Irreducible representations of the rotation group

We wish to find the complete set of objects on which linear representations the rotation group can act. A linear representation of a group is a homomorphism from the group to a set of matrices - essentially, the group multiplication becomes matrix multiplication, and we think of each matrix in the set as a distinct group element.

In general, there are many linear representations of a given group. We have seen, for example, that the (infinite-dimensional) Lie group of general coordinate transformations (the diffeomorphism group) acts multi-linearly on tensors of any rank and type, $\binom{p}{q}$. While each of these types of tensor is a linear representation of the diffeomorphism group, they are not irreducible representations, because for such tensors of rank larger than one (i.e, all except scalars and vectors) we can decompose the tensor into parts which don't mix.

Instead of the rather large groups of diffeomorphisms, let's focus on the more manageable group of rotations. These also act on tensors of all ranks and types. We can decompose a rank n matrix, M_{ij}, of type $\binom{0}{2}$, for example, into the following irreducible parts:

$$M_{ij} = \frac{1}{2}\left(M_{ij} + M_{ji} - \frac{2}{n}\delta_{ij}\delta^{mn}M_{mn}\right) + \frac{1}{2}\left(M_{ij} - M_{ji}\right) + \frac{1}{n}\delta_{ij}\delta^{mn}M_{mn}$$

To see that the parts are irreducible, consider the action of a rotation of M_{ij} on each part:

$$\tilde{M}_{ij} = \Lambda_i^k \Lambda_j^l M_{kl}$$

For the trace part,

$$\begin{aligned}
\frac{1}{n}\delta_{ij}\left(tr\tilde{M}\right) &= \Lambda_i^k \Lambda_j^l \left(\frac{1}{n}\delta_{kl}\delta^{mn}M_{mn}\right) \\
&= \Lambda_i^k \Lambda_j^l \delta_{kl}\left(\frac{1}{n}\delta^{mn}M_{mn}\right) \\
&= \delta_{ij}\left(\frac{1}{n}\delta^{mn}M_{mn}\right) \\
&= \frac{1}{n}\delta_{ij} trM
\end{aligned}$$

since δ_{ij} is a rotationally invariant tensor. Thus, trM is a scalar, invariant under rotations. For the antisymmetric part, we have

$$\begin{aligned}
\tilde{M}_{[ij]} &= \frac{1}{2}\left(\tilde{M}_{ij} - \tilde{M}_{ji}\right) \\
&= \frac{1}{2}\left(\Lambda_i^k \Lambda_j^l M_{kl} - \Lambda_j^k \Lambda_i^l M_{kl}\right) \\
&= \frac{1}{2}\left(\Lambda_i^k \Lambda_j^l M_{kl} - \Lambda_i^k \Lambda_j^l M_{lk}\right) \\
&= \Lambda_i^k \Lambda_j^l M_{[kl]}
\end{aligned}$$

so the transformation of the antisymmetric part depends only on the original antisymmetric part. Finally, the proof that the traceless, symmetric part is irreducible proceeds just like the proof for the antisymmetric part.

Our goal is to find *all* such irreducible represenations ("irreps"). So far, we have three. Each of these three representations forms a distinct vector space: the 1-dimensional space of scalars, the 3-dimensional vector space of antisymmetric matrices, and the 5-dimensional vector space of symmetric, traceless matrices.

Exercise: Show that linear combinations of antisymmetric matrices are antisymmetric.

Exercise: Check the dimension of each of these three vector spaces.

Our general proof proceeds from the properties of the Lie algebra, $so(3)$, of the rotation group, $SO(3)$.

Irreps of so(3)

We have shown that the Lie algebra, $so(3)$, (and also $su(2)$) takes the form

$$[K_i, K_j] = \varepsilon_{ijk} K_k$$

where the generators, K_i, are anti-hermitian (and therefore, if real, antisymmetric). Let us assume *only* this set of commutation relations, without any other constraint on the form of the operators. It will be simpler to work with Hermitian operators, so set

$$\mathbf{K} = -i\mathbf{J} = -i\left(J_x, J_y, J_a\right)$$

and substituting, we find

$$[J_i, J_j] = i\varepsilon_{ijk} J_k$$

We can always choose a basis such that one of these matrices is diagonal. Choose J_z to be the diagonal one. Since J_z does not commute with either J_x or J_y, we cannot simultaneously diagonalize either of those. However, consider the squared magnitude, $\mathbf{J}^2 = J_i J_i$,

$$\begin{aligned}
\left[\mathbf{J}^2, J_k\right] &= [J_i J_i, J_k] \\
&= J_i [J_i, J_k] + [J_i, J_k] J_i \\
&= i J_i \varepsilon_{ikm} J_m + i\varepsilon_{ikm} J_m J_i \\
&= i\varepsilon_{ikm} J_i J_m - i\varepsilon_{ikm} J_i J_m \\
&= 0
\end{aligned}$$

where the last nontrivial step was just a renaming of indices. Therefore, we may diagonalize $\mathbf{J}^2$ and J_z simultaneously.

Our goal is now to build vector spaces on which these operators act. I will use ket notation, $|a\rangle$, to denote these vectors. The label, a, can be anything we choose to identify the vector. I could just as well use $\mathbf{a}$, or $\vec{a}$, or a_α, but having room for labels is handy. Since we are in a basis where $\mathbf{J}^2$ and J_z are diagonal, we can label the basis with their eigenvalues. Let these eigenvalues be called α, β (for the moment), so that

$$\begin{aligned}
\mathbf{J}^2 |\alpha, \beta\rangle &= \alpha |\alpha, \beta\rangle \\
J_z |\alpha, \beta\rangle &= \beta |\alpha, \beta\rangle
\end{aligned}$$

(In quantum mechanics, the J_i are angular momentum operators, with units $\frac{kgm^2}{s}$, and the eigenvalues have an additional factor of $\hbar$. Here we may imagine that the J_i have been divided by this factor to become dimensionless). We may take the eigenvectors to be normalized to one. Writing the inner product of two vectors as

$$\langle a \mid a\rangle$$

we have

$$\langle \alpha, \beta \mid \alpha, \beta\rangle = 1$$

for all α and β.

Now consider two more operators,

$$\begin{aligned}
J_+ &= J_x + iJ_y \\
J_- &= J_x - iJ_y
\end{aligned}$$

These obviously satisfy

$$\left[\mathbf{J}^2, J_\pm\right] = 0$$

and since J_x and J_y are Hermitian we have

$$J_+^\dagger = J_-$$

but with J_z we have

$$\begin{aligned}
[J_z, J_+] &= [J_z, J_x + iJ_y] \\
&= [J_z, J_x] + i[J_z, J_y] \\
&= iJ_y + J_x \\
&= J_+
\end{aligned}$$

and similarly,

$$[J_z, J_-] = -J_-$$

We also have

$$\begin{aligned}
[J_+, J_-] &= [J_x + iJ_y, J_x - iJ_y] \\
&= [J_x, -iJ_y] + + [iJ_y, J_x] \\
&= 2J_z
\end{aligned}$$

Now, apply J_+ to an eigenvector,

$$J_+ |\alpha, \beta\rangle$$

and consider the effect of $\mathbf{J}^2$ and J_z on this new vector,

$$\begin{aligned} \mathbf{J}^2 J_+ \left|\alpha,\beta\right\rangle &= J_+ \mathbf{J}^2 \left|\alpha,\beta\right\rangle \\ &= \alpha J_+ \left|\alpha,\beta\right\rangle \end{aligned}$$

and, using the commutator to write $J_z J_+ = J_+ J_z + J_+$,

$$\begin{aligned} J_z J_+ \left|\alpha,\beta\right\rangle &= \left(J_+ J_z + J_+\right) \left|\alpha,\beta\right\rangle \\ &= \beta J_+ \left|\alpha,\beta\right\rangle + J_+ \left|\alpha,\beta\right\rangle \\ &= \left(\beta+1\right) J_+ \left|\alpha,\beta\right\rangle \end{aligned}$$

Therefore, the new vector is also an eigenvector of both $\mathbf{J}^2$ and J_z, and we have the new eigenvalue, $\beta+1$, for J_z. Therefore, we may write

$$J_+ \left|\alpha,\beta\right\rangle = A \left|\alpha,\beta+1\right\rangle$$

where we require a constant because we only know that the new vector is proportional to the given eigenstate. We can also act with J_- to reduce the eigenvalue of J_z by 1,

$$\begin{aligned} \mathbf{J}^2 J_- \left|\alpha,\beta\right\rangle &= J_- \mathbf{J}^2 \left|\alpha,\beta\right\rangle \\ &= \alpha J_- \left|\alpha,\beta\right\rangle \end{aligned}$$

and, using the commutator to write $J_z J_- = J_- J_z - J_-$,

$$\begin{aligned} J_z J_- \left|\alpha,\beta\right\rangle &= \left(J_- J_z - J_-\right) \left|\alpha,\beta\right\rangle \\ &= \beta J_- \left|\alpha,\beta\right\rangle - J_- \left|\alpha,\beta\right\rangle \\ &= \left(\beta-1\right) J_- \left|\alpha,\beta\right\rangle \end{aligned}$$

and therefore,

$$J_- \left|\alpha,\beta\right\rangle = B \left|\alpha,\beta-1\right\rangle$$

Now notice that the norm of $J_+ \left|\alpha,\beta\right\rangle$, which is necessarily non-negative, satisfies

$$\begin{aligned} 0 &\leq \left\langle\alpha,\beta\right| J_+^\dagger J_+ \left|\alpha,\beta\right\rangle \\ &= \left\langle\alpha,\beta\right| J_- J_+ \left|\alpha,\beta\right\rangle \end{aligned}$$

Similarly,

$$\begin{aligned} 0 &\leq \left\langle\alpha,\beta\right| J_-^\dagger J_- \left|\alpha,\beta\right\rangle \\ &= \left\langle\alpha,\beta\right| J_+ J_- \left|\alpha,\beta\right\rangle \end{aligned}$$

Combining these, we have

$$0 \leq \left\langle\alpha,\beta\right| \frac{1}{2}\left(J_+ J_- + J_- J_+\right) \left|\alpha,\beta\right\rangle$$

Then, using

$$\begin{aligned}
J_- J_+ &= (J_x - iJ_y)(J_x + iJ_y) \\
&= J_x^2 - iJ_yJ_x + iJ_xJ_y + J_y^2 \\
&= \mathbf{J}^2 - J_z^2 + i[J_x, J_y] \\
&= \mathbf{J}^2 - J_z^2 - J_z
\end{aligned}$$

and similarly,

$$J_+ J_- = \mathbf{J}^2 - J_z^2 + J_z$$

we have the inequality

$$\begin{aligned}
0 &\leq \langle \alpha, \beta | \frac{1}{2}(J_+J_- + J_-J_+) | \alpha, \beta \rangle \\
&= \langle \alpha, \beta | \left(\mathbf{J}^2 - J_z^2\right) | \alpha, \beta \rangle \\
&= \langle \alpha, \beta | \left(\alpha - \beta^2\right) | \alpha, \beta \rangle \\
&= \alpha - \beta^2
\end{aligned}$$

Therefore,

$$\beta^2 \leq \alpha$$

Returning to the raising and lowering operators, we see that we can only apply J_+ or J_-, a finite number of times – otherwise this bound on β would be exceeded. Therefore, there must be some vector which terminates the action of each of these operators, i.e., two nonzero vectors $|\alpha, \beta_{max}\rangle$ and $|\alpha, \beta_{min}\rangle$

$$\begin{aligned}
J_+ |\alpha, \beta_{max}\rangle &= 0 \\
J_- |\alpha, \beta_{min}\rangle &= 0
\end{aligned}$$

Acting on the first with J_-, we may write

$$\begin{aligned}
0 &= J_-J_+ |\alpha, \beta_{max}\rangle \\
&= \left(\mathbf{J}^2 - J_z^2 - J_z\right) |\alpha, \beta_{max}\rangle \\
&= \left(\alpha - \beta_{max}^2 - \beta_{max}\right) |\alpha, \beta_{max}\rangle
\end{aligned}$$

and we conclude that

$$\begin{aligned}
\alpha &= \beta_{max}^2 + \beta_{max} \\
&= \beta_{max}(\beta_{max} + 1)
\end{aligned}$$

Acting on the second with J_+ we have

$$\begin{aligned}
0 &= J_+J_- |\alpha, \beta_{min}\rangle \\
&= \left(\mathbf{J}^2 - J_z^2 + J_z\right) |\alpha, \beta_{min}\rangle \\
&= \left(\alpha - \beta_{min}^2 + \beta_{min}\right) |\alpha, \beta_{min}\rangle
\end{aligned}$$

so that

$$\alpha = \beta_{min}\left(\beta_{min} - 1\right)$$

Equating the two expressions, we require

$$\beta_{min}\left(\beta_{min} - 1\right) = \beta_{max}\left(\beta_{max} + 1\right)$$

Setting $\beta_{max} = a\beta_{min} + b$ gives

$$\begin{aligned} \beta_{min}\left(\beta_{min} - 1\right) &= \left(a\beta_{min} + b\right)\left(a\beta_{min} + b + 1\right) \\ \beta_{min}^2 - \beta_{min} &= a^2\beta_{min}^2 + ab\beta_{min} + a\left(b+1\right)\beta_{min} + b\left(b+1\right) \end{aligned}$$

The quadratic part requires $a = \pm 1$, leaving the pair of equations,

$$\begin{aligned} -\beta_{min} &= \pm b\beta_{min} \pm \left(b+1\right)\beta_{min} \\ b\left(b+1\right) &= 0 \end{aligned}$$

With $b = 0$ we find

$$-\beta_{min} = \pm\beta_{min}$$

while $b = -1$ gives

$$-\beta_{min} = \mp\beta_{min}$$

so either

$$\beta_{max} = -\beta_{min}$$

or

$$\beta_{max} = \beta_{min} - 1$$

Since the second is contradictory, we have

$$\beta_{min} = -\beta_{max}$$

and because we must be able to step from one to the other in unit steps, they must differ by an integer, $n = 0, 1, 2, \ldots$,

$$\begin{aligned} n &= \beta_{max} - \beta_{min} \\ &= 2\beta_{max} \\ \beta_{max} &= \frac{n}{2} \end{aligned}$$

Let $j = \frac{n}{2}$ be any integer multiple of $\frac{1}{2}$. Then gives

$$\begin{aligned} \alpha &= \beta_{max}\left(\beta_{max} + 1\right) \\ &= j\left(j+1\right) \end{aligned}$$

This gives a complete set of states; replacing α with j and β with m,

$$\left|\alpha, \beta\right\rangle \rightarrow \left|j, m\right\rangle$$

we rewrite the eigenvalues equations as

$$\begin{aligned}
\mathbf{J}^2 |j,m\rangle &= j(j+1)|j,m\rangle \\
J_z |j,m\rangle &= m|j,m\rangle
\end{aligned}$$

The action of $J_\pm$ on these vectors may be found from

$$\begin{aligned}
\langle j,m| J_- J_+ |j,m\rangle &= \langle j,m| \left(\mathbf{J}^2 - J_z^2 - J_z\right)|j,m\rangle \\
&= \langle j,m| (j(j+1) - m(m+1))|j,m\rangle \\
&= j(j+1) - m(m+1)
\end{aligned}$$

This gives us the squared norm of $J_+ |j,m\rangle$. Since we know that

$$J_+ |j,m\rangle = \lambda |j,m+1\rangle$$

and all of the $|j,m\rangle$ are normalized, we identify $\lambda^*\lambda = j(j+1) - m(m+1)$. Choosing the phase to be zero gives

$$J_+ |j,m\rangle = \sqrt{j(j+1) - m(m+1)}\,|j,m+1\rangle$$

A similar calculation gives

$$J_- |j,m\rangle = \sqrt{j(j+1) - m(m-1)}\,|j,m-1\rangle$$

How do the rotation generators (and therefore the rotations they generate) act on these states? We have

$$\begin{aligned}
J_x |j,m\rangle &= \frac{1}{2}(J_+ + J_-)|j,m\rangle \\
&= \frac{1}{2}\sqrt{j(j+1) - m(m+1)}\,|j,m+1\rangle \\
&\quad + \frac{1}{2}\sqrt{j(j+1) - m(m-1)}\,|j,m-1\rangle \\
J_y |j,m\rangle &= \frac{1}{2i}(J_+ - J_-)|j,m\rangle \\
&= \frac{1}{2i}\sqrt{j(j+1) - m(m+1)}\,|j,m+1\rangle \\
&= -\frac{1}{2i}\sqrt{j(j+1) - m(m-1)}\,|j,m-1\rangle \\
J_z |j,m\rangle &= m|j,m\rangle
\end{aligned}$$

The important thing to notice about these relationships is that while J_x and J_y can take $|j,m\rangle$ to any other value of m, none of the three generators changes the value of j. Since m ranges from $-j$ to $+j$ in integer steps, this means that $2j+1$ distinct basis vectors in the set

$$\{|j,-j\rangle, |j,-j+1\rangle, |j,-j+2\rangle, \ldots, |j,j-2\rangle, |j,j-1\rangle, |j,j\rangle\}$$

rotate into one another when acted on by various rotations. These are therefore the distinct vector representations on which rotations can act. Since j is half-integer, the number of independent basis vectors – hence the dimension of the representation – is $2j+1 = 2\left(\frac{n}{2}\right)+1 = 1, 2, 3, \ldots$. That is, we get representations of every integer dimension. The odd cases correspond to spin; the even cases to orbital angular momentum.