v}$ be a unit vector and let $A = I - 2\mathbf{v}\mathbf{v}^T$. Show that A has an eigenvalue equal to -1.

    Lets see what it does to $\mathbf{v}$.

$$\left(I - 2\mathbf{v}\mathbf{v}^T\right)\mathbf{v} = \mathbf{v} - 2\mathbf{v}\left(\mathbf{v}^T\mathbf{v}\right) = \mathbf{v} - 2\mathbf{v} = (-1)\,\mathbf{v}$$

    Yes. It has an eigenvector and the eigenvalue is -1.

39. Let M be an $n \times n$ matrix and suppose $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are n eigenvectors which form a linearly independent set. Form the matrix S by making the columns these vectors. Show that S^{-1} exists and that $S^{-1}MS$ is a **diagonal matrix** (one having zeros everywhere except on the main diagonal) having the eigenvalues of M on the main diagonal. When this can be done the matrix is **diagonalizable**.

    Since the vectors are linearly independent, the matrix S has an inverse. Denoting this inverse by

$$S^{-1} = \begin{pmatrix} \mathbf{w}_1^T \\ \vdots \\ \mathbf{w}_n^T \end{pmatrix}$$

    it follows by definition that

$$\mathbf{w}_i^T\mathbf{x}_j = \delta_{ij}.$$

    Therefore,

$$S^{-1}MS = S^{-1}(M\mathbf{x}_1, \cdots, M\mathbf{x}_n) = \begin{pmatrix} \mathbf{w}_1^T \\ \vdots \\ \mathbf{w}_n^T \end{pmatrix}(\lambda_1\mathbf{x}_1, \cdots, \lambda_n\mathbf{x}_n)$$

$$= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

40. Show that a matrix, M is diagonalizable if and only if it has a basis of eigenvectors. **Hint:** The first part is done in Problem 39. It only remains to show that if the matrix can be diagonalized by some matrix, S giving $D = S^{-1}MS$ for D a diagonal matrix, then it has a basis of eigenvectors. Try using the columns of the matrix S.

    The formula says that

$$MS = SD$$

    Letting $\mathbf{x}_k$ denote the k^{th} column of S, it follows from the way we multiply matrices that

$$M\mathbf{x}_k = \lambda_k\mathbf{x}_k$$

    where λ_k is the k^{th} diagonal entry on D.

41. Suppose A is an $n \times n$ matrix which is **diagonally dominant**. This means

$$|a_{ii}| > \sum_j |a_{ij}| .$$

Show that A^{-1} must exist.

The diagonally dominant condition implies that none of the Gerschgorin disks contain 0. Therefore, 0 is not an eigenvalue. Hence A is one to one, hence invertible.

42. Is it possible for a nonzero matrix to have only 0 as an eigenvalue?

Sure. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ works.

43. Let M be an $n \times n$ matrix. Then define the adjoint of M,denoted by M^* to be the transpose of the conjugate of M. For example,

$$\begin{pmatrix} 2 & i \\ 1+i & 3 \end{pmatrix}^* = \begin{pmatrix} 2 & 1-i \\ -i & 3 \end{pmatrix}.$$

A matrix, M, is self adjoint if $M^* = M$. Show the eigenvalues of a self adjoint matrix are all real. If the self adjoint matrix has all real entries, it is called symmetric.

First note that $(AB)^* = B^*A^*$. Say $M\mathbf{x} = \lambda\mathbf{x}, \mathbf{x} \neq \mathbf{0}$. Then

$$\overline{\lambda} |\mathbf{x}|^2 = \overline{\lambda}\mathbf{x}^*\mathbf{x} = (\lambda\mathbf{x})^* \mathbf{x} = (M\mathbf{x})^* \mathbf{x} = \mathbf{x}^* M^* \mathbf{x}$$

$$= \mathbf{x}^* M\mathbf{x} = \mathbf{x}^* \lambda\mathbf{x} = \lambda |\mathbf{x}|^2$$

Hence $\lambda = \overline{\lambda}$.

44. Suppose A is an $n \times n$ matrix consisting entirely of real entries but $a+ib$ is a complex eigenvalue having the eigenvector, $\mathbf{x}+i\mathbf{y}$. Here $\mathbf{x}$ and $\mathbf{y}$ are real vectors. Show that then $a-ib$ is also an eigenvalue with the eigenvector, $\mathbf{x}-i\mathbf{y}$. **Hint:** You should remember that the conjugate of a product of complex numbers equals the product of the conjugates. Here $a+ib$ is a complex number whose conjugate equals $a-ib$.

$A\mathbf{x} = (a+ib)\,\mathbf{x}$. Now take conjugates of both sides. Since A is real,

$$A\overline{\mathbf{x}} = (a-ib)\,\overline{\mathbf{x}}$$

45. Recall an $n \times n$ matrix is said to be symmetric if it has all real entries and if $A = A^T$. Show the eigenvectors and eigenvalues of a real symmetric matrix are real.

These matrices are self adjoint by definition, so this follows from the above problems.

46. Recall an $n \times n$ matrix is said to be skew symmetric if it has all real entries and if $A = -A^T$. Show that any nonzero eigenvalues must be of the form ib where $i^2 = -1$. In words, the eigenvalues are either 0 or pure imaginary. Show also that the eigenvectors corresponding to the pure imaginary eigenvalues are imaginary in the sense that every entry is of the form ix for $x \in \mathbb{R}$.

Suppose A is skew symmetric. Then what about iA?

$$(iA)^* = -iA^* = -iA^T = iA$$

and so iA is self adjoint. Hence it has all real eigenvalues. Therefore, the eigenvalues of A are all of the form $i\lambda$ where λ is real. Now what about the eigenvectors? You need

$$A\mathbf{x} = i\lambda\mathbf{x}$$

where $\lambda \neq 0$ is real and A is real. Then

$$A \operatorname{Re}(\mathbf{x}) = i\lambda \operatorname{Re}(\mathbf{x})$$

The left has all real entries and the right has all pure imaginary entries. Hence $\operatorname{Re}(\mathbf{x}) = \mathbf{0}$ and so $\mathbf{x}$ has all imaginary entries.

47. A discreet dynamical system is of the form

$$\mathbf{x}(k+1) = A\mathbf{x}(k), \ \mathbf{x}(0) = \mathbf{x}_0$$

where A is an $n \times n$ matrix and $\mathbf{x}(k)$ is a vector in $\mathbb{R}^n$. Show first that

$$\mathbf{x}(k) = A^k \mathbf{x}_0$$

for all $k \geq 1$. If A is nondefective so that it has a basis of eigenvectors, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ where

$$A\mathbf{v}_j = \lambda_j \mathbf{v}_j$$

you can write the initial condition $\mathbf{x}_0$ in a unique way as a linear combination of these eigenvectors. Thus

$$\mathbf{x}_0 = \sum_{j=1}^{n} a_j \mathbf{v}_j$$

Now explain why

$$\mathbf{x}(k) = \sum_{j=1}^{n} a_j A^k \mathbf{v}_j = \sum_{j=1}^{n} a_j \lambda_j^k \mathbf{v}_j$$

which gives a formula for $\mathbf{x}(k)$, the solution of the dynamical system.

The first formula is obvious from induction. Thus

$$A^k \mathbf{x}_0 = \sum_{j=1}^{n} a_j A^k \mathbf{v}_j = \sum_{j=1}^{n} a_j \lambda_j^k \mathbf{v}_j$$

because if $A\mathbf{v} = \lambda \mathbf{v}$, then if $A^k \mathbf{x} = \lambda^k \mathbf{x}$, do A to both sides. Thus $A^{k+1}\mathbf{x} = \lambda^{k+1}\mathbf{x}$.

48. Suppose A is an $n \times n$ matrix and let $\mathbf{v}$ be an eigenvector such that $A\mathbf{v} = \lambda \mathbf{v}$. Also suppose the characteristic polynomial of A is

$$\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0$$

Explain why

$$\left(A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I\right) \mathbf{v} = \mathbf{0}$$

If A is nondefective, give a very easy proof of the Cayley Hamilton theorem based on this. Recall this theorem says A satisfies its characteristic equation,

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0.$$

49. Suppose an $n \times n$ nondefective matrix A has only 1 and -1 as eigenvalues. Find A^{12}.

From the above formula,

$$A^{12}\mathbf{x}_0 = \sum_{j=1}^{n} a_j \lambda_j^{12} \mathbf{v}_j = \sum_{j=1}^{n} a_j \mathbf{v}_j = \mathbf{x}_0$$

Thus $A^{12} = I$.

50. Suppose the characteristic polynomial of an $n \times n$ matrix A is $1-\lambda^n$. Find A^{mn} where m is an integer. **Hint:** Note first that A is nondefective. Why?

    The eigenvalues are distinct because they are the n^{th} roots of 1. Hence from the above formula, if $\mathbf{x}$ is a given vector with
    $$\mathbf{x} = \sum_{j=1}^{n} a_j \mathbf{v}_j$$
    then
    $$A^{nm}\mathbf{x} = A^{nm}\sum_{j=1}^{n} a_j \mathbf{v}_j = \sum_{j=1}^{n} a_j A^{nm}\mathbf{v}_j = \sum_{j=1}^{n} a_j \mathbf{v}_j = \mathbf{x}$$
    so $A^{nm} = I$.

51. Sometimes sequences come in terms of a recursion formula. An example is the Fibonacci sequence.
    $$x_0 = 1 = x_1,\ x_{n+1} = x_n + x_{n-1}$$
    Show this can be considered as a discreet dynamical system as follows.
    $$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
    Now find a formula for x_n.

    What are the eigenvalues and eigenvectors of this matrix?

    $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, eigenvectors:
    $$\left\{\begin{pmatrix} \frac{1}{2}-\frac{1}{2}\sqrt{5} \\ 1 \end{pmatrix}\right\} \leftrightarrow \frac{1}{2}-\frac{1}{2}\sqrt{5}, \left\{\begin{pmatrix} \frac{1}{2}\sqrt{5}+\frac{1}{2} \\ 1 \end{pmatrix}\right\} \leftrightarrow \frac{1}{2}\sqrt{5}+\frac{1}{2}$$
    Now also
    $$\left(\frac{1}{2}-\frac{1}{10}\sqrt{5}\right)\begin{pmatrix} \frac{1}{2}-\frac{1}{2}\sqrt{5} \\ 1 \end{pmatrix} + \left(\frac{1}{10}\sqrt{5}+\frac{1}{2}\right)\begin{pmatrix} \frac{1}{2}\sqrt{5}+\frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
    Therefore, the solution is of the form
    $$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} =$$
    $$\left(\frac{1}{2}-\frac{1}{10}\sqrt{5}\right)\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^n \begin{pmatrix} \frac{1}{2}-\frac{1}{2}\sqrt{5} \\ 1 \end{pmatrix}$$
    $$+\left(\frac{1}{10}\sqrt{5}+\frac{1}{2}\right)\left(\frac{1}{2}\sqrt{5}+\frac{1}{2}\right)^n \begin{pmatrix} \frac{1}{2}\sqrt{5}+\frac{1}{2} \\ 1 \end{pmatrix}$$
    In particular,
    $$x_n = \left(\frac{1}{2}-\frac{1}{10}\sqrt{5}\right)\left(\frac{1}{2}-\frac{1}{2}\sqrt{5}\right)^n + \left(\frac{1}{10}\sqrt{5}+\frac{1}{2}\right)\left(\frac{1}{2}\sqrt{5}+\frac{1}{2}\right)^n$$

52. Let A be an $n \times n$ matrix having characteristic polynomial
    $$\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$
    Show that $a_0 = (-1)^n \det(A)$.

    The characteristic polynomial equals $\det(\lambda I - A)$. To get the constant term, you plug in $\lambda = 0$ and obtain $\det(-A) = (-1)^n \det(A)$.

53. Find $\begin{pmatrix} \frac{3}{2} & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}^{35}$. Next find

$$\lim_{n\to\infty} \begin{pmatrix} \frac{3}{2} & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}^{n}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$$

Now it follows that

$$\begin{pmatrix} \frac{3}{2} & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}^{n} = \begin{pmatrix} 2-\frac{1}{2^n} & 2-\frac{2}{2^n} \\ \frac{1}{2^n}-1 & \frac{2}{2^n}-1 \end{pmatrix}$$

Therefore, the above limit equals

$$\begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$$

54. Find e^A where A is the matrix $\begin{pmatrix} \frac{3}{2} & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$ in the above problem.

This is easy to do. It is just

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e^{1/2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 2e-e^{\frac{1}{2}} & 2e-2e^{\frac{1}{2}} \\ e^{\frac{1}{2}}-e & 2e^{\frac{1}{2}}-e \end{pmatrix}$$

B.14 Exercises 13.8

1. Here are some matrices. Label according to whether they are symmetric, skew symmetric, or orthogonal. If the matrix is orthogonal, determine whether it is proper or improper.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

This one is orthogonal and is a proper transformation.

(b) $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ -3 & 4 & 7 \end{pmatrix}$

This is symmetric.

(c) $\begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{pmatrix}$

This one is skew symmetric.

2. Show that every real matrix may be written as the sum of a skew symmetric and a symmetric matrix. **Hint:** If A is an $n \times n$ matrix, show that $B \equiv \frac{1}{2}\left(A - A^T\right)$ is skew symmetric.

$A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$.

3. Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ and consider the matrix, $I - \frac{2\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2}$. Show this matrix is both symmetric and orthogonal.

$\left(I - \frac{2\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2}\right)^T = I - \frac{2\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2}$ so it is symmetric.

$$\begin{aligned}\left(I - \frac{2\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2}\right)\left(I - \frac{2\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2}\right) &= I + 4\frac{\mathbf{x}\mathbf{x}^T\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^4} - 4\frac{\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2} \\ &= I + 4\frac{\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^4} - 4\frac{\mathbf{x}\mathbf{x}^T}{|\mathbf{x}|^2} = I\end{aligned}$$

because $\mathbf{x}^T\mathbf{x} = 1$.

4. For U an orthogonal matrix, explain why $||U\mathbf{x}|| = ||\mathbf{x}||$ for any vector, $\mathbf{x}$. Next explain why if U is an $n \times n$ matrix with the property that $||U\mathbf{x}|| = ||\mathbf{x}||$ for all vectors, $\mathbf{x}$, then U must be orthogonal. Thus the orthogonal matrices are exactly those which preserve distance.
   $\|U\mathbf{x}\|^2 = (U\mathbf{x}, U\mathbf{x}) = \left(U^T U\mathbf{x}, \mathbf{x}\right) = (I\mathbf{x}, \mathbf{x}) = \|\mathbf{x}\|^2$
   Next suppose distance is preserved by U. Then

$$\begin{aligned}(U(\mathbf{x}+\mathbf{y}), U(\mathbf{x}+\mathbf{y})) &= \|Ux\|^2 + \|Uy\|^2 + 2(Ux, Uy) \\ &= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\left(U^T Ux, \mathbf{y}\right)\end{aligned}$$

   But since U preserves distances, it is also the case that

$$(U(\mathbf{x}+\mathbf{y}), U(\mathbf{x}+\mathbf{y})) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2(\mathbf{x}, \mathbf{y})$$

   Hence

$$(\mathbf{x}, \mathbf{y}) = \left(U^T Ux, \mathbf{y}\right)$$

   and so

$$\left(\left(U^T U - I\right)\mathbf{x}, \mathbf{y}\right) = 0$$

   Since y is arbitrary, it follows that $U^T U - I = 0$. Thus U is orthogonal.

5. A quadratic form in three variables is an expression of the form $a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$. Show that every such quadratic form may be written as

$$\begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

   where A is a symmetric matrix.

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a_1 & a_4/2 & a_5/2 \\ a_4/2 & a_2 & a_6/2 \\ a_5/2 & a_6/2 & a_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

6. Given a quadratic form in three variables, x, y, and z, show there exists an orthogonal matrix, U and variables x', y', z' such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = U \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

   with the property that in terms of the new variables, the quadratic form is

$$\lambda_1 (x')^2 + \lambda_2 (y')^2 + \lambda_3 (z')^2$$

   where the numbers, λ_1, λ_2, and λ_3 are the eigenvalues of the matrix, A in Problem 5.

The quadratic form may be written as

$$\mathbf{x}^T A \mathbf{x}$$

where $A = A^T$. By the theorem about diagonalizing a symmetric matrix, there exists an orthogonal matrix U such that

$$U^T A U = D, \ A = U D U^T$$

Then the quadratic form is

$$\mathbf{x}^T U D U^T \mathbf{x} = \left(U^T \mathbf{x}\right)^T D \left(U^T \mathbf{x}\right)$$

where D is a diagonal matrix having the real eigenvalues of A down the main diagonal. Now simply let

$$\mathbf{x}' \equiv U^T \mathbf{x}.$$

7. If A is a symmetric invertible matrix, is it always the case that A^{-1} must be symmetric also? How about A^k for k a positive integer? Explain.

   If A is symmetric, then $A = U^T D U$ for some D a diagonal matrix in which all the diagonal entries are non zero. Hence $A^{-1} = U^{-1} D^{-1} U^{-T}$. Now $U^{-1} U^{-T} = \left(U^T U\right)^{-1} = I^{-1} = I$ and so $A^{-1} = Q D^{-1} Q^T$, where Q is orthogonal. Is this thing on the right symmetric? Take its transpose. This is $Q D^{-1} Q^T$ which is the same thing, so it appears that a symmetric matrix must have symmetric inverse. Now consider raising it to a power.

$$A^k = U^T D^k U$$

   and the right side is clearly symmetric.

8. If A, B are symmetric matrices, does it follow that AB is also symmetric?

   Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

   $AB = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$ which is not symmetric.

9. Suppose A, B are symmetric and $AB = BA$. Does it follow that AB is symmetric?

   $(AB)^T = B^T A^T = BA = AB$ so the answer in this case is yes.

10. Here are some matrices. What can you say about the eigenvalues of these matrices just by looking at them?

    (a) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

    The eigenvalues are all within 1 of 0, and pure imaginary or zero.

    (b) $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ -3 & 4 & 7 \end{pmatrix}$

    The eigenvalues are in $D(1,6) \cup D(7,7)$. They are also real.

    (c) $\begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{pmatrix}$

    The eigenvalues are all imaginary or 0. They are no farther than 7 from 0.

(d) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$

The eigenvalues are 1,2,2.

11. Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} c & 0 & 0 \\ 0 & 0 & -b \\ 0 & b & 0 \end{pmatrix}$. Here b, c are real numbers.

$\begin{pmatrix} c & 0 & 0 \\ 0 & 0 & -b \\ 0 & b & 0 \end{pmatrix}$, eigenvectors: $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\} \leftrightarrow c, \left\{\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}\right\} \leftrightarrow -ib, \left\{\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}\right\} \leftrightarrow ib$

12. Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} c & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix}$. Here a, b, c are real numbers.

$\begin{pmatrix} c & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix}$, eigenvectors: $\left\{\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}\right\} \leftrightarrow a - ib, \left\{\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}\right\} \leftrightarrow a + ib, \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\} \leftrightarrow c$

13. Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{pmatrix} 11 & -1 & -4 \\ -1 & 11 & -4 \\ -4 & -4 & 14 \end{pmatrix}.$$

Hint: Two eigenvalues are 12 and 18.

$\begin{pmatrix} 11 & -1 & -4 \\ -1 & 11 & -4 \\ -4 & -4 & 14 \end{pmatrix}$, eigenvectors:

$\left\{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} \leftrightarrow 6, \left\{\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\} \leftrightarrow 12, \left\{\frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}\right\} \leftrightarrow 18$

14. Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{pmatrix} 4 & 1 & -2 \\ 1 & 4 & -2 \\ -2 & -2 & 7 \end{pmatrix}.$$

Hint: One eigenvalue is 3.

$\begin{pmatrix} 4 & 1 & -2 \\ 1 & 4 & -2 \\ -2 & -2 & 7 \end{pmatrix}$, eigenvectors:

$\left\{\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} \leftrightarrow 3, \left\{\frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}\right\} \leftrightarrow 9$

15. Show that if A is a real symmetric matrix and λ and μ are two different eigenvalues, then if $\mathbf{x}$ is an eigenvector for λ and $\mathbf{y}$ is an eigenvector for μ, then $\mathbf{x} \cdot \mathbf{y} = 0$. Also all eigenvalues are real. Supply reasons for each step in the following argument. First

$$\lambda \mathbf{x}^T \overline{\mathbf{x}} = (A\mathbf{x})^T \overline{\mathbf{x}} \overset{(CD)^T = D^T C^T}{=} \mathbf{x}^T A \overline{\mathbf{x}} \overset{A \text{ is real}}{=} \mathbf{x}^T \overline{A\mathbf{x}} \overset{\lambda \text{ is eigenvalue}}{=} \mathbf{x}^T \overline{\lambda \mathbf{x}} = \overline{\lambda} \mathbf{x}^T \overline{\mathbf{x}}$$

and so $\lambda = \overline{\lambda}$. This shows that all eigenvalues are real. It follows all the eigenvectors are real. Why?

Because A is real. $A\mathbf{x} = \lambda\mathbf{x}$, $A\overline{\mathbf{x}} = \lambda\overline{\mathbf{x}}$, so $\mathbf{x}+\overline{\mathbf{x}}$ is an eigenvector. Hence it can be assumed all eigenvectors are real.

Now let $\mathbf{x}, \mathbf{y}, \mu$ and λ be given as above.

$$\lambda(\mathbf{x}\cdot\mathbf{y}) = \lambda\mathbf{x}\cdot\mathbf{y} = A\mathbf{x}\cdot\mathbf{y} = \mathbf{x}\cdot A\mathbf{y} = \mathbf{x}\cdot\mu\mathbf{y} = \mu(\mathbf{x}\cdot\mathbf{y}) = \mu(\mathbf{x}\cdot\mathbf{y})$$

and so

$$(\lambda-\mu)\,\mathbf{x}\cdot\mathbf{y} = 0.$$

Since $\lambda \neq \mu$, it follows $\mathbf{x}\cdot\mathbf{y} = 0$.

16. Suppose U is an orthogonal $n\times n$ matrix. Explain why $\operatorname{rank}(U) = n$.

    You could observe that $\det\left(UU^T\right) = (\det(U))^2 = 1$ so $\det(U) \neq 0$.

17. Show that if A is an Hermitian matrix and λ and μ are two different eigenvalues, then if $\mathbf{x}$ is an eigenvector for λ and $\mathbf{y}$ is an eigenvector for μ, then $\mathbf{x}\cdot\mathbf{y} = 0$. Also all eigenvalues are real. Supply reasons for each step in the following argument. First

$$\lambda\mathbf{x}\cdot\mathbf{x} = A\mathbf{x}\cdot\mathbf{x} \overset{A\text{ Hermitian}}{=} \mathbf{x}\cdot A\mathbf{x} = \mathbf{x}\cdot\lambda\mathbf{x} \overset{\text{rule for complex inner product}}{=} \overline{\lambda}\mathbf{x}\cdot\mathbf{x}$$

and so $\lambda = \overline{\lambda}$. This shows that all eigenvalues are real. Now let $\mathbf{x}, \mathbf{y}, \mu$ and λ be given as above.

$$\begin{aligned}\lambda(\mathbf{x}\cdot\mathbf{y}) &= \lambda\mathbf{x}\cdot\mathbf{y} = A\mathbf{x}\cdot\mathbf{y} = \\ \mathbf{x}\cdot A\mathbf{y} &= \mathbf{x}\cdot\ \mu\mathbf{y} \overset{\text{rule for complex inner product}}{=} \overline{\mu}(\mathbf{x}\cdot\mathbf{y}) = \mu(\mathbf{x}\cdot\mathbf{y})\end{aligned}$$

and so

$$(\lambda-\mu)\,\mathbf{x}\cdot\mathbf{y} = 0.$$

Since $\lambda \neq \mu$, it follows $\mathbf{x}\cdot\mathbf{y} = 0$.

18. Show that the eigenvalues and eigenvectors of a real matrix occur in conjugate pairs.

    This follows from the observation that if$A\mathbf{x} = \lambda\mathbf{x}$, then $A\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$

19. If a real matrix, A has all real eigenvalues, does it follow that A must be symmetric. If so, explain why and if not, give an example to the contrary.

    Certainly not. $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$

20. Suppose A is a 3×3 symmetric matrix and you have found two eigenvectors which form an orthonormal set. Explain why their cross product is also an eigenvector.

    There exists another eigenvector such that, with these two you have an orthonormal basis. Let the third eigenvector be $\mathbf{u}$ and the two given ones $\mathbf{v}_1, \mathbf{v}_2$. Then $\mathbf{u}$ is perpendicular to the two given vectors and so it has either the same or opposite direction to the cross product. Hence $\mathbf{u} = k\mathbf{v}_1 \times \mathbf{v}_2$ for some scalar k.

21. Study the definition of an orthonormal set of vectors. Write it from memory.

22. Determine which of the following sets of vectors are orthonormal sets. Justify your answer.

    (a) $\{(1,1), (1,-1)\}$
    This one is not orthonormal.

(b) $\left\{\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), (1,0)\right\}$
This one is not orthonormal.

(c) $\left\{\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(\frac{-2}{3}, \frac{-1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)\right\}$
$\begin{pmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ so this is an orthonormal set of vectors.

23. Show that if $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is an orthonormal set of vectors in $\mathbb{F}^n$, then it is a basis. **Hint:** It was shown earlier that this is a linearly independent set. If you wish, replace $\mathbb{F}^n$ with $\mathbb{R}^n$. Do this version if you do not know the dot product for vectors in $\mathbb{C}^n$.

    The vectors are linearly independent as shown earlier. If they do not span all of $\mathbb{F}^n$, then there is a vector $\mathbf{v} \notin \operatorname{span}(\mathbf{u}_1, \cdots, \mathbf{u}_n)$. But then $\{\mathbf{u}_1, \cdots, \mathbf{u}_n, \mathbf{v}\}$ would be a linearly independent set which has more vectors than a spanning set, $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ and this is a contradiction.

24. Fill in the missing entries to make the matrix orthogonal.

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \_ & \_ \\ \_ & \frac{\sqrt{6}}{3} & \_ \end{pmatrix}.$$

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & a \\ 0 & \frac{\sqrt{6}}{3} & b \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & a \\ 0 & \frac{\sqrt{6}}{3} & b \end{pmatrix}^T$$
$$= \begin{pmatrix} 1 & \frac{1}{3}\sqrt{3}a - \frac{1}{3} & \frac{1}{3}\sqrt{3}b - \frac{1}{3} \\ \frac{1}{3}\sqrt{3}a - \frac{1}{3} & a^2 + \frac{2}{3} & ab - \frac{1}{3} \\ \frac{1}{3}\sqrt{3}b - \frac{1}{3} & ab - \frac{1}{3} & b^2 + \frac{2}{3} \end{pmatrix}$$

Must have $a = 1/\sqrt{3}, b = 1/\sqrt{3}$

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 1/\sqrt{3} \\ 0 & \frac{\sqrt{6}}{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 1/\sqrt{3} \\ 0 & \frac{\sqrt{6}}{3} & 1/\sqrt{3} \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

25. Fill in the missing entries to make the matrix orthogonal.

$$\begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \_ & \_ \\ \_ & 0 & \_ \end{pmatrix}$$

$$\begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \frac{-\sqrt{2}}{2} & a \\ -\frac{1}{3} & 0 & b \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \frac{-\sqrt{2}}{2} & a \\ -\frac{1}{3} & 0 & b \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & \frac{1}{6}\sqrt{2}a - \frac{1}{18} & \frac{1}{6}\sqrt{2}b - \frac{2}{9} \\ \frac{1}{6}\sqrt{2}a - \frac{1}{18} & a^2 + \frac{17}{18} & ab - \frac{2}{9} \\ \frac{1}{6}\sqrt{2}b - \frac{2}{9} & ab - \frac{2}{9} & b^2 + \frac{1}{9} \end{pmatrix}$$

$a = \frac{1}{3\sqrt{2}}, b = \frac{4}{3\sqrt{2}}$

$$\begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \frac{-\sqrt{2}}{2} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \frac{-\sqrt{2}}{2} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

26. Fill in the missing entries to make the matrix orthogonal.

$$\begin{pmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & \_ \\ \frac{2}{3} & 0 & \_ \\ \_ & \_ & \frac{4}{15}\sqrt{5} \end{pmatrix}$$

Try $\begin{pmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & c \\ \frac{2}{3} & 0 & d \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{15}\sqrt{5} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & c \\ \frac{2}{3} & 0 & d \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{15}\sqrt{5} \end{pmatrix}^T$

$$= \begin{pmatrix} c^2 + \frac{41}{45} & cd + \frac{2}{9} & \frac{4}{15}\sqrt{5}c - \frac{8}{45} \\ cd + \frac{2}{9} & d^2 + \frac{4}{9} & \frac{4}{15}\sqrt{5}d + \frac{4}{9} \\ \frac{4}{15}\sqrt{5}c - \frac{8}{45} & \frac{4}{15}\sqrt{5}d + \frac{4}{9} & 1 \end{pmatrix}$$

Would require that $c = \frac{2}{3\sqrt{5}}, d = \frac{-5}{3\sqrt{5}}$

$$\begin{pmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{-5}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{15}\sqrt{5} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{-5}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{15}\sqrt{5} \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ so this seems to have worked.

27. Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Hint: One eigenvalue is -2.

$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$, eigenvectors:

$$\left\{ \begin{pmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{pmatrix} \right\} \leftrightarrow 1, \left\{ \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{pmatrix} \right\} \leftrightarrow -2$$

$$\begin{pmatrix} \sqrt{3}/3 & -\sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & \sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & 0 & \frac{1}{3}\sqrt{2}\sqrt{3} \end{pmatrix}^T \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$
$$\cdot \begin{pmatrix} \sqrt{3}/3 & -\sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & \sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & 0 & \frac{1}{3}\sqrt{2}\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

28. Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} 17 & -7 & -4 \\ -7 & 17 & -4 \\ -4 & -4 & 14 \end{pmatrix}.$$

Hint: Two eigenvalues are 18 and 24.
$\begin{pmatrix} 17 & -7 & -4 \\ -7 & 17 & -4 \\ -4 & -4 & 14 \end{pmatrix}$, eigenvectors:
$\left\{ \begin{pmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{pmatrix} \right\} \leftrightarrow 6, \left\{ \begin{pmatrix} -\frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{pmatrix} \right\} \leftrightarrow 18, \left\{ \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{pmatrix} \right\} \leftrightarrow 24$. The matrix U has these as its columns.

29. Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}.$$

Hint: Two eigenvalues are 12 and 18.
$\begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}$, eigenvectors:
$\left\{ \begin{pmatrix} -\frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{pmatrix} \right\} \leftrightarrow 6, \left\{ \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{pmatrix} \right\} \leftrightarrow 12, \left\{ \begin{pmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{pmatrix} \right\} \leftrightarrow 18$. The matrix U has these as its columns.

30. Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} -\frac{5}{3} & \frac{1}{15}\sqrt{6}\sqrt{5} & \frac{8}{15}\sqrt{5} \\ \frac{1}{15}\sqrt{6}\sqrt{5} & -\frac{14}{5} & -\frac{1}{15}\sqrt{6} \\ \frac{8}{15}\sqrt{5} & -\frac{1}{15}\sqrt{6} & \frac{7}{15} \end{pmatrix}$$

Hint: The eigenvalues are $-3, -2, 1$.
$\begin{pmatrix} -\frac{5}{3} & \frac{1}{15}\sqrt{6}\sqrt{5} & \frac{8}{15}\sqrt{5} \\ \frac{1}{15}\sqrt{6}\sqrt{5} & -\frac{14}{5} & -\frac{1}{15}\sqrt{6} \\ \frac{8}{15}\sqrt{5} & -\frac{1}{15}\sqrt{6} & \frac{7}{15} \end{pmatrix}$, eigenvectors:

$\left\{\begin{pmatrix} \frac{1}{6}\sqrt{6} \\ 0 \\ \frac{1}{6}\sqrt{5}\sqrt{6} \end{pmatrix}\right\} \leftrightarrow 1, \left\{\begin{pmatrix} -\frac{1}{3}\sqrt{2}\sqrt{3} \\ -\frac{1}{5}\sqrt{5} \\ \frac{1}{15}\sqrt{2}\sqrt{15} \end{pmatrix}\right\} \leftrightarrow -2, \left\{\begin{pmatrix} -\frac{1}{6}\sqrt{6} \\ \frac{2}{5}\sqrt{5} \\ \frac{1}{30}\sqrt{30} \end{pmatrix}\right\} \leftrightarrow -3$ These vectors are the columns of U.

31. Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

$\begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}$, eigenvectors:

$\left\{\begin{pmatrix} 0 \\ -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}\right\} \leftrightarrow 1, \left\{\begin{pmatrix} 0 \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}\right\} \leftrightarrow 2, \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\} \leftrightarrow 3$. These vectors are the columns of the matrix U.

32. Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix}.$$

$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix}$, eigenvectors:

$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\} \leftrightarrow 2, \left\{\begin{pmatrix} 0 \\ -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}\right\} \leftrightarrow 4, \left\{\begin{pmatrix} 0 \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}\right\} \leftrightarrow 6$. These vectors are the columns of U.

33. Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} \frac{4}{3} & \frac{1}{3}\sqrt{3}\sqrt{2} & \frac{1}{3}\sqrt{2} \\ \frac{1}{3}\sqrt{3}\sqrt{2} & 1 & -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{2} & -\frac{1}{3}\sqrt{3} & \frac{5}{3} \end{pmatrix}$$

Hint: The eigenvalues are $0, 2, 2$ where 2 is listed twice because it is a root of multiplicity 2.

$\begin{pmatrix} \frac{4}{3} & \frac{1}{3}\sqrt{3}\sqrt{2} & \frac{1}{3}\sqrt{2} \\ \frac{1}{3}\sqrt{3}\sqrt{2} & 1 & -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{2} & -\frac{1}{3}\sqrt{3} & \frac{5}{3} \end{pmatrix}$, eigenvectors:

$\left\{\begin{pmatrix} -\frac{1}{5}\sqrt{2}\sqrt{5} \\ \frac{1}{5}\sqrt{3}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \end{pmatrix}\right\} \leftrightarrow 0, \left\{\begin{pmatrix} \frac{1}{3}\sqrt{3} \\ 0 \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{5}\sqrt{2}\sqrt{5} \\ \frac{1}{5}\sqrt{3}\sqrt{5} \\ -\frac{1}{5}\sqrt{5} \end{pmatrix}\right\} \leftrightarrow 2$. The columns are these vectors.

34. Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix, U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{pmatrix} 1 & \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{1}{6}\sqrt{3}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{3}{2} & \frac{1}{12}\sqrt{2}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{6} & \frac{1}{12}\sqrt{2}\sqrt{6} & \frac{1}{2} \end{pmatrix}$$

Hint: The eigenvalues are $2, 1, 0$.
$\begin{pmatrix} 1 & \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{1}{6}\sqrt{3}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{3}{2} & \frac{1}{12}\sqrt{2}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{6} & \frac{1}{12}\sqrt{2}\sqrt{6} & \frac{1}{2} \end{pmatrix}$, eigenvectors:
$\left\{\begin{pmatrix} -\frac{1}{3}\sqrt{3} \\ 0 \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{pmatrix}\right\} \leftrightarrow 0, \left\{\begin{pmatrix} \frac{1}{3}\sqrt{3} \\ -\frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{6} \end{pmatrix}\right\} \leftrightarrow 1, \left\{\begin{pmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{6} \end{pmatrix}\right\} \leftrightarrow 2$. The columns are these vectors.

35. Find the eigenvalues and an orthonormal basis of eigenvectors for the matrix,

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{6}\sqrt{3}\sqrt{2} & -\frac{7}{18}\sqrt{3}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{3}{2} & -\frac{1}{12}\sqrt{2}\sqrt{6} \\ -\frac{7}{18}\sqrt{3}\sqrt{6} & -\frac{1}{12}\sqrt{2}\sqrt{6} & -\frac{5}{6} \end{pmatrix}$$

Hint: The eigenvalues are $1, 2, -2$.
$\begin{pmatrix} \frac{1}{3} & \frac{1}{6}\sqrt{3}\sqrt{2} & -\frac{7}{18}\sqrt{3}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{3}{2} & -\frac{1}{12}\sqrt{2}\sqrt{6} \\ -\frac{7}{18}\sqrt{3}\sqrt{6} & -\frac{1}{12}\sqrt{2}\sqrt{6} & -\frac{5}{6} \end{pmatrix}$, eigenvectors:
$\left\{\begin{pmatrix} -\frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{6} \end{pmatrix}\right\} \leftrightarrow 1, \left\{\begin{pmatrix} \frac{1}{3}\sqrt{3} \\ 0 \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{pmatrix}\right\} \leftrightarrow -2, \left\{\begin{pmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} \\ -\frac{1}{6}\sqrt{6} \end{pmatrix}\right\} \leftrightarrow 2$. Then the columns of U are these vectors.

36. Find the eigenvalues and an orthonormal basis of eigenvectors for the matrix,

$$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{1}{10}\sqrt{5} \\ -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{7}{5} & -\frac{1}{5}\sqrt{6} \\ \frac{1}{10}\sqrt{5} & -\frac{1}{5}\sqrt{6} & -\frac{9}{10} \end{pmatrix}$$

Hint: The eigenvalues are $-1, 2, -1$ where -1 is listed twice because it has multiplicity 2 as a zero of the characteristic equation.

$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{1}{10}\sqrt{5} \\ -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{7}{5} & -\frac{1}{5}\sqrt{6} \\ \frac{1}{10}\sqrt{5} & -\frac{1}{5}\sqrt{6} & -\frac{9}{10} \end{pmatrix}$, eigenvectors:

$\left\{ \begin{pmatrix} -\frac{1}{6}\sqrt{6} \\ 0 \\ \frac{1}{6}\sqrt{5}\sqrt{6} \end{pmatrix}, \begin{pmatrix} \frac{1}{3}\sqrt{2}\sqrt{3} \\ \frac{1}{5}\sqrt{5} \\ \frac{1}{15}\sqrt{2}\sqrt{15} \end{pmatrix} \right\} \leftrightarrow -1, \left\{ \begin{pmatrix} \frac{1}{6}\sqrt{6} \\ -\frac{2}{5}\sqrt{5} \\ \frac{1}{30}\sqrt{30} \end{pmatrix} \right\} \leftrightarrow 2$. The columns of U are these vectors.

$$\begin{pmatrix} -\frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{2}\sqrt{3} & \frac{1}{6}\sqrt{6} \\ 0 & \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} \\ \frac{1}{6}\sqrt{5}\sqrt{6} & \frac{1}{15}\sqrt{2}\sqrt{15} & \frac{1}{30}\sqrt{30} \end{pmatrix}^T \begin{pmatrix} -\frac{1}{2} & -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{1}{10}\sqrt{5} \\ -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{7}{5} & -\frac{1}{5}\sqrt{6} \\ \frac{1}{10}\sqrt{5} & -\frac{1}{5}\sqrt{6} & -\frac{9}{10} \end{pmatrix}.$$

$$\begin{pmatrix} -\frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{2}\sqrt{3} & \frac{1}{6}\sqrt{6} \\ 0 & \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} \\ \frac{1}{6}\sqrt{5}\sqrt{6} & \frac{1}{15}\sqrt{2}\sqrt{15} & \frac{1}{30}\sqrt{30} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

37. Explain why a matrix, A is symmetric if and only if there exists an orthogonal matrix, U such that $A = U^T DU$ for D a diagonal matrix.

    If A is given by the formula, then

    $$A^T = U^T D^T U = U^T DU = A$$

    Next suppose $A = A^T$. Then by the theorems on symmetric matrices, there exists an orthogonal matrix U such that

    $$UAU^T = D$$

    for D diagonal. Hence

    $$A = U^T DU$$

38. The proof of Theorem 13.3.3 concluded with the following observation. If $-ta + t^2 b \geq 0$ for all $t \in \mathbb{R}$ and $b \geq 0$, then $a = 0$. Why is this so?

    If $a \neq 0$, then the derivative of the function $f(t) = -ta + t^2 b$ is non zero at $t = 0$. However, this requires that $f(t) < 0$ for values of t near 0.

39. Using Schur's theorem, show that whenever A is an $n \times n$ matrix, $\det(A)$ equals the product of the eigenvalues of A.

    There exists U unitary such that $A = U^*TU$ such that T is uppser triangular. Thus A and T are similar. Hence they have the same determinant. Therefore, $\det(A) = \det(T)$, but $\det(T)$ equals the product of the entries on the main diagonal which are the eigenvalues of A.

40. In the proof of Theorem 13.3.7 the following argument was used. If $\mathbf{x} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in \mathbb{R}^n$, then $\mathbf{x} = \mathbf{0}$. Why is this so?

    Because you could let $\mathbf{w} = \mathbf{x}$.

41. Using Corollary 13.3.8 show that a real $m \times n$ matrix is onto if and only if its transpose is one to one.

    Let A be the matrix. Then $\mathbf{y} \in A(\mathbb{F}^n)$ if and only if $\mathbf{y} \in \ker\left(A^T\right)^{\perp}$. However, if A^T is one to one, then its kernel is 0 and so everything is in $\ker\left(A^T\right)^{\perp}$. Thus A is onto. On the other hand, if A is onto, then if $A^T\mathbf{x} = \mathbf{0}$,

    $$0 = A^T\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A\mathbf{y}$$

    Hence $\mathbf{x} \cdot \mathbf{z} = 0$ for all $\mathbf{z}$ since A is onto and so $\mathbf{x} = \mathbf{0}$. Thus A^T is one to one.

42. Suppose A is a 3×2 matrix. Is it possible that A^T is one to one? What does this say about A being onto? Prove your answer.

    A^T is a 2×3 and so it is not one to one by an earlier corollary. Not every column can be a pivot column.

43. Find the least squares solution to the following system.

    $$\begin{aligned} x + 2y &= 1 \\ 2x + 3y &= 2 \\ 3x + 5y &= 4 \end{aligned}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \end{pmatrix}^T \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 14 & 23 \\ 23 & 38 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 23 \\ 23 & 38 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 17 \\ 28 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 23 \\ 23 & 38 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 17 \\ 28 \end{pmatrix}$$

$\begin{pmatrix} 14 & 23 \\ 23 & 38 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 17 \\ 28 \end{pmatrix}$, Solution is: $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$

44. You are doing experiments and have obtained the ordered pairs,

    $$(0,1), (1,2), (2,3.5),$$

    and $(3,4)$. Find m and b such that $y = mx + b$ approximates these four points as well as possible. Now do the same thing for $y = ax^2 + bx + c$, finding a, b, and c to give the best approximation.

I will do the second of these. You really want the following

$$\begin{aligned} 0a+0b+c&=1\\ a+b+c&=2\\ 4a+2b+c&=3.5\\ 9a+3b+c&=4 \end{aligned}$$

You look for a least squares solution to this.

$$\begin{pmatrix} 0 & 0 & 1\\ 1 & 1 & 1\\ 4 & 2 & 1\\ 9 & 3 & 1 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 1\\ 1 & 1 & 1\\ 4 & 2 & 1\\ 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} a\\ b\\ c \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1\\ 1 & 1 & 1\\ 4 & 2 & 1\\ 9 & 3 & 1 \end{pmatrix}^T \begin{pmatrix} 1\\ 2\\ 3.5 \end{pmatrix}$$

$$\begin{pmatrix} 98 & 36 & 14\\ 36 & 14 & 6\\ 14 & 6 & 4 \end{pmatrix} \begin{pmatrix} a\\ b\\ c \end{pmatrix} = \begin{pmatrix} 52.0\\ 21.0\\ 10.5 \end{pmatrix}$$

$$\begin{pmatrix} 98 & 36 & 14\\ 36 & 14 & 6\\ 14 & 6 & 4 \end{pmatrix} \begin{pmatrix} a\\ b\\ c \end{pmatrix} = \begin{pmatrix} 52.0\\ 21.0\\ 10.5 \end{pmatrix}, \text{ Solution is: } \begin{pmatrix} -0.125\\ 1.425\\ 0.925 \end{pmatrix}$$

45. Suppose you have several ordered triples, (x_i, y_i, z_i). Describe how to find a polynomial,

$$z = a + bx + cy + dxy + ex^2 + fy^2$$

for example giving the best fit to the given ordered triples. Is there any reason you have to use a polynomial? Would similar approaches work for other combinations of functions just as well?

You do something similar to the above. You write an equation which would be satisfied if each (x_i, y_i, z_i) were a real solution. Then you obtain the least squares solution.

46. Find an orthonormal basis for the spans of the following sets of vectors.

(a) $(3,-4,0), (7,-1,0), (1,7,1)$.
$\begin{pmatrix} 3/5\\ -4/5\\ 0 \end{pmatrix}, \begin{pmatrix} 4/5\\ 3/5\\ 0 \end{pmatrix}, \begin{pmatrix} 0\\ 0\\ 1 \end{pmatrix}$

(b) $(3,0,-4), (11,0,2), (1,1,7)$
$\begin{pmatrix} \frac{3}{5}\\ 0\\ -\frac{4}{5} \end{pmatrix}, \begin{pmatrix} \frac{4}{5}\\ 0\\ \frac{3}{5} \end{pmatrix}, \begin{pmatrix} 0\\ 1\\ 0 \end{pmatrix}$

(c) $(3,0,-4), (5,0,10), (-7,1,1)$
$\begin{pmatrix} \frac{3}{5}\\ 0\\ -\frac{4}{5} \end{pmatrix}, \begin{pmatrix} \frac{4}{5}\\ 0\\ \frac{3}{5} \end{pmatrix}, \begin{pmatrix} 0\\ 1\\ 0 \end{pmatrix}$

47. Using the Gram Schmidt process, find an orthonormal basis for the span of the vectors, $(1,2,1), (2,-1,3)$, and $(1,0,0)$.

$$\begin{pmatrix} 1 & 2 & 1\\ 2 & -1 & 0\\ 1 & 3 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{6}\sqrt{6} & \frac{3}{10}\sqrt{2} & \frac{7}{15}\sqrt{3}\\ \frac{1}{3}\sqrt{6} & -\frac{2}{5}\sqrt{2} & -\frac{1}{15}\sqrt{3}\\ \frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{6} & \frac{1}{2}\sqrt{6} & \frac{1}{6}\sqrt{6}\\ 0 & \frac{5}{2}\sqrt{2} & \frac{3}{10}\sqrt{2}\\ 0 & 0 & \frac{7}{15}\sqrt{3} \end{pmatrix}$$

A solution is then
$$\begin{pmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{pmatrix}, \begin{pmatrix} \frac{3}{10}\sqrt{2} \\ -\frac{2}{5}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}, \begin{pmatrix} \frac{7}{15}\sqrt{3} \\ -\frac{1}{15}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} \end{pmatrix}$$
Actually, I used the QR decomposition to find this.

48. Using the Gram Schmidt process, find an orthonormal basis for the span of the vectors, $(1,2,1,0)$, $(2,-1,3,1)$, and $(1,0,0,1)$.

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{2}\sqrt{3} & \frac{5}{111}\sqrt{3}\sqrt{37} & \frac{7}{111}\sqrt{111} \\ \frac{1}{3}\sqrt{6} & -\frac{2}{9}\sqrt{2}\sqrt{3} & \frac{1}{333}\sqrt{3}\sqrt{37} & -\frac{2}{111}\sqrt{111} \\ \frac{1}{6}\sqrt{6} & \frac{5}{18}\sqrt{2}\sqrt{3} & -\frac{17}{333}\sqrt{3}\sqrt{37} & -\frac{1}{37}\sqrt{111} \\ 0 & \frac{1}{9}\sqrt{2}\sqrt{3} & \frac{22}{333}\sqrt{3}\sqrt{37} & -\frac{7}{111}\sqrt{111} \end{pmatrix}$$
$$\cdot \begin{pmatrix} \sqrt{6} & \frac{1}{2}\sqrt{6} & \frac{1}{6}\sqrt{6} \\ 0 & \frac{3}{2}\sqrt{2}\sqrt{3} & \frac{5}{18}\sqrt{2}\sqrt{3} \\ 0 & 0 & \frac{1}{9}\sqrt{3}\sqrt{37} \\ 0 & 0 & 0 \end{pmatrix}$$

Then a solution is
$$\begin{pmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{6}\sqrt{2}\sqrt{3} \\ -\frac{2}{9}\sqrt{2}\sqrt{3} \\ \frac{5}{18}\sqrt{2}\sqrt{3} \\ \frac{1}{9}\sqrt{2}\sqrt{3} \end{pmatrix}, \begin{pmatrix} \frac{5}{111}\sqrt{3}\sqrt{37} \\ \frac{1}{333}\sqrt{3}\sqrt{37} \\ -\frac{17}{333}\sqrt{3}\sqrt{37} \\ \frac{22}{333}\sqrt{3}\sqrt{37} \end{pmatrix}$$

49. The set, $V \equiv \{(x,y,z) : 2x + 3y - z = 0\}$ is a subspace of $\mathbb{R}^3$. Find an orthonormal basis for this subspace.

The subspace is of the form
$$\begin{pmatrix} x \\ y \\ 2x+3y \end{pmatrix}$$
and a basis is $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ Therefore, an orthonormal basis is
$$\begin{pmatrix} \frac{1}{5}\sqrt{5} \\ 0 \\ \frac{2}{5}\sqrt{5} \end{pmatrix}, \begin{pmatrix} -\frac{3}{35}\sqrt{5}\sqrt{14} \\ \frac{1}{14}\sqrt{5}\sqrt{14} \\ \frac{3}{70}\sqrt{5}\sqrt{14} \end{pmatrix}$$

50. The two level surfaces, $2x + 3y - z + w = 0$ and $3x - y + z + 2w = 0$ intersect in a subspace of $\mathbb{R}^4$, find a basis for this subspace. Next find an orthonormal basis for this subspace.

Using row reduced echelon form, it is easy to obtain
$$\begin{pmatrix} -2z - 7w \\ w + 5z \\ 11z \\ 11w \end{pmatrix}$$
Thus a basis is
$$\begin{pmatrix} -2 \\ 5 \\ 11 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 1 \\ 0 \\ 11 \end{pmatrix}$$

Then an orthonormal basis is

$$\begin{pmatrix} -\frac{1}{15}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \\ \frac{11}{30}\sqrt{6} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{46}{3135}\sqrt{6}\sqrt{209} \\ \frac{1}{1254}\sqrt{6}\sqrt{209} \\ -\frac{1}{330}\sqrt{6}\sqrt{209} \\ \frac{5}{209}\sqrt{6}\sqrt{209} \end{pmatrix}$$

51. Let A, B be a $m \times n$ matrices. Define an inner product on the set of $m \times n$ matrices by

$$(A, B)_F \equiv \text{trace}(AB^*).$$

Show this is an inner product satisfying all the inner product axioms. Recall for M an $n \times n$ matrix, $\text{trace}(M) \equiv \sum_{i=1}^n M_{ii}$. The resulting norm, $||\cdot||_F$ is called the Frobenius norm and it can be used to measure the distance between two matrices.

It satisfies the properties of an inner product. Note that

$$\overline{\text{trace}(AB^*)} = \overline{\sum_i \sum_k A_{ik}\overline{B_{ik}}} = \sum_k \sum_i \overline{A_{ik}} B_{ik} = \text{trace}(BA^*)$$

so

$$\overline{(A, B)_F} = (B, A)_F$$

The product is obviously linear in the first argument. If $(A, A)_F = 0$, then

$$\sum_i \sum_k A_{ik}\overline{A_{ik}} = \sum_{i,k} |A_{ik}|^2 = 0$$

52. Let A be an $m \times n$ matrix. Show

$$||A||_F^2 \equiv (A, A)_F = \sum_j \sigma_j^2$$

where the σ_j are the singular values of A.

From the singular value decomposition,

$$U^*AV = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}, \; A = U \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} V^*$$

Then

$$\begin{aligned} \text{trace}(AA^*) &= \text{trace}\left(U \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} V^*V \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} U^*\right) \\ &= \text{trace}\left(U \begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix} U^*\right) = \text{trace}\begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix} = \sum_j \sigma_j^2 \end{aligned}$$

53. The trace of an $n \times n$ matrix M is defined as $\sum_i M_{ii}$. In other words it is the sum of the entries on the main diagonal. If A, B are $n \times n$ matrices, show $\text{trace}(AB) = \text{trace}(BA)$. Now explain why if $A = S^{-1}BS$ it follows $\text{trace}(A) = \text{trace}(B)$. **Hint:** For the first part, write these in terms of components of the matrices and it just falls out.

$\text{trace}(AB) = \sum_i \sum_k A_{ik}B_{ki}$, $\text{trace}(BA) = \sum_i \sum_k B_{ik}A_{ki}$. These give the same thing. Now

$$\text{trace}(A) = \text{trace}\left(S^{-1}BS\right) = \text{trace}\left(BSS^{-1}\right) = \text{trace}(B).$$

54. Using Problem 53 and Schur's theorem, show that the trace of an $n \times n$ matrix equals the sum of the eigenvalues.

    $A = UTU^*$ for some unitary U. Then the trace of A equals the trace of T. However, the trace of T is just the sum of the eigenvalues of A.

55. If A is a general $n \times n$ matrix having possibly repeated eigenvalues, show there is a sequence $\{A_k\}$ of $n \times n$ matrices having distinct eigenvalues which has the property that the ij^{th} entry of A_k converges to the ij^{th} entry of A for all ij. **Hint:** Use Schur's theorem.

    $A = U^*TU$ where T is upper triangular and U is unitary. Change the diagonal entries of T slightly so that the resulting upper triangular matrix T_k has all distinct diagonal entries and $T_k \to T$ in the sense that the ij^{th} entry of T_k converges to the ij^{th} entry of T. Then let $A_k = U^*T_kU$. It follows that $A_k \to A$ in the sense that corresponding entries converge.

56. Prove the Cayley Hamilton theorem as follows. First suppose A has a basis of eigenvectors $\{\mathbf{v}_k\}_{k=1}^n$, $A\mathbf{v}_k = \lambda_k \mathbf{v}_k$. Let $p(\lambda)$ be the characteristic polynomial. Show $p(A)\mathbf{v}_k = p(\lambda_k)\mathbf{v}_k = \mathbf{0}$. Then since $\{\mathbf{v}_k\}$ is a basis, it follows $p(A)\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$ and so $p(A) = 0$. Next in the general case, use Problem 55 to obtain a sequence $\{A_k\}$ of matrices whose entries converge to the entries of A such that A_k has n distinct eigenvalues and therefore by Theorem 12.1.13 A_k has a basis of eigenvectors. Therefore, from the first part and for $p_k(\lambda)$ the characteristic polynomial for A_k, it follows $p_k(A_k) = 0$. Now explain why and the sense in which

    $$\lim_{k\to\infty} p_k(A_k) = p(A).$$

    First say A has a basis of eigenvectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$. $A^k\mathbf{v}_j = \lambda_j^k \mathbf{v}_j$. Then it follows that $p(A)\mathbf{v}_k = p(\lambda_k)\mathbf{v}_k$. Hence if $\mathbf{x}$ is any vector, let $\mathbf{x} = \sum_{k=1}^n x_k \mathbf{v}_k$ and it follows that

    $$\begin{aligned} p(A)\mathbf{x} &= p(A)\left(\sum_{k=1}^n x_k \mathbf{v}_k\right) = \sum_{k=1}^n x_k p(A)\mathbf{v}_k \\ &= \sum_{k=1}^n x_k p(\lambda_k)\mathbf{v}_k = \sum_{k=1}^n x_k 0 \mathbf{v}_k = \mathbf{0} \end{aligned}$$

    Hence $p(A) = 0$. Now drop the assumption that A is nondefective. From the above, there exists a sequence A_k which is non defective which converges to A and also $p_k(\lambda) \to p(\lambda)$ uniformly on compact sets because these characteristic polynomials are defined in terms of determinants of the corresponding matrix. See the above construction of the A_k. It is probably easiest to use the Frobinius norm for the last part.

    $$\|p_k(A_k) - p(A)\|_F \le \|p_k(A_k) - p(A_k)\|_F + \|p(A_k) - p(A)\|_F$$

    The first term converges to 0 because the convergence of A_k to A implies all entries of A_k lie in a compact set. The second term converges to 0 also because the entries of A_k converge to the corresponding entries of A.

57. Show that the Moore Penrose inverse A^+ satisfies the following conditions.

    $$AA^+A = A, \quad A^+AA^+ = A^+, \quad A^+A, \; AA^+ \text{ are Hermitian.}$$

    Next show that if A_0 satisfies the above conditions, then it must be the Moore Penrose inverse and that if A is an $n \times n$ invertible matrix, then A^{-1} satisfies the above conditions. Thus the Moore Penrose inverse generalizes the usual notion of inverse but does not contradict it. **Hint:** Let

    $$U^*AV = \Sigma \equiv \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$$

and suppose

$$V^{+}A_0U = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

where P is the same size as σ. Now use the conditions to identify $P = \sigma, Q = 0$ etc.

First recall what the Moore Penrose invers was in terms of the singular value decomposition. It equals $V\begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*$ where $A = U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}V^*$ with U, V unitary and of the right size. Therefore,

$$\begin{aligned} AA^{+}A &= U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}V^*V\begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}V^* \\ &= U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}V^* = A \end{aligned}$$

Next

$$\begin{aligned} A^{+}AA^{+} &= V\begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}V^*V\begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^* \\ &= V\begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^* = A^{+} \end{aligned}$$

Next,

$$A^{+}A = V\begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}V^* = V\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}V^*$$

which is clearly Hermitian.

$$AA^{+} = U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}V^*V\begin{pmatrix} \sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^* = U\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}U^*$$

which is Hermitian.

Next suppose A_0 satisfies the above Penrose conditions. Then

$$A_0 = V\begin{pmatrix} P & Q \\ R & S \end{pmatrix}U^*, \quad A = U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}V^*$$

and it is necessary to identify P, Q, R, S. Here P is the same size as σ.

$$\begin{aligned} A_0A &= V\begin{pmatrix} P & Q \\ R & S \end{pmatrix}U^*U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}V^* = V\begin{pmatrix} P\sigma & 0 \\ R\sigma & 0 \end{pmatrix}V^* \\ &= V\begin{pmatrix} \sigma P^* & \sigma R^* \\ 0 & 0 \end{pmatrix}V^* \end{aligned}$$

and so $R\sigma = 0$ which implies $R = 0$.

$$\begin{aligned} AA_0 &= U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}V^*V\begin{pmatrix} P & Q \\ 0 & S \end{pmatrix}U^* = U\begin{pmatrix} \sigma P & \sigma Q \\ 0 & 0 \end{pmatrix}U^* \\ &= U\begin{pmatrix} P^*\sigma & 0 \\ Q^*\sigma & 0 \end{pmatrix}U^* \end{aligned}$$

Hence $Q^*\sigma = 0$ so $Q = 0$.

$$\begin{aligned} A_0AA_0 &= V\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}U^*U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}V^*V\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}U^* \\ &= V\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}U^* \\ &= V\begin{pmatrix} P\sigma P & 0 \\ 0 & 0 \end{pmatrix}U^* = A_0 = V\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}U^* \end{aligned}$$

and so $S = 0$. Finally,

$$\begin{aligned} AA_0A &= U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} V^*V \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} U^*U \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} V^* \\ &= U\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} V^* \\ &= U\begin{pmatrix} \sigma P \sigma & 0 \\ 0 & 0 \end{pmatrix} V^* = U \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} V^* \end{aligned}$$

Then $\sigma P \sigma = \sigma$ and so $P = \sigma^{-1}$. Hence $A_0 = A^+$.

58. Find the least squares solution to

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1+\varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Next suppose ε is so small that all ε^2 terms are ignored by the computer but the terms of order ε are not ignored. Show the least squares equations in this case reduce to

$$\begin{pmatrix} 3 & 3+\varepsilon \\ 3+\varepsilon & 3+2\varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a+b+c \\ a+b+(1+\varepsilon)\,c \end{pmatrix}.$$

Find the solution to this and compare the y values of the two solutions. Show that one of these is -2 times the other. This illustrates a problem with the technique for finding least squares solutions presented as the solutions to $A^*A\mathbf{x} = A^*\mathbf{y}$. One way of dealing with this problem is to use the QR factorization. This is illustrated in the next problem. It turns out that this helps alleviate some of the round off difficulties of the above.

The least squares problem is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \varepsilon+1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1+\varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \varepsilon+1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

which reduces to

$$\begin{pmatrix} 3 & \varepsilon+3 \\ \varepsilon+3 & \varepsilon^2+2\varepsilon+3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a+b+c \\ a+b+c\,(\varepsilon+1) \end{pmatrix}$$

and now, since the stupid computer doesn't see ε^2, what it sees is the following.

$$\begin{pmatrix} 3 & \varepsilon+3 \\ \varepsilon+3 & 2\varepsilon+3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a+b+c \\ a+b+c\,(\varepsilon+1) \end{pmatrix}$$

This yields the solution

$$\begin{pmatrix} \frac{1}{3\varepsilon^2}\left(\varepsilon^2\,(a+b+c) - (\varepsilon+3)\,(a\varepsilon + b\varepsilon - 2c\varepsilon)\right) \\ \frac{1}{3\varepsilon^2}\,(3a\varepsilon + 3b\varepsilon - 6c\varepsilon) \end{pmatrix}$$

So what is the real least squares solution?

$$\begin{pmatrix} \frac{1}{6\varepsilon^2}\left(2\varepsilon^2\,(a+b+c) + (\varepsilon+3)\,(a\varepsilon + b\varepsilon - 2c\varepsilon)\right) \\ -\frac{1}{6\varepsilon^2}\,(3a\varepsilon + 3b\varepsilon - 6c\varepsilon) \end{pmatrix}$$

The two y values are very different one is -2 times the other.

59. Show that the equations $A^*A\mathbf{x} = A^*\mathbf{y}$ can be written as $R^*R\mathbf{x} = R^*Q^*\mathbf{y}$ where R is upper triangular and R^* is lower triangular. Explain how to solve this system efficiently. **Hint:** You first find $R\mathbf{x}$ and then you find $\mathbf{x}$ which will not be hard because R is upper triangular.

    $A = QR$ and so $A^* = R^*Q^*$. Hence the least squares problem reduces to

    $$R^*Q^*QR\mathbf{x} = R^*Q^*\mathbf{y}$$

    $$R^*R\mathbf{x} = R^*Q^*\mathbf{y}$$

    Then you can solve this by first solving $R^*\mathbf{z} = R^*Q^*\mathbf{y}$. Next, after finding $\mathbf{z}$, you would solve for $\mathbf{x}$ in $R\mathbf{x} = \mathbf{z}$.

B.15 Exercises 14.4

1. Solve the system

    $$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

    using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.

    $\begin{pmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, Solution is: $\begin{pmatrix} 0.39 \\ -0.09 \\ 0.53 \end{pmatrix}$

2. Solve the system

    $$\begin{pmatrix} 4 & 1 & 1 \\ 1 & 7 & 2 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

    using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.

    $\begin{pmatrix} 4 & 1 & 1 \\ 1 & 7 & 2 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, Solution is: $\begin{pmatrix} 5.3191 \times 10^{-2} \\ 7.4468 \times 10^{-2} \\ 0.71277 \end{pmatrix}$

3. Solve the system

    $$\begin{pmatrix} 5 & 1 & 1 \\ 1 & 7 & 2 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

    using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.

4. Solve the system

    $$\begin{pmatrix} 7 & 1 & 0 \\ 1 & 5 & 2 \\ 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

    using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.

    $\begin{pmatrix} 7 & 1 & 0 \\ 1 & 5 & 2 \\ 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, Solution is: $\begin{pmatrix} 0.10227 \\ 0.28409 \\ -0.26136 \end{pmatrix}$

5. Solve the system
$$\begin{pmatrix} 5 & 0 & 1 \\ 1 & 7 & 1 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}$$
using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.
$\begin{pmatrix} 5 & 0 & 1 \\ 1 & 7 & 1 \\ 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}$, Solution is: $\begin{pmatrix} 0.14394 \\ 0.93939 \\ 0.2803 \end{pmatrix}$

6. Solve the system
$$\begin{pmatrix} 5 & 0 & 1 \\ 1 & 7 & 1 \\ 0 & 2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
using the Gauss Seidel method and the Jacobi method. Check your answer by also solving it using row operations.
$\begin{pmatrix} 5 & 0 & 1 \\ 1 & 7 & 1 \\ 0 & 2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, Solution is: $\begin{pmatrix} 0.20521 \\ 0.11726 \\ -2.6059 \times 10^{-2} \end{pmatrix}$

7. If you are considering a system of the form $A\mathbf{x} = \mathbf{b}$ and A^{-1} does not exist, will either the Gauss Seidel or Jacobi methods work? Explain. What does this indicate about using either of these methods for finding eigenvectors for a given eigenvalue?

   It indicates that they are no good for doing it.

B.16 Exercises 15.5

1. Using the power method, find the eigenvalue correct to one decimal place having largest absolute value for the matrix $A = \begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix}$ along with an eigenvector associated with this eigenvalue.
$$\begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ -2 \end{pmatrix}$$
$$\begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -2/7 \end{pmatrix} = \begin{pmatrix} -2.8571 \\ 8.5714 \\ -1.7143 \end{pmatrix}$$
$$\begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix} \begin{pmatrix} -0.33333 \\ 1.0 \\ -0.2 \end{pmatrix} = \begin{pmatrix} -3.2 \\ 6.6667 \\ -0.53334 \end{pmatrix}$$
$$\begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix} \begin{pmatrix} -0.48000 \\ 1.0 \\ -8.0001 \times 10^{-2} \end{pmatrix} = \begin{pmatrix} -3.6800 \\ 6.2400 \\ 0.47999 \end{pmatrix}$$
$$\begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix} \begin{pmatrix} -0.58974 \\ 1.0 \\ 7.6921 \times 10^{-2} \end{pmatrix} = \begin{pmatrix} -4.3077 \\ 6.2564 \\ 1.641 \end{pmatrix}$$
$$\begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix} \begin{pmatrix} -0.68853 \\ 1.0 \\ 0.26229 \end{pmatrix} = \begin{pmatrix} -5.0492 \\ 6.4917 \\ 2.9508 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix} \begin{pmatrix} -0.77779 \\ 1.0 \\ 0.45455 \end{pmatrix} = \begin{pmatrix} -5.8182 \\ 6.8282 \\ 4.2829 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix} \begin{pmatrix} -0.85208 \\ 1.0 \\ 0.62724 \end{pmatrix} = \begin{pmatrix} -6.5090 \\ 7.1716 \\ 5.4676 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix} \begin{pmatrix} -0.90761 \\ 1.0 \\ 0.76240 \end{pmatrix} = \begin{pmatrix} -7.0496 \\ 7.4587 \\ 6.3896 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix} \begin{pmatrix} -0.94515 \\ 1.0 \\ 0.85666 \end{pmatrix} = \begin{pmatrix} -7.4266 \\ 7.6673 \\ 7.0303 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix} \begin{pmatrix} -0.96861 \\ 1.0 \\ 0.91692 \end{pmatrix} = \begin{pmatrix} -7.6677 \\ 7.8043 \\ 7.4387 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -4 & -4 \\ 7 & 10 & 5 \\ -2 & 0 & 6 \end{pmatrix} \begin{pmatrix} -0.98250 \\ 1.0 \\ 0.95315 \end{pmatrix} = \begin{pmatrix} -7.8126 \\ 7.8883 \\ 7.6839 \end{pmatrix}$$

It looks like these scaling factors are not changing much so an approximate eigenvalue is 7.88. The corresponding eigenvector is above. How well does it do?

$$7.88 \begin{pmatrix} -0.98250 \\ 1.0 \\ 0.95315 \end{pmatrix} = \begin{pmatrix} -7.7421 \\ 7.88 \\ 7.5108 \end{pmatrix}$$

so it is pretty good. The actual largest eigenvalue is 8.

2. Using the power method, find the eigenvalue correct to one decimal place having largest absolute value for the matrix $A = \begin{pmatrix} 15 & 6 & 1 \\ -5 & 2 & 1 \\ 1 & 2 & 7 \end{pmatrix}$ along with an eigenvector associated with this eigenvalue.

$$\begin{pmatrix} 15 & 6 & 1 \\ -5 & 2 & 1 \\ 1 & 2 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 22 \\ -2 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 6 & 1 \\ -5 & 2 & 1 \\ 1 & 2 & 7 \end{pmatrix} \begin{pmatrix} 22 \\ -2 \\ 10 \end{pmatrix} \frac{1}{22} = \begin{pmatrix} \frac{164}{11} \\ -\frac{52}{11} \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 6 & 1 \\ -5 & 2 & 1 \\ 1 & 2 & 7 \end{pmatrix} \begin{pmatrix} 14.909 \\ -4.7273 \\ 4.0 \end{pmatrix} \frac{1}{14.909} = \begin{pmatrix} 13.366 \\ -5.3659 \\ 2.2439 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 6 & 1 \\ -5 & 2 & 1 \\ 1 & 2 & 7 \end{pmatrix} \begin{pmatrix} 13.366 \\ -5.3659 \\ 2.2439 \end{pmatrix} \frac{1}{13.366} = \begin{pmatrix} 12.759 \\ -5.635 \\ 1.3723 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 6 & 1 \\ -5 & 2 & 1 \\ 1 & 2 & 7 \end{pmatrix} \begin{pmatrix} 12.759 \\ -5.635 \\ 1.3723 \end{pmatrix} \frac{1}{12.759} = \begin{pmatrix} 12.458 \\ -5.7757 \\ 0.86959 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 6 & 1 \\ -5 & 2 & 1 \\ 1 & 2 & 7 \end{pmatrix} \begin{pmatrix} 12.458 \\ -5.7757 \\ 0.86959 \end{pmatrix} \frac{1}{12.458} = \begin{pmatrix} 12.288 \\ -5.8574 \\ 0.56138 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 6 & 1 \\ -5 & 2 & 1 \\ 1 & 2 & 7 \end{pmatrix} \begin{pmatrix} 12.288 \\ -5.8574 \\ 0.56138 \end{pmatrix} \frac{1}{12.288} = \begin{pmatrix} 12.186 \\ -5.9077 \\ 0.36644 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 6 & 1 \\ -5 & 2 & 1 \\ 1 & 2 & 7 \end{pmatrix} \begin{pmatrix} 12.186 \\ -5.9077 \\ 0.36644 \end{pmatrix} \frac{1}{12.186} = \begin{pmatrix} 12.121 \\ -5.9395 \\ 0.24091 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 6 & 1 \\ -5 & 2 & 1 \\ 1 & 2 & 7 \end{pmatrix} \begin{pmatrix} 12.121 \\ -5.9395 \\ 0.24091 \end{pmatrix} \frac{1}{12.121} = \begin{pmatrix} 12.080 \\ -5.9602 \\ 0.15909 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 6 & 1 \\ -5 & 2 & 1 \\ 1 & 2 & 7 \end{pmatrix} \begin{pmatrix} 12.080 \\ -5.9602 \\ 0.15909 \end{pmatrix} \frac{1}{12.08} = \begin{pmatrix} 12.053 \\ -5.9736 \\ 0.10540 \end{pmatrix}$$

It looks like the eigenvalue is about 12.05 and an eigenvector is $\begin{pmatrix} 12.053 \\ -5.9736 \\ 0.10540 \end{pmatrix}$. Actually, the largest eigenvalue is 12.

3. Using the power method, find the eigenvalue correct to one decimal place having largest absolute value for the matrix $A = \begin{pmatrix} 10 & 4 & 2 \\ -3 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix}$ along with an eigenvector associated with this eigenvalue.

$$\begin{pmatrix} 10 & 4 & 2 \\ -3 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 16.0 \\ -2.0 \\ 4.0 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 4 & 2 \\ -3 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 16.0 \\ -2.0 \\ 4.0 \end{pmatrix} \frac{1}{16} = \begin{pmatrix} 10.0 \\ -3.5 \\ 1.0 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 4 & 2 \\ -3 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 10.0 \\ -3.5 \\ 1.0 \end{pmatrix} \frac{1}{10} = \begin{pmatrix} 8.8 \\ -3.8 \\ 0.4 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 4 & 2 \\ -3 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 8.8 \\ -3.8 \\ 0.4 \end{pmatrix} \frac{1}{8.8} = \begin{pmatrix} 8.3636 \\ -3.9091 \\ 0.18182 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 4 & 2 \\ -3 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 8.3636 \\ -3.9091 \\ 0.18182 \end{pmatrix} \frac{1}{8.3636} = \begin{pmatrix} 8.1739 \\ -3.9565 \\ 8.6958 \times 10^{-2} \end{pmatrix}$$

$$\begin{pmatrix} 10 & 4 & 2 \\ -3 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 8.1739 \\ -3.9565 \\ 8.6958 \times 10^{-2} \end{pmatrix} \frac{1}{8.1739} = \begin{pmatrix} 8.0851 \\ -3.9787 \\ 4.2554 \times 10^{-2} \end{pmatrix}$$

$$\begin{pmatrix} 10 & 4 & 2 \\ -3 & 2 & -1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 8.0851 \\ -3.9787 \\ 4.2554 \times 10^{-2} \end{pmatrix} \frac{1}{8.0851} = \begin{pmatrix} 8.0421 \\ -3.9895 \\ 2.1053 \times 10^{-2} \end{pmatrix}$$

It looks like the eigenvalue is about 8.04 and the eigenvector is about

$$\begin{pmatrix} 8.0421 \\ -3.9895 \\ 2.1053 \times 10^{-2} \end{pmatrix}.$$

The actual eigenvalue is 8 and an eigenvector is $\begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix}$

4. Using the power method, find the eigenvalue correct to one decimal place having largest absolute value for the matrix $A = \begin{pmatrix} 15 & 14 & -3 \\ -13 & -18 & 9 \\ 5 & 10 & -1 \end{pmatrix}$ along with an eigenvector associated with this eigenvalue.

   To make this go a little faster, I shall use a power of the given matrix.

$$\begin{pmatrix} 15 & 14 & -3 \\ -13 & -18 & 9 \\ 5 & 10 & -1 \end{pmatrix}^7 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 140\,484\,608 \\ -271\,556\,608 \\ 134\,242\,304 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 14 & -3 \\ -13 & -18 & 9 \\ 5 & 10 & -1 \end{pmatrix}^7 \begin{pmatrix} -\frac{17\,149}{33\,149} \\ 1 \\ -\frac{16\,387}{33\,149} \end{pmatrix} = \begin{pmatrix} 1.326\,3 \times 10^8 \\ -2.653\,3 \times 10^8 \\ 1.326\,8 \times 10^8 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 14 & -3 \\ -13 & -18 & 9 \\ 5 & 10 & -1 \end{pmatrix} \begin{pmatrix} 1.326\,3 \times 10^8 \\ -2.653\,3 \times 10^8 \\ 1.326\,8 \times 10^8 \end{pmatrix} \frac{1}{-2.653\,3 \times 10^8} = \begin{pmatrix} 8.002\,1 \\ -16.002 \\ 8.000\,7 \end{pmatrix}$$

$$\begin{pmatrix} 15 & 14 & -3 \\ -13 & -18 & 9 \\ 5 & 10 & -1 \end{pmatrix} \begin{pmatrix} 8.002\,1 \\ -16.002 \\ 8.000\,7 \end{pmatrix} \frac{1}{-16.002} = \begin{pmatrix} 7.998\,9 \\ -15.999 \\ 7.999\,6 \end{pmatrix}$$

   It looks like an eigenvalue is about -16 and an eigenvector is about $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. In fact, this is exactly right.

$$\begin{pmatrix} 15 & 14 & -3 \\ -13 & -18 & 9 \\ 5 & 10 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -16 \\ 32 \\ -16 \end{pmatrix}.$$

5. In Example 15.3.3 an eigenvalue was found correct to several decimal places along with an eigenvector. Find the other eigenvalues along with their eigenvectors.

   The matrix was

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

   This is a hard problem to find eigenvalues and eigenvectors. However, if the matrix is small, you can sometimes find the eigenvalues simply by graphing the characteristic equation and zooming in on the eigenvalues. I will do this first. The characteristic polynomial is $x^3 - 7x^2 + 15$. Then you can observe from the graph that there are eigenvalues near $-1.35, 1.5$, and 6.5. I will use the shifted inverse power method to get closer and to also compute the eigenvectors. First lets look for the eigenvalue closest to -1.35.

$$\left(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} + 1.35 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} 77.671 & -35.341 & -36.948 \\ -35.341 & 16.397 & 16.753 \\ -36.948 & 16.753 & 17.774 \end{pmatrix}$$

Now we use the power method on this to find the largest eigenvalue.

$$\begin{pmatrix} 77.671 & -35.341 & -36.948 \\ -35.341 & 16.397 & 16.753 \\ -36.948 & 16.753 & 17.774 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5.382 \\ -2.191 \\ -2.421 \end{pmatrix}$$

$$\begin{pmatrix} 77.671 & -35.341 & -36.948 \\ -35.341 & 16.397 & 16.753 \\ -36.948 & 16.753 & 17.774 \end{pmatrix} \begin{pmatrix} 5.382 \\ -2.191 \\ -2.421 \end{pmatrix} \frac{1}{5.382} = \begin{pmatrix} 108.68 \\ -49.552 \\ -51.763 \end{pmatrix}$$

$$\begin{pmatrix} 77.671 & -35.341 & -36.948 \\ -35.341 & 16.397 & 16.753 \\ -36.948 & 16.753 & 17.774 \end{pmatrix} \begin{pmatrix} 108.68 \\ -49.552 \\ -51.763 \end{pmatrix} \frac{1}{108.68} = \begin{pmatrix} 111.38 \\ -50.796 \\ -53.052 \end{pmatrix}$$

$$\begin{pmatrix} 77.671 & -35.341 & -36.948 \\ -35.341 & 16.397 & 16.753 \\ -36.948 & 16.753 & 17.774 \end{pmatrix} \begin{pmatrix} 1.0 \\ -0.45606 \\ -0.47632 \end{pmatrix} = \begin{pmatrix} 111.39 \\ -50.799 \\ -53.054 \end{pmatrix}$$

This is a good time to stop. The scaling factors are not changing by much so we can now find the eigenvector and eigenvalue. To get the eigenvalue, solve

$$\frac{1}{\lambda + 1.35} = 111.39$$

$\frac{1}{\lambda+1.35} = 111.39$, Solution is: $\lambda = -1.341$. The eigenvector is about $\begin{pmatrix} 1.0 \\ -0.45606 \\ -0.47632 \end{pmatrix}$. How well does it work?

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1.0 \\ -0.45606 \\ -0.47632 \end{pmatrix} = \begin{pmatrix} -1.3411 \\ 0.61156 \\ 0.63866 \end{pmatrix}$$

$$-1.341 \begin{pmatrix} 1.0 \\ -0.45606 \\ -0.47632 \end{pmatrix} = \begin{pmatrix} -1.341 \\ 0.61158 \\ 0.63875 \end{pmatrix}$$

It worked very well.

Next consider the eigenvector and eigenvalue for the eigenvalue which is near 1.5.

$$\left(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} - 1.5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} -9.5238 \times 10^{-2} & 0.7619 & -0.19048 \\ 0.7619 & 3.9048 & -2.4762 \\ -0.19048 & -2.4762 & 1.619 \end{pmatrix}$$

Now we use the power method on this.

$$\begin{pmatrix} -9.5238 \times 10^{-2} & 0.7619 & -0.19048 \\ 0.7619 & 3.9048 & -2.4762 \\ -0.19048 & -2.4762 & 1.619 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.47618 \\ 2.1905 \\ -1.0477 \end{pmatrix}$$

$$\begin{pmatrix} -9.5238 \times 10^{-2} & 0.7619 & -0.19048 \\ 0.7619 & 3.9048 & -2.4762 \\ -0.19048 & -2.4762 & 1.619 \end{pmatrix} \begin{pmatrix} 0.47618 \\ 2.1905 \\ -1.0477 \end{pmatrix} \frac{1}{2.1905}$$

$$= \begin{pmatrix} 0.8323 \\ 5.2548 \\ -3.2920 \end{pmatrix}$$

$$\begin{pmatrix} -9.5238 \times 10^{-2} & 0.7619 & -0.19048 \\ 0.7619 & 3.9048 & -2.4762 \\ -0.19048 & -2.4762 & 1.619 \end{pmatrix} \begin{pmatrix} 0.8323 \\ 5.2548 \\ -3.2920 \end{pmatrix} \frac{1}{5.2548}$$

$$= \begin{pmatrix} 0.86615 \\ 5.5768 \\ -3.5206 \end{pmatrix}$$

$$\begin{pmatrix} -9.5238 \times 10^{-2} & 0.7619 & -0.19048 \\ 0.7619 & 3.9048 & -2.4762 \\ -0.19048 & -2.4762 & 1.619 \end{pmatrix} \begin{pmatrix} 0.86615 \\ 5.5768 \\ -3.5206 \end{pmatrix} \frac{1}{5.5768}$$

$$= \begin{pmatrix} 0.86736 \\ 5.5863 \\ -3.5278 \end{pmatrix}$$

$$\begin{pmatrix} -9.5238 \times 10^{-2} & 0.7619 & -0.19048 \\ 0.7619 & 3.9048 & -2.4762 \\ -0.19048 & -2.4762 & 1.619 \end{pmatrix} \begin{pmatrix} 0.86736 \\ 5.5863 \\ -3.5278 \end{pmatrix} \frac{1}{5.5863}$$

$$= \begin{pmatrix} 0.8674 \\ 5.5868 \\ -3.5282 \end{pmatrix}$$

$$\begin{pmatrix} -9.5238 \times 10^{-2} & 0.7619 & -0.19048 \\ 0.7619 & 3.9048 & -2.4762 \\ -0.19048 & -2.4762 & 1.619 \end{pmatrix} \begin{pmatrix} 0.8674 \\ 5.5868 \\ -3.5282 \end{pmatrix} \frac{1}{5.5868}$$

$$= \begin{pmatrix} 0.86741 \\ 5.5869 \\ -3.5282 \end{pmatrix}$$

At this point, the scaling factors are not changing by much, so the eigenvalue is obtained by solving

$$\frac{1}{\lambda - 1.5} = 5.5868$$

$\frac{1}{\lambda-1.5} = 5.5868$, Solution is: 1.6790. An eigenvector is $\begin{pmatrix} 0.86741 \\ 5.5869 \\ -3.5282 \end{pmatrix}$. How well does it work?

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 0.86741 \\ 5.5869 \\ -3.5282 \end{pmatrix} = \begin{pmatrix} 1.4566 \\ 9.3804 \\ -5.9237 \end{pmatrix}$$

$$1.6790 \begin{pmatrix} 0.86741 \\ 5.5869 \\ -3.5282 \end{pmatrix} = \begin{pmatrix} 1.4564 \\ 9.3804 \\ -5.9238 \end{pmatrix}$$

Thus, it works very well indeed.

Finally, we look for the eigenvector which goes with the eigenvalue which is closest to 6.5.

$$\left(\begin{pmatrix} 1 & 2.0 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} - 6.5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^{-1} = \begin{pmatrix} 1.6735 & 1.3061 & 2.5306 \\ 1.3061 & 0.77551 & 1.8776 \\ 2.5306 & 1.8776 & 3.3878 \end{pmatrix}$$

Ans we use the power method for this last matrix.

$$\begin{pmatrix} 1.6735 & 1.3061 & 2.5306 \\ 1.3061 & 0.77551 & 1.8776 \\ 2.5306 & 1.8776 & 3.3878 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5.5102 \\ 3.9592 \\ 7.796 \end{pmatrix}$$

$$\begin{pmatrix} 1.6735 & 1.3061 & 2.5306 \\ 1.3061 & 0.77551 & 1.8776 \\ 2.5306 & 1.8776 & 3.3878 \end{pmatrix} \begin{pmatrix} 5.5102 \\ 3.9592 \\ 7.796 \end{pmatrix} \frac{1}{7.796} = \begin{pmatrix} 4.3767 \\ 3.1946 \\ 6.1300 \end{pmatrix}$$

$$\begin{pmatrix} 1.6735 & 1.3061 & 2.5306 \\ 1.3061 & 0.77551 & 1.8776 \\ 2.5306 & 1.8776 & 3.3878 \end{pmatrix} \begin{pmatrix} 4.3767 \\ 3.1946 \\ 6.1300 \end{pmatrix} \frac{1}{6.1300} = \begin{pmatrix} 4.4061 \\ 3.2143 \\ 6.1731 \end{pmatrix}$$

$$\begin{pmatrix} 1.6735 & 1.3061 & 2.5306 \\ 1.3061 & 0.77551 & 1.8776 \\ 2.5306 & 1.8776 & 3.3878 \end{pmatrix} \begin{pmatrix} 4.4061 \\ 3.2143 \\ 6.1731 \end{pmatrix} \frac{1}{6.1731} = \begin{pmatrix} 4.4052 \\ 3.2136 \\ 6.1717 \end{pmatrix}$$

The scaling factors are not changing by much so the eigenvalue is approximately the solution to

$$\frac{1}{\lambda - 6.5} = 6.1717,\ \lambda = 6.662$$

and the eigenvector is approximately

$$\begin{pmatrix} 4.4052 \\ 3.2136 \\ 6.1717 \end{pmatrix}.$$

How well does it work?

$$\begin{pmatrix} 1 & 2.0 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 4.4052 \\ 3.2136 \\ 6.1717 \end{pmatrix} = \begin{pmatrix} 29.348 \\ 21.409 \\ 41.116 \end{pmatrix}$$

$$6.662 \begin{pmatrix} 4.4052 \\ 3.2136 \\ 6.1717 \end{pmatrix} = \begin{pmatrix} 29.347 \\ 21.409 \\ 41.116 \end{pmatrix}$$

It seems to work pretty well.

6. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ numerically. In this case the exact eigenvalues are $\pm\sqrt{3}, 6$. Compare with the exact answers.

   I would like to get close to them so that the shifted inverse power method will work quickly. The characteristic polynomial is $x^3 - 6x^2 - 3x + 18$. Then you can graph this to see about where the eigenvalues are. If you do this, you find that there is one near 2, one near -2, and one near 6.

   First consider the one near 6.

   $\left(\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} - 6 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^{-1}$ does not even exist so in fact 6 is an eigenvalue. So how can I find an eigenvector? Just use row operations. In fact

   $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector.

   Now lets look for the one near -2

$\left(\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)^{-1} = \begin{pmatrix} 0.375 & -0.625 & 0.375 \\ -0.625 & 2.375 & -1.625 \\ 0.375 & -1.625 & 1.375 \end{pmatrix}$ Now you sure don't want to use $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ as an initial approximation. This cannot work because in this case, it is an eigenvector for another eigenvalue. This would be a very unlucky choice. I must try something else.

$$\begin{pmatrix} 0.375 & -0.625 & 0.375 \\ -0.625 & 2.375 & -1.625 \\ 0.375 & -1.625 & 1.375 \end{pmatrix}^6 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 450.33 \\ -1680.7 \\ 1230.3 \end{pmatrix}$$

$$\begin{pmatrix} 0.375 & -0.625 & 0.375 \\ -0.625 & 2.375 & -1.625 \\ 0.375 & -1.625 & 1.375 \end{pmatrix} \begin{pmatrix} 450.33 \\ -1680.7 \\ 1230.3 \end{pmatrix} \frac{1}{-1680.7} = \begin{pmatrix} -0.99998 \\ 3.7320 \\ -2.732 \end{pmatrix}$$

$$\begin{pmatrix} 0.375 & -0.625 & 0.375 \\ -0.625 & 2.375 & -1.625 \\ 0.375 & -1.625 & 1.375 \end{pmatrix} \begin{pmatrix} -0.99998 \\ 3.7320 \\ -2.732 \end{pmatrix} \frac{1}{3.7320} = \begin{pmatrix} -1.00000 \\ 3.732 \\ -2.732 \end{pmatrix}$$

The scaling factors have certainly settled down.

$\frac{1}{\lambda+2} = 3.732$, Solution is: -1.732 and an eigenvector is $\begin{pmatrix} -1.00000 \\ 3.732 \\ -2.732 \end{pmatrix}$.

How well does it work?

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} -1.00000 \\ 3.732 \\ -2.732 \end{pmatrix} = \begin{pmatrix} 1.732 \\ -6.464 \\ 4.732 \end{pmatrix}$$

$$-1.732 \begin{pmatrix} -1.00000 \\ 3.732 \\ -2.732 \end{pmatrix} = \begin{pmatrix} 1.732 \\ -6.4638 \\ 4.7318 \end{pmatrix}$$

This is clearly very close. Of course -1.732 is very close to $-\sqrt{3}$. Now lets look for one near 2.

$$\left(\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)^{-1} = \begin{pmatrix} -2.25 & 0.75 & 1.75 \\ 0.75 & -0.25 & -0.25 \\ 1.75 & -0.25 & -1.25 \end{pmatrix}$$

$$\begin{pmatrix} -2.25 & 0.75 & 1.75 \\ 0.75 & -0.25 & -0.25 \\ 1.75 & -0.25 & -1.25 \end{pmatrix}^7 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6272.3 \\ -1680.7 \\ -4591.7 \end{pmatrix}$$

$$\begin{pmatrix} -2.25 & 0.75 & 1.75 \\ 0.75 & -0.25 & -0.25 \\ 1.75 & -0.25 & -1.25 \end{pmatrix} \begin{pmatrix} 6272.3 \\ -1680.7 \\ -4591.7 \end{pmatrix} \frac{1}{6272.3} = \begin{pmatrix} -3.7321 \\ 1.0 \\ 2.7321 \end{pmatrix}$$

$$\begin{pmatrix} -2.25 & 0.75 & 1.75 \\ 0.75 & -0.25 & -0.25 \\ 1.75 & -0.25 & -1.25 \end{pmatrix} \begin{pmatrix} -3.7321 \\ 1.0 \\ 2.7321 \end{pmatrix} \frac{1}{-3.7321} = \begin{pmatrix} -3.7321 \\ 1.0 \\ 2.7321 \end{pmatrix}$$

To several decimal places, the scaling factors are not changing.

$\frac{1}{\lambda-2} = -3.7321$, Solution is: 1.7321. Thus an eigenvalue is about 1.7321 and an eigenvector is $\begin{pmatrix} -3.7321 \\ 1.0 \\ 2.7321 \end{pmatrix}$. How well does it work?

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} -3.7321 \\ 1.0 \\ 2.7321 \end{pmatrix} = \begin{pmatrix} -6.4642 \\ 1.7321 \\ 4.7321 \end{pmatrix}$$

$$1.7321 \begin{pmatrix} -3.7321 \\ 1.0 \\ 2.7321 \end{pmatrix} = \begin{pmatrix} -6.4644 \\ 1.7321 \\ 4.7323 \end{pmatrix}, \text{ pretty close.}$$

7. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ numerically. The exact eigenvalues are $2, 4 + \sqrt{15}, 4 - \sqrt{15}$. Compare your numerical results with the exact values. Is it much fun to compute the exact eigenvectors?

   Characteristic polynomial: $x^3 - 10x^2 + 17x - 2$. When you graph it, you find that 2 is an eigenvalue and there is also one near 0 and one near 8. First lets check the one for 2. The graph suggests that this is an eigenvalue.

   $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 1 & 3 & 0 \end{pmatrix}$. An eigenvector is $\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$. This means you would not want to use this vector as an initial gues in finding the eigenvalues and eigenvectors which go with the two other eigenvalues. First lets find the one near 0.

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 0.5 & -0.5 & 0.5 \\ -0.5 & 2.5 & -3.5 \\ 0.5 & -3.5 & 5.5 \end{pmatrix}$$

$$\begin{pmatrix} 0.5 & -0.5 & 0.5 \\ -0.5 & 2.5 & -3.5 \\ 0.5 & -3.5 & 5.5 \end{pmatrix}^{7} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 65658. \\ -3.8561 \times 10^5 \\ 5.8258 \times 10^5 \end{pmatrix}$$

$$\begin{pmatrix} 0.5 & -0.5 & 0.5 \\ -0.5 & 2.5 & -3.5 \\ 0.5 & -3.5 & 5.5 \end{pmatrix} \begin{pmatrix} 65658. \\ -3.8561 \times 10^5 \\ 5.8258 \times 10^5 \end{pmatrix} \frac{1}{5.8258 \times 10^5} = \begin{pmatrix} 0.8873 \\ -5.2111 \\ 7.873 \end{pmatrix}$$

$$\begin{pmatrix} 0.5 & -0.5 & 0.5 \\ -0.5 & 2.5 & -3.5 \\ 0.5 & -3.5 & 5.5 \end{pmatrix} \begin{pmatrix} 0.8873 \\ -5.2111 \\ 7.873 \end{pmatrix} \frac{1}{7.873} = \begin{pmatrix} 0.88730 \\ -5.2111 \\ 7.8730 \end{pmatrix}$$

   Thus the eigenvalue which is close to 0 is obtained by solving

   $\frac{1}{\lambda} = 7.8730$, Solution is: 0.12702. The eigenvalue is $\begin{pmatrix} 0.88730 \\ -5.2111 \\ 7.8730 \end{pmatrix}$. How well does it work?

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 0.88730 \\ -5.2111 \\ 7.8730 \end{pmatrix} = \begin{pmatrix} 0.1127 \\ -0.6619 \\ 1.0 \end{pmatrix}$$

$$0.12702 \begin{pmatrix} 0.88730 \\ -5.2111 \\ 7.8730 \end{pmatrix} = \begin{pmatrix} 0.1127 \\ -0.66191 \\ 1.0 \end{pmatrix}, \text{ works well.}$$

   Now lets find the one which is close to 8.

$$\left(\begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} - 8 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^{-1} = \begin{pmatrix} -1.5 & -2.5 & -1.5 \\ -2.5 & -4.8333 & -2.8333 \\ -1.5 & -2.8333 & -1.8333 \end{pmatrix}$$

$$\begin{pmatrix} -1.5 & -2.5 & -1.5 \\ -2.5 & -4.8333 & -2.8333 \\ -1.5 & -2.8333 & -1.8333 \end{pmatrix}^{6} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 41213. \\ 77190. \\ 46447. \end{pmatrix}$$

$$\begin{pmatrix} -1.5 & -2.5 & -1.5 \\ -2.5 & -4.8333 & -2.8333 \\ -1.5 & -2.8333 & -1.8333 \end{pmatrix} \begin{pmatrix} 41213. \\ 77190. \\ 46447. \end{pmatrix} \frac{1}{77190} = \begin{pmatrix} -4.2035 \\ -7.8730 \\ -4.7373 \end{pmatrix}$$

$$\begin{pmatrix} -1.5 & -2.5 & -1.5 \\ -2.5 & -4.8333 & -2.8333 \\ -1.5 & -2.8333 & -1.8333 \end{pmatrix} \begin{pmatrix} -4.2035 \\ -7.8730 \\ -4.7373 \end{pmatrix} \frac{1}{-7.8730} = \begin{pmatrix} -4.2034 \\ -7.8729 \\ -4.7373 \end{pmatrix}$$

The scaling factors are not changing by much.

$\frac{1}{\lambda-8} = -7.8729$, Solution is: 7.8730. Then an eigenvector is $\begin{pmatrix} 0.53391 \\ 1.0 \\ 0.60172 \end{pmatrix}$. How well does it work?

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 0.53391 \\ 1.0 \\ 0.60172 \end{pmatrix} = \begin{pmatrix} 4.2035 \\ 7.8730 \\ 4.7374 \end{pmatrix}$$

$7.8730 \begin{pmatrix} 0.53391 \\ 1.0 \\ 0.60172 \end{pmatrix} = \begin{pmatrix} 4.2035 \\ 7.873 \\ 4.7373 \end{pmatrix}$. Works well.

8. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ numerically. We don't know the exact eigenvalues in this case. Check your answers by multiplying your numerically computed eigenvectors by the matrix.

   The characteristic polynomial is $x^3 - 7x^2 - 4x + 1$. Now you graph this to identify roughly where the eigenvalues are.

   $x^3 - 7x^2 - 4x + 1$

   There is one near -1, one near $.25$, and one near 7.5.

$$\left(\begin{pmatrix} 0 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^{-1} = \begin{pmatrix} 3.0 & -1.0 & 0 \\ -1.0 & 0.66667 & -0.33333 \\ 0 & -0.33333 & 0.66667 \end{pmatrix}$$

$$\begin{pmatrix} 3.0 & -1.0 & 0 \\ -1.0 & 0.66667 & -0.33333 \\ 0 & -0.33333 & 0.66667 \end{pmatrix}^6 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 867.98 \\ -325.46 \\ 40.116 \end{pmatrix}$$

$$\begin{pmatrix} 3.0 & -1.0 & 0 \\ -1.0 & 0.66667 & -0.33333 \\ 0 & -0.33333 & 0.66667 \end{pmatrix} \begin{pmatrix} 867.98 \\ -325.46 \\ 40.116 \end{pmatrix} \frac{1}{867.98} = \begin{pmatrix} 3.3750 \\ -1.2654 \\ 0.15580 \end{pmatrix}$$

$$\begin{pmatrix} 3.0 & -1.0 & 0 \\ -1.0 & 0.66667 & -0.33333 \\ 0 & -0.33333 & 0.66667 \end{pmatrix} \begin{pmatrix} 3.3750 \\ -1.2654 \\ 0.15580 \end{pmatrix} \frac{1}{3.3750} = \begin{pmatrix} 3.3749 \\ -1.2653 \\ 0.15575 \end{pmatrix}$$

The scaling factors are not changing by much so

$\frac{1}{\lambda+1} = 3.3749$, Solution is: -0.70369. The eigenvectors is $\begin{pmatrix} 3.3749 \\ -1.2653 \\ 0.15575 \end{pmatrix}$. How well does it work?

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 3.3749 \\ -1.2653 \\ 0.15575 \end{pmatrix} = \begin{pmatrix} -2.3749 \\ 0.89055 \\ -0.1095 \end{pmatrix}$$

$-0.70369\begin{pmatrix} 3.3749 \\ -1.2653 \\ 0.15575 \end{pmatrix} = \begin{pmatrix} -2.3749 \\ 0.89038 \\ -0.10960 \end{pmatrix}$ This works pretty well. Of course more iterations would likely result in a better answer.

Next try the one near .25

$$\left(\begin{pmatrix} 0 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} - .25\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)^{-1}$$

$$= \begin{pmatrix} -1.6296 & -1.1852 & 2.9630 \\ -1.1852 & -3.4074 & 6.5185 \\ 2.9630 & 6.5185 & -12.296 \end{pmatrix}$$

$\begin{pmatrix} -1.6296 & -1.1852 & 2.9630 \\ -1.1852 & -3.4074 & 6.5185 \\ 2.9630 & 6.5185 & -12.296 \end{pmatrix}^8 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -2.2592 \times 10^8 \\ -4.8777 \times 10^8 \\ 9.328 \times 10^8 \end{pmatrix}$

$\begin{pmatrix} -1.6296 & -1.1852 & 2.9630 \\ -1.1852 & -3.4074 & 6.5185 \\ 2.9630 & 6.5185 & -12.296 \end{pmatrix} \begin{pmatrix} -2.2592 \times 10^8 \\ -4.8777 \times 10^8 \\ 9.328 \times 10^8 \end{pmatrix} \frac{1}{9.328 \times 10^8}$

$= \begin{pmatrix} 3.9774 \\ 8.5873 \\ -16.422 \end{pmatrix}$

$\begin{pmatrix} -1.6296 & -1.1852 & 2.9630 \\ -1.1852 & -3.4074 & 6.5185 \\ 2.9630 & 6.5185 & -12.296 \end{pmatrix} \begin{pmatrix} 3.9774 \\ 8.5873 \\ -16.422 \end{pmatrix} \frac{1}{-16.422}$

$= \begin{pmatrix} 3.9774 \\ 8.5873 \\ -16.422 \end{pmatrix}$

The eigenvalue is the solution to $\frac{1}{\lambda - .25} = -16.422$, Solution is: 0.18911. The eigenvector is $\begin{pmatrix} 3.9774 \\ 8.5873 \\ -16.422 \end{pmatrix} \frac{1}{-16.422} = \begin{pmatrix} -0.24220 \\ -0.52291 \\ 1.0 \end{pmatrix}$. How well does it work?

$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} -0.24220 \\ -0.52291 \\ 1.0 \end{pmatrix} = \begin{pmatrix} -0.04582 \\ -0.09895 \\ 0.18907 \end{pmatrix}$

$0.18911\begin{pmatrix} -0.24220 \\ -0.52291 \\ 1.0 \end{pmatrix} = \begin{pmatrix} -4.5802 \times 10^{-2} \\ -9.8888 \times 10^{-2} \\ 0.18911 \end{pmatrix}$. It works well.

Next lets find the eigenvector for the eigenvalue near 7.5

$\left(\begin{pmatrix} 0 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} - 7.5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)^{-1}$

$$= \begin{pmatrix} 5.4286 & 16.0 & 9.7143 \\ 16.0 & 46.0 & 28.0 \\ 9.7143 & 28.0 & 16.857 \end{pmatrix}$$

$$\begin{pmatrix} 5.4286 & 16.0 & 9.7143 \\ 16.0 & 46.0 & 28.0 \\ 9.7143 & 28.0 & 16.857 \end{pmatrix}^5 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6.8891 \times 10^8 \\ 1.9858 \times 10^9 \\ 1.2052 \times 10^9 \end{pmatrix}$$

$$\begin{pmatrix} 5.4286 & 16.0 & 9.7143 \\ 16.0 & 46.0 & 28.0 \\ 9.7143 & 28.0 & 16.857 \end{pmatrix} \begin{pmatrix} 0.34692 \\ 1.0 \\ 0.60691 \end{pmatrix}$$

$$= \begin{pmatrix} 23.779 \\ 68.544 \\ 41.601 \end{pmatrix}$$

$$\begin{pmatrix} 5.4286 & 16.0 & 9.7143 \\ 16.0 & 46.0 & 28.0 \\ 9.7143 & 28.0 & 16.857 \end{pmatrix} \begin{pmatrix} 0.34692 \\ 1.0 \\ 0.60692 \end{pmatrix}$$

$$= \begin{pmatrix} 23.779 \\ 68.544 \\ 41.601 \end{pmatrix}$$

The eigenvalue: $\frac{1}{\lambda - 7.5} = 68.544$, Solution is: 7.5146. The eigenvector is $\begin{pmatrix} 0.34692 \\ 1.0 \\ 0.60692 \end{pmatrix}$. How well does it work?

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 0.34692 \\ 1.0 \\ 0.60692 \end{pmatrix} = \begin{pmatrix} 2.6069 \\ 7.5146 \\ 4.5608 \end{pmatrix}$$

$7.5146 \begin{pmatrix} 0.34692 \\ 1.0 \\ 0.60692 \end{pmatrix} = \begin{pmatrix} 2.6070 \\ 7.5146 \\ 4.5608 \end{pmatrix}$. It worked well.

9. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ numerically. We don't know the exact eigenvalues in this case. Check your answers by multiplying your numerically computed eigenvectors by the matrix.

   I am tired of going through the computations. Here is the answer.

   $\begin{pmatrix} 0 & 2.0 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, eigenvectors: $\begin{pmatrix} 0.3792 \\ 0.58481 \\ 0.71708 \end{pmatrix} \leftrightarrow 4.9754$,

   $\begin{pmatrix} 0.81441 \\ 0.15694 \\ -0.55866 \end{pmatrix} \leftrightarrow -0.30056$, $\begin{pmatrix} 0.43925 \\ -0.79585 \\ 0.41676 \end{pmatrix} \leftrightarrow -2.6749$

10. Consider the matrix $A = \begin{pmatrix} 3 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 0 \end{pmatrix}$ and the vector $(1, 1, 1)^T$. Estimate the distance between the Rayleigh quotient determined by this vector and some eigenvalue of A.

Recall that there was a formula for this. Let

$$q=\frac{\begin{pmatrix}1&1&1\end{pmatrix}\begin{pmatrix}3&2&3\\2&1&4\\3&4&0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}}{3}=\frac{22}{3}$$

Then there is an eigenvalue λ such that

$$\left|\lambda-\frac{22}{3}\right|\leq\frac{\left|\begin{pmatrix}3&2&3\\2&1&4\\3&4&0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}-\frac{22}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\sqrt{3}}=\frac{1}{3}\sqrt{2}$$

11. Consider the matrix $A=\begin{pmatrix}1&2&1\\2&1&4\\1&4&5\end{pmatrix}$ and the vector $(1,1,1)^T$. Estimate the distance between the Rayleigh quotient determined by this vector and some eigenvalue of A.

$$q=\frac{\begin{pmatrix}1&1&1\end{pmatrix}\begin{pmatrix}1&2&1\\2&1&4\\1&4&5\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}}{3}=7$$

Then

$$|\lambda-7|\leq\frac{\left|\begin{pmatrix}1&2&1\\2&1&4\\1&4&5\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}-7\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\sqrt{3}}=\sqrt{6}$$

12. Using Gerschgorin's theorem, find upper and lower bounds for the eigenvalues of $A=\begin{pmatrix}3&2&3\\2&6&4\\3&4&-3\end{pmatrix}$.

From the bottom line, a lower bound is -10 From the second line, an upper bound is 12.

13. The QR algorithm works very well on general matrices. Try the QR algorithm on the following matrix which happens to have some complex eigenvalues.

$$A=\begin{pmatrix}1&2&3\\1&2&-1\\-1&-1&1\end{pmatrix}$$

Use the QR algorithm to get approximate eigenvalues and then use the shifted inverse power method on one of these to get an approximate eigenvector for one of the complex eigenvalues.

The real parts of the eigenvalues are larger than -4. I will consider the matrix which comes from adding $5I$ to the given matrix. Thus, consider

$$\begin{pmatrix}6&2&3\\1&7&-1\\-1&-1&6\end{pmatrix}$$

$$\begin{pmatrix}\frac{1}{2}\sqrt{2}&-\frac{1}{2}\sqrt{2}\\\frac{1}{2}\sqrt{2}&\frac{1}{2}\sqrt{2}\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}\sqrt{2}\\0\end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix} \begin{pmatrix} 6 & 2 & 3 \\ 1 & 7 & -1 \\ -1 & -1 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ 0 & -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -\frac{1}{2}\sqrt{2} & \frac{5}{2}\sqrt{2} \\ \sqrt{2} & \frac{15}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{11}{2} \end{pmatrix} = \begin{pmatrix} 6.0 & -0.707\,11 & 3.535\,5 \\ 1.414\,2 & 7.5 & 0.5 \\ 0 & 0.5 & 5.5 \end{pmatrix}$$

This is the upper Hessenberg matrix which goes with A.

$$\begin{pmatrix} 6.0 & -0.707\,11 & 3.535\,5 \\ 1.414\,2 & 7.5 & 0.5 \\ 0 & 0.5 & 5.5 \end{pmatrix}^{35} =$$

$$\begin{pmatrix} 3.582\,3\times 10^{29} & 4.290\,6\times 10^{29} & 6.739\,6\times 10^{29} \\ 6.190\,5\times 10^{30} & 7.422\,7\times 10^{30} & 1.165\,9\times 10^{31} \\ 1.409\,7\times 10^{30} & 1.690\,3\times 10^{30} & 2.655\,3\times 10^{30} \end{pmatrix}$$

This has a QR factorization with $Q =$

$$\begin{pmatrix} 5.633\,4\times 10^{-2} & -0.998\,39 & 7.287\,3\times 10^{-3} \\ 0.973\,49 & 5.330\,5\times 10^{-2} & -0.222\,43 \\ 0.221\,68 & 1.962\,4\times 10^{-2} & 0.974\,92 \end{pmatrix}$$

Then

$$Q^T \begin{pmatrix} 6.0 & -0.707\,11 & 3.535\,5 \\ 1.414\,2 & 7.5 & 0.5 \\ 0 & 0.5 & 5.5 \end{pmatrix} Q =$$

$$\begin{pmatrix} 7.695\,7 & -1.281\,6 & 0.230\,07 \\ 1.310\,2\times 10^{-4} & 5.898\,3 & -3.601\,3 \\ -1.781\,5\times 10^{-5} & 0.310\,73 & 5.406\,1 \end{pmatrix}$$

It is now clear that the eigenvalues are approximately those of

$$\begin{pmatrix} 5.898\,3 & -3.601\,3 \\ 0.310\,73 & 5.406\,1 \end{pmatrix}$$

and $7.695\,7$. That is,

$$5.652\,2 + 1.028\,8i, \quad 5.652\,2 - 1.028\,8i, \; 7.695\,7$$

Subtracting 5 gives the approximate eigenvalues for the original matrix,

$$0.652\,2 + 1.028\,8i, \quad 0.652\,2 - 1.028\,8i, \; 2.695\,7$$

Lets find the eigenvector which goes with the first of the above.

$$\left(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix} - (0.652\,2 + 1.028\,8i) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^{-1} =$$

$$\begin{pmatrix} -4251.5 - 8395.3i & -16391. + 3972.6i & -26508. + 5426.7i \\ 1411.0 + 4647.5i & 8687.2 - 554.36i & 13959. - 389.59i \\ 2431.6 - 3583.0i & -5253.6 - 5712.0i & -8094.1 - 9459.8i \end{pmatrix}$$

$$\begin{pmatrix} -4251.5-8395.3i & -16391.+3972.6i & -26508.+5426.7i \\ 1411.0+4647.5i & 8687.2-554.36i & 13959.-389.59i \\ 2431.6-3583.0i & -5253.6-5712.0i & -8094.1-9459.8i \end{pmatrix}^{12} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} -3.9428\times10^{51}+2.7471\times10^{51}i \\ 2.2497\times10^{51}-1.0440\times10^{51}i \\ -1.9848\times10^{51}-9.7467\times10^{50}i \end{pmatrix}$$

Then the first approximation will be

$$\begin{pmatrix} -3.9428\times10^{51}+2.7471\times10^{51}i \\ 2.2497\times10^{51}-1.0440\times10^{51}i \\ -1.9848\times10^{51}-9.7467\times10^{50}i \end{pmatrix} \frac{1}{-3.9428\times10^{51}+2.7471\times10^{51}i}$$

$$= \begin{pmatrix} 1.0 \\ -0.50831-8.9375\times10^{-2}i \\ 0.22294+0.40253i \end{pmatrix}$$

$$\begin{pmatrix} -4251.5-8395.3i & -16391.+3972.6i & -26508.+5426.7i \\ 1411.0+4647.5i & 8687.2-554.36i & 13959.-389.59i \\ 2431.6-3583.0i & -5253.6-5712.0i & -8094.1-9459.8i \end{pmatrix}\cdot$$

$$\begin{pmatrix} 1.0 \\ -0.50831-8.9375\times10^{-2}i \\ 0.22294+0.40253i \end{pmatrix} = \begin{pmatrix} -3658.8-18410.0i \\ 214.5+9684.9i \\ 6594.9-5577.1i \end{pmatrix}$$

The next approximate eigenvector is

$$\begin{pmatrix} -3658.8-18410.0i \\ 214.5+9684.9i \\ 6594.9-5577.1i \end{pmatrix} \frac{1}{-3658.8-18410.0i}$$

$$= \begin{pmatrix} 1.0 \\ -0.50831-8.9369\times10^{-2}i \\ 0.22294+0.40253i \end{pmatrix}$$

This didn't change by much. In fact, it didn't change at all. Thus the approximate eigenvalue is obtained by solving

$$\frac{1}{\lambda-(0.6522+1.0288i)} = -3658.8-18410.0i$$

Thus

$$\lambda = 0.65219+1.0289i$$

and the approximate eigenvector is

$$\begin{pmatrix} 1.0 \\ -0.50831-8.9369\times10^{-2}i \\ 0.22294+0.40253i \end{pmatrix}$$

Lets check it.

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1.0 \\ -0.50831-8.9369\times10^{-2}i \\ 0.22294+0.40253i \end{pmatrix} = \begin{pmatrix} 0.6522+1.0289i \\ -0.23956-0.58127i \\ -0.26875+0.49190i \end{pmatrix}$$

$$(0.652\,19 + 1.028\,9i)\begin{pmatrix} 1.0 \\ -0.508\,31 - 8.936\,9 \times 10^{-2}i \\ 0.222\,94 + 0.402\,53i \end{pmatrix} = \begin{pmatrix} 0.652\,19 + 1.028\,9i \\ -0.239\,56 - 0.581\,29i \\ -0.268\,76 + 0.491\,91i \end{pmatrix}$$

It worked very well. Thus you know the other eigenvalue for the other complex eigenvalue is

$$\begin{pmatrix} 1.0 \\ -0.508\,31 + 8.936\,9 \times 10^{-2}i \\ 0.222\,94 - 0.402\,53i \end{pmatrix}$$

14. Use the QR algorithm to approximate the eigenvalues of the symmetric matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -8 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Now consider the matrix. First lets find a Hessenburg matrix similar to it.
$\begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{13}\sqrt{13} & -\frac{3}{13}\sqrt{13} \\ \frac{3}{13}\sqrt{13} & \frac{2}{13}\sqrt{13} \end{pmatrix}\begin{pmatrix} \sqrt{13} & \frac{1}{13}\sqrt{13} & \frac{8}{13}\sqrt{13} \\ 0 & \frac{5}{13}\sqrt{13} & \frac{1}{13}\sqrt{13} \end{pmatrix}$ and so

$$\begin{pmatrix} \frac{2}{13}\sqrt{13} & -\frac{3}{13}\sqrt{13} \\ \frac{3}{13}\sqrt{13} & \frac{2}{13}\sqrt{13} \end{pmatrix}^T \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} \sqrt{13} & \frac{1}{13}\sqrt{13} & \frac{8}{13}\sqrt{13} \\ 0 & \frac{5}{13}\sqrt{13} & \frac{1}{13}\sqrt{13} \end{pmatrix}$$

Then the similar matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{13}\sqrt{13} & \frac{3}{13}\sqrt{13} \\ 0 & -\frac{3}{13}\sqrt{13} & \frac{2}{13}\sqrt{13} \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{13}\sqrt{13} & \frac{3}{13}\sqrt{13} \\ 0 & -\frac{3}{13}\sqrt{13} & \frac{2}{13}\sqrt{13} \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & \sqrt{13} & 0 \\ \sqrt{13} & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Note how it also placed a 0 opposite the 0 on the bottom. This must happen because the given matrix was symmetric. Now we will use this new Hessenburg matrix with the algorithm.

$$\begin{pmatrix} 1 & \sqrt{13} & 0 \\ \sqrt{13} & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}^9 = \begin{pmatrix} 1.216\,3 \times 10^6 & 1.430\,6 \times 10^6 & 2.284\,4 \times 10^5 \\ 1.430\,6 \times 10^6 & 1.676\,4 \times 10^6 & 2.700\,6 \times 10^5 \\ 2.284\,4 \times 10^5 & 2.700\,6 \times 10^5 & 42601. \end{pmatrix}$$

Now it is time to take the QR factorization of this. I am using a computer to do this of course.

$$\begin{pmatrix} 1.216\,3 \times 10^6 & 1.430\,6 \times 10^6 & 2.284\,4 \times 10^5 \\ 1.430\,6 \times 10^6 & 1.676\,4 \times 10^6 & 2.700\,6 \times 10^5 \\ 2.284\,4 \times 10^5 & 2.700\,6 \times 10^5 & 42601. \end{pmatrix}$$

$$= \begin{pmatrix} 0.643\,00 & 0.653\,69 & 0.399\,06 \\ 0.756\,29 & -0.624\,11 & -0.196\,26 \\ 0.120\,76 & 0.428\,00 & -0.895\,68 \end{pmatrix} \cdot \begin{pmatrix} 1.891\,6 \times 10^6 & 2.220\,3 \times 10^6 & 3.562\,7 \times 10^5 \\ 0 & 4492.2 & -985.67 \\ 0 & 0 & 2.806\,8 \end{pmatrix}$$

Then from the first part of this problem,

$$A_9 = \begin{pmatrix} 0.643\,00 & 0.653\,69 & 0.399\,06 \\ 0.756\,29 & -0.624\,11 & -0.196\,26 \\ 0.120\,76 & 0.428\,00 & -0.895\,68 \end{pmatrix}^T \begin{pmatrix} 1.0 & 3.605\,6 & 0 \\ 3.605\,6 & 2.0 & 1.0 \\ 0 & 1.0 & -1.0 \end{pmatrix} \cdot$$

$$\begin{pmatrix} 0.643\,00 & 0.653\,69 & 0.399\,06 \\ 0.756\,29 & -0.624\,11 & -0.196\,26 \\ 0.120\,76 & 0.428\,00 & -0.895\,68 \end{pmatrix} =$$

$$= \begin{pmatrix} 5.232\,2 & 8.540\,3 \times 10^{-3} & -2.914\,1 \times 10^{-5} \\ 8.540\,3 \times 10^{-3} & -2.453 & 3.645\,2 \times 10^{-3} \\ -2.914\,1 \times 10^{-5} & 3.645\,2 \times 10^{-3} & -0.779\,16 \end{pmatrix}$$

The eigenvalues are close to the eigenvalues of the matrix

$$\begin{pmatrix} 5.232\,2 & 0 & 0 \\ 0 & -2.453 & 0 \\ 0 & 0 & -0.779\,16 \end{pmatrix}$$

In fact, the given matrix is close to this diagonal matrix. Of course you could get closer if you did some more iterations. You could also use the shifted inverse power method to find the eigenvectors and to approximate the eigenvalues further if you want, but there are other ways which I will not go into here.

15. Try to find the eigenvalues of the matrix $\begin{pmatrix} 3 & 3 & 1 \\ -2 & -2 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ using the QR algorithm. It has eigenvalues $1, i, -i$. You will see the algorithm won't work well.

To see that the method will not work well, consider the powers of this matrix.

$$\begin{pmatrix} 3 & 3 & 1 \\ -2 & -2 & -1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 3 & 4 & 0 \\ -2 & -3 & 0 \\ -2 & -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 & 1 \\ -2 & -2 & -1 \\ 0 & 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ -2 & -3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 & 1 \\ -2 & -2 & -1 \\ 0 & 1 & 0 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus the powers of the matrix will repeat forever. Therefore, you cannot expect things to get small and be able to find the eigenvalues by looking at those of a block upper triangular matrix which has very small entries below the block diagonal. Recall that, in terms of size of the entries, you can at least theoretically consider the QR factorization of A^k and then look at Q^*AQ however, not much can happen given the fact that there are only finitely many A^k. The problem here is that there is no gap between the size of the eigenvalues. The matrix has eigenvalues equal to $i, -i$ and 1. To produce a situation in which this unfortunate situation will not occur, simply add a multiple of the identity to the matrix and use the modified one. This is always a reasonable idea. You can identify an interval, using Gerschgorin's theorem which will contain all the eigenvalues of the matrix. Then when you add a large enough multiple of the identity, the result will have all positive real parts for the eigenvalues. If you do this to this matrix, the problem goes away. The idea is to produce gaps between the absolute values of the eigenvalues.

B.17 Exercises 16.2

1. Prove the Euclidean algorithm: If m, n are positive integers, then there exist integers $q, r \geq 0$ such that $r < m$ and
$$n = qm + r$$
Hint: You might try considering
$$S \equiv \{n - km : k \in \mathbb{N} \text{ and } n - km < 0\}$$
and picking the smallest integer in S or something like this.

   The hint is a good suggestion. Pick the first thing in S. By the Archimedean property, $S \neq \emptyset$. That is $km > n$ for all k sufficiently large. Call this first thing $q + 1$. Thus $n - (q + 1) m < 0$ but $n - qm \geq 0$. Then
$$n - qm < m$$
and so
$$0 \leq r \equiv n - qm < m.$$

2. ↑The greatest common divisor of two positive integers m, n, denoted as q is a positive number which divides both m and n and if p is any other positive number which divides both m, n, then p divides q. Recall what it means for p to divide q. It means that $q = pk$ for some integer k. Show that the greatest common divisor of m, n is the smallest positive integer in the set S
$$S \equiv \{xm + yn : x, y \in \mathbb{Z} \text{ and } xm + yn > 0\}$$
Two positive integers are called relatively prime if their greatest common divisor is 1.

   First note that either m or $-m$ is in S so S is a nonempty set of positive integers. By well ordering, there is a smallest element of S, called $p = x_0 m + y_0 n$. Either p divides m or it does not. If p does not divide m, then by the above problem,
$$m = pq + r$$
where $0 < r < p$. Thus $m = (x_0 m + y_0 n) q + r$ and so, solving for r,
$$r = m (1 - x_0) + (-y_0 q) n \in S.$$
However, this is a contradiction because p was the smallest element of S. Thus $p|m$. Similarly $p|n$.Now suppose q divides both m and n. Then $m = qx$ and $n = qy$ for integers, x and y. Therefore,
$$p = mx_0 + ny_0 = x_0 qx + y_0 qy = q (x_0 x + y_0 y)$$
showing $q|p$. Therefore, $p = (m, n)$.

3. ↑A positive integer larger than 1 is called a prime number if the only positive numbers which divide it are 1 and itself. Thus 2,3,5,7, etc. are prime numbers. If m is a positive integer and p does not divide m where p is a prime number, show that p and m are relatively prime.

   Suppose r is the greatest common divisor of p and m. Then if $r \neq 1$, it must equal p because it must divide p. Hence there exist integers x, y such that
$$p = xp + ym$$
which requires that p must divide m which is assumed not to happen. Hence $r = 1$ and so the two numbers are relatively prime.

4. ↑There are lots of fields. This will give an example of a finite field. Let $\mathbb{Z}$ denote the set of integers. Thus $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$. Also let p be a prime number. We will say that two integers, a, b are equivalent and write $a \sim b$ if $a - b$ is divisible by p. Thus they are equivalent if $a - b = px$ for some integer x. First show that $a \sim a$. Next show that if $a \sim b$ then $b \sim a$. Finally show that if $a \sim b$ and $b \sim c$ then $a \sim c$. For a an integer, denote by $[a]$ the set of all integers which is equivalent to a, the equivalence class of a. Show first that is suffices to consider only $[a]$ for $a = 0, 1, 2, \cdots, p-1$ and that for $0 \leq a < b \leq p-1$, $[a] \neq [b]$. That is, $[a] = [r]$ where $r \in \{0, 1, 2, \cdots, p-1\}$. Thus there are exactly p of these equivalence classes. **Hint:** Recall the Euclidean algorithm. For $a > 0$, $a = mp + r$ where $r < p$. Next define the following operations.

$$\begin{aligned} [a] + [b] &\equiv [a+b] \\ [a][b] &\equiv [ab] \end{aligned}$$

Show these operations are well defined. That is, if $[a] = [a']$ and $[b] = [b']$, then $[a] + [b] = [a'] + [b']$ with a similar conclusion holding for multiplication. Thus for addition you need to verify $[a+b] = [a'+b']$ and for multiplication you need to verify $[ab] = [a'b']$. For example, if $p = 5$ you have $[3] = [8]$ and $[2] = [7]$. Is $[2 \times 3] = [8 \times 7]$? Is $[2+3] = [8+7]$? Clearly so in this example because when you subtract, the result is divisible by 5. So why is this so in general? Now verify that $\{[0], [1], \cdots, [p-1]\}$ with these operations is a Field. This is called the integers modulo a prime and is written $\mathbb{Z}_p$. Since there are infinitely many primes p, it follows there are infinitely many of these finite fields. **Hint:** Most of the axioms are easy once you have shown the operations are well defined. The only two which are tricky are the ones which give the existence of the additive inverse and the multiplicative inverse. Of these, the first is not hard. $-[x] = [-x]$. Since p is prime, there exist integers x, y such that $1 = px + ky$ and so $1 - ky = px$ which says $1 \sim ky$ and so $[1] = [ky]$. Now you finish the argument. What is the multiplicative identity in this collection of equivalence classes?

The only substantive issue is why $\mathbb{Z}_p$ is a field. Let $[x] \in \mathbb{Z}_p$ where $[x] \neq [0]$. Thus x is not a multiple of p. Then from the above problem, x and p are relatively prime. Hence from another of the above problems, there exist integers a, b such that

$$1 = ap + bx$$

Then

$$[1 - bx] = [ap] = 0$$

and it follows that

$$[b][x] = [1]$$

so $[b] = [x]^{-1}$.

B.18 Exercises 16.4

1. Let $M = \left\{\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : |u_1| \leq 4\right\}$. Is M a subspace? Explain.

   No. $(1, 0, 0, 0) \in M$ but $10(1, 0, 0, 0) \notin M$.

2. Let $M = \left\{\mathbf{u} = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : \sin(u_1) = 1\right\}$. Is M a subspace? Explain.

   absolutely not. Let $\mathbf{u} = \left(\frac{\pi}{2}, 0, 0, 0\right)$. Then $2\mathbf{u} \notin M$ although $\mathbf{u} \in M$.

3. If you have 5 vectors in $\mathbb{F}^5$ and the vectors are linearly independent, can it always be concluded they span $\mathbb{F}^5$? Explain.

   If not, you could add in a vector not in their span and obtain 6 vectors which are linearly independent. This cannot occur thanks to the exchange theorem.

4. If you have 6 vectors in $\mathbb{F}^5$, is it possible they are linearly independent? Explain.

   No because there exists a spanning set of 5 vectors. Note that it doesn't matter what $\mathbb{F}$ is here.

5. Show in any vector space, $\mathbf{0}$ is unique.

   Say $\mathbf{0}'$ is another one. Then $\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}'$. This happens from the definition of what it means for a vector to be an additive identity.

6. ↑In any vector space, show that if $\mathbf{x} + \mathbf{y} = \mathbf{0}$, then $\mathbf{y} = -\mathbf{x}$.

   Say $\mathbf{x} + \mathbf{y} = \mathbf{0}$. Then $-\mathbf{x} = -\mathbf{x} + (\mathbf{x} + \mathbf{y}) = (-\mathbf{x} + \mathbf{x}) + \mathbf{y} = \mathbf{0} + \mathbf{y} = \mathbf{y}$. If it acts like the additive inverse, it is the additive inverse.

7. ↑Show that in any vector space, $0\mathbf{x} = \mathbf{0}$. That is, the scalar 0 times the vector $\mathbf{x}$ gives the vector $\mathbf{0}$.

   $0\mathbf{x} = (0 + 0)\,\mathbf{x} = 0\mathbf{x} + 0\mathbf{x}$. Now add $-0\mathbf{x}$ to both sides to conclude that $0\mathbf{x} = \mathbf{0}$.

8. ↑Show that in any vector space, $(-1)\,\mathbf{x} = -\mathbf{x}$.

   Lets show $(-1)\,\mathbf{x}$ acts like the additive inverse. $\mathbf{x} + (-1)\,\mathbf{x} = (1 + (-1))\,\mathbf{x} = 0\mathbf{x} = \mathbf{0}$ from one of the above problems. Hence $(-1)\,\mathbf{x} = -\,\mathbf{x}$.

9. Let X be a vector space and suppose $\{\mathbf{x}_1, \cdots, \mathbf{x}_k\}$ is a set of vectors from X. Show that $\mathbf{0}$ is in $\operatorname{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k)$.

   Pick any vector, $\mathbf{x}_i$. Then $0\mathbf{x}_i$ is in the span, but this was shown to be $\mathbf{0}$ above.

10. Let X consist of the real valued functions which are defined on an interval $[a, b]$. For $f, g \in X$, $f + g$ is the name of the function which satisfies $(f + g)(x) = f(x) + g(x)$ and for α a real number, $(\alpha f)(x) \equiv \alpha (f(x))$. Show this is a vector space with field of scalars equal to $\mathbb{R}$. Also explain why it cannot possibly be finite dimensional.

    It doesn't matter where the functions have their values provided it is some real vector space. The axioms of a vector space are all routine because they hold for a vector space. The only thing maybe not completely obvious is the assertions about the things which are supposed to exist. $\mathbf{0}$ would be the zero function which sends everything to 0. This is an additive identity. Now if f is a function, $-f(x) \equiv (-f(x))$. Then

    $$(f + (-f))(x) \equiv f(x) + (-f)(x) \equiv f(x) + (-f(x)) = 0$$

    Hence $f + -f = \mathbf{0}$.

    For each $x \in [a, b]$, let $f_x(x) = 1$ and $f_x(y) = 0$ if $y \neq x$. Then these vectors are obviously linearly independent.

11. Let S be a nonempty set and let V denote the set of all functions which are defined on S and have values in W a vector space having field of scalars $\mathbb{F}$. Also define vector addition according to the usual rule, $(f + g)(s) \equiv f(s) + g(s)$ and scalar multiplication by $(\alpha f)(s) \equiv \alpha f(s)$. Show that V is a vector space with field of scalars $\mathbb{F}$.

    This is no different than the above. You simply understand that the vector space has the same field of scalars as the space of functions.

12. Verify that any field $\mathbb{F}$ is a vector space with field of scalars $\mathbb{F}$. However, show that $\mathbb{R}$ is a vector space with field of scalars $\mathbb{Q}$.

    A field also has multiplication. However, you can consider the elements of the field as vectors and then it satisfies all the vector space axioms. When you multiply a number (vector) in $\mathbb{R}$

by a scalar in $\mathbb{Q}$ you get something in $\mathbb{R}$. All the axioms for a vector space are now obvious. For example, if $\alpha \in \mathbb{Q}$ and $x, y \in \mathbb{R}$,

$$\alpha (x + y) = \alpha x + \alpha y$$

from the distributive law on $\mathbb{R}$.

13. Let $\mathbb{F}$ be a field and consider functions defined on $\{1, 2, \cdots, n\}$ having values in $\mathbb{F}$. Explain how, if V is the set of all such functions, V can be considered as $\mathbb{F}^n$.

    Simply let $f(i)$ be the i^{th} component of a vector $\mathbf{x} \in \mathbb{F}^n$. Thus a typical thing in $\mathbb{F}^n$ is $(f(1), \cdots, f(n))$.

14. Let V be the set of all functions defined on $\mathbb{N} \equiv \{1, 2, \cdots\}$ having values in a field $\mathbb{F}$ such that vector addition and scalar multiplication are defined by $(\mathbf{f} + \mathbf{g})(s) \equiv \mathbf{f}(s) + \mathbf{g}(s)$ and $(\alpha \mathbf{f})(s) \equiv \alpha \mathbf{f}(s)$ respectively, for $\mathbf{f}, \mathbf{g} \in V$ and $\alpha \in \mathbb{F}$. Explain how this is a vector space and show that for $\mathbf{e}_i$ given by

$$\mathbf{e}_i(k) \equiv \begin{cases} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k \end{cases},$$

    the vectors $\{\mathbf{e}_k\}_{k=1}^{\infty}$ are linearly independent.

    Say for some n, $\sum_{k=1}^{n} c_k \mathbf{e}_k = 0$, the zero function. Then pick i,

$$0 = \sum_{k=1}^{n} c_k \mathbf{e}_k(i) = c_i \mathbf{e}_i(i) = c_i$$

    Since i was arbitrary, this shows these vectors are linearly independent.

15. Suppose, in the context of Problem 10 you have smooth functions $\{y_1, y_2, \cdots, y_n\}$ (all derivatives exist) defined on an interval $[a, b]$. Then the Wronskian of these functions is the determinant

$$W(y_1, \cdots, y_n)(x) = \det \begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}$$

    Show that if $W(y_1, \cdots, y_n)(x) \neq 0$ for some x, then the functions are linearly independent.

    Say

$$\sum_{k=1}^{n} c_k y_k = 0$$

    Then taking derivatives you have

$$\sum_{k=1}^{n} c_k y_k^{(j)} = 0, \quad j = 0, 1, 2 \cdots, n-1$$

    This must hold when each equation is evaluated at x where you can pick the x at which the above determinant is nonzero. Therefore, this is a system of n equations in n variables, the c_i and the coefficient matrix is invertible. Therefore, each $c_i = 0$.

16. Give an example of two functions, y_1, y_2 defined on $[-1, 1]$ such that $W(y_1, y_2)(x) = 0$ for all $x \in [-1, 1]$ and yet $\{y_1, y_2\}$ is linearly independent.

    Let $y_1(x) = x^2$ and let $y_2(x) = x|x|$. Suppose $c_1 y_1 + c_2 y_2 = 0$ Then you could consider evaluating at 1 and get

$$c_1 + c_2 = 0$$

and then at -1 and get

$$c_1 - c_2 = 0$$

then it follows that $c_1 = c_2 = 0$. Thus, these are linearly independent. However, if $x > 0$,

$$W = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0$$

and if $x < 0$

$$W = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = 0$$

Also, $W = 0$ if $x = 0$ because at this point, $W = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0.$

17. Let the vectors be polynomials of degree no more than 3. Show that with the usual definitions of scalar multiplication and addition wherein, for $p(x)$ a polynomial, $(\alpha p)(x) = \alpha p(x)$ and for p, q polynomials $(p+q)(x) \equiv p(x) + q(x)$, this is a vector space.

    This is just a subspace of the vector space of functions because it is closed with respect to vector addition and scalar multiplication. Hence this is a vector space.

18. In the previous problem show that a basis for the vector space is $\{1, x, x^2, x^3\}$.

    This is really easy if you take the Wronskian of these functions.

$$\det \begin{pmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{pmatrix} \neq 0.$$

    See Problem 15 above.

19. Let V be the polynomials of degree no more than 3. Determine which of the following are bases for this vector space.

    (a) $\{x+1, x^3 + x^2 + 2x, x^2 + x, x^3 + x^2 + x\}$
    Lets look at the Wronskian of these functions. If it is nonzero somewhere, then these are linearly independent and are therefore a basis because there are four of them.

$$\det \begin{pmatrix} 1+x & x^3+x^2+2x & x^2+x & x^3+x^2+x \\ 1 & 3x^2+2x+2 & 2x+1 & 3x^2+2x+1 \\ 0 & 6x+2 & 2 & 6x+2 \\ 0 & 6 & 0 & 6 \end{pmatrix}$$

    Lets plug in $x = 0$ and see what happens. You only need to have it be nonzero at one point.

$$\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 6 & 0 & 6 \end{pmatrix} = 12 \neq 0$$

    so these are linearly independent.

    (b) $\{x^3 + 1, x^2 + x, 2x^3 + x^2, 2x^3 - x^2 - 3x + 1\}$
    Suppose

$$c_1(x^3+1) + c_2(x^2+x) + c_3(2x^3+x^2) + c_4(2x^3 - x^2 - 3x + 1) = 0$$

Then combine the terms according to power of x.

$$(c_1 + 2c_3 + 2c_4)\,x^3 + (c_2 + c_3 - c_4)\,x^2 + (c_2 - 3c_4)\,x + (c_1 + c_4) = 0$$

Is there a non zero solution to the system
$\begin{array}{c} c_1 + 2c_3 + 2c_4 = 0 \\ c_2 + c_3 - c_4 = 0 \\ c_2 - 3c_4 = 0 \\ c_1 + c_4 = 0 \end{array}$, Solution is: $[c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0]$ Therefore, these are linearly independent.

20. In the context of the above problem, consider polynomials

$$\left\{a_i x^3 + b_i x^2 + c_i x + d_i,\ i = 1, 2, 3, 4\right\}$$

Show that this collection of polynomials is linearly independent on an interval $[a, b]$ if and only if

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

is an invertible matrix.

Let $p_i(x)$ denote the i^{th} of these polynomials. Suppose $\sum_i C_i p_i(x) = 0$. Then collecting terms according to the exponent of x, you need to have

$$\begin{aligned} C_1 a_1 + C_2 a_2 + C_3 a_3 + C_4 a_4 &= 0 \\ C_1 b_1 + C_2 b_2 + C_3 b_3 + C_4 b_4 &= 0 \\ C_1 c_1 + C_2 c_2 + C_3 c_3 + C_4 c_4 &= 0 \\ C_1 d_1 + C_2 d_2 + C_3 d_3 + C_4 d_4 &= 0 \end{aligned}$$

The matrix of coefficients is just the transpose of the above matrix. There exists a non trivial solution if and only if the determinant of this matrix equals 0.

21. Let the field of scalars be $\mathbb{Q}$, the rational numbers and let the vectors be of the form $a + b\sqrt{2}$ where a, b are rational numbers. Show that this collection of vectors is a vector space with field of scalars $\mathbb{Q}$ and give a basis for this vector space.

This is obvious because when you add two of these you get one and when you multiply one of these by a scalar, you get another one. A basis is $\{1, \sqrt{2}\}$. By definition, the span of these gives the collection of vectors. Are they independent? Say $a + b\sqrt{2} = 0$ where a, b are rational numbers. If $a \neq 0$, then $b\sqrt{2} = -a$ which can't happen since a is rational. If $b \neq 0$, then $-a = b\sqrt{2}$ which again can't happen because on the left is a rational number and on the right is an irrational. Hence both $a, b = 0$ and so this is a basis.

22. Suppose V is a finite dimensional vector space. Based on the exchange theorem above, it was shown that any two bases have the same number of vectors in them. Give a different proof of this fact using the earlier material in the book. **Hint:** Suppose $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ and $\{\mathbf{y}_1, \cdots, \mathbf{y}_m\}$ are two bases with $m < n$. Then define

$$\phi : \mathbb{F}^n \to V, \psi : \mathbb{F}^m \to V$$

by

$$\phi(\mathbf{a}) \equiv \sum_{k=1}^{n} a_k x_k,\ \psi(\mathbf{b}) \equiv \sum_{j=1}^{m} b_j y_j$$

Consider the linear transformation, $\psi^{-1} \circ \phi$. Argue it is a one to one and onto mapping from $\mathbb{F}^n$ to $\mathbb{F}^m$. Now consider a matrix of this linear transformation and its row reduced echelon form.

Since both of these are bases, both ψ and ϕ are one to one and onto. Therefore, the given linear transformation is also one to one and onto. It has a matrix A which has more columns than rows such that multiplication by A has the same effect as doing $\psi^{-1} \circ \phi$. However, the augmented matrix $(A|0)$ has free variables because there are more columns than rows for A. Hence A cannot be one to one. Therefore, $\psi^{-1} \circ \phi$ also must fail to be one to one. This is a contradiction.

23. This and the following problems will present most of a differential equations course. To begin with, consider the scalar initial value problem

$$y' = ay,\ y(t_0) = y_0$$

When a is real, show the unique solution to this problem is $y = y_0 e^{a(t-t_0)}$. Next suppose

$$y' = (a + ib)\, y,\ y(t_0) = y_0 \tag{2.2}$$

where $y(t) = u(t) + iv(t)$. Show there exists a unique solution and it is

$$\begin{aligned} y(t) &= y_0 e^{a(t-t_0)} \left(\cos b(t - t_0) + i \sin b(t - t_0)\right) \\ &\equiv e^{(a+ib)(t-t_0)} y_0. \end{aligned} \tag{2.3}$$

Next show that for a real or complex there exists a unique solution to the initial value problem

$$y' = ay + f,\ y(t_0) = y_0$$

and it is given by

$$y(t) = e^{a(t-t_0)} y_0 + e^{at} \int_{t_0}^{t} e^{-as} f(s)\, ds.$$

Consider the first part. First you can verify that $y(t) = y_0 e^{a(t-t_0)}$ works using elementary calculus. Why is it the only one which does so? Suppose $y_1(t)$ is a solution. Then $y(t) - y_1(t) = z(t)$ solves the initial value problem

$$z'(t) = az(t),\ z(t_0) = 0.$$

Thus $z' - az = 0$. Multiply by e^{-at} on both sides. By the chain rule,

$$e^{-at}(z' - az) = \frac{d}{dt}\left(e^{-at} z(t)\right) = 0,$$

and so there is a constant C such that $e^{-at} z(t) = C$. Since $z(t_0) = 0$, this constant is 0, and so $z(t) = 0$.

Next consider the second part involving the complex stuff. You can verify that

$$y_0 e^{a(t-t_0)} \left(\cos b(t - t_0) + i \sin b(t - t_0)\right) = y(t)$$

does indeed solve the initial value problem from using elementary calculus. Now suppose you have another solution $y_1(t)$. Then let $z(t) = y(t) - y_1(t)$. It solves

$$z' = (a + ib)\, z,\ z(t_0) = 0$$

Now $z(t) = u(t) + iv(t)$. It is also clear that $\overline{z}(t) \equiv u(t) - iv(t)$ solves the equation

$$\overline{z}' = (a - ib)\, \overline{z}, \overline{z}(t_0) = 0$$

Thus from the product rule which you can easily verify from the usual proof of the product rule to be also true for complex valued functions,

$$\begin{aligned}\frac{d}{dt}|z(t)|^2 &= \frac{d}{dt}(z\overline{z}) = z'\overline{z} + z\overline{z}' = (a+ib)\,z\overline{z} + z\,(a-ib)\,\overline{z}\\ &= 2a\,|z|^2, \quad |z|\,(t_0) = 0\end{aligned}$$

Therefore, from the first part, $|z| = 0$ and so $y = y_1$.

Note that this implies that $\left(e^{i(a+ib)t}\right)' = (a+ib)\,e^{i(a+ib)t}$ where $e^{i(a+ib)t}$ is given above. Now consider the last part. Sove $y' = ay + f,\ y(t_0) = y_0$.

$$y' - ay = f, \quad y(t_0) = y_0$$

Multiply both sides by $e^{-a(t-t_0)}$

$$\frac{d}{dt}\left(e^{-a(t-t_0)}y\right) = e^{-a(t-t_0)}f(t)$$

Now integrate from t_0 to t. Then

$$e^{-a(t-t_0)}y(t) - y_0 = \int_{t_0}^{t} e^{-a(s-t_0)}f(s)\,ds$$

Hence

$$\begin{aligned}y(t) &= e^{a(t-t_0)}y_0 + e^{a(t-t_0)}\int_{t_0}^{t} e^{-a(s-t_0)}f(s)\,ds\\ &= e^{a(t-t_0)}y_0 + e^{at}\int_{t_0}^{t} e^{-as}f(s)\,ds\end{aligned}$$

24. Now consider A an $n \times n$ matrix. By Schur's theorem there exists unitary Q such that

$$Q^{-1}AQ = T$$

where T is upper triangular. Now consider the first order initial value problem

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(t_0) = \mathbf{x}_0.$$

Show there exists a unique solution to this first order system. **Hint:** Let $\mathbf{y} = Q^{-1}\mathbf{x}$ and so the system becomes

$$\mathbf{y}' = T\mathbf{y},\ \mathbf{y}(t_0) = Q^{-1}\mathbf{x}_0 \tag{2.4}$$

Now letting $\mathbf{y} = (y_1, \cdots, y_n)^T$, the bottom equation becomes

$$y_n' = t_{nn}y_n,\ y_n(t_0) = \left(Q^{-1}\mathbf{x}_0\right)_n.$$

Then use the solution you get in this to get the solution to the initial value problem which occurs one level up, namely

$$y_{n-1}' = t_{(n-1)(n-1)}y_{n-1} + t_{(n-1)n}y_n,\ y_{n-1}(t_0) = \left(Q^{-1}\mathbf{x}_0\right)_{n-1}$$

Continue doing this to obtain a unique solution to 2.4.

There isn't much to do. Just see that you understand it. In the above problem, equations of the above form are shown to have a unique solution and there is even a formula given. Thus there is a unique solution to

$$\mathbf{y}' = T\mathbf{y},\ \mathbf{y}(t_0) = Q^{-1}\mathbf{x}_0$$

Then let $\mathbf{y} = Q^{-1}\mathbf{x}$ and plug this in. Thus

$$Q^{-1}\mathbf{x}' = TQ^{-1}\mathbf{x}, \ \ Q^{-1}\mathbf{x}(t_0) = Q^{-1}\mathbf{x}_0$$

Then

$$\mathbf{x}' = QTQ^{-1}\mathbf{x} = A\mathbf{x}, \ \ \mathbf{x}(t_0) = \mathbf{x}_0.$$

The solution is unique because of uniqueness of the solution for $\mathbf{y}$. You can go backwards, starting with $\mathbf{x}$ and defining $\mathbf{y} \equiv Q^{-1}\mathbf{x}$ and then there is only one thing $\mathbf{y}$ can be and so $\mathbf{x}$ is also unique.

25. Now suppose $\Phi(t)$ is an $n \times n$ matrix of the form

$$\Phi(t) = \begin{pmatrix} \mathbf{x}_1(t) & \cdots & \mathbf{x}_n(t) \end{pmatrix} \tag{2.5}$$

where

$$\mathbf{x}_k'(t) = A\mathbf{x}_k(t).$$

Explain why

$$\Phi'(t) = A\Phi(t)$$

if and only if $\Phi(t)$ is given in the form of 2.5. Also explain why if $\mathbf{c} \in \mathbb{F}^n$,

$$\mathbf{y}(t) \equiv \Phi(t)\mathbf{c}$$

solves the equation

$$\mathbf{y}'(t) = A\mathbf{y}(t).$$

Suppose $\Phi'(t) = A\Phi(t)$ where $\Phi(t)$ is as described. This happens if and only if

$$\begin{aligned} \Phi'(t) &\equiv \begin{pmatrix} \mathbf{x}_1'(t) & \cdots & \mathbf{x}_n'(t) \end{pmatrix} = A\begin{pmatrix} \mathbf{x}_1(t) & \cdots & \mathbf{x}_n(t) \end{pmatrix} \\ &= \begin{pmatrix} A\mathbf{x}_1(t) & \cdots & A\mathbf{x}_n(t) \end{pmatrix} \end{aligned}$$

from the way we multiply matrices. Which happens if and only if

$$\mathbf{x}_k' = A\mathbf{x}_k$$

Say $\Phi'(t) = A\Phi(t)$. Then consider

$$\mathbf{y}(t) = \sum_k c_k \mathbf{x}_k(t).$$

Then

$$\mathbf{y}'(t) = \sum_k c_k \mathbf{x}_k'(t) = \sum_k c_k A\mathbf{x}_k(t) = A\left(\sum_k c_k \mathbf{x}_k(t)\right) = A\Phi(t)\mathbf{c} = A\mathbf{y}(t)$$

26. In the above problem, consider the question whether all solutions to

$$\mathbf{x}' = A\mathbf{x} \tag{2.6}$$

are obtained in the form $\Phi(t)\mathbf{c}$ for some choice of $\mathbf{c} \in \mathbb{F}^n$. In other words, is the general solution to this equation $\Phi(t)\mathbf{c}$ for $\mathbf{c} \in \mathbb{F}^n$? Prove the following theorem using linear algebra.

Theorem B.18.1 *Suppose $\Phi(t)$ is an $n \times n$ matrix which satisfies*

$$\Phi'(t) = A\Phi(t).$$

Then the general solution to 2.6 is $\Phi(t)\mathbf{c}$ if and only if $\Phi(t)^{-1}$ exists for some t. Furthermore, if $\Phi'(t) = A\Phi(t)$, then either $\Phi(t)^{-1}$ exists for all t or $\Phi(t)^{-1}$ never exists for any t.

(det $(\Phi(t))$ is called the Wronskian and this theorem is sometimes called the Wronskian alternative.)

Suppose first that $\Phi(t)\mathbf{c}$ is the general solution. Why does $\Phi(t)^{-1}$ necessarily exist? If for some t_0, $\Phi(t_0)$ fails to have an inverse, then there exists $\mathbf{c} \notin \Phi(t_0)(\mathbb{R}^n)$. However, there exists a unique a solution to the system

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{c}$$

from one of the above problems. It follows that $\mathbf{x}(t) \neq \Phi(t)\mathbf{v}$ for any choice of $\mathbf{v}$ because for any $\mathbf{v}$,

$$\mathbf{c} = \mathbf{x}(t_0) \neq \Phi(t_0)\mathbf{v}$$

Thus $\Phi(t)\mathbf{c}$ for arbitrary choice of $\mathbf{c}$ fails to deliver the general solution as assumed.

Next suppose $\Phi(t_0)^{-1}$ exists for some t_0 and let $\mathbf{x}(t)$ be any solution to the system of differential equations. Then let $\mathbf{y}(t) = \Phi(t)\Phi(t_0)^{-1}\mathbf{x}(t_0)$. It follows $\mathbf{y}' = A\mathbf{y}$ and $\mathbf{x}' = A\mathbf{x}$ and also that $\mathbf{x}(t_0) = \mathbf{y}(t_0)$ so by the uniqueness part of the above problem, $\mathbf{x} = \mathbf{y}$. Hence $\Phi(t)\mathbf{c}$ for arbitrary $\mathbf{c}$ is the general solution. Consider $\det\Phi(t)$ if it equals zero for any t_0 then $\Phi(t)\mathbf{c}$ does not deliver the general solutions. If $\det\Phi(t)$ is nonzero for any t then $\Phi(t)\mathbf{c}$ delivers the general solution. Either $\Phi(t)\mathbf{c}$ does deliver the general solution or it does not. Therefore, you cannot have two different values of t, t_1, t_2 such that $\det\Phi(t_1) = 0$ but $\det\Phi(t_2) \neq 0$. In other words $\det\Phi(t)$ either vanishes for all t or it vanishes for no t. Thus the inverse of $\Phi(t)$ either exists for all t or for no t.

27. Let $\Phi'(t) = A\Phi(t)$. Then $\Phi(t)$ is called a fundamental matrix if $\Phi(t)^{-1}$ exists for all t. Show there exists a unique solution to the equation

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f},\ \mathbf{x}(t_0) = \mathbf{x}_0 \tag{2.7}$$

and it is given by the formula

$$\mathbf{x}(t) = \Phi(t)\Phi(t_0)^{-1}\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t}\Phi(s)^{-1}\mathbf{f}(s)\,ds$$

Now these few problems have done virtually everything of significance in an entire undergraduate differential equations course, illustrating the superiority of linear algebra. The above formula is called the variation of constants formula.

It is easy to see that $\mathbf{x}$ given by the formula does solve the initial value problem. This is just calculus.

$$\begin{aligned}
\mathbf{x}'(t) &= \Phi'(t)\Phi(t_0)^{-1}\mathbf{x}_0 + \Phi'(t)\int_{t_0}^{t}\Phi(s)^{-1}\mathbf{f}(s)\,ds + \Phi(t)\Phi(t)^{-1}f(t) \\
&= A\left(\Phi(t)\Phi(t_0)^{-1}\mathbf{x}_0 + \Phi(t)\int_{t_0}^{t}\Phi(s)^{-1}\mathbf{f}(s)\,ds\right) + \mathbf{f}(t) \\
&= A\mathbf{x}(t) + \mathbf{f}(t)
\end{aligned}$$

As for the initial condition,

$$\mathbf{x}(t_0) = \Phi(t_0)\Phi(t_0)^{-1}\mathbf{x}_0 + \Phi(t_0)\int_{t_0}^{t_0}\Phi(s)^{-1}\mathbf{f}(s)\,ds = \mathbf{x}_0$$

It is also easy to see that this **must** be the solution. If you have two solutions, then let $\mathbf{u}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$ and observe that

$$\mathbf{u}'(t) = A\mathbf{u}(t), \quad \mathbf{u}(t_0) = \mathbf{0}.$$

From the above problem, there is at most one solution to this initial value problem and it is $\mathbf{u} = \mathbf{0}$.

28. Show there exists a special Φ such that $\Phi'(t) = A\Phi(t)$, $\Phi(0) = I$, and suppose $\Phi(t)^{-1}$ exists for all t. Show using uniqueness that

$$\Phi(-t) = \Phi(t)^{-1}$$

and that for all $t, s \in \mathbb{R}$

$$\Phi(t+s) = \Phi(t)\Phi(s)$$

Explain why with this special Φ, the solution to 2.7 can be written as

$$\mathbf{x}(t) = \Phi(t-t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t-s)\mathbf{f}(s)\,ds.$$

Hint: Let $\Phi(t)$ be such that the j^{th} column is $\mathbf{x}_j(t)$ where

$$\mathbf{x}_j' = A\mathbf{x}_j,\ \mathbf{x}_j(0) = \mathbf{e}_j.$$

Use uniqueness as required.

You simply let $\Phi(t) = \begin{pmatrix} \mathbf{x}_1(t) & \cdots & \mathbf{x}_n(t) \end{pmatrix}$ where

$$\mathbf{x}_k'(t) = A\mathbf{x}_k(t), \quad \mathbf{x}_k(0) = \mathbf{e}_k$$

Then $\det\Phi(0) = 1$ and so $\det\Phi(t) \neq 0$ for any t. Furthermore, there is only one solution to $\Phi'(t) = A\Phi(t)$ along with the initial condition $\Phi(0) = I$ and it is the one just described. This follows from the uniqueness of the intial value problem $\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0$ discussed earlier. Now it is interesting to note that A and $\Phi(t)$ commute. To see this, consider

$$\mathbf{y}(t) \equiv A\Phi(t)\mathbf{c} - \Phi(t)A\mathbf{c}$$

When $t = 0$ $\mathbf{y}(0) = A\mathbf{c} - A\mathbf{c} = \mathbf{0}$. What about its derivative?

$$\begin{aligned} \mathbf{y}'(t) &= A\Phi'(t)\mathbf{c} - \Phi'(t)A\mathbf{c} = A^2\Phi(t)\mathbf{c} - A\Phi(t)A\mathbf{c} \\ &= A(A\Phi(t)\mathbf{c} - \Phi(t)A\mathbf{c}) = A\mathbf{y}(t) \end{aligned}$$

By uniqueness, it follows that $\mathbf{y}(t) = \mathbf{0}$. Thus these commute as claimed, since $\mathbf{c}$ is arbitrary. Now from the product rule (the usual product rule holds when the functions are matrices by the same proof given in calculus.)

$$\begin{aligned} (\Phi(t)\Phi(-t))' &= \Phi'(t)\Phi(-t) - \Phi(t)\Phi'(-t) \\ &= A\Phi(t)\Phi(-t) - \Phi(t)A\Phi(-t) \\ &= \Phi(t)A\Phi(-t) - \Phi(t)A\Phi(-t) = 0 \end{aligned}$$

Hence $\Phi(t)\Phi(-t)$ must be a constant matrix since the derivative of each component of this product equals 0. However, this constant can only equal I because when $t = 0$ the product is I. Therefore, $\Phi(t)^{-1} = \Phi(-t)$. Next consider the claim that $\Phi(t+s) = \Phi(t)\Phi(s)$. Fix s and let t vary.

$$\Phi'(t+s) - \Phi'(t)\Phi(s) = A\Phi(t+s) - A\Phi(t)\Phi(s) = A(\Phi(t+s) - \Phi(t)\Phi(s))$$

Now $\Phi(0+s) - \Phi(0)\Phi(s) = \Phi(s) - \Phi(s) = 0$ and so, by uniqueness, $t \to \Phi(t+s) - \Phi(t)\Phi(s)$ equals 0. The rest follows from the variation of constants formula derived earlier.

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t)\Phi(0)^{-1}\mathbf{x}_0 + \Phi(t)\int_0^t \Phi(s)^{-1}\mathbf{f}(s)\,ds \\ &= \Phi(t)\mathbf{x}_0 + \Phi(t)\int_0^t \Phi(-s)\mathbf{f}(s) \\ &= \Phi(t)\mathbf{x}_0 + \int_0^t \Phi(t)\Phi(-s)\mathbf{f}(s) \\ &= \Phi(t)\mathbf{x}_0 + \int_0^t \Phi(t-s)\mathbf{f}(s) \end{aligned}$$

The reason you can take $\Phi(t)$ inside the integral is that this is constant with respect to the variable of integration. The j^{th} entry of the product is

$$\begin{aligned}\left(\Phi(t)\int_0^t \Phi(-s)\mathbf{f}(s)\right)_j &= \sum_k \Phi(t)_{jk}\int_0^t \sum_i \Phi(-s)_{ki} f_i(s)\,ds \\ &= \sum_{k,i}\int_0^t \Phi(t)_{jk}\Phi(-s)_{ki} f_i ds \\ &= \sum_i \int_0^t (\Phi(t)\Phi(-s))_{ji} f_i ds \\ &= \int_0^t \sum_i (\Phi(t)\Phi(-s))_{ji} f_i ds \\ &= \left(\int_0^t \Phi(t)\Phi(-s)\mathbf{f}(s)\,ds\right)_j\end{aligned}$$

You could also simply verify directly that this formula works much as was done earlier.

29. *Using the Lindemann Weierstrass theorem show that if σ is an algebraic number $\sin\sigma, \cos\sigma, \ln\sigma$, and e are all transcendental. **Hint:** Observe, that

$$ee^{-1} + (-1)e^0 = 0,\ 1e^{\ln(\sigma)} + (-1)\sigma e^0 = 0,$$
$$\frac{1}{2i}e^{i\sigma} - \frac{1}{2i}e^{-i\sigma} + (-1)\sin(\sigma)e^0 = 0.$$

The hint gives it away. Consider the claim about $\ln\sigma$. The equation shown does hold from the definition of $\ln\sigma$. However, if $\ln\sigma$ were algebraic, then $e^{\ln\sigma}, e^0$ would be linearly dependent with field of scalars equal to the algebraic numbers, contrary to the Lindemann Weierstrass theorem. The other instances are similar. In the case of $\cos\sigma$, you could use the identity

$$\frac{1}{2}e^{i\sigma} + \frac{1}{2}e^{-i\sigma} - e^0\cos\sigma = 0$$

contradicting independence of $e^{i\sigma}, e^{-i\sigma}, e^0$.

B.19 Exercises 16.6

1. Verify that Examples 16.5.1 - 16.5.4 are each inner product spaces.

First consider Example 16.5.1. All of the axioms of the inner product are obvious except one, the one which says that if $\langle f, f\rangle = 0$ then $f = 0$. This one depends on continuity of the functions. Suppose then that it is not true. In other words, $\langle f, f\rangle = 0$ and yet $f \neq 0$. Then for some $x \in I$, $f(x) \neq 0$. By continuity, there exists $\delta > 0$ such that if $y \in I \cap (x-\delta, x+\delta) \equiv I_\delta$, then

$$|f(y) - f(x)| < |f(x)|/2$$

It follows that for $y \in I_\delta$,

$$|f(y)| > |f(x)| - |f(x)/2| = |f(x)|/2.$$

Hence

$$\langle f, f\rangle \geq \int_{I_\delta} |f(y)|^2 p(x)\,dy \geq \left(|f(x)|^2/2\right)(\text{length of } I_\delta)(\min(p)) > 0,$$

a contradiction. Note that $\min p > 0$ because p is a continuous function defined on a closed and bounded interval and so it achieves its minimum by the extreme value theorem of calculus.

Consider the next one. All of the axioms of the inner product are obvious for this one also, except for the one which says that if $\langle f, f\rangle = 0$, then $f = 0$. Suppose than that $\langle f, f\rangle = 0$. Then f equals 0 at $n+1$ points of the interval and yet f is a polynomial of degree at most n. Therefore, this would be a contradiction unless f is identically equal to 0 for all x. This is because a polynomial of degree n has at most n zeros.

Consider the next example, inner product by decree. A generic vector of V will be denoted by $\mathbf{w} = \sum_{i=1}^{n} w_i \mathbf{v}_i$. Now

$$\langle \mathbf{w}, \mathbf{u}\rangle \equiv \sum_{k=1}^{n} w_k \overline{u_k} = \overline{\sum_{k=1}^{n} \overline{w_k} u_k} \equiv \overline{\langle \mathbf{u}, \mathbf{w}\rangle}$$

Letting a, b be scalars,

$$\begin{aligned} \langle a\mathbf{w} + b\mathbf{z}, \mathbf{u}\rangle &= \sum_k (aw_k + bz_k)\,\overline{u_k} = a\sum_k w_k \overline{u_k} + b\sum_k z_k \overline{u_k} \\ &\equiv a\langle \mathbf{w}, \mathbf{u}\rangle + b\langle \mathbf{z}, \mathbf{u}\rangle \end{aligned}$$

$$\langle \mathbf{w}, \mathbf{w}\rangle \equiv \sum_{k=1}^{n} w_k \overline{w_k} = \sum_{k=1}^{n} |w_k|^2 \geq 0$$

The only way this can equal zero is to have each $w_k = 0$. Since $\{\mathbf{v}_k\}$ is a basis, this happens if and only if $\mathbf{w} = \mathbf{0}$. Thus the inner product by decree is an inner product.

The last example is obviously an inner product as noted earlier except for needing to verify that the inner product makes sense; but this was done earlier.

2. In each of the examples 16.5.1 - 16.5.4 write the Cauchy Schwarz inequality.

   In the first example, the Cauchy Schwarz inequality says

$$\left|\int_I f(x)\,\overline{g(x)} p(x)\,dx\right| \leq \left(\int_I |f(x)|^2 p(x)\,dx\right)^{1/2} \left(\int_I |g(x)|^2 p(x)\,dx\right)^{1/2}$$

   In the next example, the Cauchy Schwarz inequality says

$$\left|\sum_{k=0}^{n} f(x_k)\,\overline{g(x_k)}\right| \leq \left(\sum_{k=0}^{n} |f(x_k)|^2\right)^{1/2} \left(\sum_{k=0}^{n} |g(x_k)|^2\right)^{1/2}$$

   In the third example, the Cauchy Schwarz inequality says

$$\left|\sum_{k=1}^{n} u_k \overline{w_k}\right| \leq \left(\sum_{k=1}^{n} |u_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} |w_k|^2\right)^{1/2}$$

   where $\mathbf{u} = \sum_k u_k \mathbf{v}_k$ and $\mathbf{w} = \sum_k w_k \mathbf{v}_k$.

   In the last example,

$$\left|\sum_{k=1}^{\infty} a_k \overline{b_k}\right| \leq \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} |b_k|^2\right)^{1/2}$$

3. Verify 16.16 and 16.17.

$$|\alpha\mathbf{z}|^2 \equiv \langle \alpha\mathbf{z}, \alpha\mathbf{z}\rangle = \alpha\langle \mathbf{z}, \alpha\mathbf{z}\rangle = \alpha\overline{\langle \alpha\mathbf{z}, \mathbf{z}\rangle} = \alpha\overline{\alpha}\langle \mathbf{z}, \mathbf{z}\rangle \equiv |\alpha|^2 |\mathbf{z}|^2$$

   and so

$$|\alpha|\,|\mathbf{z}| = |\alpha\mathbf{z}|$$

   It is clear $|\mathbf{z}| \geq 0$ because $|\mathbf{z}| = \langle \mathbf{z}, \mathbf{z}\rangle^{1/2}$. Suppose $|\mathbf{z}| = 0$. Then $\langle \mathbf{z}, \mathbf{z}\rangle = 0$ and so, by the axioms of the inner product, $\mathbf{z} = \mathbf{0}$.

4. Consider the Cauchy Schwarz inequality. Show that it still holds under the assumptions $\langle \mathbf{u},\mathbf{v}\rangle = \overline{\langle \mathbf{v},\mathbf{u}\rangle}$, $\langle (a\mathbf{u}+b\mathbf{v}),\mathbf{z}\rangle = a\langle \mathbf{u},\mathbf{z}\rangle + b\langle \mathbf{v},\mathbf{z}\rangle$, and $\langle \mathbf{u},\mathbf{u}\rangle \geq 0$. Thus it is not necessary to say that $\langle \mathbf{u},\mathbf{u}\rangle = 0$ only if $\mathbf{u}=\mathbf{0}$. It is enough to simply state that $\langle \mathbf{u},\mathbf{u}\rangle \geq 0$.

   This was all that was used in the proof. Look at the proof carefully and you will see this is the case.

5. Consider the integers modulo a prime, $\mathbb{Z}_p$. This is a field of scalars. Now let the vector space be $(\mathbb{Z}_p)^n$ where $n \geq p$. Define now
$$\langle \mathbf{z},\mathbf{w}\rangle \equiv \sum_{i=1}^{n} z_i w_i$$
   Does this satisfy the axioms of an inner product? Does the Cauchy Schwarz inequality hold for this $\langle\rangle$? Does the Cauchy Schwarz inequality even make any sense?

   It might be the case that $\langle \mathbf{z},\mathbf{z}\rangle = 0$ and yet $\mathbf{z} \neq \mathbf{0}$. Just let $\mathbf{z} = (z_1,\cdots,z_n)$ where exactly p of the z_i equal 1 but the remaining are equal to 0. Then $\langle \mathbf{z},\mathbf{z}\rangle$ would reduce to 0 in the integers mod p. Another problem is the failure to have an order on $\mathbb{Z}_p$. Consider first $\mathbb{Z}_2$. Is 1 positive or negative? If it is positive, then $1+1$ would need to be positive. But $1+1=0$ in this case. If 1 is negative, then -1 is positive, but -1 is equal to 1. Thus 1 would be both positive and negative. You can consider the general case where $p>2$ also. Simply take $a \neq 1$. If a is positive, then consider $a, a^2, a^3 \cdots$. These would all have to be positive. However, eventually a repeat will take place. Thus $a^n = a^m$ $m<n$, and so $a^m\left(a^k-1\right) = 0$ where $k = n-m$. Since $a^m \neq 0$, it follows that $a^k = 1$ for a suitable k. It follows that the sequence of powers of a must include each of $\{1,2,\cdots,p-1\}$ and all these would therefore, be positive. However, $1+(p-1)=0$ contradicting the assertion that $\mathbb{Z}_p$ can be ordered. So what would you mean by saying $\langle \mathbf{z},\mathbf{z}\rangle \geq 0$? The Cauchy Schwarz inequality would not even apply.

6. If you only know that $\langle \mathbf{u},\mathbf{u}\rangle \geq 0$ along with the other axioms of the inner product and if you define $|\mathbf{z}|$ the same way, how do the conclusions of Theorem 16.5.7 change?

   You lose the one which says that if $|\mathbf{z}| = 0$ then $\mathbf{z}=\mathbf{0}$. However, all the others are retained.

7. In an inner product space, an open ball is the set
$$B(\mathbf{x},r) \equiv \{\mathbf{y} : |\mathbf{y}-\mathbf{x}| < r\}.$$
   If $\mathbf{z} \in B(\mathbf{x},r)$, show there exists $\delta > 0$ such that $B(\mathbf{z},\delta) \subseteq B(\mathbf{x},r)$. In words, this says that an open ball is open. **Hint:** This depends on the triangle inequality.

   Let $\delta = r - |\mathbf{z}-\mathbf{x}|$. Then if $\mathbf{y} \in B(\mathbf{z},\delta)$,
$$|\mathbf{y}-\mathbf{x}| \leq |\mathbf{y}-\mathbf{z}| + |\mathbf{z}-\mathbf{x}| < \delta + |\mathbf{z}-\mathbf{x}| = r - |\mathbf{z}-\mathbf{x}| + |\mathbf{z}-\mathbf{x}| = r$$
   and so $B(\mathbf{z},\delta) \subseteq B(\mathbf{x},r)$.

8. Let V be the real inner product space consisting of continuous functions defined on $[-1,1]$ with the inner product given by
$$\int_{-1}^{1} f(x)\,g(x)\,dx$$
   Show that $\left\{1,x,x^2\right\}$ are linearly independent and find an orthonormal basis for the span of these vectors.

   Lets first find the Grammian.
$$G = \begin{pmatrix} 2 & \int_{-1}^{1} x dx & \int_{-1}^{1} x^2 \\ \int_{-1}^{1} x dx & \int_{-1}^{1} x^2 dx & \int_{-1}^{1} x^3 \\ \int_{-1}^{1} x^2 & \int_{-1}^{1} x^3 & \int_{-1}^{1} x^4 dx \end{pmatrix} = \begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix}$$

Then the inverse is

$$G^{-1} = \begin{pmatrix} \frac{9}{8} & 0 & -\frac{15}{8} \\ 0 & \frac{3}{2} & 0 \\ -\frac{15}{8} & 0 & \frac{45}{8} \end{pmatrix}$$

Now let $R = \begin{pmatrix} a & b & c \\ 0 & d & h \\ 0 & 0 & f \end{pmatrix}$. Solve

$$\begin{pmatrix} a & b & c \\ 0 & d & h \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & d & h \\ 0 & 0 & f \end{pmatrix}^T = \begin{pmatrix} \frac{9}{8} & 0 & -\frac{15}{8} \\ 0 & \frac{3}{2} & 0 \\ -\frac{15}{8} & 0 & \frac{45}{8} \end{pmatrix}$$

Thus

$$\begin{pmatrix} a^2+b^2+c^2 & bd+ch & cf \\ bd+ch & d^2+h^2 & hf \\ cf & hf & f^2 \end{pmatrix} = \begin{pmatrix} \frac{9}{8} & 0 & -\frac{15}{8} \\ 0 & \frac{3}{2} & 0 \\ -\frac{15}{8} & 0 & \frac{45}{8} \end{pmatrix}$$

Hence a solution is

$$f = \frac{3}{4}\sqrt{2}\sqrt{5}, h = 0, c = -\frac{1}{4}\sqrt{2}\sqrt{5}, d = \frac{1}{2}\sqrt{2}\sqrt{3}, b = 0, a = \frac{1}{2}\sqrt{2}$$

and so the orthonormal basis is

$$\begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{4}\sqrt{2}\sqrt{5} \\ 0 & \frac{1}{2}\sqrt{2}\sqrt{3} & 0 \\ 0 & 0 & \frac{3}{4}\sqrt{2}\sqrt{5} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2}\sqrt{3}x & \frac{3}{4}\sqrt{2}\sqrt{5}x^2 - \frac{1}{4}\sqrt{2}\sqrt{5} \end{pmatrix}$$

9. A **regular Sturm Liouville problem** involves the differential equation for an unknown function of x which is denoted here by y,

$$(p(x)y')' + (\lambda q(x) + r(x))y = 0, \ x \in [a,b]$$

and it is assumed that $p(t), q(t) > 0$ for any t along with boundary conditions,

$$\begin{aligned} C_1 y(a) + C_2 y'(a) &= 0 \\ C_3 y(b) + C_4 y'(b) &= 0 \end{aligned}$$

where

$$C_1^2 + C_2^2 > 0, \text{ and } C_3^2 + C_4^2 > 0.$$

There is an immense theory connected to these important problems. The constant λ is called an eigenvalue. Show that if y is a solution to the above problem corresponding to $\lambda = \lambda_1$ and if z is a solution corresponding to $\lambda = \lambda_2 \neq \lambda_1$, then

$$\int_a^b q(x)\,y(x)\,z(x)\,dx = 0. \tag{2.8}$$

Let y go with λ and z go with μ.

$$\begin{aligned} z(p(x)y')' + (\lambda q(x) + r(x))yz &= 0 \\ y(p(x)z')' + (\mu q(x) + r(x))zy &= 0 \end{aligned}$$

Subtract.

$$z(p(x)y')' - y(p(x)z')' + (\lambda - \mu)q(x)yz = 0$$

Now integrate from a to b. First note that

$$z\left(p\left(x\right)y'\right)' - y\left(p\left(x\right)z'\right)' = \frac{d}{dx}\left(p\left(x\right)y'z - p\left(x\right)z'y\right)$$

and so what you get is

$$p\left(b\right)y'\left(b\right)z\left(b\right) - p\left(b\right)z'\left(b\right)y\left(b\right) - \left(p\left(a\right)y'\left(a\right)z\left(a\right) - p\left(a\right)z'\left(a\right)y\left(a\right)\right)$$
$$+\left(\lambda-\mu\right)\int_a^b q\left(x\right)y\left(x\right)z\left(x\right)dx = 0$$

Look at the stuff on the top line. From the assumptions on the boundary conditions,

$$\begin{aligned} C_1 y\left(a\right) + C_2 y'\left(a\right) &= 0 \\ C_1 z\left(a\right) + C_2 z'\left(a\right) &= 0 \end{aligned}$$

and so

$$y\left(a\right)z'\left(a\right) - y'\left(a\right)z\left(a\right) = 0$$

Similarly,

$$y\left(b\right)z'\left(b\right) - y'\left(b\right)z\left(b\right) = 0$$

Hence, that stuff on the top line equals zero and so the orthogonality condition holds.

10. Using the above problem or standard techniques of calculus, show that

$$\left\{\frac{\sqrt{2}}{\sqrt{\pi}}\sin\left(nx\right)\right\}_{n=1}^{\infty}$$

are orthonormal with respect to the inner product

$$\langle f, g\rangle = \int_0^\pi f\left(x\right)g\left(x\right)dx$$

Hint: If you want to use the above problem, show that $\sin\left(nx\right)$ is a solution to the boundary value problem

$$y'' + n^2 y = 0,\ \ y\left(0\right) = y\left(\pi\right) = 0$$

It is obvious that $\sin\left(nx\right)$ solves this boundary value problem. Those boundary conditions in the above problem are implied by these. In fact,

$$\begin{aligned} 1\sin 0 + \left(0\right)\cos\left(0\right) &= 0 \\ 1\sin\pi + \left(0\right)\cos\pi &= 0 \end{aligned}$$

Then it follows that

$$\int_0^\pi \sin\left(nx\right)\sin\left(mx\right)dx = 0 \text{ unless } n = m.$$

Now also $\int_0^\pi \sin^2\left(nx\right)dx = \frac{1}{2}\pi$ and so $\left\{\frac{\sqrt{2}}{\sqrt{\pi}}\sin\left(nx\right)\right\}$ are orthonormal.

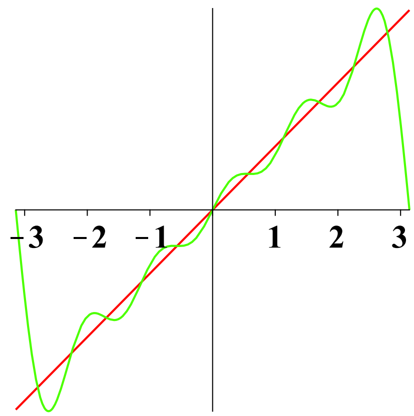
11. Find $S_5 f\left(x\right)$ where $f\left(x\right) = x$ on $\left[-\pi,\pi\right]$. Then graph both $S_5 f\left(x\right)$ and $f\left(x\right)$ if you have access to a system which will do a good job of it.

Recall that this is the partial sum of the Fourier series. $\sum_{k=-5}^{5} b_k e^{ikx}$ where

$$b_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-ikx}dx = \frac{\left(-1\right)^k}{k}i$$

Thus

$$\begin{aligned} S_5 f(x) &= \sum_{k=-5}^{5} \left(\frac{(-1)^k}{k} i \right) e^{ikx} = \sum_{k=-5}^{5} \frac{(-1)^k}{k} i \left(\cos(kx) + i \sin(kx) \right) \\ &= \sum_{k=-5}^{5} \frac{(-1)^{k+1}}{k} \sin(kx) = \sum_{k=1}^{5} \frac{2(-1)^{k+1}}{k} \sin(kx) \end{aligned}$$



12. Find $S_5 f(x)$ where $f(x) = |x|$ on $[-\pi, \pi]$. Then graph both $S_5 f(x)$ and $f(x)$ if you have access to a system which will do a good job of it.

$$b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| e^{-ikx} dx = -\frac{2}{k^2 \pi}$$

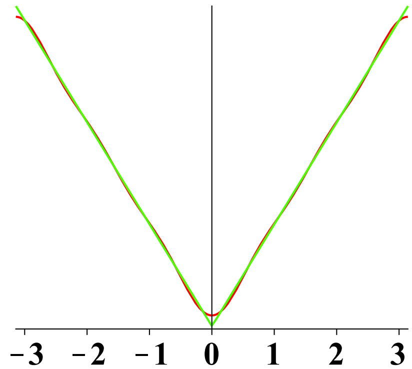
if k is odd and 0 if k is even. However, $b_0 = \frac{1}{2}\pi$ Thus the sum desired is

$$\begin{aligned} S_5 f(x) &= \sum_{k=-5}^{-1} -\frac{1}{k^2\pi} \left(1 + (-1)^{k+1}\right) \cos(kx) + \sum_{k=1}^{5} -\frac{1}{k^2\pi} \left(1 + (-1)^{k+1}\right) \cos(kx) \\ &\quad + \frac{1}{2}\pi \end{aligned}$$

since the sin terms cancel due to the fact that sin is odd. The above equals

$$\frac{1}{2}\pi - \sum_{k=0}^{2} \frac{4}{(2k+1)^2 \pi} \cos((2k+1)x)$$

$\frac{1}{2}\pi - \sum_{k=0}^{2} \frac{4}{(2k+1)^2\pi} \cos((2k+1)x)$

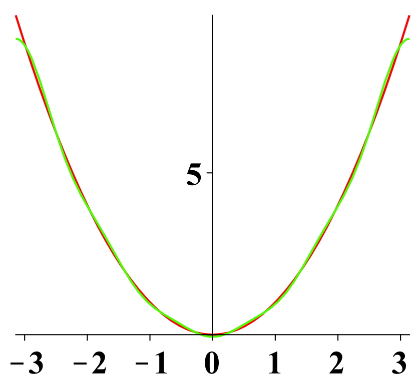


13. Find $S_5 f(x)$ where $f(x) = x^2$ on $[-\pi, \pi]$. Then graph both $S_5 f(x)$ and $f(x)$ if you have access to a system which will do a good job of it.

$$
\begin{aligned}
b_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-kix}dx = \frac{2}{k^2}(-1)^k \\
b_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{3}\pi^2
\end{aligned}
$$

Then the series is

$$
\begin{aligned}
S_5 f(x) &= \sum_{k=1}^{5} \frac{2}{k^2}(-1)^k e^{ikx} + \sum_{k=-5}^{-1} \frac{2}{k^2}(-1)^k e^{ikx} + \frac{\pi^2}{3} \\
&= \sum_{k=1}^{5} \frac{4}{k^2}(-1)^k \cos(kx) + \frac{\pi^2}{3}
\end{aligned}
$$



14. Let V be the set of real polynomials defined on $[0,1]$ which have degree at most 2. Make this into a real inner product space by defining

$$\langle f,g\rangle \equiv f(0)\,g(0)+f(1/2)\,g(1/2)+f(1)\,g(1)$$

Find an orthonormal basis and explain why this is an inner product.

It was described more generally why this was an inner product in an earlier problem. To find an orthonormal basis, consider a basis $\{1,x,x^2\}$. Then the Grammian is

$$\begin{pmatrix} 3 & 0+1/2+1 & 0+1/4+1 \\ 0+1/2+1 & 0+1/4+1 & 0+(1/2)(1/4)+1 \\ 0+1/4+1 & 0+(1/2)(1/4)+1 & 0+(1/4)^2+1 \end{pmatrix}$$

$$=\begin{pmatrix} 3 & \frac{3}{2} & \frac{5}{4} \\ \frac{3}{2} & \frac{5}{4} & \frac{9}{8} \\ \frac{5}{4} & \frac{9}{8} & \frac{17}{16} \end{pmatrix}$$

Now G^{-1} equals

$$\begin{pmatrix} 1 & -3 & 2 \\ -3 & 26 & -24 \\ 2 & -24 & 24 \end{pmatrix}$$

Let $R=\begin{pmatrix} a & b & c \\ 0 & d & h \\ 0 & 0 & f \end{pmatrix}$

$$\begin{pmatrix} a & b & c \\ 0 & d & h \\ 0 & 0 & f \end{pmatrix}\begin{pmatrix} a & b & c \\ 0 & d & h \\ 0 & 0 & f \end{pmatrix}^T=\begin{pmatrix} a^2+b^2+c^2 & bd+ch & cf \\ bd+ch & d^2+h^2 & hf \\ cf & hf & f^2 \end{pmatrix}$$

So you need to solve the following .

$$\begin{pmatrix} a^2+b^2+c^2 & bd+ch & cf \\ bd+ch & d^2+h^2 & hf \\ cf & hf & f^2 \end{pmatrix}=\begin{pmatrix} 1 & -3 & 2 \\ -3 & 26 & -24 \\ 2 & -24 & 24 \end{pmatrix}$$

$$f=2\sqrt{6},h=-2\sqrt{6},c=\frac{1}{6}\sqrt{6},d=\sqrt{2},b=-\frac{1}{2}\sqrt{2},a=\frac{1}{3}\sqrt{3}$$

So the orthonormal basis is

$$\begin{pmatrix} 1 & x & x^2 \end{pmatrix}\begin{pmatrix} \frac{1}{3}\sqrt{3} & -\frac{1}{2}\sqrt{2} & \frac{1}{6}\sqrt{6} \\ 0 & \sqrt{2} & -2\sqrt{6} \\ 0 & 0 & 2\sqrt{6} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{3}\sqrt{3} & \sqrt{2}x-\frac{1}{2}\sqrt{2} & 2\sqrt{6}x^2-2\sqrt{6}x+\frac{1}{6}\sqrt{6} \end{pmatrix}$$

15. Consider $\mathbb{R}^n$ with the following definition.

$$\langle \mathbf{x},\mathbf{y}\rangle \equiv \sum_{i=1}^{n} x_i y_i i$$

Does this define an inner product? If so, explain why and state the Cauchy Schwarz inequality in terms of sums.

It obviously defines an inner product because it satisfies all the axioms of one. The Cauchy Schwarz inequality says

$$\left|\sum_{i=1}^{n} x_i y_i i\right| \leq \left(\sum_{i=1}^{n} x_i^2 i\right)^{1/2}\left(\sum_{i=1}^{n} y_i^2 i\right)^{1/2}$$

This is a fairly surprising inequality it seems to me.

16. From the above, for f a piecewise continuous function,

$$S_n f(x) = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{ikx} \left(\int_{-\pi}^{\pi} f(y) e^{-iky} dy \right).$$

Show this can be written in the form

$$S_n f(x) = \int_{-\pi}^{\pi} f(y) D_n(x-y) \, dy$$

where

$$D_n(t) = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{ikt}$$

This is called the Dirichlet kernel. Show that

$$D_n(t) = \frac{1}{2\pi} \frac{\sin(n + (1/2)) t}{\sin(t/2)}$$

For V the vector space of piecewise continuous functions, define $S_n : V \to V$ by

$$S_n f(x) = \int_{-\pi}^{\pi} f(y) D_n(x-y) \, dy.$$

Show that S_n is a linear transformation. (In fact, $S_n f$ is not just piecewise continuous but infinitely differentiable. Why?) Explain why $\int_{-\pi}^{\pi} D_n(t) \, dt = 1$. **Hint:** To obtain the formula, do the following.

$$\begin{aligned} e^{i(t/2)} D_n(t) &= \frac{1}{2\pi} \sum_{k=-n}^{n} e^{i(k+(1/2))t} \\ e^{i(-t/2)} D_n(t) &= \frac{1}{2\pi} \sum_{k=-n}^{n} e^{i(k-(1/2))t} \end{aligned}$$

Change the variable of summation in the bottom sum and then subtract and solve for $D_n(t)$.

$$S_n f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-n}^{n} e^{ikx} \left(\int_{-\pi}^{\pi} f(y) e^{-iky} dy \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sum_{k=-n}^{n} e^{ik(x-y)} dy.$$

Let $D_n(t)$ be as described. Then the above equals

$$\int_{-\pi}^{\pi} D_n(x-y) f(y) \, dy.$$

So now you want to find a formula for $D_n(t)$. The hint is really good.

$$\begin{aligned} e^{i(t/2)} D_n(t) &= \frac{1}{2\pi} \sum_{k=-n}^{n} e^{i(k+(1/2))t} \\ e^{i(-t/2)} D_n(t) &= \frac{1}{2\pi} \sum_{k=-n}^{n} e^{i(k-(1/2))t} = \frac{1}{2\pi} \sum_{k=-(n+1)}^{n-1} e^{i(k+(1/2))t} \end{aligned}$$

$$D_n(t) \left(e^{i(t/2)} - e^{-i(t/2)} \right) = \frac{1}{2\pi} \left(e^{i(n+(1/2))t} - e^{-i(n+(1/2))t} \right)$$

$$D_n(t)\,2i\sin(t/2) = \frac{1}{2\pi}2i\sin\left(\left(n+\frac{1}{2}\right)t\right)$$

$$D_n(t) = \frac{1}{2\pi}\frac{\sin\left(t\left(n+\frac{1}{2}\right)\right)}{\sin\left(\frac{1}{2}t\right)}$$

You know that $t \to D_n(t)$ is periodic of period 2π. Therefore, if $f(y) = 1$,

$$S_n f(x) = \int_{-\pi}^{\pi} D_n(x-y)\,dy = \int_{-\pi}^{\pi} D_n(t)\,dt$$

However, it follows directly from computation that $S_n f(x) = 1$.

17. Let V be an inner product space and let U be a finite dimensional subspace with an orthonormal basis $\{\mathbf{u}_i\}_{i=1}^n$. If $\mathbf{y} \in V$, show

$$|\mathbf{y}|^2 \geq \sum_{k=1}^n |\langle \mathbf{y}, \mathbf{u}_k\rangle|^2$$

Now suppose that $\{\mathbf{u}_k\}_{k=1}^\infty$ is an orthonormal set of vectors of V. Explain why

$$\lim_{k\to\infty} \langle \mathbf{y}, \mathbf{u}_k\rangle = 0.$$

When applied to functions, this is a special case of the Riemann Lebesgue lemma.

From Lemma 16.5.11 and Theorem 16.5.12

$$\left\langle \mathbf{y} - \sum_{k=1}^n \langle \mathbf{y}, \mathbf{u}_k\rangle \mathbf{u}_k, \mathbf{w} \right\rangle = 0$$

for all $\mathbf{w} \in \operatorname{span}\left(\{\mathbf{u}_i\}_{i=1}^n\right)$. Therefore,

$$|\mathbf{y}|^2 = \left|\mathbf{y} - \sum_{k=1}^n \langle \mathbf{y}, \mathbf{u}_k\rangle \mathbf{u}_k + \sum_{k=1}^n \langle \mathbf{y}, \mathbf{u}_k\rangle \mathbf{u}_k\right|^2$$

Now if $\langle \mathbf{u}, \mathbf{v}\rangle = 0$, then you can see right away from the definition that

$$|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$$

Applying this to $\mathbf{u} = \mathbf{y} - \sum_{k=1}^n \langle \mathbf{y}, \mathbf{u}_k\rangle \mathbf{u}_k,\ \mathbf{v} = \sum_{k=1}^n \langle \mathbf{y}, \mathbf{u}_k\rangle \mathbf{u}_k$, the above equals

$$\begin{aligned} &= \left|\mathbf{y} - \sum_{k=1}^n \langle \mathbf{y}, \mathbf{u}_k\rangle \mathbf{u}_k\right|^2 + \left|\sum_{k=1}^n \langle \mathbf{y}, \mathbf{u}_k\rangle \mathbf{u}_k\right|^2 \\ &= \left|\mathbf{y} - \sum_{k=1}^n \langle \mathbf{y}, \mathbf{u}_k\rangle \mathbf{u}_k\right|^2 + \sum_{k=1}^n |\langle \mathbf{y}, \mathbf{u}_k\rangle|^2, \end{aligned}$$

the last step following because of similar reasoning to the above and the assumption that the $\mathbf{u}_k$ are orthonormal. It follows the sum $\sum_{k=1}^\infty |\langle \mathbf{y}, \mathbf{u}_k\rangle|^2$ converges and so $\lim_{k\to\infty} \langle \mathbf{y}, \mathbf{u}_k\rangle = 0$ because if a series converges, then the k^{th} term must converge to 0.

18. Let f be any piecewise continuous function which is bounded on $[-\pi, \pi]$. Show, using the above problem, that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \lim_{n\to\infty} \int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = 0$$

Let the inner product space consist of piecewise continuous bounded functions with the inner product defined by

$$\langle f, g\rangle \equiv \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$$

Then, from the above problem and the fact shown earlier that $\left\{\frac{1}{\sqrt{2\pi}}e^{ikx}\right\}_{k\in\mathbb{Z}}$ form an orthonormal set of vectors in this inner product space, it follows that

$$\lim_{n\to\infty}\left\langle f, e^{inx}\right\rangle = 0$$

without loss of generality, assume that f has real values. Then the above limit reduces to having both the real and imaginary parts converge to 0. This implies the thing which was desired. Note also that if $\alpha \in [-1, 1]$, then

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\sin((n+\alpha)t)\,dt = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\left[\sin(nt)\cos\alpha + \cos(nt)\sin\alpha\right]dt = 0$$

19. *Let f be a function which is defined on $(-\pi, \pi]$. The 2π periodic extension is given by the formula $f(x+2\pi) = f(x)$. In the rest of this problem, f will refer to this 2π periodic extension. Assume that f is piecewise continuous, bounded, and also that the following limits exist

$$\lim_{y\to 0+}\frac{f(x+y) - f(x+)}{y}, \ \lim_{y\to 0+}\frac{f(x-y) - f(x+)}{y}$$

Here it is assumed that

$$f(x+) \equiv \lim_{h\to 0+} f(x+h),\ f(x-) \equiv \lim_{h\to 0+} f(x-h)$$

both exist at every point. The above conditions rule out functions where the slope taken from either side becomes infinite. Justify the following assertions and eventually conclude that under these very reasonable conditions

$$\lim_{n\to\infty} S_n f(x) = (f(x+) + f(x-))/2$$

the mid point of the jump. In words, the Fourier series converges to the midpoint of the jump of the function.

$$S_n f(x) = \int_{-\pi}^{\pi} f(x-y)\,D_n(y)\,dy$$

$$\left|S_n f(x) - \frac{f(x+) + f(x-)}{2}\right| = \left|\int_{-\pi}^{\pi}\left(f(x-y) - \frac{f(x+) + f(x-)}{2}\right)D_n(y)\,dy\right|$$

$$\begin{aligned} = & \left|\int_0^{\pi} f(x-y)\,D_n(y)\,dy + \int_0^{\pi} f(x+y)\,D_n(y)\,dy \right.\\ & \left. - \int_0^{\pi}(f(x+) + f(x-))\,D_n(y)\,dy\right|\end{aligned}$$

$$\leq \left|\int_0^{\pi}(f(x-y) - f(x-))\,D_n(y)\,dy\right| + \left|\int_0^{\pi}(f(x+y) - f(x+))\,D_n(y)\,dy\right|$$

Now apply some trig. identities and use the result of Problem 18 to conclude that both of these terms must converge to 0.

From the definition of D_n, the top formula holds. Now observe that D_n is an even function. Therefore, the formula equals

$$\begin{aligned} S_n f(x) &= \int_0^\pi f(x-y) D_n(y)\,dy + \int_{-\pi}^0 f(x-y) D_n(y)\,dy \\ &= \int_0^\pi f(x-y) D_n(y)\,dy + \int_0^\pi f(x+y) D_n(y)\,dy \\ &= \int_0^\pi \frac{f(x+y)+f(x-y)}{2} 2D_n(y)\,dy \end{aligned}$$

Now note that $\int_0^\pi 2D_n(y) = 1$ because $\int_{-\pi}^\pi D_n(y)\,dy = 1$ and D_n is even. Therefore,

$$\begin{aligned} &\left| S_n f(x) - \frac{f(x+)+f(x-)}{2} \right| \\ = &\left| \int_0^\pi \frac{f(x+y)-f(x+)+f(x-y)-f(x-)}{2} 2D_n(y)\,dy \right| \end{aligned}$$

From the formula for $D_n(y)$ given earlier, this is dominated by an expression of the form

$$C\left| \int_0^\pi \frac{f(x+y)-f(x+)+f(x-y)-f(x-)}{\sin(y/2)} \sin((n+1/2)y)\,dy \right|$$

for a suitable constant C. The above is equal to

$$C\left| \int_0^\pi \frac{y}{\sin(y/2)} \frac{f(x+y)-f(x+)+f(x-y)-f(x-)}{y} \sin((n+1/2)y)\,dy \right|$$

and the expression $\frac{y}{\sin(y/2)}$ equals a bounded continuous function on $[0,\pi]$ except at 0 where it is undefined. This follows from elementary calculus. Therefore, changing the function at this single point does not change the integral and so we can consider this as a continuous bounded function defined on $[0,\pi]$. Also, from the assumptions on f,

$$y \to \frac{f(x+y)-f(x+)+f(x-y)-f(x-)}{y}$$

is equal to a piecewise continuous function on $[0,\pi]$ except at the point 0. Therefore, the above integral converges to 0 by the previous problem. This shows that the Fourier series generally tries to converge to the midpoint of the jump.

20. Using the Fourier series obtained in Problem 11 and the result of Problem 19 above, find an interesting formula by examining where the Fourier series converges when $x = \pi/2$. Of course you can get many other interesting formulas in the same way. **Hint:** You should get

$$S_n f(x) = \sum_{k=1}^n \frac{2(-1)^{k+1}}{k} \sin(kx)$$

Let $x = \pi/2$. Then you must have

$$\begin{aligned} \frac{\pi}{2} &= \lim_{n\to\infty} \sum_{k=1}^n \frac{2(-1)^{k+1}}{k} \sin\left(k\frac{\pi}{2}\right) = \lim_{n\to\infty} \sum_{k=1}^n \frac{2}{2k-1} \sin\left((2k-1)\frac{\pi}{2}\right) \\ &= \lim_{n\to\infty} \sum_{k=1}^n \frac{2}{2k-1} (-1)^{k+1}. \end{aligned}$$

Then, dividing by 2 you get

$$\frac{\pi}{4} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1}$$

You could use x^2 instead of x and get

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{4}{k^2} (-1)^k \cos(kx) + \frac{\pi^2}{3} = x^2$$

because the periodic extension of this function is continuous. Let $x = 0$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{4}{k^2} (-1)^k + \frac{\pi^2}{3} = 0$$

and so

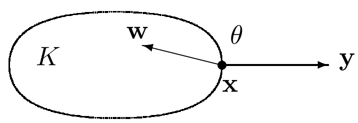
$$\frac{\pi^2}{3} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{4}{k^2} (-1)^{k+1} \equiv \sum_{k=1}^{\infty} \frac{4}{k^2} (-1)^{k+1}$$

This is one of those calculus problems where you show it converges absolutely by the comparison test with a p series. However, here is what it converges to.

21. Let V be an inner product space and let K be a convex subset of V. This means that if $\mathbf{x}, \mathbf{z} \in K$, then the line segment $\mathbf{x} + t(\mathbf{z} - \mathbf{x}) = (1-t)\mathbf{x} + t\mathbf{z}$ is contained in K for all $t \in [0,1]$. Note that every subspace is a convex set. Let $\mathbf{y} \in V$ and let $\mathbf{x} \in K$. Show that $\mathbf{x}$ is the closest point to $\mathbf{y}$ out of all points in K if and only if for all $\mathbf{w} \in K$,

$$\operatorname{Re}\langle \mathbf{y} - \mathbf{x}, \mathbf{w} - \mathbf{x}\rangle \leq 0.$$

In $\mathbb{R}^n$, a picture of the above situation where $\mathbf{x}$ is the closest point to $\mathbf{y}$ is as follows.



The condition of the above **variational inequality** is that the angle θ shown in the picture is larger than 90 degrees. Recall the geometric description of the dot product presented earlier. See Page 41.

Consider for $t \in [0,1]$ the following.

$$|\mathbf{y} - (\mathbf{x} + t(\mathbf{w} - \mathbf{x}))|^2$$

where $\mathbf{w} \in K$ and $\mathbf{x} \in K$. It equals

$$f(t) = |\mathbf{y} - \mathbf{x}|^2 + t^2 |\mathbf{w} - \mathbf{x}|^2 - 2t \operatorname{Re}\langle \mathbf{y} - \mathbf{x}, \mathbf{w} - \mathbf{x}\rangle$$

Suppose $\mathbf{x}$ is the point of K which is closest to $\mathbf{y}$. Then $f'(0) \geq 0$. However, $f'(0) = -2\operatorname{Re}\langle \mathbf{y} - \mathbf{x}, \mathbf{w} - \mathbf{x}\rangle$. Therefore, if $\mathbf{x}$ is closest to $\mathbf{y}$,

$$\operatorname{Re}\langle \mathbf{y} - \mathbf{x}, \mathbf{w} - \mathbf{x}\rangle \leq 0.$$

Next suppose this condition holds. Then you have

$$|\mathbf{y} - (\mathbf{x} + t(\mathbf{w} - \mathbf{x}))|^2 \geq |\mathbf{y} - \mathbf{x}|^2 + t^2 |\mathbf{w} - \mathbf{x}|^2 \geq |\mathbf{y} - \mathbf{x}|^2$$

By convexity of K, a generic point of K is of the form $\mathbf{x} + t(\mathbf{w} - \mathbf{x})$ for $\mathbf{w} \in K$. Hence $\mathbf{x}$ is the closest point.

22. Show that in any inner product space the parallelogram identity holds.

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

Next show that in a real inner product space, the polarization identity holds.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}\left(|\mathbf{x}+\mathbf{y}|^2 - |\mathbf{x}-\mathbf{y}|^2 \right).$$

This follows right away from the axioms of the inner product.

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 &= |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\operatorname{Re}\langle \mathbf{x}, \mathbf{y} \rangle \\ &\quad + |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\operatorname{Re}\langle \mathbf{x}, \mathbf{y} \rangle \\ &= 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2 \end{aligned}$$

Of course the same reasoning yields

$$\begin{aligned} \frac{1}{4}\left(|\mathbf{x}+\mathbf{y}|^2 - |\mathbf{x}-\mathbf{y}|^2 \right) &= \frac{1}{4}\left(|\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle - \left(|\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle \right) \right) \\ &= \langle \mathbf{x}, \mathbf{y} \rangle \end{aligned}$$

23. *This problem is for those who know about Cauchy sequences and completeness of $\mathbb{F}^p$ and about closed sets. Suppose K is a closed nonempty convex subset of a finite dimensional subspace U of an inner product space V. Let $\mathbf{y} \in V$. Then show there exists a unique point $\mathbf{x} \in K$ which is closest to $\mathbf{y}$. **Hint:** Let

$$\lambda = \inf \{ |\mathbf{y} - \mathbf{z}| : \mathbf{z} \in K \}$$

Let $\{\mathbf{x}_n\}$ be a minimizing sequence,

$$|\mathbf{y} - \mathbf{x}_n| \to \lambda.$$

Use the parallelogram identity in the above problem to show that $\{\mathbf{x}_n\}$ is a Cauchy sequence. Now let $\{\mathbf{u}_k\}_{k=1}^p$ be an orthonormal basis for U. Say

$$\mathbf{x}_n = \sum_{k=1}^{p} c_k^n \mathbf{u}_k$$

Verify that for $\mathbf{c}^n \equiv \left(c_1^n, \cdots, c_p^n \right) \in \mathbb{F}^p$

$$|\mathbf{x}_n - \mathbf{x}_m| = |\mathbf{c}^n - \mathbf{c}^m|_{\mathbb{F}^p}.$$

Now use completeness of $\mathbb{F}^p$ and the assumption that K is closed to get the existence of $\mathbf{x} \in K$ such that $|\mathbf{x} - \mathbf{y}| = \lambda$.

The hint is pretty good. Let $\mathbf{x}_k$ be a minimizing sequence. The connection between $\mathbf{x}_k$ and $\mathbf{c}^k \in \mathbb{F}^k$ is obvious because the $\{\mathbf{u}_k\}$ are orthonormal. That is,

$$|\mathbf{x}_n - \mathbf{x}_m| = |\mathbf{c}^n - \mathbf{c}^m|_{\mathbb{F}^p}.$$

Use the parallelogram identity.

$$\left| \frac{\mathbf{y} - \mathbf{x}_k - (\mathbf{y} - \mathbf{x}_m)}{2} \right|^2 + \left| \frac{\mathbf{y} - \mathbf{x}_k + (\mathbf{y} - \mathbf{x}_m)}{2} \right|^2 = 2\left| \frac{\mathbf{y} - \mathbf{x}_k}{2} \right|^2 + 2\left| \frac{\mathbf{y} - \mathbf{x}_m}{2} \right|$$

Hence

$$\begin{aligned}\frac{1}{4}\left|\mathbf{x}_m-\mathbf{x}_k\right|^2 &= \frac{1}{2}\left|\mathbf{y}-\mathbf{x}_k\right|^2+\frac{1}{2}\left|\mathbf{y}-\mathbf{x}_m\right|^2-\left|\mathbf{y}-\frac{\mathbf{x}_k+\mathbf{x}_m}{2}\right|^2 \\ &\leq \frac{1}{2}\left|\mathbf{y}-\mathbf{x}_k\right|^2+\frac{1}{2}\left|\mathbf{y}-\mathbf{x}_m\right|^2-\lambda^2\end{aligned}$$

Now the right hand side converges to 0 since $\{\mathbf{x}_k\}$ is a minimizing sequence. Therefore, $\{\mathbf{x}_k\}$ is a Cauchy sequence in U. Hence the sequence of component vectors $\{\mathbf{c}^k\}$ is a Cauchy sequence in $\mathbb{F}^n$ and so it converges thanks to completeness of $\mathbb{F}$. It follows that $\{\mathbf{x}_k\}$ also must converge to some $\mathbf{x}$. Then since K is closed, it follows that $\mathbf{x}\in K$. Hence

$$\lambda=\left|\mathbf{x}-\mathbf{y}\right|.$$

24. *Let K be a closed nonempty convex subset of a finite dimensional subspace U of a real inner product space V. (It is true for complex ones also.) For $\mathbf{x}\in V$, denote by $P\mathbf{x}$ the unique closest point to $\mathbf{x}$ in K. Verify that P is Lipschitz continuous with Lipschitz constant 1,

$$\left|P\mathbf{x}-P\mathbf{y}\right|\leq\left|\mathbf{x}-\mathbf{y}\right|.$$

Hint: Use Problem 21.

From the problem,

$$\begin{aligned}\langle P\mathbf{x}-P\mathbf{y},\mathbf{y}-P\mathbf{y}\rangle &\leq 0\\ \langle P\mathbf{y}-P\mathbf{x},\mathbf{x}-P\mathbf{x}\rangle &\leq 0\end{aligned}$$

Thus

$$\langle P\mathbf{x}-P\mathbf{y},\mathbf{x}-P\mathbf{x}\rangle\geq 0$$

Hence

$$\langle P\mathbf{x}-P\mathbf{y},\mathbf{x}-P\mathbf{x}\rangle-\langle P\mathbf{x}-P\mathbf{y},\mathbf{y}-P\mathbf{y}\rangle\geq 0$$

and so

$$\langle P\mathbf{x}-P\mathbf{y},\mathbf{x}-\mathbf{y}-(P\mathbf{x}-P\mathbf{y})\rangle\geq 0$$

$$\left|\mathbf{x}-\mathbf{y}\right|\left|P\mathbf{x}-P\mathbf{y}\right|\geq\langle P\mathbf{x}-P\mathbf{y},P\mathbf{x}-P\mathbf{y}\rangle=\left|P\mathbf{x}-P\mathbf{y}\right|^2$$

25. * This problem is for people who know about compactness. It is an analysis problem. If you have only had the usual undergraduate calculus course, don't waste your time with this problem. Suppose V is a finite dimensional normed linear space. Recall this means that there exists a norm $\|\cdot\|$ defined on V as described above,

$$\|\mathbf{v}\|\geq 0 \text{ equals } 0 \text{ if and only if } \mathbf{v}=\mathbf{0}$$

$$\|\mathbf{v}+\mathbf{u}\|\leq\|\mathbf{u}\|+\|\mathbf{v}\|,\ \|\alpha\mathbf{v}\|=|\alpha|\,\|\mathbf{v}\|.$$

Let $|\cdot|$ denote the norm which comes from Example 16.5.3, the inner product by decree. Show $|\cdot|$ and $\|\cdot\|$ are equivalent. That is, there exist constants $\delta,\Delta>0$ such that for all $\mathbf{x}\in V$,

$$\delta\left|\mathbf{x}\right|\leq\|\mathbf{x}\|\leq\Delta\left|\mathbf{x}\right|.$$

In explain why every two norms on a finite dimensional vector space must be equivalent in the above sense.

Let $\{\mathbf{u}_k\}_{k=1}^n$ be a basis for V and if $\mathbf{x}\in V$, let x_i be the components of $\mathbf{x}$ relative to this basis. Thus the x_i are defined according to

$$\sum_i x_i\mathbf{u}_i=\mathbf{x}$$

Then since $\{\mathbf{u}_i\}$ is an orthonormal basis by decree, it follows that

$$|\mathbf{x}|^2 = \sum_i |x_i|^2 .$$

Now letting $\{\mathbf{x}_k\}$ be a sequence of vectors of V let $\{\mathbf{x}^k\}$ denote the sequence of component vectors in $\mathbb{F}^n$. One direction is easy, saying that $\|\mathbf{x}\| \leq \Delta |\mathbf{x}|$. If this is not so, then there exists a sequence of vectors $\{\mathbf{x}_k\}$ such that

$$\|\mathbf{x}_k\| > k |\mathbf{x}_k|$$

dividing both sides by $\|\mathbf{x}_k\|$ it can be assumed that $1 > k |\mathbf{x}_k| = |\mathbf{x}^k|$. Hence $\mathbf{x}^k \to \mathbf{0}$ in $\mathbb{F}^k$. But from the triangle inequality,

$$\|\mathbf{x}_k\| \leq \sum_{i=1}^n x_i^k \|\mathbf{u}_i\|$$

Therefore, since $\lim_{k\to\infty} x_i^k = 0$, this is a contradiction to each $\|\mathbf{x}_k\| = 1$. It follows that there exists Δ such that for all $\mathbf{x}$,

$$\|\mathbf{x}\| \leq \Delta |\mathbf{x}|$$

Now consider the other direction. If it is not true, then there exists a sequence $\{\mathbf{x}_k\}$ such that

$$\frac{1}{k} |\mathbf{x}_k| > \|\mathbf{x}_k\|$$

Dividing both sides by $|\mathbf{x}_k|$, it can be assumed that $|\mathbf{x}_k| = |\mathbf{x}^k| = 1$. Hence, by compactness of the closed unit ball in $\mathbb{F}^n$, there exists a further subsequence, still denoted by k such that $\mathbf{x}^k \to \mathbf{a} \in \mathbb{F}^n$ and it also follows that $|\mathbf{a}|_{\mathbb{F}^n} = 1$. Also the above inequality implies $\lim_{k\to\infty} \|\mathbf{x}_k\| = 0$. Therefore,

$$\sum_{j=1}^n a_j \mathbf{u}_j = \lim_{k\to\infty} \sum_{j=1}^n x_j^k \mathbf{u}_j = \lim_{k\to\infty} \mathbf{x}_k = \mathbf{0}$$

which is a contradiction to the $\mathbf{u}_j$ being linearly independent. Therefore, there exists $\delta > 0$ such that for all $\mathbf{x}$,

$$\delta |\mathbf{x}| \leq \|\mathbf{x}\| .$$

Now if you have any other norm on this finite dimensional vector space, say $|||\cdot|||$, then from what was just shown, there exist scalars δ_i and Δ_i all positive, such that

$$\begin{aligned} \delta_1 |\mathbf{x}| &\leq \|\mathbf{x}\| \leq \Delta_1 |\mathbf{x}| \\ \delta_2 |\mathbf{x}| &\leq |||\mathbf{x}||| \leq \Delta_2 |\mathbf{x}| \end{aligned}$$

It follows that

$$|||\mathbf{x}||| \leq \frac{\Delta_2}{\delta_1} \|\mathbf{x}\| \leq \frac{\Delta_2 \Delta_1}{\delta_1} |\mathbf{x}| \leq \frac{\Delta_2 \Delta_1}{\delta_1 \delta_2} |||\mathbf{x}||| .$$

Hence

$$\frac{\delta_1}{\Delta_2} |||\mathbf{x}||| \leq \|\mathbf{x}\| \leq \frac{\Delta_1}{\delta_2} |||\mathbf{x}||| .$$

In other words, any two norms on a finite dimensional vector space are equivalent norms. What this means is that every consideration which depends on analysis or topology is exactly the same for any two norms. What might change are geometric properties of the norms.

B.20 Exercises 17.6

1. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/3$.
   $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$

2. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/4$.
   $\begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$

3. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/12$. **Hint:** Note that $\pi/12 = \pi/3 - \pi/4$.
   $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\sqrt{2}\sqrt{3}+\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2}\sqrt{3}-\frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2}\sqrt{3}-\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2}-\frac{1}{4}\sqrt{2}\sqrt{3} \end{pmatrix}$

4. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $2\pi/3$ and then reflects across the x axis.
   $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$

5. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/3$ and then reflects across the y axis.
   $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$

6. Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $5\pi/12$. **Hint:** Note that $5\pi/12 = 2\pi/3 - \pi/4$.

$$\begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} \begin{pmatrix} \cos\left(\frac{-\pi}{4}\right) & -\sin\left(\frac{-\pi}{4}\right) \\ \sin\left(\frac{-\pi}{4}\right) & \cos\left(\frac{-\pi}{4}\right) \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{4}\sqrt{2}\sqrt{3}-\frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2}\sqrt{3}-\frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2}\sqrt{3}+\frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2}\sqrt{3}-\frac{1}{4}\sqrt{2} \end{pmatrix}$$

7. Let V be an inner product space and $\mathbf{u} \neq \mathbf{0}$. Show that the function $T_{\mathbf{u}}$ defined by $T_{\mathbf{u}}(\mathbf{v}) \equiv \mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})$ is also a linear transformation. Here

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) \equiv \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{|\mathbf{u}|^2}\mathbf{u}$$

   Now show directly from the axioms of the inner product that

$$\langle T_{\mathbf{u}}\mathbf{v}, \mathbf{u} \rangle = 0$$

   It is obvious that $\mathbf{v} \to \text{proj}_{\mathbf{u}}(\mathbf{v})$ is linear from the properties of the inner product. Also $\mathbf{v} \to \mathbf{v}$ is linear. Hence $T_{\mathbf{u}}$ is linear. Also

$$\left\langle \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{|\mathbf{u}|^2}\mathbf{u}, \mathbf{u} \right\rangle = \langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{u} \rangle \frac{\langle \mathbf{u}, \mathbf{u} \rangle}{|\mathbf{u}|^2} = 0$$

8. Let V be a finite dimensional inner product space, the field of scalars equal to either $\mathbb{R}$ or $\mathbb{C}$. Verify that f given by $f\mathbf{v} \equiv \langle \mathbf{v}, \mathbf{z} \rangle$ is in $\mathcal{L}(V, \mathbb{F})$. Next suppose f is an arbitrary element of $\mathcal{L}(V, \mathbb{F})$. Show the following.

   (a) If $f = 0$, the zero mapping, then $f\mathbf{v} = \langle \mathbf{v}, \mathbf{0} \rangle$ for all $\mathbf{v} \in V$.
   This is obvious from the properties of the inner product.
   $$\langle \mathbf{v}, \mathbf{0} \rangle = \langle \mathbf{v}, \mathbf{0} + \mathbf{0} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle + \langle \mathbf{v}, \mathbf{0} \rangle$$
   so $\langle \mathbf{v}, \mathbf{0} \rangle = 0$.

   (b) If $f \neq 0$ then there exists $\mathbf{z} \neq \mathbf{0}$ satisfying $\langle \mathbf{u}, \mathbf{z} \rangle = 0$ for all $\mathbf{u} \in \ker(f)$.
   $\ker(f)$ is a subspace and so there exists $\mathbf{z}_1 \notin \ker(f)$. Then there exists a closest point of $\ker(f)$ to $\mathbf{z}_1$ called $\mathbf{x}$. Then let $\mathbf{z} = \mathbf{z}_1 - \mathbf{x}$. By the theorems on minimization, $\langle \mathbf{u}, \mathbf{z} \rangle = 0$ for all $\mathbf{u} \in \ker(f)$.

   (c) Explain why $f(\mathbf{y})\mathbf{z} - f(\mathbf{z})\mathbf{y} \in \ker(f)$.
   $f(f(\mathbf{y})\mathbf{z} - f(\mathbf{z})\mathbf{y}) = f(\mathbf{y}) f(\mathbf{z}) - f(\mathbf{z}) f(\mathbf{y}) = 0$.

   (d) Use part b. to show that there exists $\mathbf{w}$ such that $f(\mathbf{y}) = \langle \mathbf{y}, \mathbf{w} \rangle$ for all $\mathbf{y} \in V$.
   From part *b.*
   $$0 = \langle f(\mathbf{y})\mathbf{z} - f(\mathbf{z})\mathbf{y}, \mathbf{z} \rangle = f(\mathbf{y}) |\mathbf{z}|^2 - f(\mathbf{z}) \langle \mathbf{y}, \mathbf{z} \rangle$$
   and so
   $$f(\mathbf{y}) = \left\langle \mathbf{y}, \overline{\frac{f(\mathbf{z})}{|\mathbf{z}|^2}}\mathbf{z} \right\rangle$$
   so $\mathbf{w} = \overline{\frac{f(\mathbf{z})}{|\mathbf{z}|^2}}\mathbf{z}$ appears to work.

   (e) Show there is at most one such $\mathbf{w}$.
   If $\mathbf{w}_1, \mathbf{w}_2$ both work, then for every $\mathbf{y}$,
   $$0 = f(\mathbf{y}) - f(\mathbf{y}) = \langle \mathbf{y}, \mathbf{w}_1 \rangle - \langle \mathbf{y}, \mathbf{w}_2 \rangle = \langle \mathbf{y}, \mathbf{w}_1 - \mathbf{w}_2 \rangle$$
   In particular, this is true for $\mathbf{y} = \mathbf{w}_1 - \mathbf{w}_2$ and so $\mathbf{w}_1 = \mathbf{w}_2$.

   You have now proved the Riesz representation theorem which states that every $f \in \mathcal{L}(V, \mathbb{F})$ is of the form
   $$f(\mathbf{y}) = \langle \mathbf{y}, \mathbf{w} \rangle$$
   for a unique $\mathbf{w} \in V$.

9. $\uparrow$Let $A \in \mathcal{L}(V, W)$ where V, W are two finite dimensional inner product spaces, both having field of scalars equal to $\mathbb{F}$ which is either $\mathbb{R}$ or $\mathbb{C}$. Let $f \in \mathcal{L}(V, \mathbb{F})$ be given by
   $$f(\mathbf{y}) \equiv \langle A\mathbf{y}, \mathbf{z} \rangle$$
   where $\langle \rangle$ now refers to the inner product in W. Use the above problem to verify that there exists a unique $\mathbf{w} \in V$ such that $f(\mathbf{y}) = \langle \mathbf{y}, \mathbf{w} \rangle$, the inner product here being the one on V. Let $A^*\mathbf{z} \equiv \mathbf{w}$. Show that $A^* \in \mathcal{L}(W, V)$ and by construction,
   $$\langle A\mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{y}, A^*\mathbf{z} \rangle.$$
   In the case that $V = \mathbb{F}^n$ and $W = \mathbb{F}^m$ and A consists of multiplication on the left by an $m \times n$ matrix, give a description of A^*.

It is required to show that A^* is linear.

$$\begin{aligned}\langle \mathbf{y}, A^*(\alpha\mathbf{z}+\beta\mathbf{w})\rangle &\equiv \langle A\mathbf{y}, \alpha\mathbf{z}+\beta\mathbf{w}\rangle = \overline{\alpha}\langle A\mathbf{y},\mathbf{z}\rangle + \overline{\beta}\langle A\mathbf{y},\mathbf{w}\rangle \\ &\equiv \overline{\alpha}\langle \mathbf{y}, A^*\mathbf{z}\rangle + \overline{\beta}\langle \mathbf{y}, A^*\mathbf{w}\rangle \\ &= \langle \mathbf{y}, \alpha A^*\mathbf{z}\rangle + \langle \mathbf{y}, \beta A^*\mathbf{w}\rangle = \langle \mathbf{y}, \alpha A^*\mathbf{z} + \beta A^*\mathbf{w}\rangle\end{aligned}$$

Since $\mathbf{y}$ is arbitrary, this shows that A^* is linear. In case A is an $m \times n$ matrix as described,

$$A^* = \overline{(A^T)}$$

10. Let A be the linear transformation defined on the vector space of smooth functions (Those which have all derivatives) given by $Af = D^2 + 2D + 1$. Find $\ker(A)$.

    First solve $(D+1)z = 0$. This is easy to do and gives $z(t) = C_1 e^{-t}$. Now solve $(D+1)y = C_1 e^{-t}$. To do this, you multiply by an integrating factor. Thus

    $$\frac{d}{dt}\left(e^t y\right) = C_1$$

    Take an antiderivative and obtain

    $$e^t y = C_1 t + C_2$$

    Thus the desired kernel consists of all functions y which are of the form

    $$y(t) = C_1 t e^t + C_2 e^t$$

    where C_i is a constant.

11. Let A be the linear transformation defined on the vector space of smooth functions (Those which have all derivatives) given by $Af = D^2 + 5D + 4$. Find $\ker(A)$. Note that you could first find $\ker(D+4)$ where D is the differentiation operator and then consider $\ker(D+1)(D+4) = \ker(A)$ and consider Sylvester's theorem.

    In this case, the two operators $D+1$ and $D+4$ commute and are each one to one on the kernel of the other. Also, it is obvious that $\ker(D+a)$ consists of functions of the form Ce^{-at}. Therefore, $\ker(D+1)(D+4)$ consists of functions of the form

    $$y = C_1 e^{-t} + C_2 e^{-4t}$$

    where C_1, C_2 are arbitrary constants. In other words, a basis for $\ker(D+1)(D+4)$ is $\left\{e^{-t}, e^{-4t}\right\}$.

12. Suppose $A\mathbf{x} = \mathbf{b}$ has a solution where A is a linear transformation. Explain why the solution is unique precisely when $A\mathbf{x} = \mathbf{0}$ has only the trivial (zero) solution.

    Suppose first there is a unique solution $\mathbf{z}$. Then there can only be one solution to $A\mathbf{x} = \mathbf{0}$ because if this last equation had more than one solution, say $\mathbf{y}_i, i = 1,2$, then $\mathbf{y}_i + \mathbf{z}$, $i = 1,2$ would each be solutions to $A\mathbf{y} = \mathbf{b}$ because A is linear. Recall also that the general solution to $A\mathbf{x} = \mathbf{b}$ consists of the general solution to $A\mathbf{x} = \mathbf{0}$ added to any solution to $A\mathbf{x} = \mathbf{b}$. Therefore, if there is only one solution to $A\mathbf{x} = \mathbf{0}$, then there is only one solution to $A\mathbf{x} = \mathbf{b}$.

13. Verify the linear transformation determined by the matrix

    $$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$

    maps $\mathbb{R}^3$ onto $\mathbb{R}^2$ but the linear transformation determined by this matrix is not one to one.

    This is an old problem. The rank is obviously 2 so the matrix maps onto $\mathbb{R}^2$. It is not one to one because the system $A\mathbf{x} = \mathbf{0}$ must have a free variable.

14. Let L be the linear transformation taking polynomials of degree at most three to polynomials of degree at most three given by
$$D^2 + 2D + 1$$
where D is the differentiation operator. Find the matrix of this linear transformation relative to the basis $\{1, x, x^2, x^3\}$. Find the matrix directly and then find the matrix with respect to the differential operator $D + 1$ and multiply this matrix by itself. You should get the same thing. Why?

You should get the same thing because the multiplication of matrices corresponds to composition of linear transformations. Lets find the matrix for $D + 1$ first.
$$\begin{aligned} &\begin{pmatrix} 1 & x+1 & 2x+x^2 & 3x^2+x^3 \end{pmatrix} \\ =& \begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$
so the matrix is
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
Then the matrix of the desired transformation is just this one squared.
$$\begin{pmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

15. Let L be the linear transformation taking polynomials of degree at most three to polynomials of degree at most three given by
$$D^2 + 5D + 4$$
where D is the differentiation operator. Find the matrix of this linear transformation relative to the bases $\{1, x, x^2, x^3\}$. Find the matrix directly and then find the matrices with respect to the differential operators $D+1, D+4$ and multiply these two matrices. You should get the same thing. Why?

You get the same thing because the composition of linear transformations corresponds to matrix multiplication. This time, lets use the operator directly.
$$\begin{aligned} &\begin{pmatrix} 4 & 5+4x & 2+10x+4x^2 & 6x+15x^2+4x^3 \end{pmatrix} \\ =& \begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix} \begin{pmatrix} 4 & 5 & 2 & 0 \\ 0 & 4 & 10 & 6 \\ 0 & 0 & 4 & 15 \\ 0 & 0 & 0 & 4 \end{pmatrix} \end{aligned}$$
Thus the matrix is
$$\begin{pmatrix} 4 & 5 & 2 & 0 \\ 0 & 4 & 10 & 6 \\ 0 & 0 & 4 & 15 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

If you did it by looking for the matrix for $D+1$ and $D+4$ you would get for $D+4$

$$\begin{aligned} &\begin{pmatrix} 4 & 1+4x & 2x+4x^2 & 3x^2+4x^3 \end{pmatrix} \\ = &\begin{pmatrix} 1 & x & x^2 & x^3 \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \end{aligned}$$

From the above problem, the matrix of D^2+5D+4 should be the product of this matrix with the one of that problem.

$$\begin{aligned} &\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \\ = &\begin{pmatrix} 4 & 5 & 2 & 0 \\ 0 & 4 & 10 & 6 \\ 0 & 0 & 4 & 15 \\ 0 & 0 & 0 & 4 \end{pmatrix} \end{aligned}$$

which is the same thing as just obtained.

16. Show that if $L \in \mathcal{L}(V, W)$ (linear transformation) where V and W are vector spaces, then if $L\mathbf{y}_p = \mathbf{f}$ for some $\mathbf{y}_p \in V$, then the general solution of $L\mathbf{y} = \mathbf{f}$ is of the form

$$\ker(L) + \mathbf{y}_p.$$

This is really old stuff. However, here it applies to an arbitrary linear transformation. Suppose $L\mathbf{z} = \mathbf{f}$. Then $L(\mathbf{z} - \mathbf{y}_p) = L(\mathbf{z}) - L(\mathbf{y}_p) = \mathbf{f} - \mathbf{f} = \mathbf{0}$. Therefore, letting $\mathbf{y} = \mathbf{z} - \mathbf{y}_p$, it follows that $\mathbf{y}$ is a solution to $L\mathbf{y} = \mathbf{0}$ and $\mathbf{z} = \mathbf{y}_p + \mathbf{y}$. Thus every solution to $L\mathbf{y} = \mathbf{f}$ is of the form $\mathbf{y}_p + \mathbf{y}$ for some $\mathbf{y}$ which solves $L\mathbf{y} = \mathbf{0}$.

17. Let $L \in \mathcal{L}(V, W)$ where V, W are vector spaces, finite or infinite dimensional, and define $\mathbf{x} \sim \mathbf{y}$ if $\mathbf{x} - \mathbf{y} \in \ker(L)$. Show that $\sim$ is an equivalence relation. Next define addition and scalar multiplication on the space of equivalence classes as follows.

$$\begin{aligned} [\mathbf{x}] + [\mathbf{y}] &\equiv [\mathbf{x} + \mathbf{y}] \\ \alpha[\mathbf{x}] &= [\alpha\mathbf{x}] \end{aligned}$$

Show that these are well defined definitions and that the set of equivalence classes is a vector space with respect to these operations. The zero is $[\ker L]$. Denote the resulting vector space by $V/\ker(L)$. Now suppose L is onto W. Define a mapping $A : V/\ker(K) \to W$ as follows.

$$A[\mathbf{x}] \equiv L\mathbf{x}$$

Show that A is well defined, one to one and onto.

It is obvious that $\mathbf{x} \sim \mathbf{x}$. If $\mathbf{x} \sim \mathbf{y}$, then $\mathbf{y} \sim \mathbf{x}$ is also clear. If $\mathbf{x} \sim \mathbf{y}$ and $\mathbf{y} \sim \mathbf{z}$, then

$$\mathbf{z} - \mathbf{x} = \mathbf{z} - \mathbf{y} + \mathbf{y} - \mathbf{x}$$

and by assumption, both $\mathbf{z} - \mathbf{y}$ and $\mathbf{y} - \mathbf{x} \in \ker(L)$ which is a subspace. Therefore, $\mathbf{z} - \mathbf{x} \in \ker(L)$ also and so $\sim$ is an equivalence relation. Are the operations well defined? If $[\mathbf{x}] = [\mathbf{x}'], [\mathbf{y}] = [\mathbf{y}']$, is it true that $[\mathbf{x} + \mathbf{y}] = [\mathbf{y}' + \mathbf{x}']$? Of course. $\mathbf{x}' + \mathbf{y}' - (\mathbf{x} + \mathbf{y}) = (\mathbf{x}' - \mathbf{x}) + (\mathbf{y}' - \mathbf{y}) \in \ker(L)$ because $\ker(L)$ is a subspace. Similar reasoning applies to the case of scalar multiplication. Now why is A well defined? If $[\mathbf{x}] = [\mathbf{x}']$, is $L\mathbf{x} = L\mathbf{x}'$? Of course this is so. $\mathbf{x} - \mathbf{x}' \in \ker(L)$ by assumption. Therefore, $L\mathbf{x} = L\mathbf{x}'$. It is clear also that A is linear. If $A[\mathbf{x}] = \mathbf{0}$, then $L\mathbf{x} = \mathbf{0}$ and so $\mathbf{x} \in \ker(L)$ and so $[\mathbf{x}] = \mathbf{0}$. Therefore, A is one to one. It is obviously onto $L(V) = W$.

18. If V is a finite dimensional vector space and $L \in \mathcal{L}(V,V)$, show that the minimal polynomial for L equals the minimal polynomial of A where A is the $n \times n$ matrix of L with respect to some basis.

    This is really easy because of the definition of what you mean by the matrix of a linear transformation.
$$L = qAq^{-1}$$
    where q, q^{-1} are one to one and onto linear transformations from V to $\mathbb{F}^n$ or from $\mathbb{F}^n$ to V. Thus if $p(\lambda)$ is a polynomial,
$$p(L) = qp(A)q^{-1}$$
    Thus the polynomials which send L to 0 are the same as those which send A to 0.

19. Let A be an $n \times n$ matrix. Describe a fairly simple method based on row operations for computing the minimal polynomial of A. Recall, that this is a monic polynomial $p(\lambda)$ such that $p(A) = 0$ and it has smallest degree of all such monic polynomials. **Hint:** Consider $I, A^2, \cdots$. Regard each as a vector in $\mathbb{F}^{n^2}$ and consider taking the row reduced echelon form or something like this. You might also use the Cayley Hamilton theorem to note that you can stop the above sequence at A^n.

    An easy way to do this is to "unravel" the powers of the matrix making vectors in $\mathbb{F}^{n^2}$ and then making these the columns of a $n^2 \times n$ matrix. Look for linear relationships between the columns by obtaining the row reduced echelon form and using Lemma 8.2.5. As an example, consider the following matrix.
$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$
    Lets find its minimal polynomial. We have the following powers
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -3 & -2 & -3 \\ 7 & 5 & 8 \end{pmatrix}, \begin{pmatrix} -3 & -1 & -4 \\ -7 & -6 & -7 \\ 18 & 15 & 19 \end{pmatrix}$$
    By the Cayley Hamilton theorem, I won't need to consider any higher powers than this. Now I will unravel each and make them the columns of a matrix.
$$\begin{pmatrix} 1 & 1 & 0 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & -4 \\ 0 & -1 & -3 & -7 \\ 1 & 0 & -2 & -6 \\ 0 & -1 & -3 & -7 \\ 0 & 2 & 7 & 18 \\ 0 & 1 & 5 & 15 \\ 1 & 3 & 8 & 19 \end{pmatrix}$$
    Next you can do row operations and obtain the row reduced echelon form for this matrix and then look for linear relationships.
$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From this and Lemma 8.2.5, you see that for A denoting the matrix,

$$A^3 = 4A^2 - 5A + 2I$$

and so the minimal polynomial is

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2$$

No smaller degree polynomial can work either. Since it is of degree 3, this is also the characteristic polynomial. Note how we got this without expanding any determinants or solving any polynomial equations. If you factor this polynomial, you get
$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = (\lambda - 2)(\lambda - 1)^2$ so this is an easy problem, but you see that this procedure for finding the minimal polynomial will work even when you can't factor the characteristic polynomial. If you want to work with smaller matrices, you could also look at $A^k \mathbf{e}_i$ for $\mathbf{e}_1, \mathbf{e}_2$, etc., use similar techniques on each of these and then find the least common multiple of the resulting polynomials.

20. Let A be an $n \times n$ matrix which is non defective. That is, there exists a basis of eigenvectors. Show that if $p(\lambda)$ is the minimal polynomial, then $p(\lambda)$ has no repeated roots. **Hint:** First show that the minimal polynomial of A is the same as the minimal polynomial of the diagonal matrix

$$D = \begin{pmatrix} D(\lambda_1) & & \\ & \ddots & \\ & & D(\lambda_r) \end{pmatrix}$$

Where $D(\lambda)$ is a diagonal matrix having λ down the main diagonal and in the above, the λ_i are distinct. Show that the minimal polynomial is $\prod_{i=1}^{r}(\lambda - \lambda_i)$.

If two matrices are similar, then they must have the same minimal polynomial. This is obvious from the fact that for $p(\lambda)$ any polynomial and $A = S^{-1}BS$,

$$p(A) = S^{-1}p(B)S$$

So what is the minimal polynomial of the above D? It is obviously $\prod_{i=1}^{r}(\lambda - \lambda_i)$. Thus there are no repeated roots.

21. Show that if A is an $n \times n$ matrix and the minimal polynomial has no repeated roots, then A is non defective and there exists a basis of eigenvectors. Thus, from the above problem, a matrix may be diagonalized if and only if its minimal polynomial has no repeated roots. It turns out this condition is something which is relatively easy to determine. **Hint:** You might want to use Theorem 17.3.1.

If A has a minimal polynomial which has no repeated roots, say

$$p(\lambda) = \prod_{j=1}^{m}(\lambda - \lambda_i),$$

then from the material on decomposing into direct sums of generalized eigenspaces, you have

$$\mathbb{F}^n = \ker(A - \lambda_1 I) \oplus \ker(A - \lambda_2 I) \oplus \cdots \oplus \ker(A - \lambda_m I)$$

and by definition, the basis vectors for $\ker(A - \lambda_2 I)$ are all eigenvectors. Thus $\mathbb{F}^n$ has a basis of eigenvectors and is therefore diagonalizable or non defective.