Supersymmetric M2-branes and ADE

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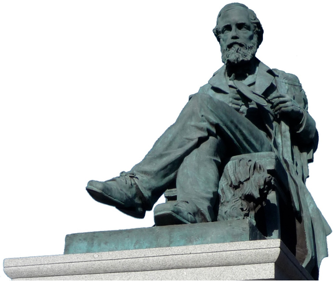
http://www.maths.ed.ac.uk/~jmf/CV/Seminars/YRM2010.pdf

1. Historical context
2. M2-brane geometries
3. ADE classification

In the beginning there was Maxwell...

Maxwell equations

$$\begin{cases} dF = 0 \\ d \star F = \star J \end{cases}$$



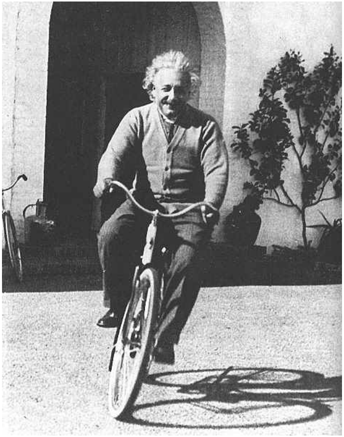
In vacuo ($J = 0$), they are Lorentz (in fact, conformally) invariant.

The birth of spacetime

Hermann Minkowski (1908)

“The views of space and time that I wish to lay before you have ***sprung from the soil of experimental physics****, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of both will retain an independent reality.”*

General Relativity



Einstein field equations

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G T_{ab}$$

GR mantra

Space tells matter how to move, matter tells space how to curve.

Geometrisation of Physics

- Both sides of the Einstein equations could not be more different.
- The LHS

$$R_{ab} - \tfrac{1}{2} R g_{ab}$$

  is geometric;
- whereas the RHS

$$8\pi G T_{ab}$$

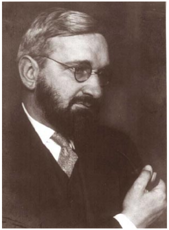
  seems put by hand.
- Geometrisation means writing the equations without a RHS!

Kaluza–Klein theory

- For Einstein–Maxwell theory

$$\begin{cases} R_{ab} - \frac{1}{2} R g_{ab} = \frac{1}{2} F_a{}^c F_{cb} - \frac{1}{8} F^{cd} F_{cd} g_{ab} \\ dF = 0 \\ d \star F = 0 \end{cases}$$

this was done independently by Nordström, Kaluza and (Oskar) Klein.


One extra dimension

- N, a five-dimensional spacetime with an isometric action of S^1: locally $N = M^4 \times S^1$, with S^1 unobservably small.
- The metric on N unpacks into a metric tensor, an electromagnetic field and a scalar field on M.
- Taking the radius of the circle to be constant, the *vacuum* Einstein field equations N are the Einstein–Maxwell equations on M.

  However taking the radius constant means that the connection must be flat!

- The electromagnetic $U(1)$ is realised as isometries of N.

... or seven!

- The gauge group of the standard model is $G = SU(3) \times SU(2) \times U(1)$.
- The smallest orbits on which G acts effectively are 7-dimensional.
- To geometrise the standard model requires at least eleven dimensions.
- (Lorentzian) supersymmetry allows at most eleven dimensions!

Eleven-dimensional supergravity

- There is a unique eleven-dimensional supersymmetric theory: eleven-dimensional supergravity.

  NAHM (1979), CREMMER+JULIA+SCHERK (1980)
- The bosonic fields are a metric g and a 3-form potential A.
- subject to the Einstein–Hilbert + Maxwell + Chern–Simons actions:

$$\int R \,\mathrm{dvol} + \int F \wedge \star F + \int F \wedge F \wedge A$$

  with $F = dA$.

Spontaneous compactification

- We seem to live in four dimensions, so a natural candidate geometry is $N^{11} = M^4 \times X^7$.
- Taking $F = \mathrm{dvol}(M)$ "geometrises" the supergravity field equations.
- The earliest such solution is $\mathrm{AdS}_4 \times S^7$.

FREUND+RUBIN (1980)

Eleven-dimensional supergravity is dead

- The sizes of AdS_4 and S^7 are roughly the same.
- Cannot obtain standard model chiral fermions from eleven dimensions.

WITTEN (1984)

- Today we can even rule out AdS_4 on empirical grounds: $\Lambda > 0$!

Long live M-theory!

Fast forward to 1995...

- Two superstring revolutions later: the strong coupling limit of IIA superstring theory is eleven-dimensional!
- Its low-energy limit has to be eleven-dimensional supergravity.
- Freund–Rubin backgrounds are back, this time as near-horizon geometries of M2-branes.
- They play a crucial rôle in the gauge/gravity correspondence.

Long live M-theory!

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Main motivation

Fast forward to 2010...

Main question

What is M-theory?

- not a theory of strings!
- a theory of membranes?
- maybe, but quantising membranes is difficult!
- AdS/CFT: try to at least understand the dual sCFT, on which much progress has been made recently.

Main motivation

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The M2-brane solution

Definition

The elementary M2-brane:

$$g = H^{-\frac{2}{3}}\, ds^2(\mathbb{R}^{2,1}) + H^{\frac{1}{3}}\, ds^2(\mathbb{R}^8)$$
$$F = \mathrm{dvol}(\mathbb{R}^{2,1}) \wedge dH^{-1},$$

where

$$H = \alpha + \frac{\beta}{r^6}\,,$$

for $\alpha, \beta \in \mathbb{R}$ not both equal to zero.

DUFF+STELLE (1991)

It is half-supersymmetric for generic α, β, i.e., $\alpha\beta \neq 0$.

Asymptotia

- $\beta \to 0$ (or $r \to \infty$):

$$g \to ds^2(\mathbb{R}^{10,1})$$
$$F \to 0$$

$\therefore$ asymptotically flat

- $\alpha \to 0$ (or $r \to 0$):

$$g \to \beta^{-\frac{2}{3}} r^4 ds^2(\mathbb{R}^{2,1}) + \beta^{\frac{1}{3}} \frac{dr^2}{r^2} + \beta^{\frac{1}{3}} ds^2(S^7)$$
$$F \to 6r^5 \beta^{-1} \,\mathrm{dvol}(\mathbb{R}^{2,1}) \wedge dr$$

$\therefore$ $AdS_4 \times S^7$, the near-horizon limit

- AdS_4 has Ricci scalar $-\frac{8}{7}$ that of the S^7: so they are of similar size.
- Both the asymptotic solution and the near-horizon solution are maximally supersymmetric.
- The M2-brane is an interpolating soliton.

GIBBONS+TOWNSEND (1993)

Killing superalgebra

- Every supersymmetric supergravity background has an associated Lie superalgebra

  JMF (1999), JMF+MEESSEN+PHILIP (2004)
- For $\mathrm{AdS}_4 \times S^7$ it is $\mathfrak{osp}(8|4)$
- The even subalgebra is

$$\mathfrak{so}(8) \oplus \mathfrak{sp}(4,\mathbb{R}) \cong \mathfrak{so}(8) \oplus \mathfrak{so}(3,2),$$

- $\mathfrak{so}(3,2)$ = isometries of AdS_4 = conformal symmetry of $\mathbb{R}^{2,1}$
- $\mathfrak{so}(8)$ = isometries of S^7 = R-symmetry of dual sCFT

Generalised M2-brane solution

- Replace the S^7 with M^7:

$$g = H^{-\frac{2}{3}}\, ds^2(\mathbb{R}^{2,1}) + H^{\frac{1}{3}}(dr^2 + r^2 ds^2(M^7))$$
$$F = \mathrm{dvol}(\mathbb{R}^{2,1}) \wedge dH^{-1},$$

- field equations $\Longrightarrow$ M is Einstein

$$R_{ab} = 6 g_{ab}$$

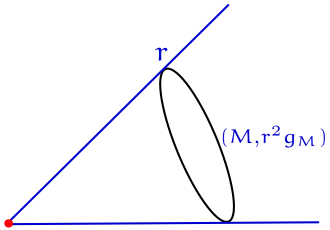
- supersymmetry $\Longrightarrow$ M admits (real) Killing spinors:

$$\nabla_m \varepsilon = \tfrac{1}{2}\Gamma_m \varepsilon$$

Bär's cone construction

Question

Which manifolds admit real Killing spinors?



- The metric cone of a riemannian manifold (M, g_M) is the manifold $C = \mathbb{R}^+ \times M$ with metric $g_C = dr^2 + r^2 g_M$
- e.g., the metric cone of the round sphere S^n is $\mathbb{R}^{n+1} \setminus \{0\}$
- (M, g_M) admits real Killing spinors if and only if (C, g_C) admits parallel spinors: $\nabla_a \varepsilon = 0$ BÄR (1993)
- It is g_C which appears in the membrane solution:

Supersymmetric M2-branes = M2-branes at a conical singularities!

ACHARYA+JMF+HULL+SPENCE, MORRISON+PLESSER (1998)

Spherical harmonics

- The cone trick is old
- Kelvin and Tait (1867) already used it to construct Laplace's spherical harmonics
- They are the restriction to the unit sphere in $\mathbb{R}^3$ of homogeneous harmonic polynomials:

$$p(\lambda x) = \lambda^\ell p(x) \qquad \text{and} \qquad \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0$$

- Bär's cone construction is the spinorial version of this.
- Unlike Kaluza–Klein, the extra dimension has no physical meaning.

Manifolds with real Killing spinors

Theorem (Gallot, 1979)

If (M, g) is complete, the cone (C, h) is either irreducible or flat.

Simply-connected 7-manifolds with real Killing spinors:

$\mathscr{N}$	Cone holonomy	7-dimensional geometry
8	$\{1\}$	sphere
3	$\mathrm{Sp}(2)$	3-Sasaki
2	$\mathrm{SU}(4)$	Sasaki-Einstein
1	$\mathrm{Spin}(7)$	weak G_2 holonomy

$\mathscr{N} = \dim\{\text{Killing spinors}\}$

M. Wang (1989)

$\mathscr{N} \geqslant 4$ and sphere quotients

- $\mathscr{N} \leqslant 3$: infinite homotopy types, hard to classify
- $\mathscr{N} \geqslant 4$: quotients S^7/Γ
- Classify finite $\Gamma < \mathrm{SO}(8)$ such that
  - Γ acts freely on S^7 (so that S^7/Γ is smooth)
  - Γ lifts to $\mathrm{Spin}(8)$ (for S^7/Γ to be spin)
  - $\dim \Delta_+^\Gamma \geqslant 4$ (for $\mathscr{N} \geqslant 4$ supersymmetry)
- It turns out there is an ADE classification... with a twist!

  DE MEDEIROS+JMF+GADHIA+MÉNDEZ-ESCOBAR (2009)

Angels and demons

(attributed to) Hermann Weyl

The angel of Geometry and the devil of Algebra fight for the soul of every mathematician.

7-dimensional spherical space forms

- Which manifolds are locally isometric to the round 7-sphere?
- Equivalently, which (finite) $\Gamma < \mathrm{SO}(8)$ act freely on the unit sphere in $\mathbb{R}^8$?
- This problem was solved (in any dimension) by Wolf, after earlier work of Vincent.
- It is published as a book: *Spaces of Constant Curvature*.
- Different editions differ in dimension 7!
- Tractable, but messy. GADHIA (2006)

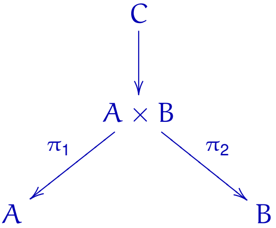
Subgroups leaving spinors invariant

- Easier to look for $\Gamma < \mathrm{SO}(8)$ such that $\mathscr{N} = \dim \Delta_+^\Gamma \geqslant 4$.
- Since $\dim \Delta_+ = 8$, $\Gamma < \mathrm{Spin}(8 - \mathscr{N})$.
- Classify $\Gamma < \mathrm{Spin}(4)$ which act freely on unit sphere of vector representation.
- $\mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$
- The action of $(u_1, u_2) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on $\mathbb{R}^8 \cong \mathbb{H} \oplus \mathbb{H}$ is

$$(u_1, u_2) \cdot (x, y) = (u_1 x, u_2 y)$$

- Γ acts freely on S^7 **iff** $(1, u), (u, 1) \in \Gamma \implies u = 1$

Goursat's lemma



$$L = \pi_1(C) \qquad R = \pi_2(C)$$

$$L_0 = C \cap \ker \pi_2 \qquad R_0 = C \cap \ker \pi_1$$

$$L_0 = \{a \in A \mid (a, 1) \in C\} \quad R_0 = \{b \in B \mid (1, b) \in C\}$$

$C \cong$ graph of $L/L_0 \xrightarrow{\cong} R/R_0$

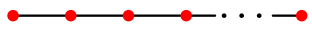
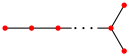
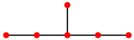
- In our case, $A = B = \mathrm{Sp}(1)$
- Γ acts freely on $S^7 \implies L_0$ and R_0 are trivial
- Therefore $L \cong R$
- $\Gamma \cong$ graph of automorphism $L \to L$, for $L < \mathrm{Sp}(1)$
- Classify pairs (L, τ), $L < \mathrm{Sp}(1)$, $\tau \in \mathrm{Aut}(L)$

Finite subgroups of rotations

- $\mathsf{Sp}(1)$ acts on $\mathsf{Im}\mathbb{H}$ by conjugation: $u \cdot x = uxu^{-1}$
- This defines double cover $\mathsf{Sp}(1) \to \mathsf{SO}(\mathsf{Im}\mathbb{H}) \cong \mathsf{SO}(3)$
- Finite $\Gamma < \mathsf{Sp}(1)$ gives finite $\overline{\Gamma} < \mathsf{SO}(3)$
- Finite subgroups of rotations: cyclic, dihedral, tetrahedral, cubic/octahedral, dodecahedral/icosahedral

We want their lifts to $\mathsf{Sp}(1)$. They are in one-to-one correspondence with the (affine) ADE Dynkin diagrams, as observed by John McKay.

The McKay correspondence

Dynkin diagram	Label	Name	Order	Presentation
	A_n	$\mathbb{Z}_{n+1}$	$n+1$	$\langle t \mid t^{n+1} = 1 \rangle$
	$D_{n+2\geqslant 4}$	$2D_{2n}$	$4n$	$\langle s, t \mid s^2 = t^n = (st)^2 \rangle$
	E_6	$2T$	24	$\langle s, t \mid s^3 = t^3 = (st)^2 \rangle$
	E_7	$2O$	48	$\langle s, t \mid s^3 = t^4 = (st)^2 \rangle$
	E_8	$2I$	120	$\langle s, t \mid s^3 = t^5 = (st)^2 \rangle$

... and the twist

- Let $\Gamma < \mathrm{Sp}(1)$ be one of the ADE subgroups
- Let $\tau \in \mathrm{Aut}(\Gamma)$ be an automorphism
- Let us embed $\Gamma \hookrightarrow \mathrm{SO}(8)$ via

$$u \cdot (x, y) = (ux, \tau(u)y) ,$$

  for $x, y \in \mathbb{H}$ and $u \in \mathrm{Sp}(1) \subset \mathbb{H}$
- Then Γ acts freely on S^7, lifts to $\mathrm{Spin}(8)$ and has $\mathcal{N} \geqslant 4$

The $\mathscr{N} \geqslant 4$ classification

The backgrounds $\mathrm{AdS}_4 \times M^7$ with $\mathscr{N} \geqslant 4$ are those with $M = S^7/\Gamma$ with $\Gamma < \mathrm{SO}(8)$ given by pairs (ADE, τ):

$\mathscr{N}$	Groups Γ
8	A_1
6	$A_{n\geqslant 2}$
5	$D_{n\geqslant 4}, E_6, E_7, E_8$
4	$(A_{n\geqslant 4, \neq 5}, r \in \mathbb{Z}_{n+1}^\times \setminus \{\pm 1\})$
4	$(D_{n\geqslant 6}, r \in \mathbb{Z}_{2(n-2)}^\times \setminus \{\pm 1\}), (E_7, \nu), (E_8, \nu)$

If $\tau = 1$ we don't write it and ν is the unique nontrivial outer automorphism of $E_{7,8}$. (The ones in red were not known.)

Open problems

- Orbifold quotients?
- $\mathscr{N} \leqslant 3$ quotients?
- Identify the dual sCFT (in most cases)
- Geometrise eleven-dimensional supergravity!

Thank you.