HOMOLOGY AND COHOMOLOGY OF COMPACT CONNECTED LIE GROUPS

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The homological properties of compact connected Lie groups over real coefficients are completely known;[1] the cohomology mod. p or over the integers of the classical groups, whose study was initiated by L. S. Pontrjagin and C. Ehresmann, has also been fully investigated.[2] In this note, we shall describe the cohomology ring mod. p (p prime), and also, partly, over the integers, of the quotient groups of the classical groups and (completing earlier results announced in (I)) of Spin(n), G_2, F_4. We shall add some information on the homology ring (Pontrjagin product), derived in part from general statements connecting cohomology and Pontrjagin product formulated in No. 1; No. 2 is devoted to a converse statement to the main theorems on transgression in universal bundles of (II). The detailed proofs of these results will appear elsewhere.

Notations and Definitions.—p denotes a prime number or zero, K_p a field of characteristic p, Z_p, ($p \neq 0$), the integers mod. p, Z_0 the rational numbers; $\bigwedge(x_1, \ldots, x_k)$ is the exterior algebra (over a field which the context will make precise) generated by the x_i (in the sense of Grassmann multiplication); in a graded module, Dx will be the degree of a homogeneous element x.

$H^*(X, A)$, (resp., $H_*(X, A)$), is the direct sum of the cohomology (resp., homology), groups $H^i(X, A)$, (resp., $H_i(X, A)$), of the space X with coefficients in A; the space X has no p-torsion ($p \neq 0$), if the torsion coefficients of $H^*(X, Z)$ are not divisible by p; by convention, X is always without 0-torsion.

G always denotes a compact connected Lie group, B_G is a classifying space for G, i. e., the base-space of a universal bundle E_G for G; an element $x \in H^*(G, A)$ is universally transgressive if it is transgressive in E_G, (see (II), §18, 19); $SU(n)$, (resp., $SO(n)$), unimodular unitary group in n complex (resp., real), variables, Sp(n) unitary group in n quaternionic variables; $V_{n,k}$, Stiefel manifold of orthonormal k-frames in Euclidian n-space. Finally we recall, in a slightly more general form, a definition introduced in (II) §6:

DEFINITION. *Let E be an associative algebra with unit over a ring A. The elements x_i ($i \in I$, I totally ordered set), form a simple system of generators of E if E is the weak direct sum of the monogeneous submodules generated by the unit and by all the products $x_{i_1}x_{i_2}\ldots x_{i_k}$ ($i_1 < i_2 < \ldots < i_k$; $k = 1, 2, \ldots$).*

1. *The Pontrjagin Product.*[3]—Let $h:G \times G \rightarrow G$ be the map defining

the product; it induces a map h_* of $H_*(G, K_p) \otimes H_*(G, K_p)$ into $H_*(G, K_p)$ and $h_*(a \otimes b)$ is called the Pontrjagin product of a and b; this product adds the degrees, is associative, distributive, possesses a unit which spans $H_0(G, K_p)$ but, unlike the cup-product, is not always anticommutative, even for Lie groups, as an example below will show. We recall that an element $x \in H^*(G, K_p)$ of positive degree is called *primitive* if $h^*(x) = x \otimes 1 + 1 \otimes x$, h^* being the homomorphism $H^*(G, K_p) \rightarrow H^*(G, K_p) \otimes H^*(G, K_p)$ induced by h.

PROPOSITION 1. *If $H^*(G, K_p)$ has a simple system of primitive generators, (x_i), $(1 \leq i \leq m)$, then $H_*(G, K_p)$ is an anticommutative exterior algebra generated by m elements u_i, $(Du_i = Dx_i)$, and conversely.*

This is a simple consequence of the fact that h^* and h_* are dual to each other. Since a universally transgressive element is primitive ((II), Proposition 20.1), the assumption of Proposition 1 is in particular fulfilled when $H^*(G, K_p)$ has a simple system of universally transgressive generators; this happens when G has no p-torsion ((II), Prop. 7.2 and Theor. 19.1), and also, mod. 2, in some other cases, e.g., for $G = SO(n)$, ((II), Prop. 23.1), or $G = \text{Spin}(n)$, $(n \leq 9)$, G_2, F_4 (see below); in particular one sees that:

If G is a classical group, $H_*(G, K_p)$ is an anticommutative exterior algebra for all p.

The degrees of the generators are given by Prop. 1 and by the results on the classical groups previously cited; e.g., for $H_*(SO(n), Z_2)$ they are equal to $1, 2, \ldots, n-1$, as was first shown by Miller, *loc. cit.*[2]

Let X be a space on which G operates; following J. Leray,[4] one can attach to each element $u \in H_s(G, K_p)$ an endomorphism ϑ_u (for the vector space structure only), of $H^*(G, K_p)$ which decreases degrees by s. The map $u \rightarrow \vartheta_u$ is a homomorphism of $H_*(G, K_p)$ into the algebra of linear endomorphisms of $H^*(X, K_p)$); if G operates on the space Y and if there is a map $f: X \rightarrow Y$ commuting with G, then ϑ_u commutes with f^*, and also acts on the spectral sequence of f; with the help of these operators one proves:

THEOREM 1. *Let Y be a space on which G operates and let $f: G \rightarrow Y$ be a map commuting with G (acting upon itself by left translations). If $H^*(G, K_p)$ has a simple system of primitive generators, then the image of f^* is a subalgebra generated by primitive elements; if moreover $p \neq 2^5$, then $H^*(Y, K_p) = N \otimes \bigwedge P'$, where f^* annihilates N and maps $\bigwedge P'$ isomorphically into $H^*(G, K_p)$.*

Applied to the particular cases where f is the inclusion of G into an overgroup, and where f is the projection of G onto a coset space, Theorem 1 generalizes a result of Leray, *loc. cit.*,[4] Note (a), as well as Prop. 21.1 and 21.2 of (II), which were extensions of theorems due to H. Samelson.

2. A Transgression Theorem in Universal Bundles.--By arguments partly analogous to but simpler than those of Chap. IV in (II), one proves the following theorem, which may be considered as a converse to Theorem 19.1 and Prop. 19.1 of (II):

THEOREM 2. If $H^*(B_G, K_p) = K_p[y_1, \ldots, y_m]$, *($Dy_i$ even), then* $H^*(G, K_p) = \bigwedge(x_1, \ldots, x_m)$, *($x_i$ universally transgressive,* $Dx_i = Dy_i - 1$); *if* $H^*(B_G, K_2) = K_2[y_1, \ldots, y_m]$, *then* $H^*(G, K_2)$ *has a simple system of universally transgressive generators* $x_1, \ldots, x_m$ $(Dx_i = Dy_i - 1)$. *In both cases* y_i *is an image of* x_i *by transgression in* E_G.

3. *Quotient Groups of the Classical Groups.*—The groups Sp(n), $SU(n)$, $SO(2n)$, $SO(2n+1)$ have cyclic centers of respective orders 2, n, 2, 1; Γ_m will denote the subgroup of order m of one of these centers. Using the known results on the classical groups, the cohomology of the cyclic groups, the spectral sequence of regular finite coverings, and the explicit determination of $\rho^*(\Gamma_m, G)$, (defined in (II), §21), one gets:

THEOREM 3. *Let n be a positive integer and s the greatest power of 2 dividing n. Then, with* $Dx = 1$, $Dx_i = i$:[6]

$$H^*(\mathrm{Sp}(n)/\Gamma_2, Z_2) \cong Z_2[x]/(x^{4s}) \otimes \bigwedge(x_3, x_7, \ldots, \hat{x}_{4s-1}, \ldots, x_{4n-1})$$

$$H^*(SO(2n)/\Gamma_2, Z_2) \cong Z_2[x]/(x^{2s}) \otimes V$$

where V is a unitary graded algebra having a simple system of $2n - 2$ generators $v_i(1 \leq i \leq 2n-1, i \neq 2s-1, Dv_i = i)$, *with the relations* $v_i \cdot v_i = v_{2i}$ *if* $2i \leq n - 1$. $v_i \cdot v_i = 0$ *otherwise.*

THEOREM 4. *Let n be a positive integer, m a divisor of n, p a prime divisor of m, and s the greatest power of p dividing n. Then (with* $Dx = 1$, $Dy = 2$, $Dx_i = i$), *for* $p \geq 3$ *or* $p = 2$, $m \equiv 0$ *mod.* 4:[6]

$$H^*(SU(n)/\Gamma_m, Z_p) \cong Z_p[y]/(y^s) \otimes \bigwedge(x_1, x_3, \ldots, \hat{x}_{2s-1}, \ldots, x_{2n-1}),$$

and, for $p = 2$, $m \equiv 2$ *mod.* 4:

$$H^*(SU(n)/\Gamma_m, Z_2) \cong Z_2[x]/(x^{2s}) \otimes \bigwedge(x_3, x_5, \ldots, \hat{x}_{2s-1}, \ldots, x_{2n-1}).$$

For the sake of completeness, we recall that if p does not divide m, G and G/Γ_m have the same cohomology mod. p, as follows from well-known theorems on finite regular coverings.

4. *The Spinor Group.*—The group Spin(n), $(n \geq 3)$, is the twofold universal covering of $SO(n)$; it admits a fibering $\mathrm{Spin}(n)/T^1 = V_{n, n-2}$; knowing the cohomology of $V_{n, n-2}$, including the Sq^i (see (II), (III), or Miller[2]), one can determine its spectral sequence and obtain not only the results formulated in (I), but the more complete:

THEOREM 5. $H^*(Spin(n), Z)$ *has torsion if and only if* $n \geq 7$, *and its torsion coefficients are then all equal to 2. Let* $s(n)$ *be the integer such that* $2^{s(n)-1} < n \leq 2^{s(n)}$ *and put* $a(n) = 2^{s(n)} - 1$. *Then* $H^*(Spin(n), Z_2)$ *has a simple system of* $n - s(n)$ *generators* $u_i(1 \leq i < n - s(n))$, *and* u, $(Du =$

$a(n)$, and the sequence $Du_1, \ldots, Du_{n-s(n)-1}$ is obtained from the sequence $3, 4, \ldots, n-1$ by erasing all powers of 2), subject to the relations:

$$\mathrm{Sq}^i u_j = \binom{Du_j}{i} u_k \text{ if } i \leq Du_j,\ i + Du_j = Du_k$$

$$\mathrm{Sq}^i u_j = 0 \text{ otherwise; } u \cdot u = 0.$$

Mod. 2, the group Spin(n) shows a rather particular behavior as regards transgression in universal bundles; in fact, by use of Theorem 2 and study of the spectral sequences of the fiberings $(Spin(n), V_{n,\, n-2}, S_1)$ and $(B_{Spin(n)}, B_{SO(n)}, B_{Z_2})$ one proves:

PROPOSITION 2. *In Theorem 5, the elements $u_i (i < n - s(n))$, may be chosen to be universally transgressive, but this is the case for u if and only if $n \leq 9$.*

In particular, $H^*(Spin(n), Z_2)$ is not generated by universally transgressive elements, and $H^*(B_{Spin(n)}, Z_2)$ is not a ring of polynomials for $n \geq 10$; these facts have also repercussions on the Pontrjagin product in $H_*(Spin(n), Z_2)$ which is not anticommutative (i.e., commutative here, since we calculate mod. 2) for $n \geq 10$. I did not completely determine $H^*(B_{Spin(n)}, Z_2)$ and $H_*(Spin(n), Z_2)$ for general n; however:

THEOREM 6. *$H^*(B_{Spin(10)}, Z_2) \cong Z_2[w_4, w_6, w_7, w_8, w_{10}, w_{32}]/(w_7 \cdot w_{10})$, with $Dw_i = i$, and $(w_7 \cdot w_{10})$ being the ideal generated by the product $w_7 \cdot w_{10}$. The algebra $H_*(Spin(10), Z_2)$ has a simple system of 6 generators $u_3, u_5, u_6, u_7, u_9, u_{15}$ $(Du_i = i)$, subject to the relations: $u_i \cdot u_i = 0$ (all i), $u_i \cdot u_j = u_j \cdot u_i$ for $i < j$ $(i, j) \neq (6, 9)$, and $u_6 \cdot u_9 = u_9 \cdot u_6 + u_{15}$.*

5. The First Two Exceptional Groups.—G_2 and F_4 denote as usual the compact exceptional groups with 14 and 52 parameters; they are necessarily simply connected.

THEOREM 7. *$H^*(G_2, Z)$ is generated by 2 elements h_3, h_{11}, $(Dh_i = i)$, with relations $h_3^4 = h_{11}^2 = h_3^2 \cdot h_{11} = 0$, such that $H^*(G_2, Z)$ is the weak direct sum of the 4 infinite cyclic groups generated by 1, h_3, h_{11}, $h_3 \cdot h_{11}$ and of the 2 cyclic groups of order 2 generated by h_3^2 and h_3^3.*

This is obtained by investigation of the fiberings $G_2/S_3 = V_{7,\,2}$ and $Spin(7)/G_2 = S_7$, which shows moreover:

THEOREM 8. *$H^*(G_2, Z_2)$ has a simple system of universally transgressive generators x_3, x_5, x_6 $(Dx_i = i)$, satisfying: $\mathrm{Sq}^2 x_3 = x_5$, $\mathrm{Sq}^3 x_3 = x_6$, $\mathrm{Sq}^1 x_5 = x_6$, $\mathrm{Sq}^i x_j = 0$ otherwise; $H^*(G_2, Z_5) = \bigwedge(y_3, y_{11})$, $(Dy_i = i)$, with $\mathcal{P}^1(y_3) = y_{11}$.*[7]

To investigate F_4 one uses, as indicated in (I), the spectral sequences of the fiberings deduced from the inclusions $F_4 \supset Spin(9) \supset Spin(8) \supset T^4$ and $F_4 \supset Spin(9) \supset Spin(7) \supset G_2$, where $F_4/Spin(9)$ is the projective plane over the Cayley numbers; to construct these spectral sequences, one needs some of the above results and one has to know that the image of the natural homomorphism of $H^8(F_4/Spin(9), Z)$ into $H^8(F_4, Z)$ is Z_3; this in turn follows from the two facts: (a) the symetric group of three objects acts faithfully on $H^8(F_4/Spin(8), Z)$, trivially on $H^8(F_4, Z)$ and commutes with the map induced by the projection; (b) $H^*(Spin(9), Z_3) = \Lambda(x_3, x_7, x_{11}, x_{15})$ with $\mathcal{P}^1 x_3 = x_7$, $\mathcal{P}^1 x_{11} = x_{15}$.[7] This leads to:

THEOREM 9. *$H^*(F_4, Z) = H^*(G_2 \times S_{15}, Z) \otimes U$, where U is a unitary graded ring defined by: $U^0 = U^{23} = Z$, $U^8 = U^{16} = Z_3$ with $U^8.U^8 = U^{16}$ and $U^i = 0$ otherwise. Thus $H^*(F_4, Z_2) = H^*(G_2 \times S_{15} \times S_{23}, Z_2)$ and for $p \neq 2, 3$, $H^*(F_4, Z_9) = \Lambda(x_3, x_{11}, x_{15}, x_{23})$, $(Dx_i = i)$. Moreover $H^*(F_4, Z_2)$ has a simple system of universally transgressive generators of degrees 3, 5, 6, 15, 23 and $H^*(F_4, Z_3) \cong Z_3[x]/(x^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15})$, $(Dx_i = i, Dx = 8)$.*

The torsion coefficients of $H^*(F_4, Z)$ are therefore equal to 2 or 3; the proof also gives the following partial results concerning reduced powers: (i) The isomorphism $H^*(F_4, Z_2) = H^*(G_2 \times S_{15} \times S_{23}, Z_2)$ is valid at least up to degree 22 for the Sq^i; (ii) mod. 3, $\mathcal{P}^1 x_3 = x_7$, $\mathcal{P}^1 x_{11} = x_{15}$ and x is obtained from x_7 by the Bockstein homomorphism (for suitable x, x_i); (iii) mod. 5 $\mathcal{P}^1 x_3 = x_{11}$, mod. 7, $\mathcal{P}^1 x_3 = x_{15}$.[7]

[1] See H. Samelson's report, *Bull. Am. Math. Soc.*, **58**, 2–37 (1952) for references.

[2] Borel, A., *Compt. rend. Acad. Sci. (Paris)*, **232**, 1628–1630 (1951); *Ann. Math.*, **57**, 115–207 (1953), Chap. III; *Comm. Math. Helv.*, **27**, 165–197 (1953), cited in the following (I), (II), (III), Borel, A., and Serre, J.-P., *Am. J. Math.*, **75**, 409–448 (1953); also, for the orthogonal group, Miller, C. E., *Ann. Math.*, **57**, 90–115 (1953).

[3] For the sake of brevity, we have stated the results in No. 1 only for Lie groups, but they are also in part valid for H-spaces with an associative product (same proofs); also, one has analogous statements over the integers, provided G, resp. G and Y, have no torsion.

[4] Leray, J., (a) *Compt. rend. Acad. Sci. (Paris)*, **228**, 1545–1547 (1949); (b) *Ibid.*, 1784–1786.

[5] $H^*(G, K_p)$ is then an exterior algebra generated by elements of odd degrees; (see (II), Proposition 6.1(b)).

[6] As usual, $\hat{}$ over a variable means that the variable has to be omitted.

[7] For suitable universally transgressive elements; $\mathcal{P}^1$ is the reduced power operation, which, mod. p, increases degrees by $2(p - 1)$; see Steenrod, N. E., PROC. NATL. ACAD. SCI., **39**, 213–223 (1953).