Symmetric spectra

This document is a preliminary and incomplete version of a book about symmetric spectra. It probably contains many typos and inconsistencies, and hopefully not too many actual mistakes. I intend to post updates regularly, so you may want to check my homepage for newer versions. I am interested in feedback.

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Introduction

This textbook is an introduction to the modern foundations of stable homotopy theory and 'algebra' over structured ring spectra, based on symmetric spectra. We begin with a quick historical review and attempt at motivation.

A crucial prerequisite for spectral algebra is an associative and commutative smash product on a good point-set level category of spectra, which lifts the well-known smash product pairing on the *homotopy category*. The first construction of what is now called 'the stable homotopy category', including its symmetric monoidal smash product, is due to Boardman [**6**] (unpublished); accounts of Boardman's construction appear in [**87**], [**84**] and [**2**, Part III] (Adams devotes more than 30 pages to the construction and formal properties of the smash product).

To illustrate the drastic simplification that occurred in the foundations in the mid-90s, let us draw an analogy with the algebraic context. Let R be a commutative ring and imagine for a moment that the notion of a chain complex (of R-modules) has not been discovered, but nevertheless various complicated constructions of the unbounded derived category $\mathcal{D}(R)$ of the ring R exist. Moreover, constructions of the *derived* tensor product on the *derived* category exist, but they are complicated and the proof that the derived tensor product is associative and commutative occupies 30 pages. In this situation, you could talk about objects A in the derived category together with morphisms $A \otimes^L_R A \longrightarrow A$, in the derived category, which are associative and unital, and possibly commutative, again in the derived category. This notion may be useful for some purposes, but it suffers from many defects – as one example, the category of modules (under derived tensor product in the derived category), does not in general form a triangulated category.

Now imagine that someone proposes the definition of a chain complex of R-modules and shows that by formally inverting the quasi-isomorphisms, one can construct the derived category. She also defines the tensor product of chain complexes and proves that tensoring with suitably nice (i.e., *homotopically projective*) complexes preserves quasi-isomorphisms. It immediately follows that the tensor product descends to an associative and commutative product on the derived category. What is even better, now one can suddenly consider differential graded algebras, a 'rigidified' version of the crude multiplication 'up-to-chain homotopy'. We would quickly discover that this notion is much more powerful and that differential graded algebras arise all over the place (while chain complexes with a multiplication which is merely associative up to chain homotopy seldom come up in nature).

Fortunately, this is not the historical course of development in homological algebra, but the development in stable homotopy theory was, in several aspects, as indicated above. In the stable homotopy category people could consider ring spectra 'up to homotopy', which are closely related to multiplicative cohomology theories. However, the need and usefulness of ring spectra with rigidified multiplications soon became apparent, and topologists developed different ways of dealing with them. One line of approach uses operads for the bookkeeping of the homotopies which encode all higher forms of associativity and commutativity, and this led to the notions of A_∞- respectively E_∞-ring spectra. Various notions of point-set level ring spectra had been used (which were only later recognized as the monoids in a symmetric monoidal model category). For example, the orthogonal ring spectra had appeared as *$\mathscr{I}_*$-prefunctors* in [**56**], the *functors with smash product* were introduced in [**8**] and symmetric ring spectra appeared as *strictly associative ring spectra* in [**31**, Def. 6.1] or as *FSPs defined on spheres* in [**33**, 2.7].

At this point it had become clear that many technicalities could be avoided if one had a smash product on a good point-set category of spectra which was associative and unital *before* passage to the homotopy category. For a long time no such category was known, and there was even evidence that it might not exist [**45**]. In retrospect, the modern spectra categories could maybe have been found earlier if Quillen's formalism of *model categories* [**62**] had been taken more seriously; from the model category perspective, one should not expect an intrinsically 'left adjoint' construction like a smash product to have a good homotopical behavior in general, and along with the search for a smash product, one should look for a compatible notion of cofibrations.

In the mid-90s, several categories of spectra with nice smash products were discovered, and simultaneously, model categories experienced a major renaissance. Around 1993, Elmendorf, Kriz, Mandell and May introduced the *S-modules* [**26**] and Jeff Smith gave the first talks about *symmetric spectra*; the details of the model structure were later worked out and written up by Hovey, Shipley and Smith [**36**]. In 1995, Lydakis [**47**] independently discovered and studied the smash product for $\mathbf{\Gamma}$-spaces (in the sense of Segal [**73**]), and a little later he developed model structures and smash product for *simplicial functors* [**48**]. Except for the *S-modules* of Elmendorf, Kriz, Mandell and May, all other known models for spectra with nice smash product have a very similar flavor; they all arise as categories of continuous (or simplicial), space-valued functors from a symmetric monoidal indexing category, and the smash product is a convolution product (defined as a left Kan extension), which had much earlier been studied by the category theorist Day [**19**]. This unifying context was made explicit by Mandell, May, Schwede and Shipley in [**53**], where another example, the *orthogonal spectra* were first worked out in detail. The different approaches to spectra categories with smash product have been generalized and adapted to equivariant homotopy theory [**21, 51, 52**] and motivic homotopy theory [**22, 37, 38**].

Why symmetric spectra? The author is a big fan of symmetric spectra; two important reasons are that symmetric spectra are easy to define and require the least amount of symmetry among the models of the stable homotopy category with smash product. A consequence of the second point is that many interesting homotopy types can be written down explicitly and in closed form. We give examples of this in Section I.1.1 right after the basic definitions, among these are the sphere spectrum, suspension spectra, Eilenberg-Mac Lane spectra, Thom spectra such as MO, MSO and MU and topological K-theory spectra.

Another consequence of 'minimal symmetry' requirements is that whenever someone writes down or constructs a model for a homotopy type in one of the other worlds of spectra, then we immediately get a model as a symmetric spectrum by applying one of the 'forgetful' functors from spectra with more symmetries which we recall in Section I.7. In fact, symmetric spectra have a certain universal property (see Shipley's paper [**76**]), making them 'initial' among stable model categories with a compatible smash product.

There are already good sources available which explain the stable homotopy category, and there are many research papers and at least one book devoted to structured ring spectra. However, my experience is that for students learning the subject it is hard to reconcile the treatment of the stable homotopy category as given, for example, in Adams' notes [**2**], with the more recent model category approaches to, say, S-modules or symmetric spectra. So one aim of this book is to provide a source where one can learn about the triangulated stable homotopy category and about stable model categories and a good point-set level smash product with just one notion of what a spectrum is.

The monograph [**26**] by Elmendorf, Kriz, Mandell and May develops the theory of one of the competing frameworks, the S-modules, in detail. It has had a big impact and is widely used, for example because many standard tools can simply be quoted from that book. The theory of symmetric spectra is by now highly developed, but the results are spread over many research papers. The aim of this book is to collect basic facts in one place, thus providing an alternative to [**26**].

Prerequisites. As a general principle, I assume knowledge of basic algebraic topology and unstable homotopy theory. I will develop in parallel the theory of symmetric spectra based on topological spaces (compactly generated and weak Hausdorff) and simplicial sets. Whenever simplicial sets are used, I assume basic knowledge of simplicial homotopy theory, as found for example in the books of Goerss and Jardine [**30**] or May [**55**]. However, the use of simplicial sets is often convenient but hardly ever essential, so not much understanding is lost by just thinking about topological spaces throughout.

On the other hand, no prior knowledge of *stable* homotopy theory is assumed. In particular, we define the stable homotopy category using symmetric spectra and develop its basic properties from scratch.

From Chapter III on I will freely use the language of Quillen's model categories and basic results of homotopical algebra. The original source is Quillen's monograph [**62**], a good introduction is the article [**23**] by Dwyer and Spalinski, and Hovey's book [**35**] is a thorough, extensive treatment.

Organization. We organize the book into chapters, each chapter into sections and some sections into subsections. The numbering scheme for referring to definitions, theorems, examples etc. is as follows. If we refer to something in the same chapter, then the reference number consists only of the arabic section number and then a running number for all kinds of environments. If the reference is to another chapter, then we add the roman chapter number in front. So 'Lemma 3.14' refers to a Lemma in Section 3 of the same chapter, with running number 14, while 'Example I.2.21' is an example from the second section of the first chapter, with running number 21.

Each chapter has a section containing exercises, which follow a seperate numbering scheme, namely the letter 'E' followed by the roman chapter number and then a running number for the exercises in that chapter. So 'Exercise E.I.13' refers to the 13th exercise in Chapter I.

In the first chapter we introduce the basic concepts of a symmetric spectrum and symmetric ring spectrum and then, before developing any extensive theory, discuss lots of examples. [stable equivalences] There is a section on the smash product where we develop its basic formal and homotopical properties. One of the few points where symmetric spectra are more complicated than other frameworks is that the usual homotopy groups can be somewhat pathological. So we spend the last section of the first chapter on the structure of homotopy groups and the notion of semistable symmetric spectra.

The second chapter is devoted to the stable homotopy category. We develop some basic theory around the stable homotopy category, such as the triangulated structure, derived smash product, homotopy (co-)limits, Postnikov sections, localization and completion, and discuss the Spanier-Whitehead category, Moore spectra and finite spectra and (Bousfield) localization and completion. In Section II.10 we discuss the mod-p Steenrod algebra and give a glimpse at the mod-p Adams spectral sequence.

In Chapter III model structures enter the scene. We start by establishing the various level model structures (projective, flat, injective, and their positive versions) for symmetric spectra of spaces and simplicial sets, and then discuss the associated, more important, stable model structures. We also develop the model structures for modules over a fixed symmetric ring spectrum and for algebras over an operad of simplicial sets. The latter includes the stable model structures for symmetric ring spectra and for commutative symmetric ring spectra.

As a general rule, I do not attribute credit for definitions and theorems in the body of the text. Instead, there is a section 'History and credits' at the end of each chapter, where I summarize, to the best of my knowledge, who contributed what. Additions and corrections are welcome.

Some conventions. Let us fix some terminology and enact several useful conventions. We think that some slight abuse of language and notation can often make statements more transparent, but when we allow ourselves such imprecision we feel obliged to state them clearly here, at the risk of being picky.

We denote by $\mathbf{T}$ the category of pointed, compactly generated topological spaces. For us a compactly generated space is a Kelley space which is also weak Hausdorff; we review the definitions and collect various properties of compactly generated spaces in Section A.2. A *map* between topological spaces always refers to a *continuous* map, unless explicitly stated otherwise. Similarly, an *action* of a group on a space refers to a *continuous action*. We denote by $\mathbf{sS}$ the category of pointed simplicial sets. We review the definitions and collect various properties of simplicial sets in Section A.3.

It will be convenient to define the n-sphere S^n as the one-point compactification of n-dimensional euclidian space $\mathbb{R}^n$, with the point at infinity as the basepoint. We will sometimes need to identify the 1-sphere with the space $|\Delta[1]/\partial\Delta[1]|$, the geometric realization of the simplicial 1-simplex $\Delta[1]$ modulo its boundary. The precise identifications do not matter, but for definiteness we fix a homeomorphism now. The realization $|\Delta[1]/\partial\Delta[1]|$ is canonically homeomorphic to the quotient space of the topological 1-simplex $\underline{\Delta}[1] = \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0, x+y=1\}$ with its endpoints identified. Our preferred homeomorphism is

$$\mathbf{t} \ : \ \underline{\Delta}[1]/\partial\underline{\Delta}[1] \ \xrightarrow{\cong} \ S^1 \ , \quad (x,y) \mapsto x/y - y/x \ . \tag{0.1}$$

Here the understanding is that the formula describes the function on the open simplex (which is mapped homeomorphically to $\mathbb{R}$), and that the map extends continuously to the quotient space by sending the identified endpoints to the point at infinity in S^1.

For $n \geq 0$, the symmetric group Σ_n is the group of bijections of the set $\{1, 2, \dots, n\}$; in particular, Σ_0 consists only of the identity of the empty set. It will often be convenient to identify the product group $\Sigma_n \times \Sigma_m$ with the subgroup of Σ_{n+m} of those permutations which take the sets $\{1, \dots, n\}$ and

$\{n+1,\dots,n+m\}$ to themselves. Whenever we do so, we implicitly use the monomorphism

$$+ \ : \ \Sigma_n \times \Sigma_m \ \longrightarrow \ \Sigma_{n+m} \ , \quad (\tau,\kappa) \mapsto \tau+\kappa$$

given by

$$(\tau+\kappa)(i) \ = \ \begin{cases} \tau(i) & \text{for } 1 \leq i \leq n, \\ \kappa(i-n)+n & \text{for } n+1 \leq i \leq n+m. \end{cases}$$

We let the symmetric group Σ_n act from the left on $\mathbb{R}^n$ by permuting the coordinates, i.e., $\gamma(x_1,\dots,x_n) = (x_{\gamma^{-1}(1)},\dots,x_{\gamma^{-1}(n)})$. This action compactifies to an action on S^n which fixes the basepoint. The 'canonical' linear isomorphism

$$\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^{n+m} \ , \quad ((x_1,\dots,x_n),(y_1,\dots,y_m)) \ \mapsto \ (x_1,\dots,x_n,y_1,\dots,y_m)$$

induces a homeomorphism $S^n \wedge S^m \longrightarrow S^{n+m}$ which is equivariant with respect to the action of the group $\Sigma_n \times \Sigma_m$, acting on the target by restriction from Σ_{n+m}.

The topological spaces we consider are usually pointed, and we use the notation $\pi_n X$ for the n-th homotopy group with respect to the distinguished basepoint, which we do not record in the notation.

We will often use the 'exponential law', i.e., the adjunction between smash product and mapping spaces. Let us fix a few conventions, in particular when we adjoint spheres. For based spaces (or simplicial sets) K, X and Z we define the *adjunction bijection*

$$a \ : \ \mathrm{map}(X \wedge K, Z) \ \cong \ \mathrm{map}(X, \mathrm{map}(K,Z)) \tag{0.2}$$

by $(a(f)(x))(k) = f(x \wedge k)$, where $f : X \wedge K \longrightarrow Z$ is a based continuous (or simplicial) map, $x \in X$ and $k \in K$.

The m-fold *loop space* of a based space (or simplicial set) Z is the space (respectively simplicial set) $\Omega^m Z = \mathrm{map}(S^m, Z)$. We use the adjunction bijection and the homeomorphism $S^m \wedge S^n \cong S^{m+n}$ to identify $\Omega^m(\Omega^n Z) = \mathrm{map}(S^m, \mathrm{map}(S^n,Z))$ with $\Omega^{m+n}Z = \mathrm{map}(S^{m+n},Z)$ without further notice. In particular, we can, and will, identify Ω^m with the m-fold iterate of Ω. The m-th *homotopy group* of a based space Z is the set of based homotopy classes of based maps $S^m \longrightarrow Z$ or, equivalently, the set of path components of the mapping space $\Omega^m Z = \mathrm{map}(S^m,Z)$. This has a natural group structure for $m \geq 1$, which is commutative for $m \geq 2$. Looping a space shifts its homotopy groups; more precisely, the adjunction bijection passes to a bijection

$$\pi_k(\Omega^m Z) = [S^k, \mathrm{map}(S^m,Z)] \ \xrightarrow{\ a\ } \ [S^{k+m}, Z] = \pi_{k+m} Z \tag{0.3}$$

on homotopy classes. So the map a takes the homotopy class of a based map $f : S^k \longrightarrow \Omega^m Z$ to the homotopy class of $a(f) : S^{k+m} \longrightarrow Z$ given by $a(f)(t \wedge s) = f(t)(s)$ for $t \in S^k$, $s \in S^m$.

Limits and colimits. Limits and colimits in a category are hardly ever unique, but the universal property which they enjoy makes then 'unique up to canonical isomorphism'. We want to fix our language for talking about this unambiguously. We recall that a *colimit* of a functor $F : I \longrightarrow \mathcal{C}$ is a pair $(\bar{F},\kappa)$ consisting of an object $\bar{F}$ of $\mathcal{C}$ and a natural transformation $\kappa : F \longrightarrow c\bar{F}$ from F to the constant functor with value $\bar{F}$ which is initial among all natural transformations from F to constant functors. We often follow the standard abuse of language and call the object $\bar{F}$ a colimit, or even *the* colimit, of the functor F and denote it by $\mathrm{colim}_I\, F$. When we need to refer to the natural transformation κ which is part of the data of a colimit, we refer to the component $\kappa_i : F(i) \longrightarrow \mathrm{colim}_I\, F$ at an object $i \in I$ as the *canonical morphism* from the object $F(i)$ to the colimit. Dually for limits. [end, coends]

[use of naive homotopy groups $\hat{\pi}_*$ versus π_* for true homotopy groups. The more important concept deserves the simpler name.]

Remark 0.4 (Manipulation rules for coordinates)**.** Natural numbers occurring as levels of a symmetric spectrum or as dimensions of homotopy groups are really placeholders for sphere coordinates. The role of the symmetric group actions on the spaces of a symmetric spectrum is to keep track of how such coordinates are shuffled. Permutations will come up over and over again in constructions and results about symmetric spectra, and there is a very useful small set of rules which predict when to expect permutations. I recommend being very picky about the order in which dimensions or levels occur when performing constructions with symmetric spectra, as this predicts necessary permutations and helps to prevent mistakes. Sometimes missing a permutation just means missing a sign; in particular missing an even permutation may not have any visible effect. But in general the issue is more serious; for symmetric spectra which are not semistable, missing a permutation typically misses a nontrivial operation.

A first example of this are the centrality and commutativity conditions for symmetric ring spectra, which use shuffle permutations $\chi_{n,1}$ and $\chi_{n,m}$. A good way of remembering when to expect a shuffle is to carefully distinguish between indices such as $n+m$ and $m+n$. Of course these two numbers are equal, but the fact that one arises naturally instead of the other reminds us that a shuffle permutation should be inserted. A shuffle required whenever identifying $n+m$ with $m+n$ is just one rule, and here are some more.

Main rule: When manipulating expressions which occur as levels of symmetric spectra or dimensions of spheres, be very attentive for how these expressions arise naturally and when you use the basic rules of arithmetic of natural numbers. When using the basic laws of addition and multiplication of natural numbers in such a context, add permutations according to the following rules (i)-(v).

(i) Do not worry about associativity of addition or multiplication, or the fact that 0 respectively 1 are units for those operations. No permutations are required.
(ii) Whenever using commutativity of addition as in $n+m=m+n$, add a shuffle permutation $\chi_{n,m} \in \Sigma_{n+m}$.
(iii) Whenever using commutativity of multiplication as in $nm=mn$, add a *multiplicative shuffle* $\chi^{\times}_{n,m} \in \Sigma_{nm}$ defined by

$$\chi^{\times}_{n,m}(j+(i-1)n) \;=\; i+(j-1)m$$

for $1 \le j \le n$ and $1 \le i \le m$.
(iv) Do not worry about left distributivity as in $p(n+m)=pn+pm$. No permutation is required.
(v) Whenever using right distributivity as in $(n+m)q=nq+mq$, add the permutation

$$(\chi^{\times}_{q,n} \times \chi^{\times}_{q,m}) \circ \chi^{\times}_{n+m,q} \;\in\; \Sigma_{(n+m)q} \;.$$

Rule (v) also requires us to throw in permutations whenever we identify a product nq with an iterated sum $q+\cdots+q$ (n copies) since we use right distributivity in the process. However, no permutations are needed when instead identifying nq with a sum of q copies of n, since that only uses left distributivity.

The heuristic rules (i) through (v) above are a great help in guessing when to expect coordinate or level permutations when working with symmetric spectra. But the rules are more than heuristics, and are based on the following rigorous mathematics. Typically, there are 'coordinate free' constructions in the background (compare Exercise E.I.5) which are indexed by finite sets A which are not identified with any of the standard finite sets $\mathbf{n}=\{1,\dots,n\}$. The outcome of such constructions may naturally be indexed by sets which are built by forming disjoint unions or products. The permutations arise because in contrast to the arithmetic rules for $+$ and $\cdot$, their analogues for disjoint union and cartesian product of sets only holds up to isomorphism, and one can arrange to make some, but not all, of the required isomorphisms be identity maps.

In more detail, when we want to restrict a 'coordinate free' construction to symmetric spectra, we specialize to standard finite sets $\mathbf{n}$; however, if the coordinate free construction involves disjoint union or cartesian product, we need to identify the unions or products of standard finite sets in a consistent way with the standard finite set of the same cardinality. A consistent way to do that amounts to what is called a structure of *bipermutative category* on the category of standard finite sets. So we define binary functors $+$ and $\cdot$ on standard finite sets resembling addition and multiplication of natural numbers as closely as possible.

We let $\mathcal{F}in$ denote the category of standard finite sets whose objects are the sets $\mathbf{n}$ for $n \ge 0$ and whose morphisms are all set maps. We define the sum functor $+ : \mathcal{F}in \times \mathcal{F}in \longrightarrow \mathcal{F}in$ by addition on objects and by 'disjoint union' on morphisms. More precisely, for morphisms $f : \mathbf{n} \longrightarrow \mathbf{n}'$ and $g : \mathbf{m} \longrightarrow \mathbf{m}'$ we define

$f+g : \mathbf{n}+\mathbf{m} \longrightarrow \mathbf{n}'+\mathbf{m}'$ by

$$(f+g)(i) \;=\; \begin{cases} f(i) & \text{if } 1 \leq i \leq n, \text{ and} \\ g(i-n)+n' & \text{if } n+1 \leq i \leq n+m. \end{cases}$$

This operation is strictly associative and the empty set $\mathbf{0}$ is a strict unit. The symmetry isomorphism is the shuffle map $\chi_{n,m} : \mathbf{n}+\mathbf{m} \longrightarrow \mathbf{m}+\mathbf{n}$.

We define the product functor $\cdot : \mathcal{F}in \times \mathcal{F}in \longrightarrow \mathcal{F}in$ by multiplication on objects and by 'cartesian product' on morphisms. To make sense of this we have to linearly order the product of the sets $\mathbf{n}$ and $\mathbf{m}$. There are two choices which are more obvious than others, namely lexicographically with either the first or the second coordinate defined as the more important one. Both choices work fine, and we will prefer the first coordinate. More precisely, for morphisms $f : \mathbf{n} \longrightarrow \mathbf{n}'$ and $g : \mathbf{m} \longrightarrow \mathbf{m}'$ we define $f \cdot g : \mathbf{n} \cdot \mathbf{m} \longrightarrow \mathbf{n}' \cdot \mathbf{m}'$ by

$$(f \cdot g)(j+(i-1)n) \;=\; f(j)+(g(i)-1)n'$$

for $1 \leq j \leq n$ and $1 \leq i \leq m$. The product $\cdot$ is also strictly associative and the set $\mathbf{1}$ is a strict unit. The commutativity isomorphism is the multiplicative shuffle $\chi^{\times}_{nm} : \mathbf{n} \cdot \mathbf{m} \longrightarrow \mathbf{m} \cdot \mathbf{n}$.

This choice of ordering the product of $\mathbf{n}$ and $\mathbf{m}$ has the effect of making $\mathbf{n} \cdot \mathbf{m}$ 'naturally' the same as $\mathbf{n}+\cdots+\mathbf{n}$ (m copies), because we have

$$f \cdot \mathrm{Id}_{\mathbf{m}} \;=\; f+\cdots+f \quad (m \text{ copies}).$$

Since $\mathbf{p}\cdot\mathbf{k}$ 'is' $\mathbf{p}+\cdots+\mathbf{p}$ (k times), we can take the left distributivity isomorphism $\mathbf{p}\cdot(\mathbf{n}+\mathbf{m}) = (\mathbf{p}\cdot\mathbf{n})+(\mathbf{p}\cdot\mathbf{m})$ as the identity (compare rule (iv)).

In contrast, $\mathrm{Id}_{\mathbf{n}} \cdot g$ is in general *not* equal to $g+\cdots+g$ (n copies), but rather we have

$$\mathrm{Id}_{\mathbf{n}} \cdot g \;=\; \chi^{\times}_{m',n}(g+\cdots+g)\chi^{\times}_{n,m}$$

for a morphism $g : \mathbf{m} \longrightarrow \mathbf{m}'$. However, then right distributivity isomorphism cannot be taken as the identity; since the coherence diagram

$$\begin{tikzcd}
\mathbf{q}\cdot(\mathbf{n}+\mathbf{m}) \arrow[r, "\chi^{\times}_{q,n+m}"] \arrow[d, equal, "\text{left dist.}"'] & (\mathbf{n}+\mathbf{m})\cdot\mathbf{q} \arrow[d, "\text{right dist.}"] \\
\mathbf{q}\cdot\mathbf{n}+\mathbf{q}\cdot\mathbf{m} \arrow[r, "\chi^{\times}_{q,n}+\chi^{\times}_{q,m}"'] & \mathbf{n}\cdot\mathbf{q}+\mathbf{m}\cdot\mathbf{q}
\end{tikzcd}$$

is supposed to commute, we are forced to define the right distributivity isomorphism $(\mathbf{n}+\mathbf{m})\cdot\mathbf{q} \cong (\mathbf{n}\cdot\mathbf{q})+(\mathbf{m}\cdot\mathbf{q})$ as $(\chi^{\times}_{q,n} \times \chi^{\times}_{q,m}) \circ \chi^{\times}_{n+m,q}$, which explains rule (v) above.

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CHAPTER I

Basics

1. Symmetric spectra

Definition 1.1. A *symmetric spectrum* consists of the following data:

- a sequence of pointed spaces X_n for $n \geq 0$
- a basepoint preserving continuous left action of the symmetric group Σ_n on X_n for each $n \geq 0$
- based maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ for $n \geq 0$.

This data is subject to the following condition: for all $n, m \geq 0$, the composite

$$(1.2)\qquad X_n \wedge S^m \xrightarrow{\sigma_n \wedge \mathrm{Id}} X_{n+1} \wedge S^{m-1} \xrightarrow{\sigma_{n+1} \wedge \mathrm{Id}} \cdots \xrightarrow{\sigma_{n+m-2} \wedge \mathrm{Id}} X_{n+m-1} \wedge S^1 \xrightarrow{\sigma_{n+m-1}} X_{n+m}$$

is $\Sigma_n \times \Sigma_m$-equivariant. Here the symmetric group Σ_m acts by permuting the coordinates of S^m, and $\Sigma_n \times \Sigma_m$ acts on the target by restriction of the Σ_{n+m}-action. We often denote the composite map (1.2) by σ^m, with the understanding that σ^0 is the identity map. We refer to the space X_n as the *n-th level* of the symmetric spectrum X.

A *morphism* $f : X \longrightarrow Y$ of symmetric spectra consists of Σ_n-equivariant based maps $f_n : X_n \longrightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge S^1)$ for all $n \geq 0$.

By our standing hypothesis, 'space' in the previous definition means a *compactly generated weak Hausdorff space*, compare Appendix A.2. The category of symmetric spectra is denoted by $\mathcal{S}p$; when we need to emphasize that we use spaces (as opposed to simplicial set), we add the index $\mathbf{T}$ (which denotes the category of compactly generated weak Hausdorff space) and use the notation $\mathcal{S}p_{\mathbf{T}}$. Symmetric spectra of simplicial sets, to be defined in 3.1 below, will accordingly be denoted $\mathcal{S}p_{\mathbf{sS}}$.

Definition 1.3. A *symmetric ring spectrum* R consists of the following data:

- a sequence of pointed spaces R_n for $n \geq 0$
- a basepoint preserving continuous left action of the symmetric group Σ_n on R_n for each $n \geq 0$
- $\Sigma_n \times \Sigma_m$-equivariant *multiplication maps*
$$\mu_{n,m} \;:\; R_n \wedge R_m \;\longrightarrow\; R_{n+m}$$
for $n, m \geq 0$, and
- two *unit maps*
$$\iota_0 \;:\; S^0 \;\longrightarrow\; R_0 \quad \text{and} \quad \iota_1 \;:\; S^1 \;\longrightarrow\; R_1\ .$$

This data is subject to the following conditions:

(Associativity) The square

$$\begin{tikzcd}
R_n \wedge R_m \wedge R_p \arrow[r, "\mathrm{Id} \wedge \mu_{m,p}"] \arrow[d, "\mu_{n,m} \wedge \mathrm{Id}"'] & R_n \wedge R_{m+p} \arrow[d, "\mu_{n,m+p}"] \\
R_{n+m} \wedge R_p \arrow[r, "\mu_{n+m,p}"'] & R_{n+m+p}
\end{tikzcd}$$

commutes for all $n, m, p \geq 0$.

(Unit) The two composites

$$R_n \cong R_n \wedge S^0 \xrightarrow{R_n \wedge \iota_0} R_n \wedge R_0 \xrightarrow{\mu_{n,0}} R_n$$

$$R_n \cong S^0 \wedge R_n \xrightarrow{\iota_0 \wedge R_n} R_0 \wedge R_n \xrightarrow{\mu_{0,n}} R_n$$

are the identity for all $n \geq 0$.

(Centrality) The diagram

$$\begin{tikzcd}
R_n \wedge S^1 \arrow[r, "R_n \wedge \iota_1"] \arrow[d, "\text{twist}"'] & R_n \wedge R_1 \arrow[r, "\mu_{n,1}"] & R_{n+1} \arrow[d, "\chi_{n,1}"] \\
S^1 \wedge R_n \arrow[r, "\iota_1 \wedge R_n"'] & R_1 \wedge R_n \arrow[r, "\mu_{1,n}"'] & R_{1+n}
\end{tikzcd}$$

commutes for all $n \geq 0$. Here $\chi_{n,m} \in \Sigma_{n+m}$ denotes the shuffle permutation which moves the first n elements past the last m elements, keeping each of the two blocks in order; in formulas,

$$\chi_{n,m}(i) \ = \ \begin{cases} i+m & \text{for } 1 \leq i \leq n, \\ i-n & \text{for } n+1 \leq i \leq n+m. \end{cases} \tag{1.4}$$

A *morphism* $f : R \longrightarrow S$ of symmetric ring spectra consists of Σ_n-equivariant based maps $f_n : R_n \longrightarrow S_n$ for $n \geq 0$, which are compatible with the multiplication and unit maps in the sense that $f_{n+m} \circ \mu_{n,m} = \mu_{n,m} \circ (f_n \wedge f_m)$ for all $n, m \geq 0$, and $f_0 \circ \iota_0 = \iota_0$ and $f_1 \circ \iota_1 = \iota_1$.

A symmetric ring spectrum R is *commutative* if the square

$$\begin{tikzcd}
R_n \wedge R_m \arrow[r, "\text{twist}"] \arrow[d, "\mu_{n,m}"'] & R_m \wedge R_n \arrow[d, "\mu_{m,n}"] \\
R_{n+m} \arrow[r, "\chi_{n,m}"'] & R_{m+n}
\end{tikzcd}$$

commutes for all $n, m \geq 0$. Note that this commutativity diagram implies the centrality condition above.

Definition 1.5. A *right module* M over a symmetric ring spectrum R consists of the following data:

- a sequence of pointed spaces M_n for $n \geq 0$
- a basepoint preserving continuous left action of the symmetric group Σ_n on M_n for each $n \geq 0$, and
- $\Sigma_n \times \Sigma_m$-equivariant *action maps* $\alpha_{n,m} : M_n \wedge R_m \longrightarrow M_{n+m}$ for $n, m \geq 0$.

The action maps have to be associative and unital in the sense that the following diagrams commute

$$\begin{tikzcd}
M_n \wedge R_m \wedge R_p \arrow[r, "M_n \wedge \mu_{m,p}"] \arrow[d, "\alpha_{n,m} \wedge R_p"'] & M_n \wedge R_{m+p} \arrow[d, "\alpha_{n,m+p}"] \\
M_{n+m} \wedge R_p \arrow[r, "\alpha_{n+m,p}"'] & M_{n+m+p}
\end{tikzcd}
\qquad
\begin{tikzcd}
M_n \cong M_n \wedge S^0 \arrow[r, "M_n \wedge \iota_0"] \arrow[dr, equal] & M_n \wedge R_0 \arrow[d, "\alpha_{n,0}"] \\
 & M_n
\end{tikzcd}$$

for all $n, m, p \geq 0$. A *morphism* $f : M \longrightarrow N$ of right R-modules consists of Σ_n-equivariant based maps $f_n : M_n \longrightarrow N_n$ for $n \geq 0$, which are compatible with the action maps in the sense that $f_{n+m} \circ \alpha_{n,m} = \alpha_{n,m} \circ (f_n \wedge R_m)$ for all $n, m \geq 0$. We denote the category of right R-modules by mod- R.

Remark 1.6. We have stated the axioms for symmetric ring spectra in terms of a minimal amount of data and conditions. Now we put these conditions into perspective. We consider a symmetric ring spectrum R.

(i) It will be useful to have the following notation for *iterated multiplication maps*. For natural numbers $n_1, \ldots, n_i \geq 0$ we denote by

$$\mu_{n_1,\ldots,n_i} \; : \; R_{n_1} \wedge \ldots \wedge R_{n_i} \; \longrightarrow \; R_{n_1+\cdots+n_i}$$

the map obtained by composing multiplication maps smashed with suitable identity maps; by associativity, the parentheses in the multiplications don't matter. More formally we can define the iterated multiplication maps inductively, setting

$$\mu_{n_1,\ldots,n_i} \; = \; \mu_{n_1,n_2+\cdots+n_i} \circ (\mathrm{Id}_{R_{n_1}} \wedge \mu_{n_2,\ldots,n_i}) \; .$$

(ii) We can define higher-dimensional unit maps $\iota_m : S^m \longrightarrow R_m$ for $m \geq 2$ as the composite

$$S^m = S^1 \wedge \ldots \wedge S^1 \xrightarrow{\iota_1 \wedge \ldots \wedge \iota_1} R_1 \wedge \ldots \wedge R_1 \xrightarrow{\mu_{1,\ldots,1}} R_m \; .$$

Centrality then implies that ι_m is Σ_m-equivariant, and it implies that the diagram

$$\begin{tikzcd} R_n \wedge S^m \arrow[r, "R_n \wedge \iota_m"] \arrow[d, "\mathrm{twist}"'] & R_n \wedge R_m \arrow[r, "\mu_{n,m}"] & R_{n+m} \arrow[d, "\chi_{n,m}"] \\ S^m \wedge R_n \arrow[r, "\iota_m \wedge R_n"'] & R_m \wedge R_n \arrow[r, "\mu_{m,n}"'] & R_{m+n} \end{tikzcd}$$

commutes for all $n, m \geq 0$, generalizing the original centrality condition.

(iii) As the terminology suggests, the symmetric ring spectrum R has an underlying symmetric spectrum. In fact, the multiplication maps $\mu_{n,m}$ make R into a right module over itself, and more generally, every right R-module M has an underlying symmetric spectrum as follows. We keep the spaces M_n and symmetric group actions and define the missing structure maps $\sigma_n : M_n \wedge S^1 \longrightarrow M_{n+1}$ as the composite $\alpha_{n,1} \circ (M_n \wedge \iota_1)$. Associativity implies that the iterated structure map $\sigma^m : M_n \wedge S^m \longrightarrow M_{n+m}$ equals the composite

$$M_n \wedge S^m \xrightarrow{M_n \wedge \iota_m} M_n \wedge R_m \xrightarrow{\alpha_{n,m}} M_{n+m} \; .$$

So the iterated structure map is $\Sigma_n \times \Sigma_m$-equivariant by part (ii) and the equivariance hypothesis on $\alpha_{n,m}$, and we have in fact obtained a symmetric spectrum.

The forgetful functors which associates to a symmetric ring spectrum or module spectrum its underlying symmetric spectrum have left adjoints. We will construct the left adjoints in Example 5.27 below after introducing the smash product of symmetric spectra. The left adjoints associate to a symmetric spectrum X the 'free R-module' $X \wedge R$ respectively the 'free symmetric ring spectrum' TX generated by X, which we will refer to as the *tensor algebra*.

(iv) Using the internal smash product of symmetric spectra introduced in Section 5, we can identify the 'explicit' definition of a symmetric ring spectrum which we just gave with a more 'implicit' definition of a symmetric spectrum R together with morphisms $\mu : R \wedge R \longrightarrow R$ and $\iota : \mathbb{S} \longrightarrow R$ (where $\mathbb{S}$ is the sphere spectrum, see Example 1.8) which are suitably associative and unital. The 'explicit' and 'implicit' definitions of symmetric ring spectra coincide in the sense that they define isomorphic categories, see Theorem 5.25.

Primary invariants of a symmetric spectrum are its homotopy groups, which come in two flavors as 'naive' and 'true' homotopy groups. The former kind is defined directly from the homotopy groups of the spaces in a spectrum: the *k-th naive homotopy group* of a symmetric spectrum X is defined as the colimit

$$\hat{\pi}_k X \; = \; \mathrm{colim}_n \; \pi_{k+n} X_n$$

taken over the *stabilization maps* $\iota : \pi_{k+n} X_n \longrightarrow \pi_{k+n+1} X_{n+1}$ defined as the composite

$$(1.7) \qquad \pi_{k+n} X_n \xrightarrow{-\wedge S^1} \pi_{k+n+1} \left(X_n \wedge S^1 \right) \xrightarrow{(\sigma_n)_*} \pi_{k+n+1} X_{n+1} \; .$$

For large enough n, the set $\pi_{k+n} X_n$ has a natural abelian group structure and the stabilization maps are homomorphisms, so the colimit $\hat{\pi}_k X$ inherits a natural abelian group structure.

As will hopefully become clear later, these naive homotopy groups are often 'wrong'; for example, the category which one obtains by localizing at the class of $\hat{\pi}_*$-isomorphisms is *not* equivalent to the stable homotopy category as we discuss it in Chapter II. However, we need the naive homotopy groups to define the more important *true homotopy groups* (see Definition 6.1 below), and also as a calculational tool to get at the true homotopy groups.

1.1. Basic examples. Before developing any more theory, we give some examples of symmetric spectra and symmetric ring spectra which represent prominent stable homotopy types. We discuss the sphere spectrum (1.8), suspension spectra (1.13), Eilenberg-Mac Lane spectra (1.14), Thom spectra (1.16, 1.18) and spectra representing topological K-theory 1.20. It is a nice feature of symmetric spectra that one can explicitly write down these examples in closed form with all the required symmetries. We also give, whenever possible, the naive homotopy groups of these symmetric spectra. It will turn out that all the examples in the section are in fact *semistable* (to be defined in Definition 3.14) and hence the naive homotopy groups coincide with the more important *true homotopy groups* (to be defined in Definition 6.1).

Example 1.8 (Sphere spectrum)**.** The symmetric *sphere spectrum* $\mathbb{S}$ is given by $\mathbb{S}_n = S^n$, where the symmetric group permutes the coordinates and $\sigma_n : S^n \wedge S^1 \longrightarrow S^{n+1}$ is the canonical homeomorphism. This is a commutative symmetric ring spectrum with identity as unit map and the canonical homeomorphism $S^n \wedge S^m \longrightarrow S^{n+m}$ as multiplication map. The sphere spectrum is the *initial* symmetric ring spectrum: if R is any symmetric ring spectrum, then a unique morphism of symmetric ring spectra $\mathbb{S} \longrightarrow R$ is given by the collection of unit maps $\iota_n : S^n \longrightarrow R_n$ (compare 1.6 (ii)). Being initial, the sphere spectrum plays the same formal role for symmetric ring spectra as the integers $\mathbb{Z}$ play for rings. This justifies the notation '$\mathbb{S}$' using the `\mathbb` font. The category of right $\mathbb{S}$-modules is isomorphic to the category of symmetric spectra, via the forgetful functor mod-$\mathbb{S} \longrightarrow \mathcal{S}p$. Indeed, if X is a symmetric spectrum then the associativity condition shows that there is at most one way to define action maps

$$\alpha_{n,m} \ : \ X_n \wedge S^m \ \longrightarrow \ X_{n+m} \ ,$$

namely as the iterated structure map σ^m, and these do define the structure of right $\mathbb{S}$-module on X.

The naive homotopy group $\hat{\pi}_k\mathbb{S} = \operatorname{colim}_n \pi_{k+n}S^n$ is called the *k-th stable homotopy group of spheres*, or the *k-th stable stem*, and will be denoted π_k^{s}. Since S^n is $(n-1)$-connected, the group π_k^{s} is trivial for negative values of k. The degree of a self-map of a sphere provides an isomorphism $\pi_0^{\mathrm{s}} \cong \mathbb{Z}$. For $k \geq 1$, the homotopy group π_k^{s} is finite. This is a direct consequence of Serre's calculation of the homotopy groups of spheres modulo torsion, which we recall without giving a proof, and Freudenthal's suspension theorem [justify].

Theorem 1.9 (Serre)**.** *Let $m > n \geq 1$. Then*

$$\pi_m S^n = \begin{cases} \text{(finite group)} \oplus \mathbb{Z} & \textit{if } n \textit{ is even and } m = 2n-1 \\ \text{(finite group)} & \textit{else.} \end{cases}$$

Thus for $k \geq 1$, the stable stem $\pi_k^{\mathrm{s}} = \hat{\pi}_k\mathbb{S}$ is finite.

As a concrete example, we inspect the colimit system defining π_1^{s}, the first stable stem. Since the universal cover of S^1 is the real line, which is contractible, the theory of covering spaces shows that the groups $\pi_n S^1$ are trivial for $n \geq 2$. The Hopf map

$$\eta \ : \ S^3 \subseteq \mathbb{C}^2 \backslash \{0\} \xrightarrow{\text{proj.}} \mathbb{C}\mathrm{P}^1 \cong S^2$$

is a locally trivial fiber bundle with fiber S^1, so it gives rise to a long exact sequence of homotopy groups. Since the fiber S^1 has no homotopy above dimension one, the group $\pi_3 S^2$ is free abelian of rank one, generated by the class of η. By Freudenthal's suspension theorem the suspension homomorphism $- \wedge S^1 : \pi_3 S^2 \longrightarrow \pi_4 S^3$ is surjective and from $\pi_4 S^3$ on the suspension homomorphism is an isomorphism. So the first stable stem π_1^{s} is cyclic, generated by the image of η, and its order equals the order of the suspension of η. On the one hand, η itself is stably essential, since the Steenrod operation Sq^2 acts non-trivially on the mod-2 cohomology of the mapping cone of η, which is homeomorphic to $\mathbb{C}\mathrm{P}^2$ (we spell this out in more detail in Example II.10.11) below.

On the other hand, twice the suspension of η is null-homotopic. To see this we consider the commutative square

$$\begin{tikzcd}
(x,y) \arrow[d, mapsto] & S^3 \arrow[r, "\eta"] \arrow[d] & \mathbb{C}\mathrm{P}^1 \arrow[d] & {[x:y]} \arrow[d, mapsto] \\
(\bar{x},\bar{y}) & S^3 \arrow[r, "\eta"'] & \mathbb{C}\mathrm{P}^1 & {[\bar{x}:\bar{y}]}
\end{tikzcd}$$

in which the vertical maps are induced by complex conjugation in both coordinates of $\mathbb{C}^2$. The left vertical map has degree 1, so it is homotopic to the identity, whereas complex conjugation on $\mathbb{C}\mathrm{P}^1 \cong S^2$ has degree -1. So $(-1)\circ\eta$ is homotopic to η. Thus the suspension of η is homotopic to the suspension of $(-1)\circ\eta$, which by the following lemma is homotopic to the negative of $\eta \wedge S^1$.

Lemma 1.10. *Let Y be a pointed space, $m \geq 0$ and $f : S^m \longrightarrow S^m$ a based map of degree k. Then for every homotopy class $x \in \pi_n(Y \wedge S^m)$ the classes $(\mathrm{Id}_Y \wedge f)_*(x)$ and $k \cdot x$ become equal in $\pi_{n+1}(Y \wedge S^{m+1})$ after one suspension.*

Proof. Let $d_k : S^1 \longrightarrow S^1$ be any pointed map of degree k. Then the maps $f \wedge S^1, S^m \wedge d_k : S^{m+1} \longrightarrow S^{m+1}$ have the same degree k, hence they are based homotopic. Suppose x is represented by $\varphi : S^n \longrightarrow Y \wedge S^m$. Then the suspension of $(Y \wedge f)_*(x)$ is represented by $(Y \wedge f \wedge S^1) \circ (\varphi \wedge S^1)$ which is homotopic to $(Y \wedge S^m \wedge d_k) \circ (\varphi \wedge S^1) = (\varphi \wedge S^1) \circ (S^n \wedge d_k)$. Precomposition with the degree k map $S^n \wedge d_k$ of S^{n+1} induces multiplication by k, so the last map represents the suspension of $k \cdot x$. □

The conclusion of Lemma 1.10 does not in general hold without the extra suspension, i.e., $(Y \wedge f)_*(x)$ need not equal $k \cdot x$ in $\pi_n(Y \wedge S^m)$: as we showed above, $(-1)\circ\eta$ is homotopic to η, which is *not* homotopic to $-\eta$ since η generates the infinite cyclic group $\pi_3 S^2$.

As far as we know, the stable homotopy groups of spheres don't follow any simple pattern. Much machinery of algebraic topology has been developed to calculate homotopy groups of spheres, both unstable and stable, but no one expects to ever get explicit formulae for all stable homotopy groups of spheres. The Adams spectral sequence based on mod-p cohomology (see Section II.10) and the Adams-Novikov spectral sequence based on MU (complex cobordism) or BP (the Brown-Peterson spectrum at a fixed prime p) are the most effective tools we have for explicit calculations as well as for discovering systematic phenomena.

Example 1.11 (Multiplication in the stable stems)**.** The stable stems $\pi_*^{\mathrm{s}} = \hat{\pi}_*\mathbb{S}$ form a graded commutative ring which acts on the naive and true homotopy groups of every other symmetric spectrum X. We denote the action simply by a 'dot'

$$\cdot \;:\; \hat{\pi}_k X \times \pi_l^{\mathrm{s}} \;\longrightarrow\; \hat{\pi}_{k+l} X \ .$$

The definition is essentially straightforward, but there is one subtlety in showing that the product is well-defined.

We first define the action of π_*^{s} on the naive homotopy groups $\hat{\pi}_* X$ of a symmetric spectrum X. Suppose $f : S^{k+n} \longrightarrow X_n$ and $g : S^{l+m} \longrightarrow S^m$ represent classes in $\hat{\pi}_k X$ respectively π_l^{s}. We denote by $f \cdot g$ the composite

$$S^{k+n+l+m} \xrightarrow{f \wedge g} X_n \wedge S^m \xrightarrow{\sigma^m} X_{n+m}$$

and then define

$$[f] \cdot [g] \;=\; (-1)^{nl} \cdot [f \cdot g] \tag{1.12}$$

in the group $\hat{\pi}_{k+l} X$. The sign can be explained by the principle that all natural number must occur in the 'natural order', compare Remark 0.4. In $f \cdot g$ the dimension of the sphere of origin is naturally $(k+n)+(l+m)$, but in order to represent an element of $\hat{\pi}_{k+l} X$ the numbers should occur in the order $(k+l)+(n+m)$. Hence a shuffle permutation is to be expected, and it enters in the disguise of the sign $(-1)^{ln}$.

We check that the multiplication is well-defined. If we replace $g : S^{l+m} \longrightarrow S^m$ by its suspension $g \wedge S^1$, then

$$f \cdot (g \wedge S^1) \;=\; \sigma^{m+1} \circ (f \wedge g \wedge S^1) \;=\; \sigma_{n+m} \circ (\sigma^m \wedge S^1) \circ (f \wedge g \wedge S^1) \;=\; \sigma_{n+m} \circ ((f \cdot g) \wedge S^1) \ .$$

Since the sign in the formula (1.12) does not change, the resulting stable class is independent of the representative g of the stable class in π_l^{s}. Independence of the representative for $\hat{\pi}_k X$ is slightly more subtle. If we replace $f : S^{k+n} \longrightarrow X_n$ by the representative $\sigma_n(f \wedge S^1) : S^{k+n+1} \longrightarrow X_{n+1}$, then $f \cdot g$ gets replaced by the lower horizontal composite in the commutative diagram

$$\begin{tikzcd}
S^{k+n+l+m+1} \arrow[r, "f\wedge g\wedge \mathrm{Id}"] \arrow[d, "\mathrm{Id}\wedge \chi_{l+m,1}"'] & X_n \wedge S^{m+1} \arrow[d, "X_n\wedge\chi_{m,1}"] & \\
S^{k+n+1+l+m} \arrow[r, "f\wedge \mathrm{Id}\wedge g"'] \arrow[rr, bend right, "\sigma_n(f\wedge S^1)\cdot g"'] & X_n \wedge S^{1+m} \arrow[r, "\sigma^{1+m}"'] & X_{n+1+m}
\end{tikzcd}$$

By Lemma 1.10 the map $X_n \wedge \chi_{m,1}$ induces multiplication by $(-1)^m$ on homotopy groups *after one suspension*. This cancels part of the sign $(-1)^{l+m}$ that is the effect of precomposition with the shuffle permutation $\chi_{l+m,1}$ on the left. So in the colimit $\hat{\pi}_{k+l}X$ we have

$$[\sigma_n(f \wedge S^1) \cdot g] \;=\; (-1)^l \cdot [\sigma^{m+1}(f \wedge g \wedge S^1)] \;=\; (-1)^l \cdot [f \cdot g] \ .$$

Since the dimension of $f \wedge S^1$ is one more than the dimension of f, the extra factor $(-1)^l$ makes sure that product $[f] \cdot [g]$ as defined in (1.12) is independent of the representative of the stable class $[f]$.

Now we verify that the dot product is biadditive. We only show the relation $(x + x') \cdot y = x \cdot y + x' \cdot y$, and additivity in y is similar. Suppose as before that $f, f' : S^{k+n} \longrightarrow X_n$ and $g : S^{l+m} \longrightarrow S^m$ represent classes in $\hat{\pi}_k X$ respectively π_l^{s}. Then the sum of f and f' in $\pi_{k+n}X_n$ is represented by the composite

$$S^{k+n} \xrightarrow{\text{pinch}} S^{k+n} \vee S^{k+n} \xrightarrow{f \vee f'} X_n \ .$$

In the square

$$\begin{tikzcd}
S^{k+l+n+m} \arrow[r, "1\wedge\chi_{l,n}\wedge 1"] \arrow[dd, "\text{pinch}"'] & S^{k+n+l+m} \arrow[d, "\text{pinch}\wedge\mathrm{Id}"'] \arrow[dr, "(f+f')\wedge g"] & \\
 & (S^{k+n} \vee S^{k+n}) \wedge S^{l+m} \arrow[r, "(f\vee f')\wedge g"'] & X_n \wedge S^m \\
S^{k+l+n+m} \vee S^{k+l+n+m} \arrow[r, "(1\wedge\chi_{l,n}\wedge 1)\vee(1\wedge\chi_{l,n}\wedge 1)"'] & S^{k+n+l+m} \vee S^{k+n+l+m} \arrow[u, "\cong"'] \arrow[ur, "(f\wedge g)\vee(f'\wedge g)"'] &
\end{tikzcd}$$

the right part commutes on the nose and the left square commutes up to homotopy. After composing with the iterated structure map $\sigma^m : X_n \wedge S^m \longrightarrow X_{n+m}$, the composite around the top of the diagram becomes $(f + f') \cdot g$, whereas the composite around the bottom represents $fg + f'g$. This proves additivity of the dot product in the left variable.

If we specialize to $X = \mathbb{S}$ then the product provides a biadditive graded pairing $\cdot : \pi_k^{\mathrm{s}} \times \pi_l^{\mathrm{s}} \longrightarrow \pi_{k+l}^{\mathrm{s}}$ of the stable homotopy groups of spheres. We claim that for every symmetric spectrum X the diagram

$$\begin{tikzcd}
\hat{\pi}_k X \times \pi_l^{\mathrm{s}} \times \pi_j^{\mathrm{s}} \arrow[r, "\cdot\times\mathrm{Id}"] \arrow[d, "\mathrm{Id}\times\cdot"'] & \hat{\pi}_{k+l} X \times \pi_j^{\mathrm{s}} \arrow[d, "\cdot"] \\
\hat{\pi}_k X \times \pi_{l+j}^{\mathrm{s}} \arrow[r, "\cdot"'] & \hat{\pi}_{k+l+j} X
\end{tikzcd}$$

commutes, so the product on the stable stems and the action on the homotopy groups of a symmetric spectrum are associative. After unraveling all the definitions, this associativity ultimately boils down to the equality

$$(-1)^{ln} \cdot (-1)^{j(n+m)} \;=\; (-1)^{jm} \cdot (-1)^{(l+j)n}$$

and commutativity of the square

$$\begin{tikzcd}
X_n \wedge S^m \wedge S^q \arrow[r, "\sigma^m \wedge \mathrm{Id}"] \arrow[d, "\mathrm{Id} \wedge \cong"'] & X_{n+m} \wedge S^q \arrow[d, "\sigma^q"] \\
X_n \wedge S^{m+q} \arrow[r, "\sigma^{m+q}"'] & X_{n+m+q}
\end{tikzcd}$$

Finally, the multiplication in the homotopy groups of spheres is commutative in the graded sense. Indeed, for representing maps $f : S^{k+n} \longrightarrow S^n$ and $g : S^{l+m} \longrightarrow S^m$ the square

$$\begin{tikzcd}
S^{k+n+l+m} \arrow[r, "f \wedge g"] \arrow[d, "\chi_{k+n,l+m}"'] & S^{n+m} \arrow[d, "\chi_{n,m}"] \\
S^{l+m+k+n} \arrow[r, "g \wedge f"'] & S^{m+n}
\end{tikzcd}$$

commutes. The two vertical coordinate permutations induce the signs $(-1)^{(k+n)(l+m)}$ respectively (after one suspension) $(-1)^{nm}$ on homotopy groups. So in the stable group we have

$$[f] \cdot [g] \;=\; (-1)^{nl} \cdot [f \cdot g] \;=\; (-1)^{kl+km} \cdot [g \cdot f] \;=\; (-1)^{kl} \cdot [g] \cdot [f]\ .$$

The following table gives the stable homotopy groups of spheres through dimension 8:

n	0	1	2	3	4	5	6	7	8
π_n^{s}	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$(\mathbb{Z}/2)^2$
generator	ι	η	η^2	ν			ν^2	σ	$\eta\sigma,\ \varepsilon$

Here ν and σ are the Hopf maps which arises unstably as fiber bundles $S^7 \longrightarrow S^4$ respectively $S^{15} \longrightarrow S^8$. The element ε in the 8-stem can be defined using Toda brackets (see Construction IV.2.5) as $\varepsilon = \eta\sigma + \langle \nu, \eta, \nu \rangle$. The table contains or determines all multiplicative relations in this range except for $\eta^3 = 12\nu$. A theorem of Nishida's [ref] says that every homotopy element of positive dimension is nilpotent. We explain in Section II.10 how this table can be obtained with the help of the Adams spectral sequence.

A word of warning: it is tempting to try to define a product on the naive homotopy groups of a symmetric ring spectrum R in a similar fashion, by smashing representatives and shuffling sphere coordinates into their natural order. This will indeed give an associative product whenever the underlying symmetric spectrum of R is semistable. However, if R is not semistable, then smashing of representatives does not descend to a well-defined product on naive homotopy groups! In that case the algebraic structure that the homotopy groups of R enjoy is more subtle, and we discuss it in Exercise E.I.67. In any case, the true homotopy groups of a symmetric ring spectrum have a natural multiplication, by Proposition 6.25 below.

Example 1.13 (Suspension spectra)**.** Every pointed space K gives rise to a *suspension spectrum* $\Sigma^\infty K$ via

$$(\Sigma^\infty K)_n \;=\; K \wedge S^n$$

with structure maps given by the canonical homeomorphism $(K \wedge S^n) \wedge S^1 \xrightarrow{\cong} K \wedge S^{n+1}$. For example, the sphere spectrum $\mathbb{S}$ is isomorphic to the suspension spectrum $\Sigma^\infty S^0$.

The naive homotopy group

$$\pi_k^{\mathrm{s}} K \;=\; \hat{\pi}_k\,(\Sigma^\infty K) \;=\; \mathrm{colim}_n\ \pi_{k+n}(K \wedge S^n)$$

is called the k-th *stable homotopy group* of K. Since $K \wedge S^n$ is $(n-1)$-connected, the suspension spectrum $\Sigma^\infty K$ is *connective*, i.e., all homotopy groups in negative dimensions are trivial. The Freudenthal suspension theorem implies that for every suspension spectrum, the colimit system for a specific homotopy group always stabilizes. A symmetric spectrum X is isomorphic to a suspension spectrum (necessarily that of its zeroth space X_0) if and only if every structure map $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ is a homeomorphism.

Example 1.14 (Eilenberg-Mac Lane spectra)**.** For an abelian group A, the *Eilenberg-Mac Lane spectrum* HA is defined by

$$(HA)_n \;=\; A[S^n] \ ,$$

the reduced A-linearization of the n-sphere. Let us review the linearization construction in some detail before defining the rest of the structure of the Eilenberg-Mac Lane spectrum.

For a general based space K, the underlying set of the A-linearization $A[K]$ is tensor product of A with the reduced free abelian group generated by the points of K. In other words, points of $A[K]$ are finite sums of points of K with coefficients in A, modulo the relation that all A-multiples of the basepoint are zero. The set $A[K]$ is topologized as a quotient space of the disjoint union of the spaces $A^n \times K^n$ (with discrete topology on A^n), via the surjection

$$\amalg_{n\geq 0}\, A^n \times K^n \;\longrightarrow\; A[K] \ , \quad (a_1,\dots,a_n,k_1,\dots,k_n) \;\mapsto\; \sum_{i=1}^n a_i\cdot k_i \ .$$

There is a natural map $\tilde H_n(K,A) \longrightarrow \pi_n(A[K],0)$ from the reduced singular homology groups of K with coefficients in A to the homotopy groups of the linearization: let $x = \sum_i a_i \cdot f_i$ be a singular chain of K with coefficients a_i in A, i.e., every $f_i : \underline{\Delta}[n] \longrightarrow K$ is a continuous map from the topological n-simplex. We use the abelian group structure of $A[K]$ and add the maps f_j pointwise and multiply by the coefficients, to get a single map $\underline{x} : \underline{\Delta}[n] \longrightarrow A[K]$, i.e., for $z \in \underline{\Delta}[n]$ we set

$$\underline{x}(z) \;=\; \sum_i a_i \cdot f_i(z) \ .$$

If the original chain x is a cycle in the singular chain complex, then the map $\underline{x}$ sends the boundary of the simplex to the neutral element 0 of $A[K]$. So $\underline{x}$ factors over a continuous based map $\underline{\Delta}[n]/\partial\underline{\Delta}[n] \longrightarrow A[K]$. After composing with a homeomorphism between the n-sphere and $\underline{\Delta}[n]/\partial\underline{\Delta}[n]$ this maps represents an element in the homotopy group $\pi_n(A[K],0)$. If K has the based homotopy type of a CW-complex, then the map $\tilde H_n(K,A) \longrightarrow \pi_n(A[K],0)$ is an isomorphism [ref]. In the special case $K = S^n$ this shows that the $A[S^n]$ has only one non-trivial homotopy group in dimension n, where it is isomorphic to A. In other words, $(HA)_n = A[S^n]$ is an Eilenberg-Mac Lane space of type (A,n).

Now we return to the definition of the Eilenberg-Mac Lane spectrum HA. The symmetric group acts on $(HA)_n = A[S^n]$ by permuting the smash factors of S^n. The structure map $\sigma_n : (HA)_n \wedge S^1 \longrightarrow (HA)_{n+1}$ is given by

$$A[S^n] \wedge S^1 \longrightarrow A[S^{n+1}] \ , \quad \bigl(\textstyle\sum_i a_i \cdot x_i\bigr) \wedge y \;\longmapsto\; \textstyle\sum_i a_i \cdot (x_i \wedge y) \ .$$

If A is not just an abelian group but also has a ring structure, then HA becomes a symmetric ring spectrum via the multiplication map

$$\begin{aligned} (HA)_n \wedge (HA)_m \;&=\; A[S^n] \wedge A[S^m] \;\longrightarrow\; A[S^{n+m}] \;=\; (HA)_{n+m} \\ \bigl(\textstyle\sum_i a_i \cdot x_i\bigr) \wedge \bigl(\textstyle\sum_j b_j \cdot y_j\bigr) \;&\longmapsto\; \textstyle\sum_{i,j} (a_i\cdot b_j)\cdot (x_i \wedge y_j) \ . \end{aligned}$$

The unit maps $S^m \longrightarrow (HA)_m$ are given by the inclusion of generators.

We shall see in Example 5.28 below that the Eilenberg-Mac Lane functor H can be made into a lax symmetric monoidal functor with respect to the tensor product of abelian groups and the smash product of symmetric spectra; this also explains why H takes rings (monoids in the category of abelian with respect to tensor product) to ring spectra (monoids in the category of symmetric spectra with respect to smash product).

Eilenberg-Mac Lane spectra enjoy a special property: the n-th space $(HA)_n$ and the loop space of the next space $(HA)_{n+1}$ are both Eilenberg-Mac Lane spaces of type (A,n), and in fact the map $\tilde{\sigma}_n : (HA)_n \longrightarrow \Omega(HA)_{n+1}$ adjoint to the structure map is a weak equivalence for all $n \geq 0$. Spectra with this property play an important role in stable homotopy theory, and they deserve a special name:

Definition 1.15. A symmetric spectrum of topological spaces X is an *Ω-spectrum* if for all $n \geq 0$ the map $\tilde{\sigma}_n : X_n \longrightarrow \Omega X_{n+1}$ which is adjoint to the structure map $\sigma_n \colon X_n \wedge S^1 \to X_{n+1}$ is a weak homotopy equivalence.

In other words, HA is an Ω-spectrum. It follows that the naive homotopy groups of the symmetric spectrum HA are concentrated in dimension zero, where we have a natural isomorphism $A = \pi_0(HA)_0 \cong \hat{\pi}_0 HA$.

Example 1.16 (Real Thom spectra)**.** We define a commutative symmetric ring spectrum MO whose stable homotopy groups are isomorphic to the ring of cobordism classes of closed smooth manifolds. We set

$$MO_n \ = \ EO(n)^+ \wedge_{O(n)} S^n \ ,$$

the Thom space of the tautological vector bundle $EO(n) \times_{O(n)} \mathbb{R}^n$ over $BO(n) = EO(n)/O(n)$. Here $O(n)$ is the n-th orthogonal group consisting of Euclidean automorphisms of $\mathbb{R}^n$. The space $EO(n)$ is the geometric realization of the simplicial space which in dimension k is the $(k+1)$-fold product of copies of $O(n)$, and where face maps are projections. Thus $EO(n)$ is contractible and has a right action by $O(n)$. The right $O(n)$-action is used to form the orbit space MO_n, where we remember that S^n is the one-point compactification of $\mathbb{R}^n$, so it comes with a left $O(n)$-action.

The symmetric group Σ_n acts on $O(n)$ by conjugation with the permutation matrices. Since the 'E'-construction is natural in topological groups, this induces an action of Σ_n on $EO(n)$. If we let Σ_n act on the sphere S^n by coordinate permutations and diagonally on $EO(n)^+ \wedge S^n$, then the action descends to the quotient space MO_n.

The unit of the ring spectrum MO is given by the maps

$$S^n \cong O(n)^+ \wedge_{O(n)} S^n \ \longrightarrow \ EO(n)^+ \wedge_{O(n)} S^n = MO_n$$

using the 'vertex map' $O(n) \longrightarrow EO(n)$. There are multiplication maps

$$MO_n \wedge MO_m \ \longrightarrow \ MO_{n+m}$$

which are induced from the identification $S^n \wedge S^m \cong S^{n+m}$ which is equivariant with respect to the group $O(n) \times O(m)$, viewed as a subgroup of $O(n+m)$ by block sum of matrices. The fact that these multiplication maps are associative and commutative uses that

- for topological groups G and H, the simplicial model of EG comes with a natural, associative and commutative isomorphism $E(G \times H) \cong EG \times EH$;
- the group monomorphisms $O(n) \times O(m) \longrightarrow O(n+m)$ by orthogonal direct sum are strictly associative, and the following diagram commutes

$$\begin{tikzcd} O(n) \times O(m) \arrow[r] \arrow[d, "\text{twist}"'] & O(n+m) \arrow[d, "\text{conj. by } \chi_{n,m}"] \\ O(m) \times O(n) \arrow[r] & O(m+n) \end{tikzcd}$$

  where the right vertical map is conjugation by the permutation matrix of the shuffle permutation $\chi_{n,m}$.

Essentially the same construction gives commutative symmetric ring spectra MSO representing oriented bordism and $MSpin$ representing spin bordism. For MSO this uses that conjugation of $O(n)$ by a permutation matrix restricts to an automorphism of $SO(n)$ and the block sum of two special orthogonal transformations is again special. For $MSpin$ it uses that the block sum pairing and the Σ_n-action uniquely lift from the groups $SO(n)$ to their universal covers $Spin(n)$.

In Example 6.43 we provide a different model of the Thom spectrum MO made from Thom spaces over Grassmannians. Moreover, the Grassmannian model admits a 'periodization', i.e., a $\mathbb{Z}$-graded commutative symmetric ring spectrum MOP whose degree 0 component is $\hat{\pi}_*$-isomorphic to MO and whose degree 1 component contains a unit of dimension 1. Any commutative symmetric ring spectrum with an odd dimensional unit satisfies $2 = 0$, and so all homotopy groups of the spectrum MO are $\mathbb{F}_2$-vector spaces (compare Corollary 6.45).

The Thom-Pontrjagin construction provides homomorphisms $\hat{\pi}_k MO \longrightarrow \mathbf{\Omega}_k^O$ from the k-th naive homotopy group of the spectrum MO to the group of cobordism classes of k-dimensional smooth closed manifolds [direction], and similarly for the other families of classical Lie groups. [structure on stable normal bundle; takes product in $\hat{\pi}_* MO$ to cartesian product of manifolds] By a theorem of Thom's [ref], the Thom-Pontrjagin map is an isomorphism. [explicit description of $\pi_* MG$, whenever known] [add $M\Sigma$, $MGl(A)$,.... Does MBr fit (how does Σ_n acts on Br_n)? MBr is stably equivalent to $H\mathbb{Z}/2$ (find explicit map), so $M\Sigma$, $MGl(A)$, MO are GEMs] We intend to discuss these and other examples of Thom spectra in more detail in a later chapter.

Example 1.17 (Real Thom spectra)**.** Now we given another commutative symmetric ring spectrum model for MO, level equivalent to the previous example. We construct this with more structure, namely as a coordinate free orthogonal spectrum. For a real inner product space V we let $L(V, V \otimes \mathbb{R}^\infty)$ be the space of linear isometric embeddings from V into $V \otimes \mathbb{R}^\infty$. If V is non-zero, then $V \otimes \mathbb{R}^\infty$ is infinite dimensional and the space $L(V, V \otimes \mathbb{R}^\infty)$ is contractible [ref]. The orthogonal group $O(V)$ acts freely and continuously from the right on $L(V, V \otimes \mathbb{R}^\infty)$ by precomposition. We can thus form the space

$$M'O(V) \;=\; L(V, V \otimes \mathbb{R}^\infty)^+ \wedge_{O(V)} S^V \ ,$$

the Thom space of the tautological vector bundle $L(V, V\otimes\mathbb{R}^\infty) \times_{O(V)} V$ over the space $L(V, V\otimes\mathbb{R}^\infty)/O(V)$, the Grassmannian of $\dim(V)$-dimensional subspaces of $V \otimes \mathbb{R}^\infty$. The group $O(V)$ also acts continuously from the left on $L(V, V \otimes \mathbb{R}^\infty)$ through its action on $V \otimes \mathbb{R}^\infty$; we give $M'O(V)$ the induced action.

If W is another finite dimensional real inner product space we can define a multiplication map

$$\mu_{V,W} \ : \ M'O(V) \wedge M'O(W) \ \longrightarrow \ M'O(V \oplus W)$$

by

$$\begin{aligned} (L(V, V \otimes \mathbb{R}^\infty)^+ \wedge_{O(V)} S^V) \wedge (L(W, W \otimes \mathbb{R}^\infty)^+ \wedge_{O(W)} S^W) \ &\longrightarrow \\ & L(V \oplus W, (V \oplus W) \otimes \mathbb{R}^\infty)^+ \wedge_{O(V\oplus W)} S^{V\oplus W} \\ [\varphi, v] \wedge [\psi, w] \qquad &\longmapsto \ [\varphi \oplus \psi,\, v \oplus w] \end{aligned}$$

Here we have implicitly identified

$$(V \otimes \mathbb{R}^\infty) \oplus (W \otimes \mathbb{R}^\infty) \quad \text{with} \quad (V \oplus W) \otimes \mathbb{R}^\infty$$

and $S^V \wedge S^W$ with $S^{V\oplus W}$. The multiplication maps are associative and commutative, and they are unital with respect to the maps

$$\iota_V = [i_V, -] \ : \ S^V \ \longrightarrow \ L(V, V \otimes \mathbb{R}^\infty)^+ \wedge_{O(V)} S^V = M'O(V)$$

where $i_V(v) = v \otimes (1, 0, 0, \dots) : V \longrightarrow V \otimes \mathbb{R}^\infty$ is the isometric embedding given by the first coordinate of $\mathbb{R}^\infty$. The same construction gives commutative symmetric ring spectrum $M'SO$ by deviding out only the action of the special orthogonal group $SO(V)$ of V.

In Example 6.43 we provide a different model of the Thom spectrum MO made from Thom spaces over Grassmannians. Moreover, the Grassmannian model admits a 'periodization', i.e., a $\mathbb{Z}$-graded commutative symmetric ring spectrum MOP whose degree 0 component is $\hat{\pi}_*$-isomorphic to MO and whose degree 1 component contains a unit of dimension 1. Any commutative symmetric ring spectrum with an odd dimensional unit satisfies $2 = 0$, and so all homotopy groups of the spectrum MO are $\mathbb{F}_2$-vector spaces (compare Corollary 6.45).

The Thom-Pontrjagin construction provides homomorphisms $\hat{\pi}_k MO \longrightarrow \mathbf{\Omega}_k^O$ from the k-th naive homotopy group of the spectrum MO to the group of cobordism classes of k-dimensional smooth closed

manifolds [direction], and similarly for the other families of classical Lie groups. [structure on stable normal bundle; takes product in $\hat{\pi}_*MO$ to cartesian product of manifolds] By a theorem of Thom's [ref], the Thom-Pontrjagin map is an isomorphism. [explicit description of π_*MG, whenever known] [add $M\Sigma$, $MGl(A)$,.... Does MBr fit (how does Σ_n acts on Br_n)? MBr is stably equivalent to $H\mathbb{Z}/2$ (find explicit map), so $M\Sigma$, $MGl(A)$, MO are GEMs]

Example 1.18 (Complex cobordism spectra)**.** The Thom ring spectra MU, MSU and MSp representing unitary, special unitary or symplectic bordism have to be handled slightly differently from real Thom spectra such as MO in the previous example. The point is that MU and MSU are most naturally indexed on 'even spheres', i.e., one-point compactifications of complex vector spaces, and MSp is most naturally indexed on spheres of dimensions divisible by 4. However, a small variation gives MU, MSU and MSp as commutative symmetric ring spectra, as we shall now explain. The complex cobordism spectrum MU plays an important role in stable homotopy theory because of its relationship to the theory of formal groups laws. Thus module and algebra spectra over MU are important, and we plan to study these in some detail later.

We first consider the collection of pointed spaces $\overline{MU}$ with

$$(\overline{MU})_n \;=\; EU(n)^+ \wedge_{U(n)} S^{\mathbb{C}^n} \ ,$$

the Thom space of the tautological complex vector bundle $EU(n) \times_{U(n)} \mathbb{C}^n$ over $BU(n) = EU(n)/U(n)$. Here $U(n)$ is the n-th unitary group consisting of Euclidean automorphisms of $\mathbb{C}^n$. The Σ_n-action arises from conjugation by permutation matrices and the permutation of complex coordinates, similarly as in the case of MO above.

There are multiplication maps

$$(\overline{MU})_p \wedge (\overline{MU})_q \;\longrightarrow\; (\overline{MU})_{p+q}$$

which are induced from the identification $\mathbb{C}^p \oplus \mathbb{C}^q \cong \mathbb{C}^{p+q}$ which is equivariant with respect to the group $U(p) \times U(q)$, viewed as a subgroup of $U(p+q)$ by direct sum of linear maps. There is a unit map $\iota_0 : S^0 \longrightarrow (\overline{MU})_0$, but instead of a unit map from the circle S^1, we only have a unit map $S^2 \longrightarrow (\overline{MU})_1$. Thus we do *not* end up with a symmetric spectrum since we only get structure maps $(\overline{MU})_n \wedge S^2 \longrightarrow (\overline{MU})_{n+1}$ involving the 2-sphere. In other words, $\overline{MU}$ has the structure of what could be called an 'even symmetric ring spectrum' ($\overline{MU}$ is really a 'unitary ring spectrum', compare Section 7.2 below).

In order to get an honest symmetric ring spectrum we now use a general construction which turns a commutative monoid R in the category of symmetric sequences into a new such monoid $\Phi(R)$ by appropriately looping all the spaces involved. We set

$$\Phi(R)_n \;=\; \mathrm{map}(S^n, R_n)$$

and let the symmetric group act by conjugation. Then the product of R combined with smashing maps gives $\Sigma_n \times \Sigma_m$-equivariant maps

$$\begin{array}{ccc} \Phi(R)_n \wedge \Phi(R)_m = \mathrm{map}(S^n, R_n) \wedge \mathrm{map}(S^m, R_m) & \longrightarrow & \mathrm{map}(S^{n+m}, R_{n+m}) = \Phi(R)_{n+m} \\ f \wedge g & \longmapsto & f \cdot g = \mu_{n,m} \circ (f \wedge g) \ . \end{array}$$

Now we apply this construction to $\overline{MU}$ and obtain a commutative monoid $MU = \Phi(\overline{MU})$ in the category of symmetric sequences. We make MU into a symmetric ring spectrum via the unit map $S^1 \longrightarrow (MU)_1 = \mathrm{map}(S^1, (\overline{MU})_1)$ which is adjoint to

$$\iota \;:\; S^2 \cong U(1)^+ \wedge_{U(1)} S^2 \;\longrightarrow\; EU(1)^+ \wedge_{U(1)} S^2 = (\overline{MU})_1$$

using the 'vertex map' $U(1) \longrightarrow EU(1)$. More precisely, we use the decomposition $\mathbb{C} = \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot i$ to view S^2 as the smash product of a 'real' and 'imaginary' circle, and then we view the source of the unit map $S^1 \longrightarrow (MU)_1 = \mathrm{map}(S^1, (\overline{MU})_1)$ as the real circle, and we think of the imaginary circle as parameterizing the loop coordinate in the target $(MU)_1$. Since the multiplication of MU is commutative, the centrality condition is automatically satisfied. Then the iterated unit map

$$S^n \;\longrightarrow\; (MU)_n = \Omega^n(\overline{MU})_n$$

is given by

$$(x_1,\ldots,x_n) \quad \longmapsto \quad ((y_1,\ldots,y_n)\mapsto \mu(\iota(x_1,y_1),\ldots,\iota(x_n,y_n)))$$

where $\mu : (\overline{MU})_1^{\wedge n} \longrightarrow (\overline{MU})_n$ is the iterated multiplication map.

The naive (and true) homotopy groups of MU are given by

$$\hat{\pi}_k MU = \operatorname{colim}_n \pi_{k+n} \operatorname{map}(S^n, (\overline{MU}_n)) \cong \operatorname{colim}_n \pi_{k+2n}(EU(n)^+ \wedge_{U(n)} S^{2n}) \ ;$$

so by Thom's theorem they are isomorphic to the ring of cobordism classes of stably almost complex k-manifolds. So even though the individual spaces $MU_n = \operatorname{map}(S^n, EU(n)^+ \wedge_{U(n)} S^{2n})$ are not Thom spaces, the symmetric spectrum which they form altogether has the 'correct' homotopy groups (and in fact, the correct stable homotopy type). [$\pi_* MU$; semistable since orthogonal spectrum]

Essentially the same construction gives a commutative symmetric ring spectrum MSU. The symplectic bordism and MSp can also be handled similarly: it first arises as a commutative monoid $\overline{MSp}$ in symmetric sequences with structure maps $(\overline{MSp})_n \wedge S^4 \longrightarrow (\overline{MSp})_{n+1}$ and a unit map $S^4 \longrightarrow (\overline{MSp})_1$. If we apply the construction Φ three times, we obtain a commutative symmetric ring spectrum $MSp = \Phi^3(\overline{MSp})$ representing symplectic bordism.

A symmetric spectrum X is a *positive Ω-spectrum* if the map $\tilde{\sigma}_n : X_n \longrightarrow \Omega X_{n+1}$ is a weak equivalence for all positive values of n (but not necessarily for $n = 0$). Examples which arise naturally as positive Ω-spectra are the spectra of topological K-theory (Example 1.20) and algebraic K-theory $K(\mathcal{C})$ (Example 3.50) and spectra arising from special (but not necessarily very special) $\mathbf{\Gamma}$-spaces by evaluation on spheres.

For every Ω-spectrum X and all $k, n \geq 0$, the canonical map $\pi_k X_n \longrightarrow \hat{\pi}_{k-n} X$ is a bijection. Indeed, the homotopy groups of ΩX_{n+1} are isomorphic to the homotopy groups of X_{n+1}, shifted by one dimension. So the colimit system which defines $\hat{\pi}_{k-n} X$ is isomorphic to the colimit system

$$\pi_k\, X_n \ \longrightarrow\ \pi_k\,(\Omega X_{n+1})\ \longrightarrow\ \pi_k\,(\Omega^2 X_{n+2})\ \longrightarrow\ \cdots\ , \tag{1.19}$$

where the maps in the system are induced by the maps $\tilde{\sigma}_n$ adjoint to the structure maps. In an Ω-spectrum, the maps $\tilde{\sigma}_n$ are weak equivalences, so all maps in the sequence (1.19) are bijective, hence so is the map from each term to the colimit $\hat{\pi}_{k-n} X$.

Example 1.20 (Topological K-theory)**.** We define the commutative symmetric ring spectrum ku of *connective complex topological K-theory*. We need some preparations. We let $\mathcal{U}$ be a complex hermitian inner product space $\mathcal{U}$ of finite or countably infinite dimension. For a finite based set A we denote by $\Lambda(A,\mathcal{U})$ the space of of tuples (V_a), indexed by the non-basepoint elements of A, of finite dimensional, pairwise orthogonal complex subvector spaces of $\mathcal{U}$. The topology on this space is as a disjoint union, indexed over the dimension vectors $(\dim(V_a)) \in \mathbb{N}^A$; for a fixed dimension vector, the topology is that of a subspace of a product of Grassmannians of $\mathcal{U}$. The basepoint of $\Lambda(A,\mathcal{U})$ is the tuple where each V_a is the zero subspace. For a based map $\alpha : A \longrightarrow B$ an induced continuous map $\alpha_* : \Lambda(A,\mathcal{U}) \longrightarrow \Lambda(B,\mathcal{U})$ sends (V_a) to (W_b) where

$$W_b \ = \ \bigoplus_{\alpha(a)=b} V_a \ .$$

Now we let K be a based topological space. We define the value $\Lambda(K,\mathcal{U})$ by

$$\Lambda(K,\mathcal{U}) \ = \ \left(\amalg_{n\geq 0} K^n \times \Lambda(n^+,\mathcal{U})\right) / \sim \ ,$$

where the equivalence relation (This is in fact a coend, over the category $\mathbf{\Gamma}$ of standard finite based sets [...]). So an element of $\Lambda(K,\mathcal{U})$ is represented by a tuple $(k_1,\ldots,k_n)$ of points of K 'labelled' with pairwise orthogonal vector subspaces $(V_1,\ldots,V_n)$ in $\mathcal{U}$ for some n. The topology is such that, informally speaking, the vector spaces V_i and V_j are summed up whenever two points k_i and k_j collide and V_i disappears when k_i approaches the basepoint of K.

If $\mathcal{U}'$ is another complex inner product space of at most countable dimension and B another finite based set we define a multiplication map

$$\Lambda(A,\mathcal{U}) \wedge \Lambda(B,\mathcal{U}') \ \longrightarrow\ \Lambda(A\wedge B,\, \mathcal{U}\otimes\mathcal{U}')\ , \quad (V_a)\wedge(W_b)\ \longmapsto\ (V_a\otimes W_b)_{a\wedge b}\ .$$

The multiplication maps $\mu_{n,m}$ are associative and commutative in the sense that the squares

$$\begin{tikzcd}
\Lambda(A,\mathcal{U}) \wedge \Lambda(B,\mathcal{U}') \wedge \Lambda(C,\mathcal{U}'') \arrow[r] \arrow[d] & \Lambda(A,\mathcal{U}) \wedge \Lambda(B \wedge C, \mathcal{U}' \otimes \mathcal{U}'') \arrow[d] \\
\Lambda(A \wedge B, \mathcal{U} \otimes \mathcal{U}') \wedge \Lambda(C,\mathcal{U}'') \arrow[r] & \Lambda(A \wedge B \wedge C, \mathcal{U} \otimes \mathcal{U}' \otimes \mathcal{U}'')
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda(A,\mathcal{U}) \wedge \Lambda(B,\mathcal{U}') \arrow[r] \arrow[d] & \Lambda(A \wedge B, \mathcal{U} \otimes \mathcal{U}') \arrow[d] \\
\Lambda(B,\mathcal{U}') \wedge \Lambda(A,\mathcal{U}) \arrow[r] & \Lambda(B \wedge A, \mathcal{U}' \otimes \mathcal{U})
\end{tikzcd}$$

commute.

Given a finite dimensional real inner product space V we let

$$\mathrm{Sym}(V) \;=\; \bigoplus_{n\geq 0} \mathbb{C} \otimes_{\mathbb{R}} (V^{\otimes n})^{\Sigma_n}$$

denote the complexification of the symmetric algebra of V. This has a preferred hermitian inner product such that the defining action of $O(V)$ on V extends to a unitary action of $O(V)$ on $\mathrm{Sym}(V)$. Moreover, the natural isomorphism

$$\mathrm{Sym}(V) \otimes_{\mathbb{C}} \mathrm{Sym}(W) \;\cong\; \mathrm{Sym}(V \oplus W)$$

respects the inner products and is $O(V) \times O(W)$-equivariant.

We can now define an orthogonal spectrum ku. The value of ku on a real inner product space V is

$$ku(V) \;=\; \Lambda(S^V, \mathrm{Sym}(V)) \;,$$

the configuration space of the V-sphere with labels in orthogonal subspaces of the symmetric algebra $\mathrm{Sym}(V)$. We let the orthogonal group $O(V)$ act diagonally, via the action on the sphere S^V and the action on unitary action on $\mathrm{Sym}(V)$. Explicitly, given an orthogonal automorphism $\varphi : V \longrightarrow V$, elements $v_1, \dots, v_n$ of S^V and pairwise orthogonal subspaces $X_1, \dots X_n$ of $\mathrm{Sym}(V)$, we set

$$\varphi \cdot [v_1, \dots, v_n;\, X_1, \dots, X_n] \;=\; [\varphi(v_1), \dots, \varphi(v_n);\, \varphi_*(X_1), \dots, \varphi_*(X_n)] \;.$$

We define an $O(V) \times O(W)$-equivariant multiplication map

$$\begin{aligned}
\mu_{V,W} \;:\; ku(V) \wedge ku(W) \;&=\; \Lambda(S^V, \mathrm{Sym}(V)) \wedge \Lambda(S^W, \mathrm{Sym}(W)) \\
&\longrightarrow \Lambda(S^V \wedge S^W, \mathrm{Sym}(V) \otimes_{\mathbb{C}} \mathrm{Sym}(W)) \;\cong\; \Lambda(S^{V\oplus W}, \mathrm{Sym}(V \oplus W)) = ku(V \oplus W) \;.
\end{aligned}$$

The maps $\mu_{V,W}$ are associative and commutative. Now we define an $O(V)$-equivariant unit map

$$\iota_V \;:\; S^V \;\longrightarrow\; \Lambda(S^V, \mathrm{Sym}(V)) \;, \quad v \;\longmapsto\; [v, \mathbb{C}\cdot 1] \;,$$

where $\mathbb{C} \cdot 1$ is the line in $\mathrm{Sym}(V)$ spanned by the unit element 1 of the symmetric algebra.

If the vector space V is non-zero, then the symmetric algebra $\mathrm{Sym}(V)$ in infinite dimensional. We will show in [...] below that then the $\mathbf{\Gamma}$-space $\Lambda(-,\mathcal{U})$ is 'special' and so [...] Hence the orthogonal spectrum ku is a positive Ω-spectrum.

A morphism of symmetric ring spectra $\dim : ku \longrightarrow H\mathbb{Z}$ to the integral Eilenberg-Mac Lane spectrum (see Example 1.14)is given by the dimension function, i.e., on a real inner product spaces V the map

$$\dim \;:\; ku(V) \;=\; \Lambda(S^V, \mathrm{Sym}(V)) \;\longrightarrow\; \mathbb{Z}[S^V] \;=\; H\mathbb{Z}(V)$$

is given by

$$\dim[\lambda_1, \dots, \lambda_n;\, V_1, \dots, V_n] \;=\; \sum_{i=1}^{n} (\dim V_i) \cdot \lambda_i \;.$$

The dimension morphism $\dim : ku \longrightarrow H\mathbb{Z}$ induces an isomorphism on π_0.

We can identify level 0 and level 1 of ku very explicitly. The symmetric algebra $\mathrm{Sym}(0)$ of the zero vector space is 1-dimensional, generated by the unit, and hence the space $ku_0 = \Lambda(S^0, \mathrm{Sym}(0))$ has two points, the basepoint and the configuration $[0, \mathbb{C}\cdot 1]$ where 0 is the non-basepoint of S^0. So the unit map $\iota_0 : S^0 \longrightarrow ku_0$ is a homeomorphism. We choose a homeomorphism of S^1 (the one-point compactification of $\mathbb{R}$) with the unit sphere $S(\mathbb{C}) = \{z \in \mathbb{C} \mid |z| = 1\}$ of the complex number, taking the basepoint of S^1 at infinity to $1 \in \mathbb{C}$. This induces a homeomorphism of configuration spaces

$$ku_1 = \Lambda(S^1, \mathrm{Sym}(\mathbb{R})) \;\cong\; \Lambda(S(\mathbb{C}), \mathbb{C}^\infty) \;. \tag{1.21}$$

Given a tuple $(\lambda_1, \dots, \lambda_n) \in S(\mathbb{C})^n$ and a tuple $(V_1, \dots, V_n)$ of pairwise orthogonal subspaces of $\mathbb{C}^\infty$ we let $\psi(\lambda_1, \dots, \lambda_n; V_1, \dots, V_n)$ be the unitary transformation of $\mathbb{C}^\infty$ that is multiplication by λ_i on V_i and the identity on the orthogonal complement of $\bigoplus_{i=1}^n V_i$. As n varies, these maps are compatible with the equivalence relation and so they assemble into a continuous map

$$\psi \; : \; \Lambda(S(\mathbb{C}), \mathbb{C}^\infty) \; \longrightarrow \; U \; .$$

This map is bijective because every unitary transformation is diagonalizable with finitely many eigenvalues in the unit circle and pairwise orthogonal eigenspaces.

We identify the homotopy groups of the spectrum ku in low dimensions. The space

$$ku_n \; = \; \Lambda(S^n, \mathrm{Sym}(\mathbb{R}^n))$$

is the values of a $\mathbf{\Gamma}$-space on S^n, so it is $(n-1)$-connected [ref]. In particular, the spectrum ku_n is connective, i.e., its homotopy groups vanish in negative dimensions. The inclusion $U(2) \longrightarrow U$ is 4-connected and $U(2)$ is homeomorphic to $S^1 \times S^3$, so the first and third homotopy group of U, and hence of ku_1, are free abelian of rank 1, and the second homotopy group of ku_1 is trivial. Specific generators of $\pi_1(ku_1)$ and $\pi_3(ku_1)$ are given by the unit map $\iota_1 : S^1 \longrightarrow ku_1$ respectively the composite

$$u \; : \; S^3 \cong SU(2) \xrightarrow{\text{incl.}} U \xrightarrow[(1.21)]{\simeq} ku_1 \; .$$

Since ku is a positive Ω-spectrum the natural map $\pi_{k+1}(ku_1) \longrightarrow \hat{\pi}_k(ku)$ is an isomorphism for all $k \geq 0$. Altogether we have shown that $\pi_0(ku) \cong \pi_2(ku) \cong \mathbb{Z}$, generated by the classes of $\iota_1 : S^1 \longrightarrow ku_1$ respectively $u : S^3 \longrightarrow ku_1$, and $\pi_1(ku)$ is trivial.

The class in $\pi_2(ku)$ represented by u is called the *Bott class*; we will abuse notation and denote this class also by u. The *Bott periodicity theorem* constructs a homotopy equivalence $\Omega^2 BU \; \simeq \; \mathbb{Z} \times BU$. An incarnation of this Bott periodicity in our context is the fact that the map

$$ku_1 \; \longrightarrow \; \Omega^3(ku_2)$$

which is adjoint to multiplication by u,

$$ku_1 \wedge S^3 \xrightarrow{ku_1 \wedge u} ku_1 \wedge ku_1 \xrightarrow{\mu_{1,1}} ku_2$$

is a weak equivalence [ref]. Since ku_1 is also weakly equivalent to $\Omega(ku_2)$, we conclude that ku_1 and $\Omega^2(ku_n)$ are weakly equivalent. This form of Bott periodicity implies that the homotopy ring of ku is polynomial on the Bott class, i.e., $\hat{\pi}_*(ku) = \mathbb{Z}[u]$ as a graded ring. The symmetric spectrum KU of of *periodic complex topological K-theory* is constructed from the connective version ku by 'inverting the Bott class'. We spell out this process of inverting a homotopy class in Example 6.58 below. We will discuss another model for connective topological K-theory in Example 7.10.

The complex K-theory spectra ku and KU of the previous example have real variants ko and KO; in the definitions we simply have to replace complex universes by real universes. [spell out?] Both ko and KO are commutative symmetric ring spectra and their underlying symmetric spectra are positive Ω-spectra. The real version has a Bott-periodicity of order 8, i.e., there is a homotopy equivalence

$$\Omega^8 BO \; \simeq \; \mathbb{Z} \times BO \; ;$$

in the spectra ko and KO this periodicity is realized by a Bott class $\beta \in \hat{\pi}_8(ko)$ which becomes invertible in the homotopy ring of KO [define this] and arises from a specific map $S^9 \longrightarrow ko_1$. The following table (and Bott periodicity) gives the homotopy groups of the spectra ko and KO:

n	0	1	2	3	4	5	6	7	8
$\hat{\pi}_n(ko)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
generator	ι	η	η^2		ξ				β

Here η is the Hurewicz image of the Hopf map, i.e., the image of the class $\eta \in \hat{\pi}_1\mathbb{S}$ under the unique homomorphism of ring spectra $\mathbb{S} \longrightarrow ko$. There are 'complexification maps', i.e., homomorphisms of ring spectra $ko \longrightarrow ku$ and $KO \longrightarrow KU$, which are injective on homotopy groups in dimensions divisible by 4, and bijective in dimensions divisible by 8. The elements ξ and β can be defined by the property that they hit $2u^2 \in \hat{\pi}_4(ku)$ respectively $u^4 \in \hat{\pi}_8(ku)$ under this complexification map. Thus there is the multiplicative relation $\xi^2 = 4\beta$ in $\hat{\pi}_8(ko)$.

As in the complex case, the periodic ring spectrum KO can be obtained from the connective ring spectrum ko by inverting a geometric representative of the Bott class, see Example 6.58. [Is there a self-conjugate version KT, or, even better, the real, C_2-equivariant version KR?]

2. Properties of naive homotopy groups

In Section 3.2 we will discuss various constructions which one can do to a symmetric spectrum. Whenever possible we want to say how a construction effects the naive homotopy groups. So in this section we develop a few general properties of naive homotopy groups. More specifically, we construct long exact sequences of naive homotopy groups from a morphism of symmetric spectra and we identify the naive homotopy groups of a wedge and a finite product of spectra.

The naive homotopy groups of a symmetric spectrum do not depend on the symmetric group actions, so they are really defined for 'sequential spectra', i.e., 'symmetric spectra without symmetric group actions'. We develop the basic properties of naive homotopy groups in this more general context. This extra generality is useful because a symmetric spectrum sometimes decomposes into simpler pieces *after forgetting the symmetric group actions*, but the levelwise decompositions are not equivariant. Since the naive homotopy groups don't care about the symmetries, such a non-equivariant splitting still gives information about naive homotopy groups. An example of this strategy is the decomposition of the naive homotopy groups of a twisted smash product, see Example 3.27.

Definition 2.1. A *sequential spectrum* consists of a sequence of pointed spaces X_n and based maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ for $n \geq 0$. A *morphism* $f : X \longrightarrow Y$ of sequential spectra consists of based maps $f_n : X_n \longrightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge \mathrm{Id}_{S^1})$ for all $n \geq 0$. The category of sequential spectra is denoted by $\mathcal{S}p^{\mathbb{N}}$.

We refer to the space X_n as the *n-th level* of the sequential spectrum X. The *k-th naive homotopy group* of a sequential spectrum X is defined as the colimit

$$\hat{\pi}_k X \; = \; \mathrm{colim}_n \; \pi_{k+n} X_n$$

taken over the *stabilization maps* $\iota : \pi_{k+n} X_n \longrightarrow \pi_{k+n+1} X_{n+1}$ defined as for symmetric spectra as the composite (1.7).

2.1. Loop and suspension. The *loop spectrum* ΩX of a symmetric or sequential spectrum X is defined by

$$(\Omega X)_n \; = \; \Omega(X_n) \; ,$$

the based mapping space from the circle S^1 to the n-th level of X. In the symmetric (as opposed to 'sequential') case, the symmetric group Σ_n acts on $\Omega(X_n)$ through the given action on X_n and trivially on the circle. The structure map is the composite

$$(\Omega X)_n \wedge S^1 \; = \; \Omega(X_n) \wedge S^1 \; \longrightarrow \; \Omega(X_n \wedge S^1) \; \xrightarrow{\Omega(\sigma_n)} \; \Omega(X_{n+1}) \; = \; (\Omega X)_{n+1}$$

where the first map takes $l \wedge t \in \Omega(X_n) \wedge S^1$ to the loop $S^1 \longrightarrow X_n \wedge S^1$ which sends s to $l(s) \wedge t$.

The *suspension* $S^1 \wedge X$ of a symmetric or sequential spectrum X is defined by

$$(S^1 \wedge X)_n \; = \; S^1 \wedge X_n \; ,$$

the smash product of the circle with the n-th level of X. In the symmetric (as opposed to 'sequential') case, the symmetric group Σ_n acts on $S^1 \wedge X_n$ through the given action on X_n and trivially on the circle. The structure map is the composite

$$(S^1 \wedge X)_n \wedge S^1 \; = \; S^1 \wedge X_n \wedge S^1 \; \xrightarrow{\mathrm{Id} \wedge \sigma_n} \; S^1 \wedge X_{n+1} \; = \; (S^1 \wedge X)_{n+1} \; .$$

We define an adjunction

$$\hat{} \; : \; \mathcal{S}p^{\mathbb{N}}(X, \Omega Y) \; \xrightarrow{\cong} \; \mathcal{S}p^{\mathbb{N}}(S^1 \wedge X, Y) \tag{2.2}$$

which takes a morphism $f : X \longrightarrow \Omega Y$ to the morphism $\hat{f} : S^1 \wedge X \longrightarrow Y$ whose n-th level $\hat{f}_n : S^1 \wedge X_n \longrightarrow Y_n$ is given by $\hat{f}_n(t \wedge x) = f_n(x)(t)$. If X and Y are symmetric spectra, then the adjunction bijection (2.2) restricts to a similar adjunction bijection $\hat{} : \mathcal{S}p(X, \Omega Y) \cong \mathcal{S}p(S^1 \wedge X, Y)$ between the morphism sets of *symmetric* spectra.

Now we show that looping and suspending a spectrum shifts the naive homotopy groups. The loop homomorphism starts from the isomorphism

$$\alpha \ : \ \pi_{k+n}(\Omega X_n) \ \cong \ \pi_{1+k+n} X_n \tag{2.3}$$

that is defined by the same adjunction as above, i.e., the class represented by a continuous map $f : S^{k+n} \longrightarrow \Omega X_n$ is sent to the class of the map $\hat{f} : S^{1+k+n} \longrightarrow X_n$ given by $\hat{f}(s \wedge t) = f(t)(s)$, where $s \in S^1$, $t \in S^{k+n}$. As n varies, these particular isomorphisms are compatible with stabilization maps, so they induce an isomorphism

$$\alpha \ : \ \hat{\pi}_k(\Omega X) \ \xrightarrow{\cong} \ \hat{\pi}_{1+k} X \tag{2.4}$$

on colimits. (Note that the identification α differs from the adjunction isomorphism (0.3) by precomposition with the coordinate permutation $\chi_{k+n,1} : S^{k+n+1} \longrightarrow S^{1+k+n}$; the adjunction isomorphisms themselves are not compatible with stabilization.)

The maps $S^1 \wedge - : \pi_{k+n} X_n \longrightarrow \pi_{1+k+n}(S^1 \wedge X_n)$ given by smashing from the left with the identity of the circle are compatible with the stabilization process for the homotopy groups of X respectively $S^1 \wedge X$, so upon passage to colimits they induce a natural map of naive homotopy groups

$$S^1 \wedge - \ : \ \hat{\pi}_k X \ \longrightarrow \ \hat{\pi}_{1+k}(S^1 \wedge X) \ ,$$

which we call the *suspension homomorphism.*

We let $\eta : X \longrightarrow \Omega(S^1 \wedge X)$ and $\epsilon : S^1 \wedge \Omega X \longrightarrow X$ denote the unit respectively counit of the adjunction (2.2). Then for every map $f : S^{k+n} \longrightarrow \Omega X_n$ we have $\hat{f} = \epsilon_n \circ (S^1 \wedge f)$ and for every map $g : S^{k+n} \longrightarrow X_n$ we have $S^1 \wedge g = \widehat{\eta_n \circ g}$. This means that the two triangles

$$\begin{tikzcd} \hat{\pi}_k(\Omega X) \arrow[rr, "\alpha"] \arrow[dr, "S^1\wedge -"'] & & \hat{\pi}_{1+k} X \\ & \hat{\pi}_{1+k}(S^1 \wedge \Omega X) \arrow[ur, "\hat{\pi}_{1+k}\epsilon"'] & \end{tikzcd} \qquad \begin{tikzcd} \hat{\pi}_k X \arrow[rr, "S^1\wedge -"] \arrow[dr, "\hat{\pi}_k \eta"'] & & \hat{\pi}_{1+k}(S^1 \wedge X) \\ & \hat{\pi}_k(\Omega(S^1 \wedge X)) \arrow[ur, "\alpha"'] & \end{tikzcd} \tag{2.5}$$

commute.

Proposition 2.6. *For every sequential spectrum X and integer k the loop and suspension homomorphisms*

$$\alpha \ : \ \hat{\pi}_k(\Omega X) \ \longrightarrow \ \hat{\pi}_{1+k} X \qquad \text{and} \qquad S^1 \wedge - \ : \ \hat{\pi}_k X \ \longrightarrow \ \hat{\pi}_{1+k}(S^1 \wedge X) \tag{2.7}$$

are isomorphisms of naive homotopy groups. Moreover, the unit $\eta : X \longrightarrow \Omega(S^1 \wedge X)$ and counit $\epsilon : S^1 \wedge \Omega X \longrightarrow X$ of the adjunction (2.2) are $\hat{\pi}_*$-isomorphisms.

For every sequential spectrum X the Σ_n-action on the sphere coordinates of the spectrum $S^n \wedge X$ induces the sign action on naive homotopy groups.

Proof. We already argued that the loop homomorphism α on naive homotopy groups is bijective since it is the colimit of compatible bijections. The case of the suspension homomorphism $S^1 \wedge -$ is slightly more involved. We show injectivity first. Let $f : S^{k+n} \longrightarrow X_n$ represent an element in the kernel of the suspension homomorphism. By stabilizing, if necessecary, we can assume that the suspension $S^1 \wedge f : S^{1+k+n} \longrightarrow S^1 \wedge X_n$ is nullhomotopic. Then $\sigma_n \tau(S^1 \wedge f) : S^{1+k+n} \longrightarrow X_{n+1}$ is also nullhomotopic, where $\tau : S^1 \wedge X \cong X \wedge S^1$ is the twist homeomorphism. The maps $\sigma_n \tau(S^1 \wedge f)$ and $\sigma_n(f \wedge S^1)$, the stabilization of f, only differ by a coordinate permutation of the source sphere, hence the stabilization of f is nullhomotopic. So f represents the trivial element in $\hat{\pi}_k X$, which shows that the suspension homomorphism is injective.

It remains to show that the suspension homomorphism is surjective. Let $g : S^{1+k+n} \longrightarrow S^1 \wedge X_n$ be a map which represents a class in $\hat{\pi}_{1+k}(S^1 \wedge X)$. We consider the map $f = \sigma_n \tau g : S^{1+k+n} \longrightarrow X_{n+1}$ where τ

is again the twist homeomorphism. We claim that $(-1)^{k+n}(S^1 \wedge f) : S^{1+1+k+n} \longrightarrow S^1 \wedge X_{n+1}$ represents the same class as g in $\hat{\pi}_{1+k}(S^1 \wedge X)$. To see this, we contemplate the diagram

$$\begin{tikzcd}
S^{1+k+n+1} \arrow[rrr, "g \wedge S^1"] \arrow[d, "S^1 \wedge \chi_{k+n,1}"'] & & & S^1 \wedge X_n \wedge S^1 \arrow[d, "S^1 \wedge \sigma_n"] \\
S^{1+1+k+n} \arrow[r, "S^1 \wedge g"] \arrow[rrr, bend right, "S^1 \wedge f"'] & S^1 \wedge S^1 \wedge X_n \arrow[r, "S^1 \wedge \tau"] & S^1 \wedge X_n \wedge S^1 \arrow[r, "S^1 \wedge \sigma_n"] & S^1 \wedge X_{n+1}
\end{tikzcd}$$

The composite through the upper right corner is the stabilization of g and the composite through the lower left corner represents $(-1)^{k+n}(S^1 \wedge f)$. However, this diagram does *not* commute! The two composites from $S^{1+k+n+1}$ to $S^1 \wedge X_n \wedge S^1$ differ by the automorphisms of $S^{1+k+n+1}$ and $S^1 \wedge X_n \wedge S^1$ which interchanges the outer two sphere coordinates in each case. This coordinate change in the source induces multiplication by -1; the coordinate change in the target is a map of degree -1, so after a single suspension it also induces multiplication by -1 on homotopy groups (see Lemma 1.10). Altogether this shows that the diagram above commutes up to homotopy *after one suspension*, and so the suspension map on naive homotopy groups is also surjective.

Since loops and suspension homomorphism are bijective and the triangles (2.5) commute, the unit and counit of the adjunction are $\hat{\pi}_*$-isomorphisms.

For the last claim we note that the iterated suspension isomorphism

$$S^m \wedge - \ : \ \hat{\pi}_k X \ \longrightarrow \ \hat{\pi}_{m+k}(S^m \wedge X)$$

can be made Σ_m-equivariant as follows. We let Σ_m act trivially on the source; we let Σ_m act by conjugation on the target, i.e., by permuting the first m coordinates in the source sphere and the coordinates of S^m in the target of

$$\pi_{m+k+n}(S^m \wedge X_n) \ = \ [S^{m+k+n}, S^m \wedge X] \ ,$$

and then pass to the colimit. Permuting source coordinates acts by sign on unstable homotopy groups, hence also on $\hat{\pi}_{m+k}(S^m \wedge X)$. To compensate this effect, the action on S^m in the colimit must also be by sign. $\square$

2.2. Mapping cone and homotopy fiber. Now we review the mapping cone and the homotopy fiber of a map of based spaces in some detail, along with their relationships to one another and to suspension and loop space. The *(reduced) mapping cone* $C(f)$ of a morphism of based spaces $f : A \longrightarrow B$ is defined by

$$C(f) \ = \ ([0,1] \wedge A) \cup_f B \ . \tag{2.8}$$

Here the unit interval $[0,1]$ is pointed by $0 \in [0,1]$, so that $[0,1] \wedge A$ is the reduced cone of A. The mapping cone comes with an inclusion $i : B \longrightarrow C(f)$ and a projection $p : C(f) \longrightarrow S^1 \wedge A$; the projection sends B to the basepoint and is given on $[0,1] \wedge A$ by $p(x \wedge a) = \mathbf{t}(x) \wedge a$ where

$$\mathbf{t} : [0,1] \longrightarrow S^1 \qquad \text{is defined as} \qquad \mathbf{t}(x) = \frac{2x-1}{x(1-x)} \ .$$

What is relevant about the map $\mathbf{t}$ is not the precise formula, but that it passes to a homeomorphism between the quotient space $[0,1]/\{0,1\}$ and the circle S^1, and that it satisfies $\mathbf{t}(1-x) = -\mathbf{t}(x)$.

We observe that an iteration of the mapping cone construction yields the suspension of A, up to homotopy.

Lemma 2.9. *Let $f : A \longrightarrow B$ be any continuous based map.*

(i) *The collapse map*

$$* \cup p \ : \ C(i) = ([0,1] \wedge B) \cup_i C(f) \ \longrightarrow \ S^1 \wedge A$$

is a based homotopy equivalence.

(ii) *The square*

$$\begin{tikzcd} C(i) \arrow[r, "p\cup *"] \arrow[d, "*\cup p"'] & S^1 \wedge B \arrow[d, "\tau\wedge B"] \\ S^1 \wedge A \arrow[r, "S^1\wedge f"'] & S^1 \wedge B \end{tikzcd}$$

commutes up to natural, based homotopy, where τ is the involution of S^1 given by $\tau(x) = -x$.

(iii) *Let $\beta : Z \longrightarrow B$ be a continuous based map such that the composite $i\beta : Z \longrightarrow C(f)$ is null-homotopic. Then there exists a based map $h : S^1 \wedge Z \longrightarrow S^1 \wedge A$ such that $(S^1 \wedge f) \circ h : S^1 \wedge Z \longrightarrow S^1 \wedge B$ is homotopic to $S^1 \wedge \beta$.*

Proof. (i) A homotopy inverse $r : S^1 \wedge A \longrightarrow ([0,1] \wedge B) \cup_i C(f)$ of $* \cup p$ is defined by the formula

$$r(x \wedge a) \;=\; \begin{cases} 2x \wedge a & \text{in } C(f) \text{ for } 0 \le x \le 1/2, \text{ and} \\ (2-2x) \wedge f(a) & \text{in } [0,1] \wedge B \text{ for } 1/2 \le x \le 1. \end{cases}$$

We give explicit based homotopies between the two composites r and $*\cup p$ and the respective identitiy maps. The space $C(i) = ([0,1] \wedge B) \cup_i C(f)$ is homeomorphic to the quotient of the disjoint union of $[0,1] \wedge B$ and $[0,1] \wedge A$ by the equivalence relation that identifies $1 \wedge f(a)$ in $[0,1] \wedge B$ with $1 \wedge a$ in $[0,1] \wedge A$ for all $a \in A$. So we can define a homotopy on the space $C(i)$ by gluing two compatible homotopies. The homotopy

$$[0,1] \times ([0,1] \wedge B) \;\longrightarrow\; C(i) \;, \quad (t, x \wedge b) \longmapsto (1-t)x \wedge b \quad \text{in } [0,1] \wedge B \;.$$

and the homotopy

$$[0,1] \times ([0,1] \wedge A) \;\longrightarrow\; C(i) \;, \quad (t, x \wedge a) \mapsto \begin{cases} (1+t)x \wedge a & \text{in } C(f) \text{ for } 0 \le x \le 1/(1+t), \text{ and} \\ (2 - x(1+t)) \wedge f(a) & \text{in } [0,1] \wedge B \text{ for } 1/(1+t) \le x \le 1, \end{cases}$$

are compatible, and the combined homotopy starts at $t = 0$ with the identity and ends at $t = 1$ with the map $r \circ (* \cup p)$.

A homotopy from the identity of $S^1 \wedge A$ to $(* \cup p) \circ r$ is given by

$$[0,1] \times (S^1 \wedge A) \;\longrightarrow\; S^1 \wedge A \;, \quad (t,\; x \wedge a) \;\longmapsto\; (1+t)x \wedge a$$

which is to be interpreted as the basepoint if $(1+t)x \ge 1$.

(ii) Again we glue the desired homotopy from two pieces, namely

$$[0,1] \times ([0,1] \wedge B) \;\longrightarrow\; S^1 \wedge B \;, \quad (t,\; x \wedge b) \;\longmapsto\; (1+t-x) \wedge b \;,$$

which has to be interpreted as the basepoint if $x \le t$ and

$$[0,1] \times ([0,1] \wedge A) \;\longrightarrow\; S^1 \wedge B \;, \quad (t,\; x \wedge a) \;\longmapsto\; (t+x-1) \wedge f(a)$$

which has to be interpreted as the basepoint if $t + x \le 1$. The two homotopies are compatible and the combined homotopy starts with the map $(\tau \wedge B) \circ (p \cup *)$ for $t = 0$ and it ends with the map $(S^1 \wedge f) \circ (* \cup p)$ for $t = 1$.

(iii) Let $H : [0,1] \wedge Z \longrightarrow C(f)$ be a based null-homotopy of the composite $i\beta : Z \longrightarrow C(f)$, i.e., $H(1 \wedge x) = i(\beta(x))$ for all $x \in Z$. The composite $p_A H : [0,1] \wedge Z \longrightarrow S^1 \wedge A$ then factors as $p_A H = h p_Z$ for a unique map $h : S^1 \wedge Z \longrightarrow S^1 \wedge A$.

To analyze $(S^1 \wedge f) \circ h$ we compose it with the map $* \cup p_Z : ([0,1] \wedge Z) \cup_{1 \times Z} ([0,1] \wedge Z) \longrightarrow S^1 \wedge Z$ which collapses the second cone and which is a homotopy equivalence by (i). We obtain a sequence of equalities and homotopies

$$\begin{aligned} (S^1 \wedge f) \circ h \circ (* \cup p_Z) \;&=\; (S^1 \wedge f) \circ (* \cup p_A) \circ (([0,1] \wedge \beta) \cup H) \\ &\simeq\; (\tau \wedge B) \circ (p_B \cup *) \circ (([0,1] \wedge \beta) \cup H) \\ &=\; (\tau \wedge B) \circ (S^1 \wedge \beta) \circ (p_Z \cup *) \\ &=\; (S^1 \wedge \beta) \circ (\tau \wedge Z) \circ (p_Z \cup *) \;\simeq\; (S^1 \wedge \beta) \circ (* \cup p_Z) \end{aligned}$$

Here $([0,1] \wedge \beta) \cup H : CZ \cup_{1 \times Z} CZ \longrightarrow CB \cup_i C(f) = C(i)$. The two homotopies result from part (ii) applied to f respectively the identity of Z. Since the map $* \cup p_Z$ is a homotopy equivalence, this proves that $(S^1 \wedge f) \circ h$ is homotopic to $S^1 \wedge \beta$. □

Now we can introduce mapping cones for (symmetric and sequential) *spectra*. The *mapping cone* $C(f)$ of a morphism of symmetric or sequential spectra $f : X \longrightarrow Y$ is defined by

$$C(f)_n \;=\; C(f_n) \;=\; ([0,1] \wedge X_n) \cup_f Y_n \;, \tag{2.10}$$

the reduced mapping cone of $f_n : X_n \longrightarrow Y_n$. In the symmetric (as opposed to 'sequential') case, the symmetric group Σ_n acts on $C(f)_n$ through the given action on X_n and Y_n and trivially on the interval. The inclusions $i_n : Y_n \longrightarrow C(f)_n$ and projections $p_n : C(f)_n \longrightarrow S^1 \wedge X_n$ assemble into morphisms of symmetric (respectively sequential) spectra $i : Y \longrightarrow C(f)$ and $p : C(f) \longrightarrow S^1 \wedge X$.

We define a *connecting homomorphism* $\delta : \hat{\pi}_{1+k} C(f) \longrightarrow \hat{\pi}_k X$ as the composite

$$\hat{\pi}_{1+k} C(f) \xrightarrow{\;p_*\;} \hat{\pi}_{1+k}(S^1 \wedge X) \xrightarrow{\;S^{-1}\wedge -\;} \hat{\pi}_k X \;, \tag{2.11}$$

where the second map is the inverse of the suspension isomorphism $S^1 \wedge - : \hat{\pi}_k X \longrightarrow \hat{\pi}_{1+k}(S^1 \wedge X)$. If we unravel all the definition, we see that δ sends the class represented by a based map $\varphi : S^{1+k+n} \longrightarrow C(f)_n$ to $(-1)^{k+n}$ times the class of the composite

$$S^{1+k+n} \xrightarrow{\;\varphi\;} C(f)_n \xrightarrow{\;p_n\;} S^1 \wedge X_n \xrightarrow{\text{twist}} X_n \wedge S^1 \xrightarrow{\;\sigma_n\;} X_{n+1} \;.$$

Proposition 2.12. *For every morphism $f : X \longrightarrow Y$ of sequential spectra the long sequence of abelian groups*

$$\cdots \longrightarrow \hat{\pi}_k X \xrightarrow{\;f_*\;} \hat{\pi}_k Y \xrightarrow{\;i_*\;} \hat{\pi}_k C(f) \xrightarrow{\;\delta\;} \hat{\pi}_{k-1} X \longrightarrow \cdots$$

is exact.

Proof. We start with exactness at $\hat{\pi}_k Y$. The composite of $f : X \longrightarrow Y$ and the inclusion $Y \longrightarrow C(f)$ is null-homotopic, so it induces the trivial map on $\hat{\pi}_k$. It remains to show that every element in the kernel of i_* is in the image of f_*. Let $\beta : S^{k+n} \longrightarrow Y_n$ represent an element in the kernel. By increasing n, if necessary, we can assume that $i\beta : S^{k+n} \longrightarrow C(f_n)$ is null-homotopic. By Lemma 2.9 (iii) there is a based map $h : S^{1+k+n} \longrightarrow S^1 \wedge X_n$ such that $(S^1 \wedge f_n) \circ h$ is homotopic to $S^1 \wedge \beta$. The composite

$$\tilde{h} \;:\; S^{k+n+1} \xrightarrow{\;\chi_{k+n,1}\;} S^{1+k+n} \xrightarrow{\;\;h\;\;} S^1 \wedge X_n \xrightarrow{\;\tau_{S^1, X_n}\;} X_n \wedge S^1$$

then has the property that $(f_n \wedge S^1) \circ \tilde{h}$ is homotopic to $\beta \wedge S^1$. The map $\sigma_n \circ \tilde{h} : S^{k+n+1} \longrightarrow X_{n+1}$ represents a homotopy class in $\hat{\pi}_k X$ and we have

$$f_*[\sigma_n \circ \tilde{h}] \;=\; [f_{n+1} \circ \sigma_n \circ \tilde{h}] \;=\; [\sigma_n \circ (f_n \wedge S^1) \circ \tilde{h}] \;=\; [\sigma_n \circ (\beta \wedge S^1)] \;=\; [\beta] \;.$$

So the class represented by β is in the image of $f_* : \hat{\pi}_k X \longrightarrow \hat{\pi}_k Y$.

We now deduce the exactness at $\hat{\pi}_k C(f)$ and $\hat{\pi}_{k-1} X$ by comparing the mapping cone sequence for $f : X \longrightarrow Y$ to the mapping cone sequence for the morphism $i : Y \longrightarrow C(f)$ (shifted to the left). The collapse map

$$* \cup p \;:\; C(i) = CY \cup_i C(f) \;\longrightarrow\; S^1 \wedge X$$

is levelwise a homotopy equivalence by Lemma 2.9 (i), and thus induces an isomorphism of naive homotopy groups. Now we consider the diagram

$$\begin{tikzcd} C(f) \arrow[r, "i_i"] \arrow[dr, "p"'] & C(i) \arrow[r, "p \cup *"] \arrow[d, "* \cup p"] & S^1 \wedge Y \arrow[d, "\tau \wedge Y"] \\ & S^1 \wedge X \arrow[r, "S^1 \wedge f"'] & S^1 \wedge Y \end{tikzcd}$$

whose upper row is part of the mapping cone sequence for the morphism $i : Y \longrightarrow C(f)$. The left triangle commutes on the nose and the right triangle commutes up to based homotopy by Lemma 2.9 (ii). The involution $\tau : S^1 \longrightarrow S^1$ has degree -1, so the automorphism $\tau \wedge Y$ of $S^1 \wedge Y$ induces multiplication by -1 on naive homotopy groups. We get a commutative diagram

$$\begin{tikzcd}
\hat{\pi}_k Y \arrow[r, "i_*"] \arrow[d, equal] & \hat{\pi}_k C(f) \arrow[r, "(i_i)_*"] \arrow[d, equal] & \hat{\pi}_k C(i) \arrow[r, "\delta"] \arrow[d, "(S^{-1}\wedge -)\circ(*\cup p)_*"', "\cong"] & \hat{\pi}_{k-1} Y \arrow[d, "(-1)\cdot"] \\
\hat{\pi}_k Y \arrow[r, "i_*"'] & \hat{\pi}_k C(f) \arrow[r, "\delta"'] & \hat{\pi}_{k-1} X \arrow[r, "f_*"'] & \hat{\pi}_{k-1} Y
\end{tikzcd}$$

(using for the right square the naturality of the suspension isomorphism). By the previous paragraph, applied to $i : Y \longrightarrow C(f)$ instead of f, the upper row is exact at $\hat{\pi}_k C(f)$. Since all vertical maps are isomorphisms, the original lower row is exact at $\hat{\pi}_k C(f)$. But the morphism f was arbitrary, so when applied to $i : Y \longrightarrow C(f)$ instead of f, we obtain that the upper row is exact at $\hat{\pi}_k C(i)$. Since all vertical maps are isomorphisms, the original lower row is exact at $\hat{\pi}_{k-1} X$. This finishes the proof. □

A continuous map $f : A \longrightarrow B$ of spaces is an *h-cofibration* if it has the homotopy extension property, i.e., given a continuous map $\varphi : B \longrightarrow X$ and a homotopy $H : [0,1] \times A \longrightarrow X$ such that $H(0,-) = \varphi f$, there is a homotopy $\bar{H} : [0,1] \times B \longrightarrow X$ such that $\bar{H} \circ ([0,1] \times f) = H$ and $\bar{H}(0,-) = \varphi$. An equivalent condition is that the map $[0,1] \times A \cup_{0\times f} B \longrightarrow [0,1] \times B$ has a retraction. For every h-cofibration the map $C(f) \longrightarrow B/A$ which collapses the cone of A to a point is a based homotopy equivalence (see Corollary 2.2 of Appendix A).

Let $f : X \longrightarrow Y$ be a morphism of sequential spectra that is levelwise an h-cofibration. Then by the above, the morphism $c : C(f) \longrightarrow Y/X$ that collapses the cone of X is a level equivalence, and so it induces an isomorphism of homotopy groups. We can thus define another connecting map

$$\delta \ : \ \hat{\pi}_k(Y/X) \ \longrightarrow \ \hat{\pi}_{k-1} X$$

as the composite of the inverse of the isomorphism $c_* : \hat{\pi}_k C(f) \longrightarrow \hat{\pi}_k(Y/X)$ and the connecting homomorphism $\hat{\pi}_k C(f) \longrightarrow \hat{\pi}_{k-1} X$ defined in (2.11).

Corollary 2.13. *Let $f : X \longrightarrow Y$ be a morphism of sequential spectra that is levelwise an h-cofibration and denote by $q : Y \longrightarrow Y/X$ the quotient map. Then the long sequence of naive homotopy groups*

$$\cdots \ \longrightarrow \ \hat{\pi}_k X \ \xrightarrow{f_*} \ \hat{\pi}_k Y \ \xrightarrow{q_*} \ \hat{\pi}_k(Y/X) \ \xrightarrow{\delta} \ \hat{\pi}_{k-1} X \ \longrightarrow \ \cdots$$

is exact.

Now we discuss the *homotopy fiber*, a construction 'dual' to the mapping cone. The homotopy fibre of a morphism $f : A \longrightarrow B$ of based spaces is the fiber product

$$F(f) \ = \ * \times_B B^{[0,1]} \times_B A \ = \ \{(\lambda, a) \in B^{[0,1]} \times A \mid \lambda(0) = *,\ \lambda(1) = f(a)\} \ ,$$

i.e., the space of paths in B starting at the basepoint and equipped with a lift of the endpoint to A. As basepoint of the homotopy fiber we take the pair consisting of the constant path at the basepoint of B and the basepoint of A. The homotopy fiber comes with maps

$$\Omega B \xrightarrow{i} F(f) \xrightarrow{p} A \ ;$$

the map p is the projection to the second factor and the value of the map i on a based loop $\omega : S^1 \longrightarrow B$ is $i(\omega) = (\omega \circ \mathbf{t}, *)$.

We can apply the homotopy fiber levelwise to a morphism of spectra. Let $f : X \longrightarrow Y$ be a morphism between sequential or symmetric spectra. The homotopy fiber $F(f)$ is the spectrum defined by

$$F(f)_n \ = \ F(f_n) \ ,$$

the homotopy fiber of $f_n : X_n \longrightarrow Y_n$. In the symmetric (as opposed to 'sequential') case, the symmetric group Σ_n acts on $F(f)_n$ through the given action on X_n and Y_n and trivially on the interval. The inclusions

$i_n : \Omega(Y_n) \longrightarrow F(f)_n$ and projections $p_n : F(f)_n \longrightarrow X_n$ assemble into morphisms of symmetric (respectively sequential) spectra $i : \Omega Y \longrightarrow F(f)$ and $p : F(f) \longrightarrow X$. Put another way, the homotopy fiber is the fibre product

$$F(f) \;=\; * \times_Y Y^{[0,1]} \times_Y X$$

i.e., the pullback in the cartesian square of spectra:

(2.14)
$$\begin{tikzcd} F(f) \arrow[r, "p"] \arrow[d] & X \arrow[d, "{(*,f)}"] \\ Y^{[0,1]} \arrow[r, "{(\mathrm{ev}_0, \mathrm{ev}_1)}"'] & Y \times Y \end{tikzcd}$$

Here $\mathrm{ev}_i : Y^{[0,1]} \longrightarrow Y$ for $i = 0, 1$ is the ith evaluation map which takes a path $\omega \in Y^{[0,1]}$ to $\omega(i)$, i.e., the start or endpoint.

We define a *connecting homomorphism* $\delta : \hat{\pi}_{1+k} Y \longrightarrow \hat{\pi}_k F(f)$ as the composite

(2.15)
$$\hat{\pi}_{1+k} Y \xrightarrow{\alpha^{-1}} \hat{\pi}_k(\Omega Y) \xrightarrow{i_*} \hat{\pi}_k F(f) ,$$

where $\alpha : \hat{\pi}_k(\Omega Y) \longrightarrow \hat{\pi}_{1+k} Y$ is the loop isomorphism.

We can compare the mapping cone and homotopy fibre as follows. For a map $f : A \longrightarrow B$ of based spaces we define a map $\bar{h} : [0,1] \times F(f) \longrightarrow ([0,1] \wedge A) \cup_f B = C(f)$ by

$$(t, \lambda, a) \;\longmapsto\; \begin{cases} 2t \wedge a & \text{for } 0 \leq t \leq 1/2, \text{ and} \\ \lambda(2-2t) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

We note that the two formulas match at $t = 1/2$ because $\lambda(1) = f(a) = 1 \wedge a$ in $C(f)$. Since $\bar{h}(0, \lambda, a)$ and $\bar{h}(1, \lambda, a)$ are the basepoint of the mapping cone for all (λ, a) in $F(f)$, the map $\bar{h}$ factors over a based map

$$h \;:\; S^1 \wedge F(f) \;\longrightarrow\; C(f) ,$$

which satisfies $h \circ p = \bar{h}$ and is natural in f. So for a morphism $f : X \longrightarrow Y$ of spectra, sequential or symmetric, the maps h for the various levels together form a natural morphism of (sequential respectively symmetric) spectra

(2.16)
$$h \;:\; S^1 \wedge F(f) \;\longrightarrow\; C(f) .$$

Proposition 2.17. *For every morphism $f : X \longrightarrow Y$ of sequential spectra the long sequence of abelian groups*

$$\cdots \longrightarrow \hat{\pi}_k F(f) \xrightarrow{p_*} \hat{\pi}_k X \xrightarrow{f_*} \hat{\pi}_k Y \xrightarrow{\delta} \hat{\pi}_{k-1} F(f) \xrightarrow{p_*} \hat{\pi}_{k-1} X \longrightarrow \cdots$$

is exact and the morphism $h : S^1 \wedge F(f) \longrightarrow C(f)$ is a $\hat{\pi}_*$-isomorphism.

Proof. The long sequence is exact because it is obtained from the unstable long exact sequences for the homotopy fiber sequences $F(f_n) \longrightarrow X_n \longrightarrow Y_n$ by passage to the colimit (which is exact).

For showing that h is a $\hat{\pi}_*$-isomorphism it suffices to show that the composite $h_* \circ (S^1 \wedge -) : \hat{\pi}_k F(f) \longrightarrow \hat{\pi}_{1+k} C(f)$ is an isomorphism. We claim that the diagram

$$\begin{tikzcd} \hat{\pi}_{1+k} Y \arrow[r, "\delta"] \arrow[d, "(-1)\cdot"'] & \hat{\pi}_k F(f) \arrow[r, "p_*"] \arrow[d, "h_* \circ (S^1 \wedge -)"'] & \hat{\pi}_k X \arrow[d, equal] \\ \hat{\pi}_{1+k} Y \arrow[r, "i_*"'] & \hat{\pi}_{1+k} C(f) \arrow[r, "\delta"'] & \hat{\pi}_k X \end{tikzcd}$$

commutes. The morphism $h_* \circ (S^1 \wedge -) : \hat{\pi}_k F(f) \longrightarrow \hat{\pi}_{1+k} C(f)$ and the identity maps of the naive homotopy groups of X and Y thus give a natural map from the long exact sequence of the homotopy fiber to the long exact sequence of the mapping cone, with an extra sign. A sign does not affect exactness of a sequence, and so the five lemma shows that $h_* \circ (S^1 \wedge -)$ is an isomorphism. Hence h is a $\hat{\pi}_*$-isomorphism.

We still have to justify the commutativity of the previous diagram. For the right square this is the definition of the connecting homomorphism, naturality of the suspension isomorphism and the fact that the composite

$$S^1 \wedge F(f) \xrightarrow{h} C(f) \xrightarrow{p} S^1 \wedge X$$

is homotopic to $S^1 \wedge p$ via the homotopy

$$[0,1] \times (S^1 \wedge F(f)) \longrightarrow S^1 \wedge X \ , \quad (t,\ x \wedge (\lambda, a)) \longmapsto \begin{cases} 2x/(2-t) \wedge a & \text{for } 0 \leq x \leq 1 - t/2, \text{ and} \\ * & \text{for } 1 - t/2 \leq x \leq 1. \end{cases}$$

(to be interpreted levelwise). For the left square we need that the diagram

$$\begin{tikzcd} S^1 \wedge \Omega Y \arrow[r, "\tau \wedge i"] \arrow[d, "\epsilon"'] & S^1 \wedge F(f) \arrow[d, "h"] \\ Y \arrow[r, "i"'] & C(f) \end{tikzcd}$$

commutes up to based homotopy, where ϵ is the adjunction counit. One possible such homotopy is

$$\begin{aligned} [0,1] \times (S^1 \wedge \Omega Y) &\longrightarrow \quad C(f) \\ (t,\ x \wedge \omega) \quad &\longmapsto \begin{cases} * & \text{for } 0 \leq x \leq t/2, \text{ and} \\ \omega(2(1-t)/(2-s)) & \text{for } t/2 \leq x \leq 1. \end{cases} \end{aligned}$$

Given this, we have

$$\begin{aligned} h_*(S^1 \wedge \delta(y)) &= h_*(S^1 \wedge i_*(\alpha^{-1}(y))) = (h \circ (S^1 \wedge i))_*(S^1 \wedge \alpha^{-1}(y)) \\ &= -(i \circ \epsilon)_*(S^1 \wedge \alpha^{-1}(y)) =_{(2.5)} -i_*(y) \end{aligned}$$

and this finishes the proof. □

For every Serre fibration $\varphi : E \longrightarrow B$ of topological spaces the map $c : F \longrightarrow F(\varphi)$ from the strict fiber to the homotopy fiber that sends $e \in F$ to (const_*, e). is a weak equivalence. We let $f : X \longrightarrow Y$ be a morphism of sequential spectra that is levelwise a Serre fibration; then by the above the morphism $c : F \longrightarrow F(f)$ from the strict fibre to the homotopy fiber of f is a level equivalence. So we can define another connecting morphism

$$\delta \ : \ \pi_k Y \ \longrightarrow \ \hat{\pi}_{k-1} F$$

as the composite of the connecting homomorphism $\hat{\pi}_k Y \longrightarrow \hat{\pi}_{k-1} F(f)$ defined in (2.15) and the inverse of the isomorphism $c_* : \hat{\pi}_{k-1} F(f) \longrightarrow \hat{\pi}_{k-1} F$.

Corollary 2.18. *Let $f : X \longrightarrow Y$ be a morphism of sequential spectra that is levelwise a Serre fibration and denote by $i : F \longrightarrow X$ the inclusion of the fiber over the basepoint. Then the long sequence of naive homotopy groups*

$$\cdots \longrightarrow \hat{\pi}_k F \xrightarrow{i_*} \hat{\pi}_k X \xrightarrow{f_*} \hat{\pi}_k Y \xrightarrow{\delta} \hat{\pi}_{k-1} F \longrightarrow \cdots$$

is exact.

We draw some consequences of previous propositions.

Proposition 2.19. (i) *For every family of sequential spectra $\{X^i\}_{i \in I}$ and every integer k the canonical map*

$$\bigoplus_{i \in I} \hat{\pi}_k X^i \longrightarrow \hat{\pi}_k \left(\bigvee_{i \in I} X^i \right)$$

is an isomorphism of abelian groups.

(ii) *For every finite indexing set I, every family $\{X^i\}_{i\in I}$ of sequential spectra and every integer k the canonical map*
$$\hat{\pi}_k\left(\prod_{i\in I} X^i\right) \longrightarrow \prod_{i\in I}\hat{\pi}_k X^i$$
is an isomorphism of abelian groups.

(iii) *For every family of sequential spectra the canonical morphism from the wedge to the weak product is a $\hat{\pi}_*$-isomorphism. In particular, for every finite family of symmetric spectra the canonical morphism from the wedge to the product is a $\hat{\pi}_*$-isomorphism.*

(iv) *For every based CW-complex K the functor $K \wedge -$ preserves $\hat{\pi}_*$-isomorphisms of sequential spectra.*

(v) *For every finite based CW-complex K the functor* $\mathrm{map}(K,-)$ *preserves $\hat{\pi}_*$-isomorphisms of sequential spectra.*

PROOF. (i) We first show the special case of two summands. If A and B are two symmetric spectra, then the wedge inclusion $i_A : A \longrightarrow A \vee B$ has a retraction. So the associated long exact homotopy group sequence of Proposition 2.12 splits into short exact sequences
$$0 \longrightarrow \hat{\pi}_k A \xrightarrow{(i_A)_*} \hat{\pi}_k(A \vee B) \xrightarrow{i_*} \hat{\pi}_k(C(i_A)) \longrightarrow 0\ .$$
The mapping cone $C(i_A)$ is isomorphic to $(CA) \vee B$ and thus homotopy equivalent to B. So we can replace $\hat{\pi}_k(C(i_A))$ by $\hat{\pi}_k B$ and conclude that $\hat{\pi}_k(A \vee B)$ splits as the sum of $\hat{\pi}_k A$ and $\hat{\pi}_k B$, via the canonical map. The case of a finite indexing set I now follows by induction, and the general case follows since homotopy groups of symmetric spectra commute with filtered colimits [more precisely, the image of every compact space in an infinite wedge lands in a finite wedge].

(ii) Unstable homotopy groups commute with products, which for finite indexing sets are also sums, which commute with filtered colimits.

(iii) This is a direct consequence of (i) and (ii). More precisely, for finite indexing set I and every integer k the composite map
$$\bigoplus_{i\in I}\hat{\pi}_k X^i \longrightarrow \hat{\pi}_k(\bigvee_{i\in I} X^i) \longrightarrow \hat{\pi}_k(\prod_{i\in I} X^i) \longrightarrow \prod_{i\in I}\hat{\pi}_k X^i$$
is an isomorphism, where the first and last maps are the canonical ones. These canonical maps are isomorphisms by parts (i) respectively (ii), hence so is the middle map.

For a finite CW-complex K we prove parts (iv) and (v) simultaneously by induction on the number of cells. If K consists only of the basepoint, then $K \wedge X$ and $\mathrm{map}(K,X)$ are trivial and the claim is trivially true. Now suppose we have shown the claim for K and K' is obtained from K by attaching an n-cell. Then the mapping cone $C(i)$ of the inclusion $i : K \longrightarrow K'$ is based homotopy equivalent to an n-sphere. Hence smashing with $C(i)$ shifts the naive homotopy groups by Proposition 2.6, and thus preserves $\hat{\pi}_*$-isomophisms. The mapping cone of the morphism $i \wedge X : K \wedge X \longrightarrow K' \wedge X$ is naturally isomorphic to $C(i) \wedge X$; since $K \wedge -$ and $C(i) \wedge -$ preserve $\hat{\pi}_*$-isomorphisms, the long exact sequence of Proposition 2.12 and the five lemma show that $K' \wedge X$ preserves $\hat{\pi}_*$-isomorphisms.

The induction step for $\mathrm{map}(K,X)$ is exactly dual. Since $C(i)$ is homotopy equivalent to S^n, the spectrum $\mathrm{map}(C(i),X)$ is homotopy equivalent to $\Omega^n X$, and hence the functor $\mathrm{map}(C(i),-)$ preserves $\hat{\pi}_*$-isomophisms by Proposition 2.6. The homotopy fiber of the morphism $\mathrm{map}(i,X) : \mathrm{map}(K',X) \longrightarrow \mathrm{map}(K,X)$ is naturally isomorphic to $\mathrm{map}(C(i),X)$; since $\mathrm{map}(K,-)$ and $\mathrm{map}(C(i),-)$ preserve $\hat{\pi}_*$-isomorphisms, the long exact sequence of Proposition 2.17 and the five lemma show that $\mathrm{map}(K',-)$ preserves $\hat{\pi}_*$-isomorphisms. This finishes the proof of claims (iv) and (v) for finite CW-complexes.

An infinite CW-complex is the filtered colimit, along h-cofibrations, of its finite subcomplexes. Since homotopy groups commute with such filtered colimits, the general case of part (iv) follows from the special case above. □

The previous Proposition is stated for sequential spectra. However, the forgetful functor from symmetric to sequential spectra preserves limits, colimits and smash products with and maps from a based spaces. So all the conclusion of the corollary also hold for symmetric spectra.

Remark 2.20. The restriction to *finite* indexing sets in parts (ii) of the previous corollary is essential, and it ultimately comes from the fact that infinite products do not in general commute with sequential colimits. Here is an explicit example: we consider the symmetric spectra $\mathbb{S}^{\leq i}$ obtained by truncating the sphere spectrum above level i, i.e.,

$$(\mathbb{S}^{\leq i})_n \;=\; \begin{cases} S^n & \text{for } n \leq i, \\ * & \text{for } n \geq i+1 \end{cases}$$

with structure maps as a quotient spectrum of $\mathbb{S}$. Then $\mathbb{S}^{\leq i}$ has trivial homotopy groups for all i. The 0th naive homotopy group of the product $\prod_{i\geq 1}\mathbb{S}^{\leq i}$ is the colimit of the sequence of maps

$$\prod_{i\geq n}\pi_n S^n \;\longrightarrow\; \prod_{i\geq n+1}\pi_{n+1}S^{n+1}$$

which first projects away from the factor indexed by $i = n$ and then takes a product of the suspensions homomorphisms $-\wedge S^1 : \pi_n S^n \longrightarrow \pi_{n+1}S^{n+1}$. The colimit is thus isomorphic to the quotient of an infinite product of copies of the group $\mathbb{Z}$ by the direct sum of the same number of copies of $\mathbb{Z}$. Hence the right hand side of the canonical map

$$\hat{\pi}_0\left(\prod_{i\geq 1}\mathbb{S}^{\leq i}\right) \;\longrightarrow\; \prod_{i\geq 1}\hat{\pi}_0(\mathbb{S}^{\leq i})$$

is trivial, while the left hand side is not.

In Example 4.33 below we give a different example for the fact that a product of $\hat{\pi}_*$-isomorphisms need not be a $\hat{\pi}_*$-isomorphism, namely a spectrum X all of whose naive homotopy groups are trivial, but such that $\hat{\pi}_0(X^{\mathbb{N}})$ is non-zero. That example shows in particular that the restriction to *finite* K in Proposition 2.19 (v) is essential.

Example 2.21 (Telescope and diagonal of a sequence)**.** We will sometimes be confronted with a sequence of morphisms of sequential or symmetric spectra

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} X^2 \xrightarrow{f^2} \cdots \tag{2.22}$$

of which we want to take a kind of colimit in a homotopy invariant way, and such that the homotopy groups of the 'colimit' are the colimits of the homotopy groups. We describe two constructions which do this job, the mapping telescope and the diagonal.

The *mapping telescope* $\mathrm{tel}_i\, X^i$ of the sequence (2.22) is a classical construction for spaces which we apply levelwise to symmetric spectra. It is defined as the coequalizer of two maps of (sequential or symmetric) spectra

$$\bigvee_{i\geq 0} X^i \quad \rightrightarrows \quad \bigvee_{i\geq 0}\, [i,i+1]^+ \wedge X^i$$

Here $[i,i+1]$ denotes a copy of the unit interval (when in the context of spaces) respectively the 1-simplex $\Delta[1]$ (when in the context of simplicial sets). One of the morphisms takes X^i to $\{i+1\}^+\wedge X^i$ by the identity, the other one takes X^i to $\{i+1\}^+\wedge X^{i+1}$ by the morphism f^i. [sequential vs symmetric]

The *diagonal* $\mathrm{diag}_i\, X^i$ of the sequence (2.22) is the spectrum given by

$$(\mathrm{diag}_i\, X^i)_n \;=\; X^n_n\ ,$$

i.e., we take the n-th level of the n-th (sequential or symmetric) spectrum. In the symmetric case, we use given Σ_n-action on this space as part of the symmetric spectrum X^n. The structure map $(\mathrm{diag}_i\, X^i)_n \wedge S^1 \longrightarrow$

$(\mathrm{diag}_i X^i)_{n+1}$ is the composite around either way in the commutative square:

$$\begin{tikzcd} X^n_n \wedge S^1 \arrow[r, "\sigma^n_n"] \arrow[d, "f^n_n \wedge \mathrm{Id}"'] & X^n_{n+1} \arrow[d, "f^n_{n+1}"] \\ X^{n+1}_n \wedge S^1 \arrow[r, "\sigma^{n+1}_n"'] & X^{n+1}_{n+1} \end{tikzcd}$$

Lemma 2.23. *For every sequence of sequential spectra* (2.22)*, there is a chain of two natural $\hat{\pi}_*$-isomorphisms between the diagonal* $\mathrm{diag}_i X^i$ *and the mapping telescope* $\mathrm{tel}_i X^i$ *of the sequence. In particular this gives natural isomorphisms of naive homotopy groups*

$$\hat{\pi}_k(\mathrm{diag}_i X^i) \cong \mathrm{colim}_i\ \hat{\pi}_k(X^i)\ .$$

If the sequence consists of morphisms of symmetric spectra, then the chain is through morphisms of symmetric spectra as well.

Proof. We use the 'partial telescopes' $\mathrm{tel}_{[0,n]} X^i$, the coequalizer of two maps of (symmetric or sequential) spectra

$$\bigvee_{i=0}^{n-1} X^i \quad \rightrightarrows \quad \bigvee_{i=0}^{n} [i,i+1]^+ \wedge X^i$$

defined as before. The spectrum $\mathrm{tel}_{[0,n]} X^i$ includes into the next spectrum $\mathrm{tel}_{[0,n+1]} X^i$ with (categorical) colimit the mapping telescope. The morphism $c_n : \mathrm{tel}_{[0,n]} X^i \longrightarrow X^n$ which projects each wedge summand $[i,i+1]^+ \wedge X^i$ onto X^i and then applies the morphism $f^{n-1} \cdots f^i : X^i \longrightarrow X^n$ is a homotopy equivalence. The commutative diagram of spectra

$$\begin{tikzcd} \mathrm{tel}_{[0,0]} X^i \arrow[r] \arrow[d, "c_0"'] & \mathrm{tel}_{[0,1]} X^i \arrow[r] \arrow[d, "c_1"'] & \mathrm{tel}_{[0,2]} X^i \arrow[r] \arrow[d, "c_2"] & \cdots \\ X^0 \arrow[r, "f^0"'] & X^1 \arrow[r, "f^1"'] & X^2 \arrow[r, "f^2"'] & \cdots \end{tikzcd}$$

induces a morphism

$$\mathrm{diag}_n(\mathrm{tel}_{[0,n]} X^i) \longrightarrow \mathrm{diag}_n X^n$$

on diagonals which is thus levelwise a homotopy equivalence, hence a $\hat{\pi}_*$-isomorphism. On the other hand we have a morphism of symmetric spectra

$$\mathrm{diag}_n(\mathrm{tel}_{[0,n]} X^i) \longrightarrow \mathrm{tel}_i X^i \tag{2.24}$$

which is levelwise given by the inclusion of a partial telescope in the full mapping telescope. This morphism is a $\hat{\pi}_*$-isomorphism by 'cofinality': indeed, any basepoint in a compactly generated weak Hausdorff space is closed and the telescope has the weak topology with the respect to the filtration by the closed subspaces $\mathrm{tel}_{[0,n]} X^i$. So every continous map from a compact space to the reduced mapping telescope of a sequence of based compactly generated weak Hausdorff spaces has its image in one of the final stages. In particular, the unstable homotopy group of the mapping telescope is the colimit of the sequence of homotopy groups A.2.8. So the right hand side of (2.24) is a sequential colimit of groups which are themselves sequential colimits, and it is thus the colimit over the partially ordered set $\mathbb{N} \times \mathbb{N}$ of the functor $(n,i) \mapsto \pi_{k+n}(X^i_n)$. The group $\hat{\pi}_k(\mathrm{diag}_i \mathrm{tel}_{[0,n]} X^i)$ is isomorphic to the colimit over the diagonal terms in this system. Since the diagonal embedding $\mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ is cofinal, the colimit over the diagonal terms is isomorphic to the colimit over $\mathbb{N} \times \mathbb{N}$, which proves the isomorphism (2.24). □

Let me point out an advantage of the diagonal construction over the mapping telescope of a sequence of spectra: the diagonal construction has nicer formal and in particular multiplicative properties, as we shall see, for example, in Examples 6.53 and 3.54.

3. Basic constructions

In this section we discuss various constructions involving symmetric spectra and symmetric ring spectra. We start by introducing symmetric spectra of simplicial sets, a close relative of the category of symmetric spectra of spaces. In Section 3.2 we discuss constructions which produce new symmetric spectra from old ones. We define limits and colimits (3.5), smash product with and functions from a space (3.6), shifts (3.9), induction (3.17), free (3.20) and semifree symmetric spectra (3.23), twisted smash product with a Σ_m-space (3.27), mapping spaces (3.36) and internal Hom spectra (3.38).

In Section 3.3 we explain some elementary constructions involving ring spectra: endomorphism ring spectra (3.41), monoid ring spectra (3.42), matrix ring spectra (3.44), inverting an integer (3.47) or an element in π_0 of a symmetric ring spectrum (3.48) and adjoining roots of unity to a symmetric ring spectrum (3.49) and algebraic K-theory spectra (3.50).

3.1. Symmetric spectra of simplicial sets. After the interlude about naive homotopy groups of sequential spectra we return to symmetric spectra. We will often use a variation on the notions of symmetric spectrum and symmetric ring spectrum where topological spaces are replaced by simplicial sets. We can go back and forth between the two concepts using the adjoint functors of geometric realization and singular complex, as we explain below.

Definition 3.1. A *symmetric spectrum of simplicial sets* consists of the following data:

- a sequence of pointed simplicial sets X_n for $n \geq 0$
- a basepoint preserving simplicial left action of the symmetric group Σ_n on X_n for each $n \geq 0$
- pointed morphisms $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ for $n \geq 0$,

such that for all $n, m \geq 0$, the composite

$$X_n \wedge S^m \xrightarrow{\sigma_n \wedge \mathrm{Id}} X_{n+1} \wedge S^{m-1} \xrightarrow{\sigma_{n+1} \wedge \mathrm{Id}} \cdots \xrightarrow{\sigma_{n+m-2} \wedge \mathrm{Id}} X_{n+m-1} \wedge S^1 \xrightarrow{\sigma_{n+m-1}} X_{n+m}$$

is $\Sigma_n \times \Sigma_m$-equivariant. Here S^1 denotes the 'small simplicial circle' $S^1 = \Delta[1]/\partial\Delta[1]$ and $S^m = S^1 \wedge \ldots \wedge S^1$ is the m-th smash power, with Σ_m permuting the factors.

Morphisms of symmetric spectra of simplicial sets are defined just as for symmetric spectra of spaces. We denote the category of symmetric spectra of simplicial sets by $\mathcal{S}p_{\mathbf{sS}}$. There are many situations in which symmetric spectra of spaces and simplicial sets can be used interchangeably. We then often use the term 'symmetric spectrum' and the notation $\mathcal{S}p$ without an index $\mathbf{T}$ or $\mathbf{sS}$ as a generic term/symbol for either the category of symmetric spectra of spaces or simplicial sets.

We similarly define a *symmetric ring spectrum of simplicial sets* by replacing 'space' by 'simplicial set' in Definition 1.3, while also replacing the topological circle S^1 by the simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$ and replacing S^m by the m-fold smash power $S^m = S^1 \wedge \ldots \wedge S^1$.

As we already mentioned we can apply the adjoint functors 'geometric realization', denoted $|-|$, and 'singular complex', denoted $\mathcal{S}$, levelwise to go back and forth between topological and simplicial symmetric spectra. We recall the definitions and main properties of these functors in Appendix A.3. We use that geometric realization is a 'strong symmetric monoidal' functor, i.e., there is natural, unital, associative and commutative homeomorphism

$$r_{A,B} \ : \ |A| \wedge |B| \ \cong \ |A \wedge B| \tag{3.2}$$

for pointed simplicial sets A and B. Indeed, the canonical continuous map $|A \times B| \longrightarrow |A| \times |B|$ is a homeomorphism (since we work in the category of compactly generated topological spaces) and the homeomorphism $r_{A,B}$ is obtained from there by passing to quotients.

We already allowed ourselves the freedom to use the same symbols for the topological and simplicial spheres. The justification is that the geometric realization of the simplicial S^m is homeomorphic to the topological S^m. Let us be completely explicit about how we identify these two spaces. Since the simplicial set $S^1 = \Delta[1]/\partial\Delta[1]$ is generated by its non-degenerate 1-simplex, its realization $|S^1|$ is a quotient space of the topological 1-simplex $\underline{\Delta}[1] = \{(x, y) \in \mathbb{R}^2 \,|\, x, y \geq 0, x + y = 1\}$. We agree to use the homeomorphism

$h : |S^1| \longrightarrow S^1$ which sends the open simplex inside $|S^1|$ to $\mathbb{R}$ by $(x, y) \mapsto x/y - y/x$. Then we obtain a Σ_m-equivariant homeomorphism as the composite

$$(3.3) \qquad |S^m| = |S^1 \wedge \cdots \wedge S^1| \xrightarrow{r^{-1}_{S^1,\dots,S^1}} |S^1| \wedge \cdots \wedge |S^1| \xrightarrow{h^{(m)}} S^1 \wedge \cdots \wedge S^1 \cong S^m \ .$$

Now we can define the adjoint functors 'geometric realization' and 'singular complex' for symmetric spectra. If Y is a symmetric spectrum of simplicial sets we define a symmetric spectrum $|Y|$ of topological spaces by $|Y|_n = |Y_n|$ with structure maps

$$|Y_n| \wedge S^1 \xrightarrow{\mathrm{Id} \wedge h^{-1}} |Y_n| \wedge |S^1| \xrightarrow{r_{Y_n,S^1}} |Y_n \wedge S^1| \xrightarrow{|\sigma_n|} |Y_{n+1}| \ .$$

Commutativity of the isomorphism (3.2) guarantees that the equivariance condition for the iterated structure map σ^m is inherited by the realization $|Y|$.

Adjoint to the homeomorphism (3.2) is a 'lax symmetric monoidal' transformation of pointed simplicial sets, i.e., a natural, unital, associative and commutative morphism $\hat{r}_{X,Y} : \mathcal{S}(X) \wedge \mathcal{S}(Y) \longrightarrow \mathcal{S}(X \wedge Y)$ for pointed spaces X and Y. So if X is a symmetric spectrum of topological spaces, then we get a symmetric spectrum $\mathcal{S}(X)$ of simplicial sets by $\mathcal{S}(X)_n = \mathcal{S}(X_n)$ with structure map

$$\mathcal{S}(X_n) \wedge S^1 \xrightarrow{\mathrm{Id} \wedge \hat{h}} \mathcal{S}(X_n) \wedge \mathcal{S}(S^1) \xrightarrow{\hat{r}_{X_n,S^1}} \mathcal{S}(X_n \wedge S^1) \xrightarrow{\mathcal{S}(\sigma_n)} \mathcal{S}(X_{n+1}) \ .$$

Here $\hat{h} : S^1 \longrightarrow \mathcal{S}(S^1)$ is the morphism of pointed simplicial sets which is adjoint to the $h : |S^1| \longrightarrow S^1$. We use the adjunction unit and counit between $|-|$ and $\mathcal{S}$ levelwise to make geometric realization and singular complex into adjoint functors between topological and simplicial symmetric spectra.

Geometric realization and singular complex are lax symmetric monoidal functors with respect to the smash products of pointed spaces and pointed simplicial sets (geometric realization is even strong symmetric monoidal, i.e., commutes with the smash product up to homeomorphism). So both constructions preserve multiplications, so they take ring spectra to ring spectra and preserve commutativity.

The homotopy groups of a symmetric spectrum based on simplicial sets Y are defined as the homotopy groups of the geometric realization $|Y|$.

A symmetric spectrum of simplicial sets Y is an *Ω-spectrum* respectively *positive Ω-spectrum* if the geometric realization $|Y|$ is an Ω-spectrum, respectively positive Ω-spectrum, of topological spaces. A symmetric spectrum of simplicial sets Y is thus an Ω-spectrum if and only if for all $n \geq 0$ the map $|Y_n| \longrightarrow \Omega|Y_{n+1}|$ which is adjoint to the composite

$$|Y_n| \wedge S^1 \xrightarrow{\cong} |Y_n \wedge S^1| \xrightarrow{|\sigma_n|} |Y_{n+1}|$$

is a weak homotopy equivalence. Our definition of 'Ω-spectrum' differs slightly from other sources in that we do *not* require that each simplicial set Y_n has to be a Kan complex. If Y is a symmetric spectrum of simplicial sets in which all the Y_n's are Kan, then the natural maps $|\Omega Y_n| \longrightarrow \Omega|Y_n|$ adjoint to

$$|\Omega Y_n| \wedge S^1 \longrightarrow |(\Omega Y_n) \wedge S^1| \xrightarrow{|\text{evaluate}|} |Y_n|$$

are weak equivalences, and so Y is an Ω-spectrum in our sense if and only if the morphisms of simplicial sets $\tilde{\sigma}_n : Y_n \longrightarrow \Omega Y_{n+1}$ adjoint to the structure maps are weak equivalences.

Proposition 3.4. *For every sequential or symmetric spectrum of spaces A the natural map $S^1 \wedge \mathcal{S}(A) \longrightarrow \mathcal{S}(S^1 \wedge A)$ is a $\hat{\pi}_*$-isomorphism. For every morphism $f : A \longrightarrow B$ of sequential or symmetric spectra of spaces the natural map $C\mathcal{S}(f) \longrightarrow \mathcal{S}(Cf)$ is a $\hat{\pi}_*$-isomorphism.*

3.2. Constructions. We discuss various constructions which produce new symmetric spectra from old ones. Whenever possible, we describe the effect that a certain construction has on the naive homotopy groups.

Example 3.5 (Limits and colimits)**.** The category of symmetric spectra has all limits and colimits, and they are defined levelwise. Let us be a bit more precise and consider a functor $F : J \longrightarrow \mathcal{S}p$ from a small category J to the category of symmetric spectra (of spaces or simplicial sets). Then we define a symmetric spectrum $\operatorname{colim}_J F$ in level n by

$$(\operatorname{colim}_J F)_n \;=\; \operatorname{colim}_{j\in J} F(j)_n \;,$$

the colimit being taken in the category of pointed Σ_n-spaces (or pointed Σ_n-simplicial sets). The structure map is the composite

$$(\operatorname{colim}_{j\in J} F(j)_n)\wedge S^1 \cong \operatorname{colim}_{j\in J}(F(j)_n \wedge S^1) \xrightarrow{\operatorname{colim}_J \sigma_n} \operatorname{colim}_{j\in J} F(j)_{n+1} \;;$$

here we exploit that smashing with S^1 is a left adjoint, and thus the natural map $\operatorname{colim}_{j\in J}(F(j)_n\wedge S^1) \longrightarrow (\operatorname{colim}_{j\in J} F(j)_n)\wedge S^1$ is an isomorphism, whose inverse is the first map above.

The argument for inverse limits is similar, but we have to use that structure maps can also be defined in the adjoint form. We can take

$$(\lim_J F)_n \;=\; \lim_{j\in J} F(j)_n \;,$$

and the structure map is adjoint to the composite

$$\lim_{j\in J} F(j)_n \xrightarrow{\lim_J \hat\sigma_n} \lim_{j\in J}\Omega(F(j)_{n+1}) \cong \Omega\left(\lim_{j\in J} F(j)_{n+1}\right) \;.$$

The inverse limit, calculated levelwise, of a diagram of symmetric *ring* spectra and homomorphisms is again a symmetric ring spectrum. In other words, symmetric ring spectra have limits and the forgetful functor to symmetric spectra preserves them. Symmetric ring spectra also have *co*-limits, but they are not preserved by the forgetful functor.

Example 3.6 (Smash products with and functions from spaces)**.** If K is pointed space and X a symmetric spectrum, we can define two new symmetric spectra $K\wedge X$ and X^K by smashing with K or taking maps from K levelwise; the structure maps and symmetric group actions do not interact with K.

In more detail we set

$$(K\wedge X)_n \;=\; K\wedge X_n \qquad \text{respectively} \qquad (X^K)_n \;=\; X_n^K \;=\; \operatorname{map}(K, X_n)$$

for $n\geq 0$. The symmetric group Σ_n acts through its action on X_n. The structure map is given by the composite

$$(K\wedge X)_n\wedge S^1 \;=\; K\wedge X_n\wedge S^1 \xrightarrow{\operatorname{Id}\wedge\sigma_n} K\wedge X_{n+1} \;=\; (K\wedge X)_{n+1}$$

respectively by the composite

$$X_n^K\wedge S^1 \;\longrightarrow\; (X_n\wedge S^1)^K \xrightarrow{\sigma_n^K} X_{n+1}^K$$

where the first map is adjoint to the evaluation map $X_n^K\wedge S^1\wedge K\longrightarrow X_n\wedge S^1$ and the second is application of $\operatorname{map}(K,-)$ to the structure map of X. For example, the spectrum $K\wedge\mathbb{S}$ is equal to the suspension spectrum $\Sigma^\infty K$.

Just as the functors $K\wedge -$ and $\operatorname{map}(K,-)$ are adjoint on the level of based spaces (or simplicial sets), the two functors just introduced are an adjoint pair on the level of symmetric spectra. The adjunction

$$\hat{}\;:\; \mathcal{S}p(X, Y^K) \xrightarrow{\cong} \mathcal{S}p(K\wedge X, Y) \tag{3.7}$$

takes a morphism $f : X\longrightarrow Y^K$ to the morphism $\hat f : K\wedge X\longrightarrow Y$ whose n-th level $\hat f_n : K\wedge X_n\longrightarrow Y_n$ is given by $\hat f_n(k\wedge x) = f_n(x)(k)$.

We note that if X is an Ω-spectrum, then so is X^K, provided we also assume that

- K is cofibrant (for example a CW-complex) when in the context of topological spaces, or
- X is levelwise a Kan complex when in the context of simplicial sets.

Indeed, under either hypothesis, the mapping space functor $\operatorname{map}(K,-)$ takes the weak equivalence $\tilde\sigma_n : X_n\longrightarrow\Omega X_{n+1}$ to a weak equivalence

$$X_n^K = \operatorname{map}(K, X_n) \xrightarrow{\operatorname{map}(K,\tilde\sigma_n)} \operatorname{map}(K,\Omega X_{n+1}) \cong \Omega(X_{n+1}^K) \;.$$

Loops and suspensions are the special case $K = S^1$ of this discussion (where S^1 denotes the onepoint compactification of $\mathbb{R}$ in the topological context, and the simplicial set $\Delta[1]/\partial\Delta[1]$ in the simplicial context).

We obtain two adjoint constructions, the suspension $S^1 \wedge X$ and the loop spectrum $\Omega X = X^{S^1}$ of a symmetric spectrum X. These spectra a levelwise given by $(S^1 \wedge X)_n = S^1 \wedge X_n$ respectively $(\Omega X)_n = \Omega(X_n)$ where the structure maps and symmetric groups actions do not interact with the new suspension respectively loop coordinate. As a special case of (3.7) we have an adjunction $\hat{}: \mathcal{S}p(X, \Omega Y) \cong \mathcal{S}p(S^1 \wedge X, Y)$ that already showed up in (2.2) during the discussion of the loop and suspension isomorphism.

Proposition 3.8. *Let $f : S^1 \wedge A \longrightarrow X$ be a morphism of symmetric spectra. In the simplicial context, suppose also that X is levelwise Kan. Then f is a $\hat{\pi}_*$-isomorphism if and only if its adjoint $\hat{f} : A \longrightarrow \Omega X$ is a $\hat{\pi}_*$-isomorphism.*

Proof. The morphism f and its ajoint are related by $f = \epsilon \circ (S^1 \wedge \hat{f})$ where $\epsilon : S^1 \wedge \Omega X$ is the counit of the adjunction. In the context of spectra of spaces, the counit is a $\hat{\pi}_*$-isomorphism by Proposition 2.6. In the context of spectra of simplicial sets the morphism $|\Omega X| \longrightarrow \Omega |X|$ is a level equivalence since X is levelwise Kan, so the counit is also a $\hat{\pi}_*$-isomorphism. We conclude that the morphism f is a $\hat{\pi}_*$-isomorphism if and only if the morphism $S^1 \wedge \hat{f} : S^1 \wedge A \longrightarrow S^1 \wedge \Omega X$ is. Since suspension shifts homotopy groups, this happens if and only if f is a $\hat{\pi}_*$-isomorphism. □

Example 3.9 (Shift)**.** The *shift* of a symmetric spectrum X is given by

$$(\mathrm{sh}\, X)_n \ = \ X_{1+n}$$

with action of Σ_n by restriction of the Σ_{1+n}-action on X_{1+n} along the monomorphism $(1+-) : \Sigma_n \longrightarrow \Sigma_{1+n}$ which is explicitly given by $(1+\gamma)(1) = 1$ and $(1+\gamma)(i) = \gamma(i-1)+1$ for $2 \leq i \leq 1+n$. The structure maps of $\mathrm{sh}\, X$ are the reindexed structure maps for X. As an example, the shift of a suspension spectrum is another suspension spectrum, $\mathrm{sh}(\Sigma^\infty K) = \Sigma^\infty(K \wedge S^1)$.

For any symmetric spectrum X, integer k and large enough n we have

$$\pi_{(k+1)+n}(\mathrm{sh}\, X)_n \ = \ \pi_{k+(1+n)} X_{1+n} \ ,$$

and the maps in the colimit system for $\hat{\pi}_{k+1}(\mathrm{sh}\, X)$ are the same as the maps in the colimit system for $\hat{\pi}_k X$. Thus we get the equality of naive homotopy groups $\hat{\pi}_{k+1}(\mathrm{sh}\, X) = \hat{\pi}_k X$.

We can iterate the shift construction and get $(\mathrm{sh}^m X)_n = X_{m+n}$. In every level of the symmetric spectrum $\mathrm{sh}^m X$ the symmetric group Σ_m acts via the 'inclusion' $(-+n) : \Sigma_m \longrightarrow \Sigma_{m+n}$, and these actions are compatible with the structure maps. So in this way $\mathrm{sh}^m X$ becomes a Σ_m-symmetric spectrum. We note that for $k, m \geq 0$ we have

$$(\mathrm{sh}^k(\mathrm{sh}^m X))_n \ = \ (\mathrm{sh}^m X)_{k+n} \ = \ X_{m+k+n} \ = \ (\mathrm{sh}^{m+k} X)_n \ .$$

We observe that the symbols k and m have switched places along the way, which suggests that we should write

$$\mathrm{sh}^k(\mathrm{sh}^m X) \ = \ \mathrm{sh}^{m+k} X \tag{3.10}$$

and that we should avoid writing $\mathrm{sh}^k(\mathrm{sh}^m X)$ as $\mathrm{sh}^{k+m} X$. Of course, $m+k = k+m$, and so $\mathrm{sh}^{m+k} X$ equals $\mathrm{sh}^{k+m} X$; but for $x \in X_n$, $\gamma \in \Sigma_k$ and $\kappa \in \Sigma_m$ we have the relation

$$\gamma \cdot (\kappa \cdot x) = (\kappa + \gamma) \cdot x \ .$$

So the interpretation (3.10) is better because it records the equivariance properties correctly.

Example 3.11 (Semistable symmetric spectra)**.** We have already seen in Proposition 2.6 that the loop and suspension constructions shift the homotopy groups; these phenomena depend only on the underlying sequential spectra. In the situation of symmetric spectra, there is a natural morphism $\lambda_X : S^1 \wedge X \longrightarrow \mathrm{sh}\, X$ whose level n component $\lambda_n : S^1 \wedge X_n \longrightarrow X_{1+n}$ is the composite

$$S^1 \wedge X_n \xrightarrow[\text{twist}]{\cong} X_n \wedge S^1 \xrightarrow{\sigma_n} X_{n+1} \xrightarrow[=\chi_{n,1}]{(1,\dots,n+1)} X_{1+n} \ . \tag{3.12}$$

One should note that the shuffle permutation is necessary to get a morphism, even of sequential spectra; for sequential spectra this is not available, and in fact there is no natural morphism from the suspension to the shift of a *sequential* spectrum.

Proposition 3.13. *For every symmetric spectrum X the morphism $\lambda_X : S^1 \wedge X \longrightarrow \operatorname{sh} X$ induces a monomorphism on all naive homotopy groups.*

Proof. Since the suspension homomorphism $S^1 \wedge - : \hat{\pi}_k X \longrightarrow \hat{\pi}_{1+k}(S^1 \wedge X)$ is bijective (see Proposition 2.6), it suffices to show that the composite

$$\hat{\pi}_{1+k}(\lambda_X) \circ (S^1 \wedge -) \ : \ \hat{\pi}_k X \ \longrightarrow \ \hat{\pi}_{1+k}(\operatorname{sh} X)$$

is injective for all integers k. We let $f : S^{k+n} \longrightarrow X_n$ represent an element in the kernel of this composite. By increasing n, if necessary, we can assume without loss of generality that the composite $(\lambda_X)_n \circ (S^1 \wedge f) : S^{1+k+n} \longrightarrow X_{1+n} = (\operatorname{sh} X)_n$ is nullhomotopic. This composite and the map

$$\sigma_n \circ (f \wedge S^1) \ : \ S^{k+n+1} \ \longrightarrow \ X_{n+1}$$

differ by precomposition with the coordinate permutation $\chi_{1,k+n} : S^{1+k+n} \longrightarrow S^{k+n+1}$ and postcomposition with the homeomorphism $\chi_{n,1} : X_{n+1} \longrightarrow X_{1+n}$. So $\sigma_n \circ (f \wedge S^1)$ is also nullhomotopic. Since $\sigma_n \circ (f \wedge S^1)$ and f represent the same class in $\hat{\pi}_k X$, the class of f in $\hat{\pi}_k X$ is zero. □

The map induced by the morphism λ_X on naive homotopy groups is not in general surjective. The symmetric spectra for which $(\lambda_X)_*$ is surjective (hence bijective) play an important role and deserve a special name.

Definition 3.14. A symmetric spectrum X is *semistable* if the morphism $\lambda_X : S^1 \wedge X \longrightarrow \operatorname{sh} X$ is a $\hat{\pi}_*$-isomorphism.

The class of semistable symmetric spectra is closed under various constructions:

Proposition 3.15. (i) *A symmetric spectrum of simplicial sets is semistable if and only if its geometric realization is. A symmetric spectrum of spaces is semistable if and only if its singular complex is.*

(ii) *Let $f : A \longrightarrow B$ be a $\hat{\pi}_*$-isomorphism of symmetric spectra. Then A is semistable if and only if B is semistable.*

(iii) *For a symmetric spectrum X the following are equivalent:*
- *the spectrum X is semistable;*
- *the suspended spectrum $S^1 \wedge X$ is semistable;*
- *the shifted spectrum $\operatorname{sh} X$ is semistable.*

In the context of spaces, or if X is levelwise Kan, these conditions are furthermore equivalent to
- *the morphism $\tilde{\lambda}_X : X \longrightarrow \Omega \operatorname{sh} X$ is a $\hat{\pi}_*$-isomorphism;*
- *the looped spectrum ΩX is semistable.*

Proof. (i) Both shift and suspension commute with geometric realization. So for a symmetric spectrum of simplicial sets Y the morphism $\lambda_{|Y|}$ is isomorphic to $|\lambda_Y|$, and the claim follows. [...]

(ii) Since f is a $\hat{\pi}_*$-isomorphism, both the maps $S^1 \wedge f : S^1 \wedge A \longrightarrow S^1 \wedge B$ and $\operatorname{sh} f : \operatorname{sh} A \longrightarrow \operatorname{sh} B$ are $\hat{\pi}_*$-isomorphisms. We have $\lambda_B(S^1 \wedge f) = (\operatorname{sh} f)\lambda_A$, so λ_A is a $\hat{\pi}_*$-isomorphisms if and only if λ_B is.

(iii) We have $S^1 \wedge (\operatorname{sh} X) = \operatorname{sh}(S^1 \wedge X)$ and the morphisms $S^1 \wedge \lambda_X$ and $\lambda_{S^1 \wedge X}$ differ by the twist isomorphism of the two circles. So $\lambda_{S^1 \wedge X}$ if a $\hat{\pi}_*$-isomorphism if and only if $S^1 \wedge \lambda_X$ is. Since suspension shifts homotopy groups, this happens if and only if λ_X is a $\hat{\pi}_*$-isomorphism. So X is semistable if and only if its suspensions $S^1 \wedge X$ is. Similarly, the morphisms $\operatorname{sh}(\lambda_X)$ and $\lambda_{\operatorname{sh} X}$ differ by the automorphism of $\operatorname{sh}(\operatorname{sh} X)$ that interchanges the two shifted coordinates. So $\lambda_{\operatorname{sh} X}$ if a $\hat{\pi}_*$-isomorphism if and only if $\operatorname{sh}(\lambda_X)$ is. Since shifting shifts homotopy groups, this happens if and only if λ_X is a $\hat{\pi}_*$-isomorphism. So X is semistable if and only if its shift is.

In the context of spaces, or if X is levelwise Kan, then Proposition 3.8 showa that λ_X is a $\hat{\pi}_*$-isomorphism if and only if its adjoint is. Also, now the adjunction counit $S^1 \wedge \Omega X \longrightarrow X$ is a $\hat{\pi}_*$-isomorphism (see Proposition 2.6, so X is semistable if and only if $S^1 \wedge \Omega X$ is, and by the above this happens if and only if ΩX is semistable. □

Proposition 3.16. (i) *A wedge of semistable symmetric spectra is semistable.*

(ii) *A finite product of semistable symmetric spectra is semistable.*

(iii) *If X is a semistable symmetric spectrum and K a based CW-complex (respectively simplicial set), then $K \wedge X$ is semistable.*
(iv) *Let X be a semistable symmetric spectrum; in the simplicial context suppose also that X is levelwise Kan. If K is a finite based CW-complex (respectively finite simplicial set), then* $\mathrm{map}(K, X)$ *is semistable.*
(v) *Let $f : X \longrightarrow Y$ be a morphism between semistable symmetric spectra. The mapping cone $C(f)$ is semistable. In the context of spaces, or if X and Y are levelwise Kan, the homotopy fiber $F(f)$ is again semistable.*
(vi) *Let X be a symmetric spectrum such that all even permutations in Σ_n induce the identity map on the homotopy groups of X_n. Then X is semistable.*

Proof. (i) Let $\{X_i\}_{i \in I}$ by a family of semistable symmetric spectra. Since each morphism λ_{X_i} is a $\hat{\pi}_*$-isomorphism so is their wedge (by Proposition 2.19 (i)). Suspension and shift preserves wedge, so $\lambda_{\bigvee X_i} : S^1 \wedge (\bigvee X_i) \longrightarrow \mathrm{sh}(\bigvee X_i)$ is isomorphic to the wedge of the morphisms λ_{X_i}, and hence a $\hat{\pi}_*$-isomorphism.

(ii) It suffices to consider a product of two semistable symmetric spectra X and Y. The morphism $\lambda_{X \times Y} : S^1 \wedge (X \times Y) \longrightarrow \mathrm{sh}(X \times Y)$ equals the composite

$$S^1 \wedge (X \times Y) \longrightarrow (S^1 \wedge X) \times (S^1 \wedge Y) \xrightarrow{\lambda_X \times \lambda_Y} (\mathrm{sh}\, X) \times (\mathrm{sh}\, Y) = \mathrm{sh}(X \times Y)$$

where the first morphisms is the canonical one. Suspension shifts homotopy groups and homotopy groups commute with finite products (by Proposition 2.19 (ii)), so the first morphism is a $\hat{\pi}_*$-isomorphism. As a product of two $\hat{\pi}_*$-isomorphisms, the morphism $\lambda_X \times \lambda_Y$ is also a $\hat{\pi}_*$-isomorphism. So $\lambda_{X \times Y}$ is a $\hat{\pi}_*$-isomorphism, i.e., the product $X \times Y$ is semistable.

(iii) Smashing with any based space or simplicial set commutes with suspension and shift. Smashing with a based CW-complex (or any simplicial sets) preserves $\hat{\pi}_*$-isomorphisms by Proposition 2.19 (iv). So $\lambda_{K \wedge X}$ is a $\hat{\pi}_*$-isomorphism because $K \wedge \lambda_X$ is, hence $K \wedge X$ is again semistable.

(iv) Since X is semistable, the morphism $\tilde{\lambda}_X : X \longrightarrow \Omega\, \mathrm{sh}\, X$ is a $\hat{\pi}_*$-isomorphism (see Proposition 3.15 (iii)). The morphism $\tilde{\lambda}_{\mathrm{map}(K,X)} : \mathrm{map}(K, X) \longrightarrow \Omega\, \mathrm{sh}(\mathrm{map}(K, X))$ equals the composite

$$\mathrm{map}(K, X) \xrightarrow{\mathrm{map}(K, \tilde{\lambda}_X)} \mathrm{map}(K, \Omega\, \mathrm{sh}\, X) \cong \Omega\, \mathrm{sh}(\mathrm{map}(K, X)) ,$$

where the second morphism is an assembly isomorphism. Since $\tilde{\lambda}_X$ is a $\hat{\pi}_*$-isomorphism, so is $\mathrm{map}(K, \tilde{\lambda}_X)$ (by Proposition 2.19 (v)). So $\tilde{\lambda}_{\mathrm{map}(K,X)}$ is a $\hat{\pi}_*$-isomorphism, hence $\mathrm{map}(K, X)$ is again semistable.

(v) Both suspension and shift commute with mapping cones. So the morphisms λ_X, λ_Y and $\lambda_{C(f)}$ related the long exact sequences of homotopy groups (compare Proposition 2.12) for the mapping cones of $S^1 \wedge f : S^1 \wedge X \longrightarrow S^1 \wedge Y$ and $\mathrm{sh}\, f : \mathrm{sh}\, X \longrightarrow \mathrm{sh}\, Y$. Since λ_X and λ_Y are $\hat{\pi}_*$-isomorphisms, so is $\lambda_{C(f)}$. So the mapping cone is again semistable.

Since the symmetric spectrum $S^1 \wedge F(f)$ is $\hat{\pi}_*$-isomorphic to the mapping cone $C(f)$, it is semistable by the previous paragraph and part (ii) of Proposition 3.15. So $F(f)$ is itself semistable by part (i) of Proposition 3.15.

(vi) We show that for every integer k the composite

$$\hat{\pi}_k X \xrightarrow{S^1 \wedge -} \hat{\pi}_k(S^1 \wedge X) \xrightarrow{(\lambda_X)_*} \hat{\pi}_{1+k}(\mathrm{sh}\, X) = \hat{\pi}_k X$$

is multiplication by $(-1)^k$. Since the suspension homomorphism $S^1 \wedge - : \hat{\pi}_k X \longrightarrow \hat{\pi}_{1+k}(S^1 \wedge X)$ is bijective (see Proposition 2.6), this shows that λ_X induces isomorphisms of all naive homotopy groups. We let $f : S^{k+n} \longrightarrow X_n$ represent an element in $\hat{\pi}_k X$. By increasing n, if necessary, we can assume without loss of generality that n is even. The class $(\lambda_X)_*[S^1 \wedge f]$ is represented by the lower horizontal map in the commutative square:

$$\begin{tikzcd} S^{k+n+1} \arrow[r, "\sigma_n^X\circ(f\wedge S^1)"] \arrow[d, "\chi_{k+n,1}"'] & X_{n+1} \arrow[d, "\chi_{n,1}"] \\ S^{1+k+n} \arrow[r, "(\lambda_X)_n\circ(S^1\wedge f)"'] & X_{1+n} \end{tikzcd}$$

The upper horizontal map represents the same class as f. The permutation $\chi_{n,1}$ is even, so the map $\chi_{n,1} : X_{n+1} \longrightarrow X_{1+n}$ induces the identity on homotopy groups. Precomposition with the permutation $\chi_{k+n,1}$ on the sphere induces multiplication by its degree, i.e., by $(-1)^k$. So $(\lambda_X)_n \circ (S^1 \wedge f)$ is homotopic to $(-1)^k \cdot [\sigma_n^X \circ (f \wedge S^1)] = (-1)^k \cdot [f]$. □

If $f : X \longrightarrow Y$ is an h-cofibration (in the context of spaces) respectively an injective morphism (in the simplicial context), then the mapping cone $C(f)$ is level equivalent, thus $\hat{\pi}_*$-isomorphic, to the quotient Y/X. Thus if two of the spectra X, Y and Y/X are semistable, then so is the third. [same with homotopy fiber]

Examples of spectra that satisfy the hypothesis (vi) of the previous proposition are all symmetric spectra X such that action of Σ_n on X_n extends to a continuous action of the orthogonal group $O(n)$ (over the embedding $\Sigma_n \longrightarrow O(n)$ as permutation matrices), because then the action of every even permutations is homotopic to the identity. Examples are given by suspension spectra (1.13), Eilenberg-Mac Lane spectra (1.14), the various Thom spectra MO, MSO and $MSpin$ of Example 1.16 or MU, MSU and MSp of Example 1.18, or more generally all symmetric spectra that extend to orthogonal spectra (in the sense of Definition 7.2 below).

As we show in Proposition 5.57 below, the smash product of two semistable symmetric spectra is semistable (under mild 'flatness' hypotheses). In Theorem 8.25 below we provide various characterizations of semistable spectra. In Proposition 8.26 (ii) below we show that a symmetric spectrum X is semistable if the naive homotopy groups of X are dimensionwise finitely generated as abelian groups.

Examples of symmetric spectra that are not semistable can be obtained via induction as in the next Example 3.17, and by free and semifree symmetric spectra generated in positive levels (Examples 3.20 and 3.23).

Example 3.17 (Induction)**.** The shift functor introduced in Example 3.9 has a left and a right adjoint. We use the notation '$\triangleright$' for the left adjoint and refer to it as *induction* because it induces group actions from one symmetric group to the next. This construction is a special case of the more general construction $L \triangleright_m X$ of the twisted smash product of a Σ_m-space with a spectrum, see Example 3.27 below.

The induced spectrum $\triangleright X$ is trivial in level 0 and is given in positive levels by

$$(\triangleright X)_{1+n} \;=\; \Sigma_{1+n}^+ \wedge_{1\times\Sigma_n} X_n \ .$$

So ignoring the action of the symmetric group, the space (or simplicial set) $(\triangleright X)_{1+n}$ is a wedge of $n+1$ copies of X_n. The structure map is obtained from the structure map of X and the 'inclusion' of Σ_{1+n} into Σ_{1+n+1}, i.e.,

$$\Sigma_{1+n}^+ \wedge_{1\times\Sigma_n} X_n \wedge S^1 \;\longrightarrow\; \Sigma_{1+n+1}^+ \wedge_{1\times\Sigma_{n+1}} X_{n+1} \ , \quad \gamma \wedge a \wedge t \longmapsto (\gamma+1)\wedge \sigma_n(a\wedge t) \ .$$

We claim that the naive homotopy groups of $\triangleright X$ are a countably infinite sum of shifted copies of the naive homotopy groups of X. For any $m \geq 1$ we let $(\triangleright^{(m)} X)_{1+n}$ denote the wedge summand of $(\triangleright X)_{1+n} = \Sigma_{1+n}^+ \wedge_{1\times\Sigma_n} X_n$ indexed by the $(1\times\Sigma_n)$-coset containing the transposition $(1,m)$ if $1+n \geq m$, and a one-point space for $1+n < m$. The structure map takes $(\triangleright^{(m)} X)_{1+n} \wedge S^1$ to $(\triangleright^{(m)} X)_{1+n+1}$, so as n varies we obtain a sequential subspectrum $\triangleright^{(m)} X$ of $\triangleright X$. The action of the symmetric groups, however, does *not* stabilize $\triangleright^{(m)} X$, so $\triangleright^{(m)} X$ is not a *symmetric* subspectrum. As a sequential spectrum, $\triangleright X$ is moreover the internal wedge of the spectra $\triangleright^{(m)} X$,

$$\triangleright X \;=\; \bigvee_{m\geq 1} \triangleright^{(m)} X \ . \tag{3.18}$$

So by Proposition 2.19 (i) the natural map

$$\bigoplus_{m\geq 1} \hat{\pi}_k\left(\triangleright^{(m)}X\right) \longrightarrow \hat{\pi}_k\left(\bigvee_{m\geq 1} \triangleright^{(m)} X\right) = \hat{\pi}_k(\triangleright X)$$

is an isomorphism. The shifted sequential spectrum $\mathrm{sh}\left(\triangleright^{(m)}X\right)$ is isomorphic to X, at least from level m on, and so the group k-th naive homotopy group of $\triangleright^{(m)}X$ is isomorphic to the $(k+1)$-th naive homotopy group of X. So altogether we have established an isomorphism between $\hat{\pi}_k(\triangleright X)$ and a countably infinite sum of copies of $\hat{\pi}_{k+1}X$. This calculation is a special case of the more general result for the naive homotopy groups of a twisted smash product $L \triangleright_m X$, see (3.30). We will review the calculation of $\hat{\pi}_k(\triangleright X)$ in a more structured way in (8.22) below.

Now we observe that induction is indeed left adjoint to the shifting. The adjunction unit $\eta_X : X \longrightarrow \mathrm{sh}(\triangleright X)$ is given in level n as the map

$$[1\wedge -] \; : \; X_n \longrightarrow \Sigma^+_{1+n} \wedge_{1\times\Sigma_n} X_n \; .$$

We omit the verification that for every morphism of symmetric spectra $f : X \longrightarrow \mathrm{sh}\, Z$ there is a unique morphism $\hat{f} : \triangleright X \longrightarrow Z$ such that $f = (\mathrm{sh}\,\hat{f})\eta_X$. The adjunction counit $\epsilon : \triangleright(\mathrm{sh}\, Z) \longrightarrow Z$ is given in level $1+n$ by

$$\Sigma^+_{1+n} \wedge_{1\times\Sigma_n} X_{1+n} \longrightarrow X_{1+n} \; , \quad [\gamma\wedge a] \longmapsto \gamma\cdot a \; .$$

The adjunction unit is in fact a wedge summand inclusion and $\mathrm{sh}(\triangleright X)$ splits as $X \vee \triangleright(\mathrm{sh}\, X)$. We define a morphism $\rho : \triangleright(\mathrm{sh}\, X) \longrightarrow \mathrm{sh}(\triangleright X)$ in level $1+n$ as the map

$$\Sigma^+_{1+n} \wedge_{1\times\Sigma_n} X_{1+n} \longrightarrow \Sigma^+_{2+n} \wedge_{1\times\Sigma_{1+n}} X_{1+n} \; , \quad [\gamma\wedge a] \longmapsto [(1+\gamma)(1\,2)\wedge a] \; .$$

We observe here that for every permutation $\sigma \in \Sigma_n$ the element $(1+1+\sigma) \in \Sigma_{2+n}$ commutes with the transposition $(1\,2)$, so that the formula above is well-defined. Now we claim that the map

$$\eta_X \vee \rho \; : \; X \vee \triangleright(\mathrm{sh}\, X) \longrightarrow \mathrm{sh}(\triangleright X) \tag{3.19}$$

is an isomorphism. To prove the claim, let us consider the symmetric group Σ_{1+n} as a Σ_n-biset by restricting the translation acts on both sides via the monomorphism $1+- : \Sigma_n \longrightarrow \Sigma_{1+n}$. Then Σ_{1+n} splits as $S \cup T$, the disjoint union of the two Σ_n-bisets

$$S = \{\gamma \in \Sigma_{1+n} \,|\, \gamma(1) = 1\} \quad \text{respectively} \quad T = \{\gamma\in\Sigma_{1+n} \,|\, \gamma(1) \neq 1\} \; .$$

So $\mathrm{sh}(\triangleright X)$ splits levelwise as

$$(\mathrm{sh}(\triangleright X))_n = \Sigma^+_{1+n} \wedge_{1\times\Sigma_n} X_n = (S^+ \wedge_{\Sigma_n} X_n) \vee (T^+ \wedge_{\Sigma_n} X_n) \; .$$

Since S is free and transitive as a right Σ_n-set, the summand $(S^+ \wedge_{\Sigma_n} X_n)$ is a single copy of X_n, which is precisely the image in level n of the adjunction unit η_X. The other summand T is also free a right Σ_n-set, but it consists of n right Σ_n-orbits, each containing precisely one of the transpositions $(1\,i)$ for $i = 2, \ldots, 1+n$. So $(T^+ \wedge_{\Sigma_n} X_n)$ is a wedge of n copies of X_n, and this is precisely the image in level n of the morphism ρ.

If we compare the splitting (3.18) of $\triangleright X$ in the category of sequential spectra with the splitting (3.19) of $\mathrm{sh}(\triangleright X)$ in the category of symmetric spectra, we see that the image of the unit $\eta_X : X \longrightarrow \mathrm{sh}(\triangleright X)$ is precisely the shift of $\triangleright^{(1)}X$, whereas the image of $\rho : \triangleright(\mathrm{sh}\, X) \longrightarrow \mathrm{sh}(\triangleright X)$ is the wedge of the shifts of $\triangleright^{(m)}X$ for $m \geq 2$.

The morphism $\lambda_{\triangleright X} : S^1 \wedge \triangleright X \longrightarrow \mathrm{sh}(\triangleright X)$ factors as the composite

$$S^1 \wedge \triangleright X \cong \triangleright(S^1 \wedge X) \xrightarrow{\triangleright(\lambda_X)} \triangleright(\mathrm{sh}\, X) \xrightarrow{\rho} \mathrm{sh}(\triangleright X) \; .$$

So under the splitting (3.19) of $\mathrm{sh}(\triangleright X)$ into the wedge of X and $\triangleright(\mathrm{sh}\, X)$, $\lambda_{\triangleright X}$ lands in the second summand. If X has at least on non-trivial naive homotopy group, then the map ρ is not surjective on naive homotopy groups, hence $\lambda_{\triangleright X}$ is not surjective on naive homotopy groups either. We conclude that an induced spectrum $\triangleright X$ is not semistable as soon as X has at least one non-trivial naive homotopy group.

Example 3.20 (Free symmetric spectra)**.** Given a based space (or simplicial set) K and $m \geq 0$, we define a symmetric spectrum $F_m K$ which is 'freely generated by K in level m'. The spectrum $F_m K$ is trivial below level m and is given by

$$(F_m K)_{m+n} \;=\; \Sigma^+_{m+n} \wedge_{1\times\Sigma_n} K \wedge S^n$$

in levels m and above. Here $1 \times \Sigma_n$ is the subgroup of Σ_{m+n} of permutations which fix the first m elements. The structure map $\sigma_{m+n} : (F_m K)_{m+n} \wedge S^1 \longrightarrow (F_m K)_{m+n+1}$ is given by smashing the 'inclusion' $-+1 : \Sigma_{m+n} \longrightarrow \Sigma_{m+n+1}$ with the identity of K and the preferred isomorphism $S^n \wedge S^1 \cong S^{n+1}$.

Induction of free spectra. Free symmetric spectra generated in level zero are just suspension spectra, i.e., there is a natural isomorphism $F_0 K \cong \Sigma^\infty K$. Inducing a free symmetric spectrum gives a free spectrum generated on level higher, i.e., there is a natural isomorphism

$$\triangleright (F_m K) \;\cong\; F_{1+m} K \;, \tag{3.21}$$

where $\triangleright$ is the induction discussed in the previous examples. Indeed, an isomorphism which immediately meets the eye is given in level $1+m+n$ by

$$\begin{array}{ccc} \Sigma^+_{1+m+n} \wedge_{1\times\Sigma_{m+n}} (\Sigma^+_{m+n} \wedge_{1\times\Sigma_n} K \wedge S^n)_{m+n} & \longrightarrow & \Sigma^+_{1+m+n} \wedge_{1\times\Sigma_n} K \wedge S^n \\ [\gamma \wedge [\tau \wedge x]] & \longmapsto & [\gamma(1+\tau)) \wedge x] \end{array}$$

for $\gamma \in \Sigma_{1+m+n}$, $\tau \in \Sigma_{m+n}$ and $x \in K \wedge S^n$.

Freeness property. The 'freeness' property of $F_m K$ is made precise by the following fact: for every based continuous map $f : K \longrightarrow X_m$ (or morphism of based simplicial sets) there is a unique morphism of symmetric spectra $\hat{f} : F_m K \longrightarrow X$ such that the composite

$$K \xrightarrow{k \mapsto 1\wedge k \wedge 0} \Sigma^+_m \wedge K \wedge S^0 = (F_m K)_m \xrightarrow{\hat{f}_m} X_m$$

equals f (here $1 \in \Sigma_m$ is the unit element and $0 \in S^0$ is the non-basepoint). So technically speaking this bijection makes $F_m : \mathbf{T} \longrightarrow \mathcal{S}p$ into a left adjoint of the forgetful functor which takes a symmetric spectrum X to the pointed space X_m. The freeness property can be obtained as a special case of semifree symmetric spectra below [ref], or it can be established by induction on m, using the isomorphsm $\triangleright(F_m K) \cong F_{1+m} K$ above and the adjunction between shift and $\triangleright$ [ref]. In any case, the morphism $\hat{f} : F_m K \longrightarrow X$ corresponding to $f : K \longrightarrow X_m$ is given in level $m+n$ as the composite

$$\Sigma^+_{m+n} \wedge_{1\times\Sigma_n} K \wedge S^n \xrightarrow{\mathrm{Id}\wedge\sigma^n(f\wedge\mathrm{Id})} \Sigma^+_{m+n} \wedge_{1\times\Sigma_n} X_{m+n} \xrightarrow{\text{act}} X_{m+n}$$

where $\sigma^n : X_m \wedge S^n \longrightarrow X_{m+n}$ is the iterated structure map of X.

Right Σ_m-action. The free symmetric spectrum $F_m K$ comes naturally with a right action of the symmetric group Σ_m; we will refer to this as the *'right action on the free coordinates'*. In level $m+n$ a permutation $\sigma \in \Sigma_m$ acts on $(F_m K)_{m+n} \;=\; \Sigma^+_{m+n} \wedge_{1\times\Sigma_n} K \wedge S^n$ by

$$[\gamma \wedge x] \cdot \sigma \;=\; [\gamma(\sigma+1) \wedge x] \;, \tag{3.22}$$

where $\gamma \in \Sigma_{m+n}$ and $x \in K \wedge S^n$. This right Σ_m-action is well defined and commutes with the left action of Σ_{m+n} and the structure maps, so the action is indeed by automorphisms of symmetric spectra. The isomorphism (3.21) between $\triangleright(F_m K)$ and $F_{1+m} K$ is Σ_m-equivariant when we let Σ_m acts on $F_{1+m} K$ by restriction of the action along the homomorphism $1 + - : \Sigma_m \longrightarrow \Sigma_{1+m}$. We claim that moreover, the adjunction bijection

$$\mathbf{T}(K, X_m) \;\cong\; \mathrm{Spec}(F_m K, X) \;, \quad f \longmapsto \hat{f}$$

is equivariant for the two natural left actions of Σ_m; these actions are given on the left hand side by $(\sigma \cdot f)(k) = \sigma \cdot (f(k))$ for $f : K \longrightarrow X_m$ and $k \in K$, and on the right hand side by $(\sigma \hat{f})(z) = \hat{f}(z \cdot \sigma)$ for $\hat{f} : F_m K \longrightarrow X$. And indeed: we have

$$(\sigma \cdot \hat{f})_m [1 \wedge k \wedge 0] \;=\; (\hat{f})_m [\sigma \wedge k \wedge 0] \;=\; \sigma \cdot \Big(\hat{f}_m [1 \wedge k \wedge 0] \Big) \;,$$

so $\sigma \cdot \hat{f}$ has the defining property of $\widehat{\sigma f}$, and hence these two homomorphisms are equal. One final remark about the right Σ_m-action on the free coordinates of $F_m K$: the symmetric group Σ_{m+n} is free as a right

$\Sigma_m \times \Sigma_n$-set. So the underlying space of $(F_m K)_{m+n}$ is isomorphic to a wedge of $(m+n)!/n!$ copies of $K \wedge S^n$, and the right action of Σ_m freely permutes the wedge summands. Hence the Σ_m-action on the free coordinates is levelwise free away from the basepoint.

Naive homotopy groups. Using that $F_m K$ is isomorphic to the m-fold iterate of $\triangleright$ applied to the suspension spectrum $\Sigma^\infty K$ allows us to identify the naive homotopy groups of the free spectrum. Indeed, by the calculation of the naive homotopy groups of $\triangleright A$ in the previous example and induction, $\hat{\pi}_k(F_m K)$ is isomorphic to a countable sum of copies of $\hat{\pi}_{k+m}(\Sigma^\infty K) = \pi^{\mathrm{s}}_{k+m} K$, as long as $m \geq 1$.

We point out a special case which will be relevant later. The symmetric spectrum $F_1 S^1$ freely generated by the circle S^1 in level 1 ought to be a desuspension of the suspension spectrum of the circle. And indeed, we shall see in Example 4.26 that $F_1 S^1$ *stably equivalent* (to be defined in Section 4) to the sphere spectrum $\mathbb{S}$. More generally, the free symmetric spectrum $F_m K$ is stably equivalent to $\Omega^m(\Sigma^\infty K)$, the m-fold loop spectrum of the suspension spectrum of K, compare Example 4.35.

However, the naive homotopy groups of $F_1 S^1$ are a countable sum of copies of the stable stems, so $F_1 S^1$ is not $\hat{\pi}_*$-isomorphic to the sphere spectrum $\mathbb{S}$, whose zeroth homotopy group is a single copy of the integers.

Example 3.23 (Semifree symmetric spectra)**.** There are somewhat 'less free' symmetric spectra which start from a pointed Σ_m-space (or Σ_m-simplicial set) L; we want to install L in level m, and then fill in the remaining data of a symmetric spectrum as freely as possible. In other words, we claim that the forgetful *evaluation functor*

$$\mathrm{ev}_m \; : \; \mathcal{S}p_{\mathbf{T}} \longrightarrow \Sigma_m\text{-}\mathbf{T} \; , \quad X \longmapsto X_m$$

(and its analog for symmetric spectra of simplicial sets) has a left adjoint which we denote G_m; we refer to $G_m L$ as the *semifree symmetric spectrum* generated by L in level m. (The evaluation functor ev_m also has a right adjoint which will feature in Example 4.2.) The spectrum $G_m L$ is trivial below level m, and otherwise given by

$$(G_m L)_{m+n} \; = \; \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge S^n \; .$$

The structure map $\sigma_{m+n} : (G_m L)_{m+n} \wedge S^1 \longrightarrow (G_m L)_{m+n+1}$ is defined by smashing the 'inclusion' $-+1 : \Sigma_{m+n} \longrightarrow \Sigma_{m+n+1}$ with the identity of L and the preferred isomorphism $S^n \wedge S^1 \cong S^{n+1}$.

Free versus semifree spectra. Free and semifree symmetric spectra can be obtained from each other as follows. On the one hand, every free symmetric spectrum is semifree, i.e., there is a natural isomorphism $F_m K \cong G_m(\Sigma^+_m \wedge K)$ by 'cancelling Σ_m'. On the other hand, a semifree spectrum $G_m L$ can be obtained from the free spectrum $F_m L$ by coequalizing the right Σ_m-action on the free coordinates (3.22) and the given left Σ_m-action on L. Indeed, in the relevant levels $m+n$, the only difference between $F_m L$ and $G_m L$ is that the action of a larger group ($\Sigma_m \times \Sigma_n$ as opposed to $1 \times \Sigma_n$) is divided out in the semifree case; so the natural projection

$$(F_m L)_{m+n} = \Sigma^+_{m+n} \wedge_{1 \times \Sigma_n} L \wedge S^n \; \longrightarrow \; \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge S^n = (G_m L)_{m+n}$$

relates the two constructions by a morphism of symmetric spectra and factors over an isomorphism

$$(F_m L)/\Sigma_m \; \longrightarrow \; G_m L \tag{3.24}$$

where the left hand side denotes the quotient (in level $m+n$) by the equivalence relation

$$[\gamma(\sigma+1) \wedge l \wedge x] = [\gamma \wedge l \wedge x] \cdot \sigma \quad \sim \quad [\gamma \wedge (\sigma l) \wedge x]$$

for $\gamma \in \Sigma_{m+n}$, $\sigma \in \Sigma_m$, $l \in L$ and $x \in S^n$.

Semifreeness property. The 'semifreeness' property of $G_m L$, or more technically the adjunction bijection, works as follows: we claim that for every morphism of based Σ_m-spaces (or Σ_m-simplicial sets) $f : L \longrightarrow X_m$ there is a unique morphism of symmetric spectra $\hat{f} : G_m L \longrightarrow X$ such that the composite

$$L \xrightarrow{1 \wedge -} \Sigma^+_m \wedge_{\Sigma_m} L = (G_m L)_m \xrightarrow{\hat{f}_m} X_m$$

equals f. The requirements of equivariance and compatibility with structure maps imply that there is at most one morphism with this property. Given f, we define the corresponding morphism $\hat{f} : G_m L \longrightarrow X$ in level $m+n$ as the composite

$$\Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge S^n \xrightarrow{\mathrm{Id} \wedge \sigma^n (f \wedge \mathrm{Id})} \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} X_{m+n} \xrightarrow{\text{act}} X_{m+n}$$

where $\sigma^n : X_m \wedge S^n \longrightarrow X_{m+n}$ is the iterated structure map of X. We omit the straightforward verification that $\hat{f}$ is indeed a morphism.

Building symmetric spectra from semifree pieces. Semifree spectra are the basic building blocks in the theory of symmetric spectra. This slogan can be made precise in various ways. The first, and almost tautological, way takes the form of a natural coequalizer diagram

$$\bigvee_{n \geq 0} G_{n+1}(\Sigma^+_{n+1} \wedge_{\Sigma_n \times 1} X_n \wedge S^1) \overset{\sigma}{\underset{I}{\rightrightarrows}} \bigvee_{m \geq 0} G_m X_m \longrightarrow X \ . \tag{3.25}$$

The upper map σ to be coequalized takes the nth wedge summand to the $(n+1)$st wedge summand by the adjoint of

$$\sigma_n \ : \ X_n \wedge S^1 \ \longrightarrow \ X_{n+1} \ = \ (G_{n+1} X_{n+1})_{n+1} \ .$$

The other map I takes the nth wedge summand to the nth wedge summand by the adjoint of the wedge summand inclusion

$$X_n \wedge S^1 \ \longrightarrow \ \Sigma^+_{n+1} \wedge_{\Sigma_n \times \Sigma_1} (X_n \wedge S^1) \ = \ (G_n X_n)_{n+1}$$

indexed by the identity of Σ_{n+1}. The morphism to X is the wedge over the adjoints of the identity maps of the spaces X_m.

The fact that G_m is left adjoint to evaluation at level m directly implies that X has the universal property of a coequalizer of σ and I. Indeed, a morphism $f : \bigvee_{m \geq 0} G_m X_m \longrightarrow Z$ corresponds bijectively to a family of equivariant based maps $f_m : X_m \longrightarrow Z_m$. The morphism f coequalizes the maps σ and I if and only if the maps f_m satisfy $f_{m+1} \circ \sigma_m = \sigma_m \circ (f_m \wedge \mathrm{Id}_{S^1})$ for all m, i.e., if they form a morphism of symmetric spectra from X to Z.

The coequalizer (3.25) can be used to reduce certain statements about general symmetric spectra to the special case of semifree spectra. In Construction 5.29 we will discuss a different way in which a general symmetric spectrum is 'built up' from semifree symmetric spectra.

Shifts of semifree spectra. We identify the shift of a semifree symmetric spectrum. For this purpose it will be more convenient to shift the indexing and start with a based Σ_{1+m}-space (or simplicial set) L, as opposed to a Σ_m-space. We denote by $\mathrm{sh}\, L$ the restriction of L to a Σ_m-space along the monomorphism $1+- : \Sigma_m \longrightarrow \Sigma_{1+m}$. We define a morphism $\xi : G_m(\mathrm{sh}\, L) \longrightarrow \mathrm{sh}(G_{1+m} L)$ as adjoint to the Σ_m-equivariant map $[1 \wedge -] : \mathrm{sh}\, L \longrightarrow \Sigma^+_{1+m} \wedge_{\Sigma_{1+m}} L = (\mathrm{sh}(G_{1+m} L))_m$; this morphism can be wedged together with the morphism $\lambda_{G_{1+m} L}$ defined in (3.12) to yield a morphism

$$\lambda_{G_{1+m} L} \vee \ \xi : \ (S^1 \wedge G_{1+m} L) \ \vee \ G_m(\mathrm{sh}\, L) \ \longrightarrow \ \mathrm{sh}(G_{1+m} L) \ . \tag{3.26}$$

As we discuss in Exercise E.I.9 the map (3.26) is an isomorphism. The map (3.26) is even an isomorphism for $m = -1$, if we interpret G_{-1} as a trivial spectrum; in other words, in this case $\lambda_{G_0 L} : S^1 \wedge G_0 L \longrightarrow \mathrm{sh}(G_0 L)$ is an isomorphism.

We can specialize to free symmetric spectra. Using $F_{1+m} K = G_{1+m}(\Sigma^+_{1+m} \wedge K)$ and the fact that $\mathrm{sh}(\Sigma_{1+m})$ is the disjoint union of $(1+m)$ free transitive Σ_m-sets we obtain

$$(S^1 \wedge F_{1+m} K) \ \vee \ \bigvee_{i=1}^{1+m} F_m K \ \cong \ \mathrm{sh}(F_{1+m} K) \ .$$

This wedge decomposition is hiding some of the symmetries. Indeed, the free spectrum $F_{1+m} K$, and hence its shift, has a right action of the symmetric groups Σ_{1+m} as in (3.22). To make the decomposition equivariant we have to rewrite the second summand of the left hand side as the spectrum $F_m K \wedge_{\Sigma_m} \Sigma^+_{1+m}$ induced from the right Σ_m-spectrum $F_m K$ along the homomorphism $1+- : \Sigma_m \longrightarrow \Sigma_{1+m}$. Then the map

$$\lambda_{F_{1+m} K} \vee \xi \ : \ (S^1 \wedge F_{1+m} K) \ \vee \ (F_m K \wedge_{\Sigma_m} \Sigma^+_{1+m}) \ \longrightarrow \ \mathrm{sh}(F_{1+m} K)$$

is a Σ_{1+m}-equivariant isomorphism.

One can describe the naive homotopy groups of the semifree spectrum $G_m L$ functorially in terms of the stable homotopy groups of L and the induced Σ_m-action. This is most naturally done with all the structure present in the naive homotopy groups, and so we defer this discussion to Example 8.18. As for the stable homotopy type represented by a semifree spectrum, we refer to Example 4.36.

Example 3.27 (Twisted smash product)**.** The twisted smash product starts from a number $m \geq 0$, a pointed Σ_m-space (or Σ_m-simplicial set) L and a symmetric spectrum X and produces a new symmetric spectrum which we denote $L \rhd_m X$. This construction is a simultaneous generalization of the smash product of a space and a spectrum (Example 3.6), induction (Example 3.17), and semifree symmetric spectra (Example 3.23). Once the internal smash product of symmetric spectra is available, we will identify the twisted smash product $L \rhd_m X$ with the smash product of the semifree spectrum $G_m L$ and X, see Proposition 5.13 below.

We define the twisted smash product $L \rhd_m X$ as a point in levels smaller than m and in general by

$$(L \rhd_m X)_{m+n} \ = \ \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge X_n \ .$$

The structure map $\sigma_{m+n} : (L \rhd_m X)_{m+n} \wedge S^1 \longrightarrow (L \rhd_m X)_{m+n+1}$ is obtained from $\mathrm{Id} \wedge \sigma_n : L \wedge X_n \wedge S^1 \longrightarrow L \wedge X_{n+1}$ by inducing up.

Here are some special cases. Taking $X = \mathbb{S}$ gives semifree symmetric spectra as $G_m L = L \rhd_m \mathbb{S}$ and so a free spectrum $F_m K$ is isomorphic to the twisted smash product $(\Sigma^+_m \wedge K) \rhd_m \mathbb{S}$. For $m = 0$ we get

$$K \rhd_0 X \ = K \wedge X \ ,$$

the levelwise smash product of K and X. Induction can be recovered as $\rhd X = S^0 \rhd_1 X$. The twisted smash product has an associativity property in the form of a natural isomorphism

$$L \rhd_m (L' \rhd_n X) \ \cong \ (\Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge L') \rhd_{m+n} X \ .$$

The twisted smash product is related by various adjunctions to other constructions. As we noted at the end of Example 3.9, the m-fold shift of a symmetric spectrum Z has an action of Σ_m through spectrum automorphisms, i.e., $\mathrm{sh}^m Z$ is a Σ_m-symmetric spectrum. The levelwise smash product $L \wedge X$ (in the sense of Example 3.6) of the underlying space of L and X also is a Σ_m-symmetric spectrum through the action on L. Given a morphism $f : L \rhd_m X \longrightarrow Z$ of symmetric spectra, we can restrict the component in level $m+n$ to the summand $1 \wedge L \wedge X_n$ in $(L \rhd_m X)_{m+n}$ and obtain a $\Sigma_m \times \Sigma_n$-equivariant based map $\hat{f}_n = f_{m+n}(1 \wedge -) : L \wedge X_n \longrightarrow Z_{m+n} = (\mathrm{sh}^m Z)_n$. The compatibility of the f_{m+n}'s with the structure maps translates into the property that the maps $\hat{f} = \{\hat{f}_n\}_{n \geq 0}$ form a morphism of Σ_m-symmetric spectra from $L \wedge X$ to $\mathrm{sh}^m Z$. Conversely, every Σ_m-equivariant morphism $L \wedge X \longrightarrow \mathrm{sh}^m Z$ arises in this way from a morphism $f : L \rhd_m X \longrightarrow Z$. We can rephrase this as slightly in a form resembling an adjunction, using the morphism of Σ_m-symmetric spectra

$$\eta_{L,X} \ : \ L \wedge X \ \longrightarrow \ \mathrm{sh}^m (L \rhd_m X) \tag{3.28}$$

defined in level n as the map

$$[1 \wedge -] \ : \ L \wedge X_n \ \longrightarrow \ \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge X_n = (L \rhd_m X)_{m+n} = \mathrm{sh}^m((L \rhd_m X))_n \ .$$

Then for any $f : L \rhd_m X \longrightarrow Z$ as above, $\hat{f} = (\mathrm{sh}^m f) \circ \eta_{L,X}$, and so the assignment

$$\mathcal{S}p(L \rhd_m X, Z) \ \longrightarrow \ \Sigma_m\text{-}\mathcal{S}p(L \wedge X, \mathrm{sh}^m Z) \ , \quad f \ \longmapsto \ \hat{f} = (\mathrm{sh}^m f) \circ \eta_{L,X} \tag{3.29}$$

is a bijection, natural in all three variables.

The case $m = 1$ and $L = S^0$ gives a bijection,

$$\mathcal{S}p(\rhd X, Z) \ \cong \ \mathcal{S}p(X, \mathrm{sh}\, Z) \ ,$$

natural in the symmetric spectra X and Z, which shows that $X \mapsto \rhd X$ is left adjoint to shifting.

Now we express the naive homotopy groups of a twisted smash product $L \rhd_m X$ in terms of the naive homotopy groups of the spectrum $L \wedge X$, the levelwise smash product of the underlying based space (or simplicial set) of L and X. For this calculation we decompose the sequential spectrum which underlies $L \rhd_m X$; the wedge decomposition of the induced spectrum $\rhd X$ in Example 3.17 is a special case.

We denote by O_m the set of order preserving injections $f : \mathbf{m} \longrightarrow \omega$ from the set $\mathbf{m} = \{1, \dots, m\}$ to the set $\omega = \{1, 2, \dots\}$ of positive integers. Every such $f \in O_m$ gives rise to a sequential subspectrum (not a *symmetric* subspectrum) $L \triangleright_m^f X$ of $L \triangleright_m X$ as follows. We recall that a permutation $\gamma \in \Sigma_{m+n}$ is an (m, n)-*shuffle* if the restriction of γ to $\{1, \dots, m\}$ and the restriction to $\{m+1, \dots, m+n\}$ are monotone. For $f \in O_m$ we denote by $|f|$ the maximum of the values of f. If the image of f is contained in $\mathbf{m} + \mathbf{n}$ for a particular $n \geq 0$, then there is a unique (m, n)-shuffle $\chi(f)$ which agrees with f on $\mathbf{m}$. Conversely, every (m, n)-shuffle arises in this way from a unique $f \in O_m$. The sequential spectrum $L \triangleright_m^f X$ is given by

$$(L \triangleright_m^f X)_{m+n} \;=\; \begin{cases} \big(\chi(f) \cdot (\Sigma_m \times \Sigma_n)\big)^+ \wedge_{\Sigma_m \times \Sigma_n} (L \wedge X_n) & \text{if } \mathrm{Im}(f) \subset \mathbf{m} + \mathbf{n} \\ \qquad\qquad\qquad * & \text{else.} \end{cases}$$

Note that $L \triangleright_m^f X$ is also trivial below level m.

Since the (m, n)-shuffles provide a set of right coset representatives for the group $\Sigma_m \times \Sigma_n$ in Σ_{m+n}, we have an internal wedge decomposition

$$(L \triangleright_m X)_{m+n} \;=\; \bigvee_{f \in O_m} (L \triangleright_m^f X)_{m+n} \ .$$

The monomorphism $- + 1 : \Sigma_{m+n} \longrightarrow \Sigma_{m+n+1}$ takes (m, n)-shuffles to $(m, n+1)$-shuffle and satisfies $\chi(f)+1 = \chi(f)$ whenever $\mathrm{Im}(f) \subset \mathbf{m} + \mathbf{n}$. The structure map $\sigma_{m+n} : (L \triangleright_m X)_{m+n} \wedge S^1 \longrightarrow (L \triangleright_m X)_{m+n+1}$ thus preserves the wedge decomposition and so the underlying sequential spectrum of $L \triangleright_m X$ decomposes as an internal wedge of the sequential spectra $L \triangleright_m^f X$ for $f \in O_m$. For $m = 1$ and $L = S^0$ we have $S^0 \triangleright_1 X = \triangleright X$, the induced spectrum. In this case the splitting specializes to the splitting (3.18) of $\triangleright X$.

A consequence of the above splitting is that the underlying sequential spectrum of $L \triangleright_m X$, and hence its naive homotopy groups do not depend on the Σ_m-action on L. Since the naive homotopy groups of a wedge are the direct sum of the naive homotopy groups (Proposition 2.19 (i)), the inclusions induce an isomorphism

$$\bigoplus_{f \in O_m} \hat{\pi}_k(L \triangleright_m^f X) \;\xrightarrow{\cong}\; \hat{\pi}_k(L \triangleright_m X) \ .$$

For $f \in O_m$ let $n \geq 0$ be the smallest number such that $\mathrm{Im}(f) \subset \mathbf{m} + \mathbf{n}$ (i.e., the difference of the maximum of f and m). Then the shifted sequential spectrum $\mathrm{sh}^{m+n}(L \triangleright_m^f X)$ is isomorphic to the underlying sequential spectrum of $\mathrm{sh}^n(L \wedge X)$, so the group $\hat{\pi}_k(L \triangleright_m^f X) = \hat{\pi}_{k+m+n}(\mathrm{sh}^{m+n}(L \triangleright_m^f X))$ is isomorphic to $\hat{\pi}_{k+m+n}(\mathrm{sh}^n(L \wedge X)) = \hat{\pi}_{k+m}(L \wedge X)$. Combining these two isomorphisms gives

$$\bigoplus_{f \in O_m} \hat{\pi}_{k+m}(L \wedge X) \;\cong\; \hat{\pi}_k(L \triangleright_m X) \ . \tag{3.30}$$

We emphasize that (3.30) is in general only an isomorphism of abelian group, but that it does not preserve certain extra structure which is available on the naive homotopy groups and which we discuss in Section 8.1. We return to this point in Example 8.15 below. A consequence of the isomorphism (3.30) is:

Proposition 3.31. *Let L be a cofibrant based Σ_m-space (respectively a based Σ_m-simplicial set). Then the twisted smash product functor $L \triangleright_m -$ preserves $\hat{\pi}_*$-isomorphisms. For every symmetric spectrum of spaces A and every cofibrant based Σ_m-space L, the natural map*

$$\mathcal{S}(L) \triangleright_m \mathcal{S}(A) \;\longrightarrow\; \mathcal{S}(L \triangleright_m A)$$

is a $\hat{\pi}_*$-isomorphism.

Proof. First we consider a symmetric spectrum C such that all naive homotopy groups $\hat{\pi}_k C$ vanish. Then for every cofibrant based space K (or any based simplicial set K) the naive homotopy groups of the symmetric spectrum $K \wedge C$ also vanish by Proposition 2.19 (iv). By (3.30) the naive homotopy group $\hat{\pi}_{k+m}(L \triangleright_m C)$ is isomorphic to a direct sum of copies of the naive homotopy group $\hat{\pi}_k(L \wedge C)$. Hence all the groups $\hat{\pi}_{k+m}(L \triangleright_m C)$ are trivial by the above (in the context of spaces we use that the underlying based space of a based Σ_m-CW-complex is cofibrant).

If $f : X \longrightarrow Y$ is a $\hat{\pi}_*$-isomorphism, then its mapping cone $C(f)$ has trivial naive homotopy groups by the long exact sequence of Proposition 2.12. By the last paragraph the symmetric spectrum $L \triangleright_m C(f)$ has trivial naive homotopy groups. Since the twisted smash product functor commutes with taking mapping cones, the mapping cone of the morphism $L \triangleright_m f : L \triangleright_m X \longrightarrow L \triangleright_m Y$ has trivial naive homotopy groups. So $L \triangleright_m f$ is a $\hat{\pi}_*$-isomorphism.

For the second statement we use that the map in question is a $\hat{\pi}_*$-isomorphism if and only if its adjoint is. This adjoint factors as the composite

$$|\mathcal{S}(L) \triangleright_m \mathcal{S}(A)| \;\cong\; |\mathcal{S}(L)| \triangleright_m |\mathcal{S}(A)| \xrightarrow{\epsilon \triangleright_m |\mathcal{S}(A)|} L \triangleright_m |\mathcal{S}(A)| \xrightarrow{L \triangleright_m \epsilon} L \triangleright_m A \ .$$

The first map is a level equivalence because $\epsilon : \mathcal{S}(L) \longrightarrow L$ is a weak equivalence between Σ_m-CW-complexes and the symmetric spectrum $|\mathcal{S}(A)|$ is levelwise of the homotopy type of a CW-complex. The adjunction counit $\epsilon : |\mathcal{S}(A)| \longrightarrow A$ is a level equivalence, hence $\hat{\pi}_*$-isomorphism, so the second map $L \triangleright_m \epsilon$ is a $\hat{\pi}_*$-isomorphism by the above. $\square$

We can also identify the shift of a twisted smash product. Firstly, we have $\mathrm{sh}(L \triangleright_0 X) = \mathrm{sh}(L \wedge X) = L \wedge \mathrm{sh}\, X$. In general, the shift of a twisted smash product decomposes into two pieces which are themselves twisted smash products: in Exercise E.I.9 we construct a natural isomorphism

$$\mathrm{sh}(L \triangleright_{1+m} X) \quad\cong\quad (\mathrm{sh}\, L) \triangleright_m X \;\vee\; L \triangleright_{1+m} (\mathrm{sh}\, X)$$

for any pointed Σ_{1+m}-space (or simplicial set) L, where $\mathrm{sh}\, L$ denotes the restriction of L to a Σ_m-space along the monomorphism $1 + - : \Sigma_m \longrightarrow \Sigma_{1+m}$. A way to remember this is to say that 'shifting is a derivation with respect to twisted smash product'. For $m = 0$ and $L = S^0$ we have $S^0 \triangleright_1 X = \triangleright X$, the induction of X. So in this case the splitting of $\mathrm{sh}(S^0 \triangleright_1 X)$ recovers the splitting $\mathrm{sh}(\triangleright X) \cong X \vee \triangleright(\mathrm{sh}\, X)$ of (3.19). If $X = \mathbb{S}$ is the sphere spectrum, we have $L \triangleright_{1+m} \mathbb{S} = G_{1+m} L$ and $\mathrm{sh}\, \mathbb{S} = S^1 \wedge \mathbb{S}$, so as another special case of the splitting of $\mathrm{sh}(L \triangleright_m X)$ we obtain a wedge decomposition of the shift of a semifree symmetric spectrum

$$\mathrm{sh}(G_{1+m} L) \;\cong\; G_m(\mathrm{sh}\, L) \;\vee\; (S^1 \wedge G_{1+m} L) \ .$$

We can specialize even further to free symmetric spectra. Using $F_{1+m} K = G_{1+m}(\Sigma^+_{1+m} \wedge K)$ and the fact that $\mathrm{sh}(\Sigma_{1+m})$ is the disjoint union of $(1 + m)$ free transitive Σ_m-sets we obtain

$$\mathrm{sh}(F_{1+m} K) \;\cong\; \bigvee_{i=1}^{1+m} F_m K \;\vee\; F_{1+m}(S^1 \wedge K) \ .$$

Example 3.32 (Equivariant function spectrum)**.** We define an equivariant function spectrum which generalizes the function spectrum X^K discussed in Example 3.6. This construction takes as input a based Σ_m-space (or Σ_m-simplicial set) L and a symmetric spectrum X. The *equivariant function spectrum* $\triangleright^m(L, X)$ is defined in level n as

$$\triangleright^m(L, X)_n \;=\; \mathrm{map}^{\Sigma_m}(L, X_{m+n}) \ ,$$

the space (or simplicial set) of Σ_m-equivariant maps from L to X_{m+n}, where Σ_m acts on the target via the 'inclusion' $- + 1 : \Sigma_m \longrightarrow \Sigma_{m+n}$. The Σ_n-action on $(\triangleright^m(L, X))_n$ is obtained from the action on the target X_{m+n} by restriction along the homomorphism $1 + - : \Sigma_n \longrightarrow \Sigma_{m+n}$. The structure map $\sigma_n : \triangleright^m(L, X)_n \wedge S^1 \longrightarrow \triangleright^m(L, X)_{n+1}$ is the composite

$$\mathrm{map}^{\Sigma_m}(L, X_{m+n}) \wedge S^1 \xrightarrow{\text{assembly}} \mathrm{map}^{\Sigma_m}(L, X_{m+n} \wedge S^1) \xrightarrow{\mathrm{map}^{\Sigma_m}(L, \sigma_{m+n})} \mathrm{map}^{\Sigma_m}(L, X_{m+n+1}) \ ,$$

where the first map is of assembly type and is obtained from

$$\mathrm{map}(L, X_{m+n}) \wedge S^1 \;\longrightarrow\; \mathrm{map}(L, X_{m+n} \wedge S^1) \ , \quad f \wedge t \;\longmapsto\; [x \mapsto f(x) \wedge t]$$

by restriction to the subspaces of Σ_m-equivariant maps. In the special case $m = 0$ we have $\triangleright^0(L, X) = X^L$, i.e., the equivariant function spectrum reduced to the (non-equivariant) function spectrum in the sense of Example 3.6.

Since $X_{m+n} = (\mathrm{sh}^m X)_n$, the nth level of $\triangleright^m(L,X)$ equals the space of Σ_m-equivariant maps from L to $(\mathrm{sh}^m X)_n$. So we can view $\triangleright^m(L,X)$ as the symmetric spectrum of Σ_m-equivariant maps from L to $\mathrm{sh}^m X$, the m-fold shift of X. This explains the name 'equivariant function spectrum'.

Equivariant function spectra commute with shift in the second variable, up to a natural isomorphism involving a shuffle permutation. More precisely, for a given level n we consider the isomorphism

$$\begin{aligned}\triangleright^m(L,\mathrm{sh}\, X)_n \;&=\; \mathrm{map}^{\Sigma_m}(L,(\mathrm{sh}\, X)_{m+n}) \;=\; \mathrm{map}^{\Sigma_m}(L, X_{1+m+n}) \\ &\xrightarrow{\mathrm{map}^{\Sigma_m}(L,\chi_{1,m}+1_n)} \mathrm{map}^{\Sigma_m}(L,X_{m+1+n}) \;=\; \triangleright^m(L,X)_{1+n} \;=\; \mathrm{sh}(\triangleright^m(L,X))_n \ .\end{aligned}$$

As n varies, these maps constitute an isomorphism of symmetric spectra

$$\chi \;:\; \triangleright^m(L,\mathrm{sh}\, X) \xrightarrow{\;\cong\;} \mathrm{sh}(\triangleright^m(L,X)) \ .$$

For the Σ_{1+m}-space $\Sigma_{1+m}^+ \wedge_{1\times\Sigma_m} L$ induced from the Σ_m-space L along the homomorphism $1+- : \Sigma_m \longrightarrow \Sigma_{1+m}$, the collection of adjunction isomorphisms

$$\mathrm{map}^{\Sigma_{1+m}}(\Sigma_{1+m}^+ \wedge_{1\times\Sigma_m} L, X_{1+m+n}) \;\cong\; \mathrm{map}^{\Sigma_m}(L,X_{1+m+n}) = \mathrm{map}^{\Sigma_m}(L,(\mathrm{sh}\, X)_{m+n})$$

constitute an isomorphism of symmetric spectra

$$\triangleright^{1+m}(\Sigma_{1+m}^+ \wedge_{1\times\Sigma_m} L, X) \;\cong\; \triangleright^m(L,\mathrm{sh}\, X) \ .$$

[If we use the 'shift adjoint' notation $\triangleright L = \Sigma_{1+m}^+ \wedge_{1\times\Sigma_m} L$, this becomes $\triangleright^{1+m}(\triangleright L, X) \;\cong\; \triangleright^m(L,\mathrm{sh}\, X)$]

Finally, we can combine all the above to a natural isomorphism

$$\triangleright^m(\Sigma_m^+ \wedge K, X) \;\cong\; (\mathrm{sh}^m X)^K$$

where K is any (non-equivariant) based space (or simplicial set).

The twisted smash product $L \triangleright_m -$ (Example 3.27) with a Σ_m-space L (or simplicial set) is left adjoint to the equivariant function spectrum $\triangleright^m(L,-)$, we shall now explain. We summarize the adjunction isomorphisms in the commutative triangle, where X and Z are arbitrary symmetric spectra:

(3.33)
$$\begin{tikzcd} \mathcal{S}p(L \triangleright_m X, Z) \arrow[rr, "\triangleright^m(L,-)\circ\mu_{X,L}"] \arrow[dr, "(\mathrm{sh}^m -)\circ\eta_{L,X}"'] & & \mathcal{S}p(X, \triangleright^m(L,Z)) \arrow[dl, "\mathrm{ev}_{L,Z}\circ(L\wedge -)"] \\ & \Sigma_m\text{-}\mathcal{S}p(L\wedge X, \mathrm{sh}^m Z) & \end{tikzcd}$$

The bottom entry is the set of Σ_m-equivariant morphisms of symmetric spectra from $L\wedge X$ (with Σ_m-action through L) to the m-fold shift $\mathrm{sh}^m Z$ (with Σ_m-action on the shifted coordinates). The maps are defined with the help of the homomorphisms of Σ_m-symmetric spectra

$$\eta_{L,X} \;:\; L\wedge X \;\longrightarrow\; \mathrm{sh}^m(L\triangleright_m X) \ , \quad \mathrm{ev}_{L,Z} \;:\; L\wedge \triangleright^m(L,Z) \;\longrightarrow\; \mathrm{sh}^m Z$$

and the homomorphism of symmetric spectra

$$\mu_{L,X} \;:\; X \;\longrightarrow\; \triangleright^m(L, L\triangleright_m X) \ ;$$

the latter is the unit of the adjunction for the functor pair $(L\triangleright_m -, \triangleright^m(L,-))$. The morphism $\eta_{L,X}$ was defined in (3.28). The morphism $\mathrm{ev}_{L,Z}$ is defined in level n as the map evaluation map

(3.34)
$$L\wedge \triangleright^m(L,Z)_n \;=\; L\wedge \mathrm{map}^{\Sigma_m}(L, Z_{m+n}) \;\longrightarrow\; Z_{m+n} \;=\; (\mathrm{sh}^m Z)_n \ .$$

The morphism $\mu_{L,X}$ is defined in level n as the map

$$X_n \;\longrightarrow\; \mathrm{map}^{\Sigma_m}(L, \Sigma_{m+n}^+ \wedge_{\Sigma_m\times\Sigma_n} L\wedge X_n)$$

given by

$$(\eta_{L,X})_n(x)(l) \;=\; [1\wedge l\wedge x] \ .$$

The vertical map labelled 'ev_m' is essentially evaluation at level m and is bijective by the semifreeness property of the symmetric spectrum $G_m L$ (compare Example 3.23) and the definition of $\mathrm{Hom}(X,Z)_m$ as $\mathrm{map}(X, \mathrm{sh}^m Z)$. The relation

$$\mathrm{ev}_{L, L\rhd_m X} \circ (L \wedge \mu_{X,L}) \ = \ \eta_{L,X} \ : \ L \wedge X \ \longrightarrow \ \mathrm{sh}^m (L \rhd_m X)$$

and the naturality of $\mathrm{ev}_{L,Z}$ imply that the triangle (3.33) commutes.

All these maps are natural in L, X and Z. We omit the straightforward, but boring, verification that all maps are bijective. In particular, the lower horizontal bijection makes the twisted smash product functor $L \rhd_m -$ a left adjoint of the equivariant function spectrum functor $\rhd^m(L, -)$.

There are two more adjunctions: (the semifree symmetric spectrum $G_m L$ (Example 3.23))

$$(3.35) \qquad \Sigma_m\text{-}\mathcal{S}p(L \wedge X, \mathrm{sh}^m Z) \ \xrightarrow{\cong} \ \Sigma_m\text{-}\mathbf{T}(L, \mathrm{map}(X, \mathrm{sh}^m Z)) \ \xleftarrow{\mathrm{ev}_m} \ \mathcal{S}p(G_m L, \mathrm{Hom}(X, Z))$$

[mapping spaces are only introduced later...] The adjunction bijections can be promoted to 'enriched adjunctions', where isomorphisms of mapping spaces or even symmetric function spectra take the place of bijections. For the precise formulation we refer to Exercise E.I.11.

Example 3.36 (Mapping spaces)**.** There is a whole space, respectively simplicial set, of morphisms between two symmetric spectra. For symmetric spectra X and Y, every morphism $f : X \longrightarrow Y$ consists of a family of based maps $\{f_n : X_n \longrightarrow Y_n\}_{n \geq 0}$ which satisfy some conditions. So the set of morphisms from X to Y is a subset of the product of mapping spaces $\prod_{n\geq 0} \mathrm{map}(X_n, Y_n)$ and we give it the subspace topology of the (compactly generated) product topology. We denote this mapping space by $\mathrm{map}(X, Y)$.

Now suppose that X and Y are symmetric spectra of simplicial sets. Then the mapping space $\mathrm{map}(X, Y)$ is the simplicial set whose n-simplices are the spectrum morphisms from $\Delta[n]^+ \wedge X$ to Y. For a monotone map $\alpha : [n] \longrightarrow [m]$ in the simplicial category Δ, the map $\alpha^* : \mathrm{map}(X,Y)_m \longrightarrow \mathrm{map}(X,Y)_n$ is given by precomposition with $\alpha_* \wedge \mathrm{Id} : \Delta[n]^+ \wedge X \longrightarrow \Delta[m]^+ \wedge X$. The morphism space has a natural basepoint, namely the trivial map from $\Delta[0]^+ \wedge X$ to Y. We can, and will, identify the vertices of $\mathrm{map}(X, Y)$ with the morphisms from X to Y using the natural isomorphism $\Delta[0]^+ \wedge X \cong X$.

Furthermore, for a pointed space K and topological symmetric spectra X and Y we have adjunction homeomorphisms

$$\mathrm{map}(K, \mathrm{map}(X, Y)) \ \cong \ \mathrm{map}(K \wedge X, Y) \ \cong \ \mathrm{map}(X, Y^K) \ ,$$

where the first mapping space is taken in the category $\mathbf{T}$ of compactly generated spaces. In the context of symmetric spectra of simplicial sets, the analogous isomorphisms of mapping simplicial sets hold as well.

The topological and simplicial mapping spaces are related by various adjunctions. We list some of these. For a simplicial spectrum X and a topological spectrum Y there is a natural isomorphism of simplicial sets

$$\mathrm{map}(X, \mathcal{S}(Y)) \ \cong \mathcal{S}(\mathrm{map}(|X|, Y))$$

which on vertices specializes to the adjunction between singular complex and geometric realization. Indeed, an n-simplex of the left hand side is a morphism $\Delta[n]^+ \wedge X \longrightarrow \mathcal{S}(Y)$ of symmetric spectra of simplicial sets. We can pass to the adjoint morphism $|\Delta[n]^+ \wedge X| \longrightarrow Y$ of symmetric spectra of spaces, exploit that realization commutes with smash products and then identify the realization $|\Delta[n]|$ with the topological n-simplex $\underline{\Delta}[n]$ as in (3.1) of Appendix A. The outcome is a morphism $\underline{\Delta}[n]^+ \wedge |X| \longrightarrow Y$ whose adjoint $\underline{\Delta}[n] \longrightarrow \mathrm{map}(|X|, Y)$, a continuous map of unbased spaces, is an n-simplex of the singular complex of the mapping space $\mathrm{map}(|X|, Y)$.

For free symmetric spectra we have Σ_m-equivariant isomorphisms

$$(3.37) \qquad \mathrm{map}(F_m K, Y) \ \cong \ \mathrm{map}(K, Y_m) \ .$$

In more detail: in the context of spectra of spaces, the adjunction bijection between mapping sets which we specified in Example 3.20 is indeed a homeomorphism. In the context of spectra of simplicial sets, we can use the adjunction bijection for $\Delta[n]^+ \wedge K$ and exploit $F_m(\Delta[n]^+ \wedge K) = \Delta[n]^+ \wedge F_m K$ and get a natural bijection between the n-simplices of the two mapping simplicial sets. For $K = S^0$ isomorphism (3.37) specializes to an isomorphism $\mathrm{map}(F_m S^0, Y) \cong Y_m$.

We have associative and unital composition maps

$$\mathrm{map}(Y,Z)\wedge \mathrm{map}(X,Y) \longrightarrow \mathrm{map}(X,Z) \ .$$

Indeed, for symmetric spectra of topological spaces this is just the observation that composition of morphisms is continuous for the mapping space topology. For symmetric spectra of simplicial sets the composition maps are given on n-simplices by

$$\begin{array}{rcl} \mathcal{S}p(\Delta[n]^+\wedge Y, Z) \ \wedge \ \mathcal{S}p(\Delta[n]^+\wedge X, Y) & \longrightarrow & \mathcal{S}p(\Delta[n]^+\wedge X, Z) \\ g\wedge f & \longmapsto & g\circ(\Delta[n]^+\wedge f)\circ(\mathrm{diag}\wedge X) \end{array}$$

where $\mathrm{diag}:\Delta[n]^+\longrightarrow \Delta[n]^+\wedge\Delta[n]^+$ is the diagonal map.

Example 3.38 (Internal Hom spectra)**.** Symmetric spectra have internal function objects: for symmetric spectra X and Y we define a symmetric spectrum $\mathrm{Hom}(X,Y)$ in level n by

$$\mathrm{Hom}(X,Y)_n \ = \ \mathrm{map}(X, \mathrm{sh}^n Y)$$

with Σ_n-action induced by the action on $\mathrm{sh}^n Y$ as described in Example 3.9. The structure map $\sigma_n : \mathrm{Hom}(X,Y)_n\wedge S^1 \longrightarrow \mathrm{Hom}(X,Y)_{n+1}$ is the composite

$$\mathrm{map}(X,\mathrm{sh}^n Y)\wedge S^1 \xrightarrow{\text{assembly}} \mathrm{map}(X, S^1\wedge \mathrm{sh}^n Y) \xrightarrow{\mathrm{map}(X,\lambda_{\mathrm{sh}^n Y})} \mathrm{map}(X,\mathrm{sh}^{n+1} Y) \ ;$$

here the first map is of 'assembly type', i.e., it takes $f\wedge t$ to the map which sends $x\in X$ to $t\wedge f(x)$ (for $f: X\longrightarrow \mathrm{sh}^n Y$ and $t\in S^1$), and $\lambda_{\mathrm{sh}^n Y} : S^1\wedge \mathrm{sh}^n Y \longrightarrow \mathrm{sh}(\mathrm{sh}^n Y) = \mathrm{sh}^{n+1} Y$ is the natural morphism defined in (3.12).

In order to verify that this indeed gives a symmetric spectrum we describe the iterated structure map. Let us denote by $\lambda_Y^{(m)} : S^m\wedge Y\longrightarrow \mathrm{sh}^m Y$ the morphism whose n-th level is the composite

(3.39) $$S^m\wedge Y_n \xrightarrow{\text{twist}} Y_n\wedge S^m \xrightarrow{\sigma^m} Y_{n+m} \xrightarrow{\chi_{n,m}} Y_{m+n} = (\mathrm{sh}^m Y)_n \ .$$

For $m=0$ this is the canonical isomorphism $S^0\wedge Y\cong Y$ and for $m=1$ this specializes to the morphism $\lambda_Y : S^1\wedge Y\longrightarrow \mathrm{sh}\, Y$ of (3.12); in general $\lambda_Y^{(m)}$ is a morphism of Σ_m-symmetric spectra. Then for all $k,m\geq 0$ the diagram

$$\begin{tikzcd} S^k\wedge S^m\wedge Y \arrow[r, "\mathrm{Id}\wedge\lambda_Y^{(m)}"] \arrow[d, "\cong"'] & S^k\wedge \mathrm{sh}^m Y \arrow[r, "\lambda^{(k)}_{\mathrm{sh}^m Y}"] & \mathrm{sh}^k(\mathrm{sh}^m Y) \arrow[d, equal] \\ S^{k+m}\wedge Y \arrow[r, "\chi_{k,m}\wedge\mathrm{Id}"'] & S^{m+k}\wedge Y \arrow[r, "\lambda_Y^{(m+k)}"'] & \mathrm{sh}^{m+k} Y \end{tikzcd}$$

commutes. This implies that the iterated structure map of the spectrum $\mathrm{Hom}(X,Y)$ equals the composite

$$\mathrm{map}(X,\mathrm{sh}^n Y)\wedge S^m \xrightarrow{\text{assembly}} \mathrm{map}(X, S^m\wedge \mathrm{sh}^n Y) \xrightarrow{\mathrm{map}(X,\lambda^{(m)}_{\mathrm{sh}^n Y})} \mathrm{map}(X,\mathrm{sh}^{n+m} Y)$$

and is thus $\Sigma_n\times\Sigma_m$-equivariant. The first map is again of 'assembly type', i.e., for $f: X\longrightarrow \mathrm{sh}^n Y$ and $t\in S^m$ it takes $f\wedge t$ to the map which sends $x\in X$ to $t\wedge f(x)$.

Taking function spectrum commutes 'on the nose' with shifting in the second variable, i.e., we have

(3.40) $$\mathrm{Hom}(X,\mathrm{sh}\, Y) \ = \ \mathrm{sh}\left(\mathrm{Hom}(X,Y)\right) \ ,$$

where we really mean equality, not just isomorphism. Indeed, in level n we have

$$\begin{aligned} \mathrm{Hom}(X,\mathrm{sh}\, Y)_n \ &= \ \mathrm{map}(X,\mathrm{sh}^n(\mathrm{sh}\, Y)) \ = \ \mathrm{map}(X,\mathrm{sh}^{1+n} Y) \\ &= \ \mathrm{Hom}(X,Y)_{1+n} \ = \ \left(\mathrm{sh}\left(\mathrm{Hom}(X,Y)\right)\right)_n \ . \end{aligned}$$

The symmetric group actions and structure maps coincide as well; as we explained in (3.10), simplifying the expression $\mathrm{sh}^n(\mathrm{sh}\, Y)$ to $\mathrm{sh}^{1+n} Y$ (rather than $\mathrm{sh}^{n+1} Y$) is the right way to get the group actions straight.

A natural isomorphism of symmetric spectra $\mathrm{Hom}(F_m S^0, Y) \cong \mathrm{sh}^m Y$ is given at level n by

$$\mathrm{Hom}(F_m S^0, Y)_n = \mathrm{map}(F_m S^0, \mathrm{sh}^n Y) \;\cong\; (\mathrm{sh}^n Y)_m \;=\; Y_{n+m} \xrightarrow{\chi_{n,m}} Y_{m+n} = (\mathrm{sh}^m Y)_n$$

where the second map is the adjunction bijection described in Example 3.20. This isomorphism is equivariant for the left actions of Σ_m induced on the source from the right Σ_m-action on a free spectrum described in Example 3.20. In the special case $m = 0$ we have $F_0 S^0 = \mathbb{S}$, which gives a natural isomorphism of symmetric spectra $\mathrm{Hom}(\mathbb{S}, Y) \cong Y$.

The internal function spectrum functor $\mathrm{Hom}(X, -)$ is right adjoint to the internal smash product $- \wedge X$ of symmetric spectra, to be discussed in Section 5.

3.3. Constructions involving ring spectra.

Example 3.41 (Endomorphism ring spectra)**.** For every symmetric spectrum X, the symmetric function spectrum $\mathrm{Hom}(X, X)$ defined in Example 3.38 has the structure of a symmetric ring spectrum which we call the *endomorphism ring spectrum* of X. The multiplication map $\mu_{n,m} : \mathrm{Hom}(X, X)_n \wedge \mathrm{Hom}(X, X)_m \longrightarrow \mathrm{Hom}(X, X)_{n+m}$ is defined as the composite

$$\begin{aligned} \mathrm{map}(X, \mathrm{sh}^n X) \wedge \mathrm{map}(X, \mathrm{sh}^m X) &\xrightarrow{\mathrm{sh}^m \wedge \mathrm{Id}} \mathrm{map}(\mathrm{sh}^m X, \mathrm{sh}^m(\mathrm{sh}^n X)) \wedge \mathrm{map}(X, \mathrm{sh}^m X) \\ = \mathrm{map}(\mathrm{sh}^m X, \mathrm{sh}^{n+m} X) \wedge \mathrm{map}(X, \mathrm{sh}^m X) &\xrightarrow{\;\circ\;} \mathrm{map}(X, \mathrm{sh}^{n+m} X) \end{aligned}$$

where the second map is the composition pairing of Example 3.36. We refer to (3.10) for why it is 'right' to identify $\mathrm{sh}^m(\mathrm{sh}^n X)$ with $\mathrm{sh}^{n+m} X$ (note the orders in which m and n occur), so that no shuffle permutation is needed.

While this construction always works on the pointset level, one can only expect $\mathrm{Hom}(X, X)$ to be homotopically meaningful under certain conditions on X. The stable model structures which we discuss in Section III.4 will explain which conditions are sufficient.

In much the same way as above we can define associative and unital action maps $\mathrm{Hom}(X, Z)_n \wedge \mathrm{Hom}(X, X)_m \longrightarrow \mathrm{Hom}(X, Z)_{n+m}$ and $\mathrm{Hom}(X, X)_n \wedge \mathrm{Hom}(Z, X)_m \longrightarrow \mathrm{Hom}(Z, X)_{n+m}$ for any other symmetric spectrum Z. This makes $\mathrm{Hom}(X, Z)$ and $\mathrm{Hom}(Z, X)$ into right respectively left modules over the endomorphism ring spectrum of X.

Example 3.42 (Spherical monoid ring)**.** If M is a topological or simplicial monoid (depending on the kind of symmetric spectra under consideration), we define the *spherical monoid ring* $\mathbb{S}M$ by

$$(\mathbb{S}M)_n \;=\; M^+ \wedge S^n$$

with symmetric group actions and structure maps only on the spheres; here M^+ denotes the underlying space (or simplicial set) of M with a disjoint basepoint added. In other words, the underlying symmetric spectrum of the spherical monoid ring is the suspension spectrum, as defined in Example 1.13, of M^+. The spherical monoid ring becomes a symmetric ring spectrum as follows: the unit maps of $\mathbb{S}M$ are the maps $1 \wedge - : S^n \longrightarrow M^+ \wedge S^n = (\mathbb{S}M)_n$ which include via the unit element 1 of the monoid M. The multiplication map $\mu_{n,m}$ is given by the composite

$$(M^+ \wedge S^n) \wedge (M^+ \wedge S^m) \cong (M \times M)^+ \wedge (S^n \wedge S^m) \xrightarrow{\mathrm{mult.} \wedge \mu_{n,m}} M^+ \wedge S^{n+m} \ .$$

A right module over the spherical monoid ring $\mathbb{S}M$ is 'the same as' a symmetric spectrum with a (continuous or simplicial) right action by the monoid M through endomorphisms.

If R is a symmetric ring spectrum, then the zeroth space (or simplicial set) R_0 is a topological (or simplicial) monoid via the composite

$$R_0 \times R_0 \;\longrightarrow\; R_0 \wedge R_0 \xrightarrow{\mu_{0,0}} R_0 \ .$$

The monoid R_0 is commutative if the ring spectrum R is. Given any monoid homomorphism $f : M \longrightarrow R_0$ we define a homomorphism of symmetric spectra $\hat{f} : \mathbb{S}M \longrightarrow R$ in level n as the composite

$$M^+ \wedge S^n \xrightarrow{f \wedge \mathrm{Id}} R_0^+ \wedge S^n \;\longrightarrow\; R_0 \wedge S^n \xrightarrow{\sigma^n} R_n \ .$$

This morphism is in fact unital and associative, i.e., a morphism of symmetric ring spectra. Moreover, $\hat{f}$ is uniquely determined by the property that it gives back f in level 0. We can summarize this as saying that the construction of the monoid ring over $\mathbb{S}$ is left adjoint to the functor which takes a symmetric ring spectrum R to the (topological or simplicial) monoid R_0. [pointed monoids]

Example 3.43 (Monoid ring spectra)**.** The previous construction works more generally when we start with a symmetric ring spectrum R instead of the sphere spectrum. If R is a symmetric ring spectrum and M a topological or simplicial monoid (depending on the kind of symmetric spectra), we can define a symmetric ring spectrum RM by $RM = M^+ \wedge R$, i.e., the levelwise smash product with M with a disjoint basepoint added. The unit map is the composite of the unit map of R and the morphism $R \cong \{1\}^+ \wedge R \longrightarrow M^+ \wedge R$ induced by the unit of M. The multiplication map $\mu_{n,m}$ is given by the composite

$$(M^+ \wedge R_n) \wedge (M^+ \wedge R_m) \cong (M \times M)^+ \wedge (R_n \wedge R_m) \xrightarrow{\text{mult.} \wedge \mu_{n,m}} M^+ \wedge R_{n+m} \ .$$

If both R and M are commutative, then so is RM. A right module over the symmetric ring spectrum RM amounts to the same data as an R-module together with a continuous (or simplicial) right action of the monoid M by R-linear endomorphisms.

As we shall see later, the homotopy groups of RM are the R-homology groups of the underlying space of M, with the Pontryagin product as multiplication. In the special case of a *discrete* spherical monoid ring, the homotopy groups are the monoid ring, in the ordinary sense, of the homotopy groups, i.e., there is a natural isomorphism of graded rings

$$\pi_* (RM) \ \cong \ (\pi_* R)M \ .$$

[add this below]

Example 3.44 (Matrix ring spectra)**.** If R is a symmetric ring spectrum and $k \geq 1$ we define the symmetric ring spectrum $M_k(R)$ of $k \times k$ *matrices* over R by

$$M_k(R) \ = \ \mathrm{map}(k^+, k^+ \wedge R) \ .$$

Here $k^+ = \{0, 1, \dots, k\}$ with basepoint 0, and so $M_k(R)$ is a k-fold product of a k-fold coproduct (wedge) of copies of R. So 'elements' of $M_k(R)$ are more like matrices which in each row have at most one nonzero entry. The multiplication

$$\mu_{n,m} \ : \ \mathrm{map}(k^+, k^+ \wedge R_n) \ \wedge \ \mathrm{map}(k^+, k^+ \wedge R_m) \ \longrightarrow \ \mathrm{map}(k^+, k^+ \wedge R_{n+m})$$

sends $f \wedge g$ to the composite

$$k^+ \xrightarrow{g} k^+ \wedge R_m \xrightarrow{f \wedge \mathrm{Id}} k^+ \wedge R_n \wedge R_m \xrightarrow{k^+ \wedge \mu_{n,m}} k^+ \wedge R_{n+m} \ .$$

We shall see below that homotopy groups take wedges and products to direct sums; this implies a natural isomorphism of graded rings

$$\pi_*(M_k(R)) \ \cong \ M_k(\pi_* R) \ .$$

We revisit this in more detail in Example 6.29

Example 3.45 (Opposite ring spectrum)**.** For every symmetric ring spectrum R we can define the *opposite* ring spectrum R^{op} by keeping the same spaces (or simplicial sets), symmetric group actions and unit maps, but with new multiplication $\mu^{\mathrm{op}}_{n,m}$ on R^{op} given by the composite

$$R^{\mathrm{op}}_n \wedge R^{\mathrm{op}}_m = R_n \wedge R_m \xrightarrow{\text{twist}} R_m \wedge R_n \xrightarrow{\mu_{m,n}} R_{m+n} \xrightarrow{\chi_{m,n}} R_{n+m} = R^{\mathrm{op}}_{n+m} \ .$$

As a consequence of centrality of ι_1, the higher unit maps for R^{op} agree with the higher unit maps for R. By definition, a symmetric ring spectrum R is commutative if and only if $R^{\mathrm{op}} = R$. In the internal form, the multiplication μ^{op} is obtained from the multiplication $\mu : R \wedge R \longrightarrow R$ as the composite

$$R \wedge R \ \xrightarrow{\tau_{R,R}} \ R \wedge R \ \xrightarrow{\mu} \ R \ .$$

For example, we have $(HA)^{\mathrm{op}} = H(A^{\mathrm{op}})$ for the Eilenberg-Mac Lane ring spectra (Example 1.14) of an ordinary ring A and its opposite, and $(RM)^{\mathrm{op}} = (R^{\mathrm{op}})M^{\mathrm{op}}$ for the monoid ring spectra (Example 3.42) of a simplicial or topological monoid M and its opposite.

By the centrality of the unit, the underlying symmetric spectra of R and R^{op} are equal (not just isomorphic), hence R and R^{op} have the same (not just isomorphic) naive and true homotopy groups. As we discuss in Example 6.30 below, we have

$$\pi_*(R^{\mathrm{op}}) \ = \ (\pi_* R)^{\mathrm{op}}$$

(again equality) as graded rings, where the right hand side is the graded-opposite ring, i.e., the graded abelian group $\pi_* R$ with new product $x \cdot_{\mathrm{op}} y = (-1)^{kl} \cdot y \cdot x$ for $x \in \pi_k R$ and $y \in \pi_l R$.

Example 3.46 (Function ring spectra)**.** If R is a symmetric ring spectrum and L an *unpointed* space, then the mapping spectrum R^{L^+} (compare Example 3.6) is again a symmetric ring spectrum. The multiplication maps $R_n^{L^+} \wedge R_m^{L^+} \longrightarrow R_{n+m}^{L^+}$ are the composites

$$\mathrm{map}(L^+, R_n) \wedge \mathrm{map}(L^+, R_m) \ \xrightarrow{\ \wedge\ } \ \mathrm{map}(L^+ \wedge L^+, R_n \wedge R_m) \ \xrightarrow{\mathrm{map}(\mathrm{diag}, \mu_{n,m})} \ \mathrm{map}(L^+, R_{n+m})$$

using the diagonal map $L^+ \longrightarrow L^+ \wedge L^+$. Associativity of the multiplication on R^{L^+} comes from associativity of R and coassociativity of the diagonal map. The unit map $\iota_n : S^n \longrightarrow R_n^{L^+}$ is the composite of the unit map of R with the map $R_n \longrightarrow R_n^{L^+}$ that takes a point (or simplex) $x \in R_n$ to the constant map that sends all of L to x. If the multiplication of R is commutative, then so is the multiplication of R^{L^+}, since the diagonal map is cocommutative.

A concrete example is $X = HA$, the Eilenberg-Mac Lane spectrum of an abelian group A. Then

$$\hat{\pi}_{-k}(HA^{L^+}) \ \cong \ H^k(L, A) \ ,$$

a natural isomorphism of abelian groups, where the right hand side is the singular cohomology of L with coefficients in A. Indeed, since HA is an Ω-spectrum, so is HA^{L^+} for any cofibrant space or any simplicial set. So the canonical map $[L, A[S^k]] = \pi_0 \,\mathrm{map}(L^+, A[S^k]) = \pi_0 (HA^{L^+})_k \longrightarrow \hat{\pi}_{-k}(HA^{L^+})$ is bijective, where $[L, A[S^k]]$ is the set of homotopy classes of (unbased) maps. Since $A[S^k]$ is an Eilenberg-Mac Lane space of type (A, k), evaluation at the fundamental cohomology class $\iota_k \in H^k(A[S^k], A)$ is an isomorphism

$$[L, A[S^k]] \ \longrightarrow \ H^k(L, A) \ , \quad [f] \longmapsto f^*(\iota_k) \ .$$

If A is a ring, then HA becomes a ring spectrum and this isomorphism takes the product of homotopy groups to the cup product in singular cohomology.

Example 3.47 (Inverting m)**.** We consider an integer m and define $\mathbb{S}[1/m]$, the *sphere spectrum with m inverted* by starting from the sphere spectrum (of topological spaces) and using a map $\varphi_m : S^1 \longrightarrow S^1$ of degree m as the new unit map ι_1. Since the multiplication on $\mathbb{S}$ is commutative, centrality is automatic. So $\mathbb{S}[1/m]$ has the same spaces and symmetric group actions as $\mathbb{S}$, but the n-th unit map ι_n of $\mathbb{S}[1/m]$ is the n-fold smash power of φ_m, which is a self map of S^n of degree mn. The unit maps form a morphism $\mathbb{S} \longrightarrow \mathbb{S}[1/m]$ of symmetric ring spectra which on homotopy groups induces an isomorphism

$$\hat{\pi}_* \, \mathbb{S}[1/m] \ \cong \ \hat{\pi}_* \mathbb{S} \otimes \mathbb{Z}[1/m] \ .$$

For $m = 0$, the homotopy groups are thus trivial and for $m = 1$ or $m = -1$ the unit morphism $\mathbb{S} \longrightarrow \mathbb{S}[1/m]$ is a $\hat{\pi}_*$-isomorphism.

Example 3.48 (Inverting homotopy elements)**.** Let R be a symmetric ring spectrum and let $x : S^1 \longrightarrow R_1$ be a central map of pointed spaces (or simplicial sets), i.e., such that the square

$$\begin{tikzcd}
R_n \wedge S^1 \arrow[r, "\mathrm{Id} \wedge x"] \arrow[d, "\tau"'] & R_n \wedge R_1 \arrow[r, "\mu_{n,1}"] & R_{n+1} \arrow[d, "\chi_{n,1}"] \\
S^1 \wedge R_n \arrow[r, "x \wedge \mathrm{Id}"'] & R_1 \wedge R_n \arrow[r, "\mu_{1,n}"'] & R_{1+n}
\end{tikzcd}$$

commutes for all $n \geq 0$. We observe that if R is commutative, then any map from S^1 to R_1 is central.

We define a new symmetric ring spectrum $R[1/x]$ as follows. For $n \geq 0$ we set

$$R[1/x]_n \;=\; \mathrm{map}(S^n, R_{2n}) \ ,$$

the n-fold loop space of R_{2n}. In order to guess the correct action and multiplication maps it is helpful to think of $2n$ as $n+n$, and not as the n-fold sum of 2's. The group Σ_n acts on S^n by coordinate permutations, on R_{n+n} via restriction along the diagonal embedding

$$\Delta \;:\; \Sigma_n \longrightarrow \Sigma_{2n} \ , \quad \Delta(\gamma) = \gamma + \gamma \ ,$$

and by conjugation on the whole mapping space. The multiplication $\mu_{n,m} : R[1/x]_n \wedge R[1/x]_m \longrightarrow R[1/x]_{n+m}$ is the map

$$\begin{aligned} \mathrm{map}(S^n, R_{2n}) \wedge \mathrm{map}(S^m, R_{2m}) &\longrightarrow \mathrm{map}(S^{n+m}, R_{2(n+m)}) \\ f \wedge g &\longmapsto (1_n + \chi_{n,m} + 1_m) \circ \mu_{2n,2m} \circ (f \wedge g) \ . \end{aligned}$$

Note that the interpretation $2n = n + n$ lets us predict the shuffle permutation: the natural target of $\mu_{2n,2m} \circ (f \wedge g)$ is R_{2n+2m}, whose index expands to $(n+n)+(m+m)$. The visual difference to $2(n+m) = (n+m) + (n+m)$ suggest to insert the permutation $1_n + \chi_{n,m} + 1_m$ which moves the indices into the correct order. The multiplication map is associative since smashing of maps and the product of R are, and because the relation

$$(1_{n+m} + \chi_{n+m,k} + 1_k)(1_n + \chi_{n,m} + 1_m + 1_{2k}) \;=\; (1_n + \chi_{n,m+k} + 1_{m+k})(1_{2n} + 1_m + \chi_{m,k} + 1_k)$$

holds in the group $\Sigma_{2n+2m+2k}$. The multiplication map is $\Sigma_n \times \Sigma_m$-equivariant since the original multiplication maps are equivariant and since the diagonal embeddings satisfy

$$\Delta(\gamma + \tau)(1_n + \chi_{n,m} + 1_m) \;=\; (1_n + \chi_{n,m} + 1_m)(\Delta(\gamma) + \Delta(\tau))$$

for $\gamma \in \Sigma_n$ and $\tau \in \Sigma_m$. We have $R[1/x]_0 = R_0$ and the 0th unit map for $R[1/x]$ is the same as for R.

Next we define based maps $j_n : R_n \longrightarrow \mathrm{map}(S^n, R_{2n})$ as the adjoints of the maps

$$R_n \wedge S^n \xrightarrow{\mathrm{Id} \wedge x^{\wedge n}} R_n \wedge R_1^{\wedge n} \xrightarrow{\mu_{n,1,\dots,1}} R_{n+n}$$

Since the map $\mu_{n,1,\dots,1} \circ x^{\wedge n} : R_n \wedge S^n \longrightarrow R_{n+n}$ is $\Sigma_n \times \Sigma_n$-equivariant, the adjoint j_n is Σ_n-equivariant. The maps j_n are multiplicative in the sense of the relation $\mu_{n,m}(j_n \wedge j_m) = j_{n+m}\mu_{n,m}$ holds.

We define unit maps $\iota_n : S^n \longrightarrow R[1/x]_n$ as the composite of the unit map of R with j_n (which agrees with the n-fold power of the map $j_1 : S^1 \longrightarrow R[1/x]_1$). This finishes the definition of $R[1/x]$ which is again a symmetric ring spectrum and comes with a morphism of symmetric ring spectra $j : R \longrightarrow R[1/x]$.

We note that the central map x does not enter in the definition of the spaces $R[1/x]_n$, and it is not used in defining the multiplication of $R[1/x]$, but it enters in the definition of the morphism j and hence the unit map of the ring spectrum $R[1/x]$. Since $R[1/x]_n$ is the n-fold loop space of R_{2n}, the homotopy group $\pi_{k+n}(R[1/x]_n)$ is isomorphic to $\pi_{k+2n}R_{2n}$; thus the colimit system which defines the naive homotopy group $\hat{\pi}_k(R[1/x])$ involves 'half of' the groups which define $\hat{\pi}_k R$, but the effect of the map x is twisted into the morphisms in the sequence, and so the homotopy groups of R and $R[1/x]$ are typically different. We show in Proposition 6.56 below that if R is semistable, then so is $R[1/x]$ and the effect of the morphism $j : R \longrightarrow R[1/x]$ on the graded rings of homotopy groups is precisely inverting the class in $\hat{\pi}_0 R$ represented by the map x.

Example 3.49 (Adjoining roots of unity)**.** As an application of the localization construction of Example 3.48 we construct a commutative symmetric ring spectrum which models the 'Gaussian integers over $\mathbb{S}$' with 2 inverted. We start with the spherical group ring $\mathbb{S}C_4$ of the cyclic group of order 4, a commutative symmetric ring spectrum as in Example 3.42. We invert the element

$$1 - t^2 \;\in\; \mathbb{Z}C_4 \;=\; \hat{\pi}_0(\mathbb{S}C_4)$$

where $t \in C_4$ is a generator, and define

$$\mathbb{S}[1/2, i] \;=\; \mathbb{S}C_4[1/(1-t^2)] \ .$$

In more detail, the space $(\mathbb{S}C_4)_1 = C_4^+ \wedge S^1$ is a wedge of 4 circles and the map from $\pi_1(\mathbb{S}C_4)_1$ to the stable group $\hat{\pi}_0(\mathbb{S}C_4)$ is surjective. So $1 - t^2 \in \hat{\pi}_0(\mathbb{S}C_4)$ can be represented by a based map $x : S^1 \longrightarrow (\mathbb{S}C_4)_1$ to which we apply Example 3.48. The monoid ring spectrum $\mathbb{S}C_4$ is commutative and semistable, and so Proposition 6.56 below shows that the graded ring of homotopy groups of $\mathbb{S}C_4[1/(1-t^2)]$ is obtained from the ring $\pi_*(\mathbb{S}C_4)$ by inverting the class $1 - t^2$ in $\hat{\pi}_0$.

Because $(1+t^2)(1-t^2) = 0$ in the group ring $\mathbb{Z}C_4$, inverting $1-t^2$ forces $1+t^2 = 0$, so t becomes a square root of -1. Since $(1-t^2)^2 = 2(1-t^2)$, inverting $1-t^2$ also inverts 2, and in fact

$$\pi_0\, \mathbb{S}[1/2, i] \;\cong\; \mathbb{Z}C_4[1/(1-t^2)] \;=\; \mathbb{Z}\,[1/2, i]\;,$$

where i is the image of t. The ring spectrum $\mathbb{S}[1/2, i]$ is $\hat{\pi}_*$-isomorphic as a symmetric spectrum to a wedge of 2 copies of $\mathbb{S}[1/2]$, and thus deserves to be called the 'Gaussian integers over $\mathbb{S}$' with 2 inverted. Moreover, $\mathbb{S}[1/2, i]$ is a Moore spectrum for the ring $\mathbb{Z}[1/2, i]$, i.e., its integral homology is concentrated in dimension zero (compare Section II.6.3).

If p is a prime number and $n \geq 1$, we can similarly adjoin a primitive p^n-th root of unity to the sphere spectrum, provided we are also willing to invert p in the homotopy groups. We first form the monoid ring spectrum $\mathbb{S}C_{p^n}$ of the cyclic group of order p^n, let $t \in C_{p^n}$ denote a generator and invert the element $f = p - (t^{q(p-1)} + t^{q(p-2)} + \cdots + t^q + 1)$ in $\mathbb{Z}C_{p^n} = \hat{\pi}_0(\mathbb{S}C_{p^n})$, where $q = p^{n-1}$. This defines

$$\mathbb{S}[1/p, \zeta] \;=\; \mathbb{S}C_{p^n}[1/f]\;.$$

We have $f^2 = pf$, so inverting f also inverts the prime p and forces the expression $p - f$ to become 0 in the localized ring. If we let ζ denote the image of t in the localized ring, then the latter says that ζ is a root of the cyclotomic polynomial, i.e.,

$$\zeta^{q(p-1)} + \zeta^{q(p-2)} + \cdots + \zeta^q + 1 \;= 0$$

where again $q = p^{n-1}$. In fact we have $\mathbb{Z}C_{p^n}[1/f] = \mathbb{Z}[1/p, \zeta]$ where ζ is a primitive p^n-th root of unity; moreover the commutative symmetric ring spectrum $\mathbb{S}[1/p, \zeta]$ is a Moore spectrum for the ring $\mathbb{Z}[1/p, \zeta]$.

We can do the same constructions starting with any semistable commutative symmetric ring spectrum R instead of the sphere spectrum, yielding a new commutative symmetric ring spectrum $R[1/p, \zeta]$. If p is already invertible and the cyclotomic polynomial above is irreducible in $\hat{\pi}_0 R$, then this adjoins a primitive p^n-th root of unity to the homotopy ring of R.

These examples are a special case of a much more general phenomenon: every number ring can be 'lifted' to an extension of the sphere spectrum by a commutative symmetric ring spectrum, provided we also invert the ramified primes. However, the only proofs of this general fact that I know use obstruction theory, and so we cannot give a construction which is as explicit and simple as the one above for adjoining roots of unity.

Example 3.50 (Algebraic K-theory)**.** There are various formalisms which associate to a category with suitable extra structure an algebraic K-theory space. These spaces are typically infinite loop spaces in a natural way, i.e., they arise from an Ω-spectrum. One very general framework is Waldhausen's *$S_\cdot$-construction* which accepts *categories with cofibrations and weak equivalences* as input and which produces symmetric spectra which are positive Ω-spectra.

We consider a category $\mathcal{C}$ with cofibrations and weak equivalences in the sense of Waldhausen [**88**]. For any finite set Q we denote by $\mathcal{P}(Q)$ the power set of Q viewed as a partially ordered set under inclusions, and thus as a category. A *Q-cube* in $\mathcal{C}$ is a functor $X : \mathcal{P}(Q) \longrightarrow \mathcal{C}$. Such a Q-cube X is a *cofibration cube* if for all $S \subset T \subset Q$ the canonical map

$$\operatorname{colim}_{S \subseteq U \subsetneq T} X(U) \;\longrightarrow\; X(T)$$

is a cofibration in $\mathcal{C}$. (The colimit on the left can be formed by iterated pushouts along cofibrations, so it exists in $\mathcal{C}$.)

We view the ordered set $[n] = \{0 < 1 < \cdots < n\}$ as a category. If $n = \{n_s\}_{s \in Q}$ is a Q-tuple of non-negative integers, we denote by $[n]$ the product category of the categories $[n_s]$, $s \in Q$. For a morphism

$i \to j$ in $[n]$ and a subset $U \subset Q$ we let $(i \to j)_U$ be the new morphisms in $[n]$ whose sth component is $i_s \to j_s$ if $s \in U$ and the identity $i_s \to i_s$ if $s \notin U$. Then for each morphism $i \to j$ in $[n]$, the assignment

$$U \ \mapsto \ (i \to j)_U$$

defines a Q-cube in the arrow category $\mathrm{Ar}[n]$.

For a finite set Q and a Q-indexed tuple $n = \{n_s\}_{s \in Q}$ we define a category $S_n^Q\mathcal{C}$ as the full subcategory of the category of functors from $\mathrm{Ar}[n]$ to $\mathcal{C}$ consisting of the functors

$$A : \mathrm{Ar}[n] \longrightarrow \mathcal{C} \ , \quad (i \to j) \mapsto A_{i \to j}$$

with the following properties:

(i) if some component $i_s \to j_s$ of $i \to j$ is an identity (i.e., if $i_s = j_s$ for some $s \in Q$), then $A_{i \to j} = *$ is the distinguished zero object of $\mathcal{C}$;

(ii) for every pair of composable morphisms $i \to j \to k$ the cube

$$U \mapsto A_{(j \to k)_U \circ (i \to j)}$$

is a cofibration cube

(iii) for every pair of composable morphisms $i \to j \to k$ the square

$$\begin{tikzcd} \operatorname{colim}_{U \subsetneq Q} A_{(j \to k)_U \circ (i \to j)} \arrow[r] \arrow[d] & A_{i \to k} \arrow[d] \\ * \arrow[r] & A_{j \to k} \end{tikzcd}$$

is a pushout in $\mathcal{C}$.

The category $S_n^Q\mathcal{C}$ depends contravariantly on $[n]$, so that as $[n]$ varies, we get a Q-simplicial category $S_\cdot^Q\mathcal{C}$. We can make $S_\cdot^Q\mathcal{C}$ into a Q-simplicial object of categories with cofibrations and weak equivalences as follows. A morphism $f : A \longrightarrow A'$ is a *cofibration* in $S_n^Q\mathcal{C}$ if for every pair of composable morphisms $i \to j \to k$ the induced map of Q-cubes

$$\left(U \mapsto A_{(j \to k)_U \circ (i \to j)}\right) \ \longrightarrow \ \left(U \mapsto A'_{(j \to k)_U \circ (i \to j)}\right)$$

is a cofibration cube when viewed as a $(|Q|+1)$-cube in $\mathcal{C}$. A morphism $f : A \longrightarrow A'$ is a *weak equivalence* in $S_n^Q\mathcal{C}$ if for every morphism $i \longrightarrow j$ in $[n]$ the morphism $f_{i \longrightarrow j}$ is a weak equivalence in $\mathcal{C}$. If Q has one element, the $S_\cdot^Q\mathcal{C}$ is isomorphic to $S_\cdot\mathcal{C}$ as defined by Waldhausen [**88**]. If $P \subset Q$ there is an isomorphism of Q-simplicial categories with cofibrations and weak equivalences

$$S_\cdot^Q\mathcal{C} \ \cong \ S_\cdot^{Q-P}(S_\cdot^P\mathcal{C})$$

[define]. So a choice of linear ordering of the set Q specifies an isomorphism of categories

$$S_\cdot^Q\mathcal{C} \ \cong \ S_\cdot \cdots S_\cdot\mathcal{C}$$

to the $|Q|$-fold iterate of the $S_\cdot$-construction. Note that the permutation group of the set Q acts on $S_\cdot^Q\mathcal{C}$ by permuting the indices.

Now we are ready to define the *algebraic K-theory spectrum* $K(\mathcal{C})$ of the category with cofibrations and weak equivalences $\mathcal{C}$. (This is really naturally a coordinate free symmetric spectrum in the sense of Exercise E.I.5.) It is the symmetric spectrum of simplicial sets with nth level given by

$$K(\mathcal{C})_n \ = \ N_\cdot\left(wS_\cdot^{\{1,\dots,n\}}\mathcal{C}\right) \ ,$$

i.e., the nerve of the subcategory of weak equivalences in $S_\cdot^Q\mathcal{C}$ for the special case $Q = \{1, \dots, n\}$. The basepoint is the object of $S_\cdot^{\{1,\dots,n\}}\mathcal{C}$ given by the constant functor with values the distinguished zero object. The group Σ_n of permutations of the set $\{1, \dots, n\}$, acts on $S_\cdot^{\{1,\dots,n\}}\mathcal{C}$ preserving weak equivalences, so it acts on the simplicial set $K(\mathcal{C})_n$. Note that $K(\mathcal{C})_0$ is the nerve of the category $w\mathcal{C}$ of weak equivalences in $\mathcal{C}$.

We still have to define the structure maps

$$\sigma_n \ : \ K(\mathcal{C})_n \wedge S^1 \ \longrightarrow \ K(\mathcal{C})_{n+1} \ .$$

[...] By a theorem of Waldhausen [ref], the symmetric spectrum $K(\mathcal{C})$ is a positive Ω-spectrum.

Pairings of exact categories give rise to pairings of K-theory spectra. Consider a biexact functor $\wedge : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$ between categories with cofibrations and weak equivalences. For disjoint finite subsets Q and Q' we obtain a biexact functor of $(Q \cup Q')$-simplicial categories with cofibrations and weak equivalences

$$\wedge \ : \ S^Q_\cdot \mathcal{C} \times S^{Q'}_\cdot \mathcal{D} \ \longrightarrow \ S^{Q\cup Q'}_\cdot \mathcal{E}$$

by assigning

$$(A \wedge A')_{i\cup i' \to j \cup j'} \ = \ A_{i\to j} \wedge A'_{i'\to j'} \ .$$

We specialize to $Q = \{1, \dots, n\}$ and $Q' = \{n+1, \dots, n+m\}$, restrict to weak equivalences and take nerves. This yields a $\Sigma_n \times \Sigma_m$-equivariant map $K(\mathcal{C})_n \times K(\mathcal{D})_m \longrightarrow K(\mathcal{E})_{n+m}$ which factors as

$$K(\mathcal{C})_n \wedge K(\mathcal{D})_m \ \longrightarrow \ K(\mathcal{E})_{n+m} \ .$$

These maps are associative for strictly associative pairings [explain].

An example of a category with cofibrations and weak equivalences is the category $\mathbf{\Gamma}$ of finite pointed sets $n^+ = \{0, 1, \dots, n\}$ with 0 as basepoint, and pointed set maps. Here the cofibrations are the injective maps and the weak equivalences are the bijections. The 'smash product' functor

$$\wedge \ : \ \mathbf{\Gamma} \times \mathbf{\Gamma} \ \longrightarrow \ \mathbf{\Gamma} \ , \ (n^+, m^+) \mapsto (nm)^+$$

is biexact and strictly associative so it makes the symmetric sequence $\{K(\mathbf{\Gamma})_n\}_{n\geq 0}$ into a strict monoid of symmetric sequences. Here we identify $n^+ \wedge m^+$ with $(nm)^+$ using the lexicographic ordering. The object 1^+ of $\mathbf{\Gamma}$ gives a 0-simplex in $K(\mathcal{C})_0$; a theorem by Barratt-Priddy-Quillen asserts that the morphism $\mathbb{S} \longrightarrow K(\mathbf{\Gamma})$ adjoint to this is a $\hat{\pi}_*$-isomorphism. [compare with Jardine's 'The K-theory presheaf of spectra']

3.4. Symmetric spectra and I-spaces. Symmetric spectra are intimately related to the category $\mathbf{I}$ of (standard) finite sets and injective maps. We denote by $\mathbf{I}$ the category with objects the sets $\mathbf{n} = \{1, \dots, n\}$ for $n \geq 0$ (where $\mathbf{0}$ is the empty set) and with morphisms all injective maps. In other words, $\mathbf{I}$ is the subcategory of the category $\mathcal{F}in$ of standard finite sets (compare Remark 0.4) with only injective maps as morphisms. We denote by $\mathbf{T}^\mathbf{I}$ and $\mathbf{sS}^\mathbf{I}$ the categories of $\mathbf{I}$-*spaces*, i.e., covariant functors from $\mathbf{I}$ to the category of pointed spaces or simplicial sets.

Example 3.51 (Smash product with **I**-spaces)**.** Given an $\mathbf{I}$-space $T : \mathbf{I} \longrightarrow \mathbf{T}$ and a symmetric spectrum X, we can form a new symmetric spectrum $T \wedge X$ by setting

$$(T \wedge X)_n \ = \ T(\mathbf{n}) \wedge X_n$$

with diagonal action of Σ_n (which equals the monoid of endomorphism of the object $\mathbf{n}$ of $\mathbf{I}$). The structure map is given by

$$(T \wedge X)_n \wedge S^1 \ = \ T(\mathbf{n}) \wedge X_n \wedge S^1 \ \xrightarrow{T(\iota)\wedge\sigma_n} \ T(\mathbf{n+1}) \wedge X_{n+1} \ = \ (T \wedge X)_{n+1}$$

where $\iota : \mathbf{n} \longrightarrow \mathbf{n+1}$ is the inclusion. If K is a pointed space and T the constant functor with value K, then $T \wedge X$ is equal to $K \wedge X$, i.e., this construction reduces to the pairing of Example 3.6.

Example 3.52. The collection of spheres can be used to construct adjoint pairs of functors

$$\mathbf{T}^\mathbf{I} \ \underset{\Omega^\bullet}{\overset{\Sigma^\infty}{\rightleftarrows}} \ \mathcal{S}p_\mathbf{T} \qquad \text{and} \qquad \mathbf{sS}^\mathbf{I} \ \underset{\Omega^\bullet}{\overset{\Sigma^\infty}{\rightleftarrows}} \ \mathcal{S}p_{\mathbf{sS}}$$

between $\mathbf{I}$-spaces and symmetric spectra. The left adjoint Σ^∞ is the same as $- \wedge \mathbb{S}$, the smash product of an $\mathbf{I}$-space with the sphere spectrum as in Example 3.51. So explicitly, we have

$$(\Sigma^\infty T)_n \ = \ T(\mathbf{n}) \wedge S^n$$

with diagonal action of Σ_n; the structure map is given by

$$(\Sigma^\infty T)_n \wedge S^1 \;=\; T(\mathbf{n}) \wedge S^n \wedge S^1 \xrightarrow{T(\iota)\wedge\cong} T(\mathbf{n+1}) \wedge S^{n+1} \;=\; (\Sigma^\infty T)_{n+1}$$

where $\iota : \mathbf{n} \longrightarrow \mathbf{n+1}$ is the inclusion, where the isomorphism is the canonical one. We refer to $\Sigma^\infty T$ as the *suspension spectrum* of the **I**-space T. This generalizes the suspension spectrum of a based space or simplicial set (compare Example 1.13), which we recover for a constant **I**-space T.

The right adjoint $\Omega^\bullet$ is defined as follows. If X is a symmetric spectrum, we set

$$(\Omega^\bullet X)(\mathbf{n}) \;=\; \mathrm{map}(S^n, X_n)$$

on objects, where the symmetric group Σ_n acts by conjugation, i.e., $(\gamma_* f)(x) = \gamma f(\gamma^{-1}x)$ for $f : S^n \longrightarrow X_n$ and $\gamma \in \Sigma_n$. If $\alpha : \mathbf{n} \longrightarrow \mathbf{n+m}$ is an injective map then $\alpha_* : \mathrm{map}(S^n, X_n) \longrightarrow \mathrm{map}(S^{n+m}, X_{n+m})$ is given as follows. We choose a permutation $\gamma \in \Sigma_{n+m}$ such that $\gamma(i) = \alpha(i)$ for all $i = 1, \dots, n$ and set

$$\alpha_*(f) \;=\; \gamma_*(\sigma^m(f \wedge S^m)) \;,$$

i.e., we let γ acts as just defined on the composite

$$S^{n+m} \xrightarrow{f\wedge S^m} X_n \wedge S^m \xrightarrow{\sigma^m} X_{n+m} \;.$$

If $\bar\gamma \in \Sigma_{n+m}$ is another permutation that agrees with α on $\mathbf{n}$, then $\gamma^{-1}\bar\gamma$ fixes the set $\mathbf{n}$ elementwise, so $\gamma^{-1}\bar\gamma = 1_{\mathbf{n}} + \tau$ for a unique permutation $\tau \in \Sigma_m$. This gives

$$\bar\gamma_*(\sigma^m(f \wedge S^m)) \;=\; \gamma_*(1_{\mathbf{n}} + \tau)_*(\sigma^m(f \wedge S^m)) \;=\; \gamma_*(\sigma^m(f \wedge S^m))$$

because the iterated structure map σ^m is $\Sigma_n \times \Sigma_m$-equivariant. So the definition of $\alpha_* f$ is independent of the choice of permutation. Functoriality of the assignment $\alpha \mapsto \alpha_*$ is then straightforward.

The adjunction bijection

$$\mathcal{S}p_{\mathbf{T}}(\Sigma^\infty T, X) \;\cong\; \mathbf{T}^I(T, \Omega^\bullet X)$$

takes a morphism $\varphi : \Sigma^\infty T \longrightarrow X$ to the natural transformation $\hat\varphi : T \longrightarrow \Omega^\bullet X$ whose value at the object $\mathbf{n}$ is the adjoint $T(\mathbf{n}) \longrightarrow \mathrm{map}(S^n, X_n)$ of $\varphi_n : T(\mathbf{n}) \wedge S^n \longrightarrow X_n$.

Now suppose that we are either in the context of simplicial set or all the maps $T(\iota) : T(\mathbf{n}) \longrightarrow T(\mathbf{n+1})$ are h-cofibrations. The sphere spectrum is semistable, so if the external $\mathcal{M}$-action on the stable homotopy groups $\pi_*^{\mathrm{s}}(T(\omega)) = \hat\pi_*(T(\omega) \wedge \mathbb{S})$ is trivial, Proposition **??** applies. We can then conclude that the symmetric spectrum $\Sigma^\infty T$ is semistable and there is a chain of two stable equivalences between $\Sigma^\infty T$, the suspension spectrum of the **I**-space T, and $\Sigma^\infty T$, the suspension spectrum of the colimit space $T(\omega)$.

[for a symmetric ring spectrum R, the **I**-space $\Omega^\bullet R$ has a product]

Example 3.53. The free symmetric spectrum $F_m S^m$ generated by an m-sphere in level m (compare Example 3.20) is isomorphic to the suspension spectrum of a representable **I**-space. Indeed, we let $\mathbf{I}(\mathbf{m}, -)^+$ denote the **I**-space that is given by the representable functor of the object $\mathbf{m}$ with a disjoint basepoint. Here the set $\mathbf{I}(\mathbf{m}, \mathbf{m})$ is viewed as a discrete space respectively a constant simplicial set.

Indeed, both $F_m S^m$ and $\mathbf{I}(\mathbf{m}, -)^+ \wedge \mathbb{S}$ consist only of basepoints below level m. In level m and above, an isomorphism

$$\begin{aligned} \Sigma_{m+n} \wedge_{1\times\Sigma_n} S^m \wedge S^n \;&\longrightarrow\; \mathbf{I}(\mathbf{m}, \mathbf{m+n})^+ \wedge S^{m+n} = (\mathbf{I}(\mathbf{m}, -)^+ \wedge \mathbb{S})_{m+n} \\ [\gamma \wedge x] \;&\longmapsto\; \gamma|_{\mathbf{m}} \wedge x \;, \end{aligned}$$

where $\gamma|_{\mathbf{m}} : \mathbf{m} \longrightarrow \mathbf{m+n}$ is the restriction of a permutation of $\mathbf{m+n}$ to the subset $\mathbf{m}$.

Example 3.54 (Ring spectra from multiplicative **I**-spaces)**.** We can use the construction which pairs an **I**-space with a symmetric spectrum (see Example 3.51) to produce symmetric ring spectra which model the suspension spectra of certain infinite loop spaces such as BO, the classifying space of the infinite orthogonal group, even if these do not have a strictly associative multiplication. This works for infinite loop spaces which can be represented as 'monoids of **I**-spaces', as we now explain.

The symmetric monoidal sum operation restricts from the category $\mathcal{F}in$ of standard finite sets to the category $\mathbf{I}$. Thus $\mathbf{I}$ has a symmetric monoidal product '+' given by addition on objects and defined for morphisms $f : \mathbf{n} \longrightarrow \mathbf{n}'$ and $g : \mathbf{m} \longrightarrow \mathbf{m}'$ we define $f + g : \mathbf{n} + \mathbf{m} \longrightarrow \mathbf{n}' + \mathbf{m}'$ by

$$(f+g)(i) \;=\; \begin{cases} f(i) & \text{if } 1 \leq i \leq n, \text{ and} \\ g(i-n)+n' & \text{if } n+1 \leq i \leq n+m. \end{cases}$$

The product $+$ is strictly associative and has the object $\mathbf{0}$ as a strict unit. The symmetry isomorphism is the shuffle map $\chi_{n,m} : \mathbf{n} + \mathbf{m} \longrightarrow \mathbf{m} + \mathbf{n}$.

Consider an $\mathbf{I}$-space $T : \mathbf{I} \longrightarrow \mathbf{T}$ with a pairing, i.e., an associative and unital natural transformation $\mu_{\mathbf{n},\mathbf{m}} : T(\mathbf{n}) \wedge T(\mathbf{m}) \longrightarrow T(\mathbf{n}+\mathbf{m})$. If R is a symmetric ring spectrum, then the smash product $T \wedge R$ (see Example 3.51) becomes a symmetric ring spectrum with respect to the multiplication map

$$(T \wedge R)_n \wedge (T \wedge R)_m \;\longrightarrow\; (T \wedge R)_{n+m}$$

defined as the composite

$$T(\mathbf{n}) \wedge R_n \wedge T(\mathbf{m}) \wedge R_m \xrightarrow{\mathrm{Id} \wedge \mathrm{twist} \wedge \mathrm{Id}} T(\mathbf{n}) \wedge T(\mathbf{m}) \wedge R_n \wedge R_m \xrightarrow{\mu_{\mathbf{n},\mathbf{m}} \wedge \mu_{n,m}} T(\mathbf{n}+\mathbf{m}) \wedge R_{n+m} \; .$$

If the transformation μ is commutative in the sense that the square

$$\begin{tikzcd} T(\mathbf{n}) \wedge T(\mathbf{m}) \arrow[r, "\mu_{\mathbf{n},\mathbf{m}}"] \arrow[d, "\mathrm{twist}"'] & T(\mathbf{n}+\mathbf{m}) \arrow[d, "T(\chi_{n,m})"] \\ T(\mathbf{m}) \wedge T(\mathbf{n}) \arrow[r, "\mu_{\mathbf{m},\mathbf{n}}"'] & T(\mathbf{m}+\mathbf{n}) \end{tikzcd}$$

commutes for all $n, m \geq 0$ and if the multiplication of R is commutative, then the product of $T \wedge R$ is also commutative. This construction generalizes monoid ring spectra (see Example 3.42): if M is a topological (respectively simplicial) monoid, then the constant $\mathbf{I}$-functor with values M^+ inherits an associative and unital product from M which is commutative if M is. The smash product of a ring spectrum R with such a constant multiplicative functor equals the monoid ring spectrum $R[M]$. The construction commutes with taking opposite multiplications (compare Example 3.45): we have $(T \wedge R)^{\mathrm{op}} = T^{\mathrm{op}} \wedge R^{\mathrm{op}}$ for the smash product of an $\mathbf{I}$-space with multiplication and a ring spectrum.

A more interesting instance of this construction is a commutative symmetric ring spectrum which models the suspension spectrum of the space BO^+, the classifying space of the infinite orthogonal group, supplied with a disjoint basepoint. Here we start with the '$\mathbf{I}$-topological group' $\mathbf{O}$, a functor from $\mathbf{I}$ to topological groups whose value at $\mathbf{n}$ is $O(n)$, the n-th orthogonal group. The behavior on morphisms is determined by requiring that a permutation $\gamma \in \Sigma_n$ acts as conjugation by the permutation matrix associated to γ and the inclusion $\iota : \mathbf{n} \longrightarrow \mathbf{n}+\mathbf{1}$ induces

$$\iota_* \;:\; O(n) \;\longrightarrow\; O(n+1) \;, \quad A \;\longmapsto\; \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \; .$$

A general injective set map $\alpha : \mathbf{n} \longrightarrow \mathbf{m}$ then induces the group homomorphism $\alpha_* : O(n) \longrightarrow O(m)$ given by

$$(\alpha_* A)_{i,j} \;=\; \begin{cases} A_{\alpha^{-1}(i),\alpha^{-1}(j)} & \text{if } i,j \in \mathrm{Im}(\alpha), \\ 1 & \text{if } i = j \text{ and } i \notin \mathrm{Im}(\alpha), \\ 0 & \text{if } i \neq j \text{ and } i \text{ or } j \text{ is not contained in } \mathrm{Im}(\alpha). \end{cases}$$

Orthogonal sum of matrices gives a natural transformation of group valued functors

$$\mathbf{O}(\mathbf{n}) \times \mathbf{O}(\mathbf{m}) \;\longrightarrow\; \mathbf{O}(\mathbf{n}+\mathbf{m}) \;, \quad (A,B) \;\longmapsto\; \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \; .$$

This transformation is unital, associative and commutative, in a sense which by now is hopefully clear. The classifying space functor B takes topological groups to topological spaces and commutes with products up to unital, associative and commutative homeomorphism. So by taking classifying spaces objectwise we obtain an $\mathbf{I}$-space $\mathbf{BO}$ with values $\mathbf{BO}(\mathbf{n}) = BO(n)$. This $\mathbf{I}$-space inherits a unital, associative and commutative product in the sense discussed above, but with respect to the cartesian product, as opposed to the smash product, of spaces. So if we add disjoint basepoints and perform the construction above, we obtain a symmetric spectrum $\Sigma^\infty \mathbf{BO}^+$ whose value in level n is the space $BO(n)^+ \wedge S^n$. By our discussion above, this is a commutative symmetric ring spectrum. Since the action of the symmetric group on $BO(n)^+$ extends to an action of the n-th orthogonal group, the symmetric ring spectrum $\Sigma^\infty \mathbf{BO}^+$ is semistable (by Proposition 3.16 (vi)). [and related by a chain of two stable equivalences to $\Sigma^\infty BO^+$, the unreduced suspension spectrum of the space BO, the classifying space of the infinite orthogonal group. The upshot of all of this is that $\Sigma^\infty \mathbf{BO}^+$ is a commutative symmetric ring spectrum which is stably equivalent (even $\hat{\pi}_*$-isomorphic) to the suspension spectrum of the space BO^+.

This construction can be adapted to yield commutative symmetric ring spectra which model the unreduced suspension spectra of $BSO, BSpin, BU, BSU$ and BSp. In each case, the respective family of classical groups fits into an '$\mathbf{I}$-topological group' with commutative product, and from there we proceed as for the orthogonal groups. More examples of the same kind are obtained from families of discrete groups which fit into '$\mathbf{I}$-groups' with products, for example symmetric groups, alternating groups or general or special linear groups over some ring. In those cases, we generally do not obtain semistable spectra, however, and it is more subtle to analyse the stable homotopy type of the construction.

4. Stable equivalences

In this section we introduce and discuss the important notion of a stable equivalence of symmetric spectra, see Definition 4.11. An important result is that a morphism of symmetric spectra which induces isomorphisms on naive homotopy groups is a stable equivalence (see Theorem 4.23). So for morphisms of symmetric spectra we have the implications

$$\text{homotopy equivalence} \implies \text{level equivalence} \implies \hat{\pi}_*\text{-isomorphism} \implies \text{stable equivalence.}$$

In general, the reverse implications do not hold. However, for certain classes of spectra, one can argue in the other direction:

- every stable equivalence between semistable spectra is a $\hat{\pi}_*$-isomorphism (see Proposition 6.3);
- every $\hat{\pi}_*$-isomorphism between Ω-spectra is a level equivalence;
- in the context of simplicial sets, every level equivalence between injective spectra is a homotopy equivalence (see Proposition 4.6).

In Proposition 4.17 below we give a list of several equivalent characterizations of stable equivalences. In Proposition 4.31 we prove that stable equivalences are closed under various constructions such as suspensions, loop, shift adjoint, wedges, and finite products. Up to stable equivalence, every symmetric spectrum can be replaced by an Ω-spectrum (Proposition 4.39). The ultimate consequence will be that the stable homotopy category arises as the localization of the category of symmetric spectra obtained by 'inverting stable equivalences', compare Theorem 1.6 of Chapter II.

4.1. Injective spectra.

The notion of an *injective* symmetric spectrum of simplicial sets is needed below to define stable equivalences of symmetric spectra.

Definition 4.1. A symmetric spectrum of simplicial sets X is *injective* if for every monomorphism $i : A \longrightarrow B$ which is also a level equivalence and every morphism $f : A \longrightarrow X$ there exists an extension $g : B \longrightarrow X$ with $f = gi$.

Injective spectra do not arise 'in nature' very often, so we give some examples arising as co-free and co-semifree symmetric spectra. Moreover, we prove in Proposition 4.10 below that injectivity can always be arranged up to level equivalence.

Example 4.2 (Co-free and co-semifree symmetric spectra)**.** In Example 3.23 we discussed that the evaluation functor $\mathrm{ev}_m : \mathcal{S}p_{\mathbf{sS}} \longrightarrow \Sigma_m\mathbf{sS}$ at level m has a left adjoint G_m, whose values are the semifree symmetric spectra. But this evaluation functor (and its analog for symmetric spectra of spaces) also has a right adjoint

$$P_m \ : \ \Sigma_m\mathbf{sS} \ \longrightarrow \ \mathcal{S}p_{\mathbf{sS}} \ ,$$

which we construct now. For a based Σ_m-simplicial set (or Σ_m-space) L we refer to $P_m L$ as the *co-semifree* symmetric spectrum generated by L in level m. The spectrum $P_m L$ consists only of a point above level m and for $n \leq m$ we have

$$(P_m L)_n \;=\; \mathrm{map}^{1\times\Sigma_{m-n}}(S^{m-n}, L)\ ,$$

the subspace of $(1\times\Sigma_{m-n})$-equivariant maps in $\mathrm{map}(S^{m-n}, L)$, with restricted Σ_n-action from L. The structure map $\sigma_n : (P_m L)_n \wedge S^1 \longrightarrow (P_m L)_{n+1}$ is adjoint to the map

$$\mathrm{map}^{1\times\Sigma_{m-n}}(S^{m-n}, L) \xrightarrow{\text{incl.}} \mathrm{map}^{1\times\Sigma_{m-n-1}}(S^{m-n}, L) \;\cong\; \Omega\left(\mathrm{map}^{1\times\Sigma_{m-n-1}}(S^{m-n-1}, L)\right)\ .$$

The forgetful functors $\Sigma_m\mathbf{sS} \longrightarrow \mathbf{sS}$ and $\Sigma_m\mathbf{T} \longrightarrow \mathbf{T}$ which forget the group action also have a right adjoint given by $K \mapsto \mathrm{map}(\Sigma_m^+, K)$, the function space from the set Σ_m into K (i.e., a product of $m!$ copies of K). So the composite forgetful functors $\mathcal{S}p_{\mathbf{sS}} \longrightarrow \mathbf{sS}$ and $\mathcal{S}p_{\mathbf{T}} \longrightarrow \mathbf{T}$ which take X to X_m have right adjoints R_m given by $R_m K = P_m(\mathrm{map}(\Sigma_m^+, K))$.

In the context of simplicial sets certain co-free and co-semifree symmetric spectra provide examples of injective spectra. We start from the fact that every Kan simplicial set K has the extension property with respect to all injective weak equivalences of simplicial sets. So by adjointness, the co-free symmetric spectrum $R_m K$ is injective whenever K is a Kan simplicial set.

More generally, a co-semifree symmetric spectrum $P_m L$ is injective whenever the based Σ_m-simplicial set L is 'strongly fibrant' in the Σ_m-equivariant sense and has a special Σ_m-equivariant homotopy type, in the sense of the following definition.

Definition 4.3. A Σ_m-simplicial set L is *strictly Σ_m-fibrant* if for every subgroup H of Σ_m the H-fixed points L^H are a Kan simplicial set and the map $L^H \longrightarrow L^{hH}$ from the fixed points to the homotopy fixed points is a weak equivalence.

As we recall in Proposition A.4.5, strictly Σ_m-fibrant Σ_m-simplicial sets are the fibrant objects in the 'mixed' equivariant model structure, and they can be characterized by the extension property (also known as 'right lifting property') with respect to all monomorphisms of Σ_m-simplicial sets which are also weak equivalences after forgetting the group action. So if L is strictly Σ_m-fibrant, again by adjointness the co-semifree symmetric spectrum $P_m L$ is injective.

Now we deduce various properties that injective spectra have.

Proposition 4.4. *Let X be an injective symmetric spectrum of simplicial sets.*

(i) *For every monomorphism $i : A \longrightarrow B$ of symmetric spectra of simplicial sets the map*

$$\mathrm{map}(i, X) \;:\; \mathrm{map}(B, X) \;\longrightarrow\; \mathrm{map}(A, X)$$

is a Kan fibration. If in addition i is a level equivalence, then $\mathrm{map}(i, X)$ *is a weak equivalence.*

(ii) *For every symmetric spectrum B of simplicial sets the function space* $\mathrm{map}(B, X)$ *is a Kan complex.*

(iii) *For every $m \geq 0$ the Σ_m-simplicial set X_m is strictly Σ_m-fibrant. In particular, the underlying simplicial set of X_m is a Kan complex.*

(iv) *For every based Σ_m-simplicial set L, the equivariant function spectrum $\triangleright^m(L, X)$ is injective. In particular, the function spectrum X^K for any based simplicial set K, the loop spectrum ΩX and the shifted spectrum* $\mathrm{sh}\, X$ *are injective.*

Proof. (i) We check that $\mathrm{map}(i, X)$ has the right lifting property with respect to every acyclic cofibration $j : K \longrightarrow L$ of simplicial sets. By the adjunction between the smash pairing and mapping spaces, a lifting problem in the form of a commutative square

$$\begin{tikzcd} K \arrow[r] \arrow[d, "j"'] & \mathrm{map}(B, X) \arrow[d, "\mathrm{map}(i,X)"] \\ L \arrow[r] & \mathrm{map}(A, X) \end{tikzcd}$$

corresponds to a morphism $K \wedge B \cup_{K\wedge A} L \wedge A \longrightarrow X$, and a lifting corresponds to a morphism $L \wedge B \longrightarrow X$ which restricts to the previous morphism along the 'pushout product' map $j \wedge i : K \wedge B \cup_{K\wedge A} L \wedge A \longrightarrow L \wedge B$. Since j is an acyclic cofibration and i is a level cofibration, the pushout product morphism $j \wedge i$ is levelwise an acyclic cofibration by the pushout product property in $\mathbf{T}$ respectively $\mathbf{sS}$. So the lifting exists since we assumed that X is injective.

The second part is very similar. If i is levelwise an acyclic cofibration, then for every cofibration $j : K \longrightarrow L$ (not necessarily a weak equivalence) of pointed spaces (simplicial sets), the pushout product map $j \wedge i$ is levelwise an acyclic cofibration. So $\mathrm{map}(i, X)$ has the right lifting property with respect to all cofibration pointed spaces (simplicial sets).

Part (ii) is the special case of (i) where A is the trivial spectrum so that $\mathrm{map}(A, X)$ is a one-point simplicial set.

For part (iii) is an exercise in juggling adjunctions. We need to show that X_m has the extension property for monomorphisms $i : A \longrightarrow B$ of Σ_m-simplicial sets which are non-equivariant weak equivalences. Evaluation at level m has the semifree functor G_m as left adjoint (compare Example 3.23), so it suffices to show that $G_m i : G_m A \longrightarrow G_m B$ is a monomorphism and a level equivalence of symmetric spectra. The explicit form of the $(m+n)$th level as

$$(G_m A)_{m+n} \ = \ \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} A \wedge S^n$$

shows that $(G_m A)_{m+n}$ is (non-equivariantly) a wedge of $\binom{m+n}{n}$ copies of $A \wedge S^n$. But smashing with S^n and taking wedges preserves monomorphisms and weak equivalences of simplicial sets, so $G_m i : G_m A \longrightarrow G_m B$ is indeed a monomorphism and a level equivalence.

Property (iv) can be derived as follows. The functor which sends a symmetric spectrum X to the equivariant function spectrum $\triangleright^m(L, X)$ is right adjoint to $A \mapsto L \triangleright_m A$ the twisted smash product with L (compare (3.33)). In level $m + n$ the spectrum $L \triangleright_m A$ is given by $\Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge A_n$. Since Σ_{m+n} is free as right $\Sigma_m \times \Sigma_n$, the simplicial set $(L \triangleright_m A)_{m+n}$ is naturally a wedge of $\binom{m+n}{n}$ copies of $L \wedge A_n$. The processes of smashing with any based simplicial set and taking wedges preserves monomorphisms and weak equivalences of based simplicial sets. So altogether we deduce that the twisted smash product functor $L \triangleright_m -$ preserves monomorphisms and level equivalences of symmetric spectra of simplicial sets.

Now we are ready to show that the spectrum $\triangleright^m(L, X)$ is injective whenever X is: given a monomorphism $i : A \longrightarrow B$ which is also a level equivalence, the morphism $L \triangleright_m i : L \triangleright_m A \longrightarrow L \triangleright_m B$ is again a monomorphism and a level equivalence. Any morphism $f : A \longrightarrow \triangleright^m(L, X)$ has an adjoint morphism $\hat{f} : L \triangleright_m A \longrightarrow X$ which has an extension $\hat{g} : L \triangleright_m B \longrightarrow X$ with $\hat{f} = \hat{g} \circ (L \triangleright_m i)$ since X is injective. The adjoint $g : B \longrightarrow \triangleright^m(L, X)$ is then the required extension of f.

If we specialize to $m = 0$ and $L = K$, we obtain that function spectrum X^K is injective. For $K = S^1$ this shows that the loop spectrum ΩX is injective. For the final claim we specialize to $m = 1$ and $L = S^0$, where $\triangleright^1(S^0, X)$ specializes to the shift of X. □

We now get a criterion for level equivalence by testing against injective spectra. The criterion involves homotopy classes of morphisms, which we define first.

Definition 4.5 (Homotopy relation)**.** Two morphisms of symmetric spectra $f_0, f_1 : A \to X$ are called *homotopic* if there is a morphism

$$H \ : \ I^+ \wedge A \ \longrightarrow \ X \ ,$$

called a *homotopy*, such that $f_0 = H \circ i_0$, and $f_1 = H \circ i_1$. Here I is either the unit interval $[0, 1]$ when we are in the context of symmetric spectra of spaces, or the simplicial 1-simplex $\Delta[1]$ when in the context of simplicial sets. The morphisms $i_j : A \longrightarrow I^+ \wedge A$ for $j = 0, 1$ are the 'end point inclusions' which are given levelwise by $i_j(a) = j \wedge a$ (in the context of spaces) or are induced by the face morphisms $d_j : \Delta[0] \longrightarrow \Delta[1]$ (in the context of simplicial sets) and the identification $A \cong \Delta[0]^+ \wedge A$.

A homotopy between spectrum morphisms is really the same as levelwise based homotopies between $(f_0)_n$ and $(f_1)_n : A_n \to X_n$ compatible with the Σ_n-actions and structure maps. In particular, homotopic morphisms induce the same map of naive homotopy groups.

Homotopies can equivalently be given in two adjoint forms. By the adjunction (3.7) a homotopy $H : I^+ \wedge A \longrightarrow X$ from f_0 to f_1 is adjoint to a morphism $\hat{H} : A \longrightarrow X^{I^+}$ such that $\mathrm{ev}_0 \circ \hat{H} = f_0$ and $\mathrm{ev}_1 \circ \hat{H} = f_1$ where $\mathrm{ev}_j : X^{I^+} \longrightarrow X$ for $j = 0, 1$ is given levelwise by evaluation at $j \in [0,1]$ (respectively the two vertices of $\Delta[1]$) . Finally, the homotopy H is also adjoint to a morphism of (unbased) spaces or simplicial sets $I \longrightarrow \mathrm{map}(A, X)$, which is either a path or a 1-simplex in the mapping space. So in the context of spectra of spaces, two morphisms are homotopic if and only if they lie in the same path component of the mapping space $\mathrm{map}(A, X)$. For spectra of simplicial sets, a morphism f_0 is homotopic to a morphism f_1 if and only if there exists a 1-simplex $H \in \mathrm{map}(A, X)_1$ satisfying $d_1(H) = f_0$ and $d_0(H) = f_1$.

For symmetric spectra of spaces, homotopy is an equivalence relation; for symmetric spectra of simplicial sets, 'homotopy' is in general neither symmetric nor transitive. The homotopy relation is an equivalence relation when the target is injective: vertices of the simplicial set $\mathrm{map}(A, X)$ correspond bijectively to morphisms $A \longrightarrow X$ in such a way that 1-simplices correspond to homotopies. By Proposition 4.4 (ii), the simplicial set $\mathrm{map}(A, X)$ is a Kan complex whenever X is injective. In every Kan complex, the relation $x \sim y$ on vertices defined by existence of a 1-simplex z with $d_1 z = x$ and $d_0 z = y$ is an equivalence relation, hence the homotopy relation for morphisms from A to X is an equivalence relation.

In any of the two worlds we denote by $[A, X]$ the set of homotopy classes of morphisms from A to X, i.e., the classes under the equivalence relation generated by homotopy. The natural morphisms $\Delta[1]^+ \wedge \mathcal{S}(X) \longrightarrow \mathcal{S}([0,1]^+ \wedge X)$ and $[0,1]^+ \wedge |Y| \cong |\Delta[1]^+ \wedge Y|$, compatible with the end point inclusions, show that the singular complex and geometric realization functor preserve the homotopy relation.

A morphism $f : A \longrightarrow B$ of symmetric spectra is a *homotopy equivalence* if there exists a morphism $g : B \longrightarrow A$ such that gf and fg are homotopic to the respective identity morphisms. Hence every homotopy equivalence of symmetric spectra is in particular levelwise a homotopy equivalence of spaces or simplicial sets, thus a level equivalence, but the converse is not true in general.

Proposition 4.6. *A morphism $f : A \longrightarrow B$ of symmetric spectra of simplicial sets is a level equivalence if and only if for every injective spectrum X the induced map $[f, X] : [B, X] \longrightarrow [A, X]$ on homotopy classes of morphisms is bijective. Every level equivalence between injective spectra of simplicial sets is a homotopy equivalence.*

Proof. Suppose first that f is a level equivalence. We replace f by the inclusion of A into the mapping cylinder of f, which is homotopy equivalent to B. This way we can assume without loss of generality that f is a monomorphism. By part (i) of Proposition 4.4 the map $\mathrm{map}(f, X) : \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$ is then a weak equivalence of simplicial sets, so in particular a bijection of components. Since the set $\pi_0 \,\mathrm{map}(B, X)$ of path components of the mapping spaces is in natural bijection with the set $[B, X]$, and similarly for A, this proves the claim.

Now suppose conversely that $[f, X] : [B, X] \longrightarrow [A, X]$ is bijective for every injective spectrum X. If K is a pointed Kan complex and $m \geq 0$, then the co-free symmetric spectrum $R_m K$ of Example 4.2 is injective. The adjunction for morphisms and homotopies provides a natural bijection $[A, R_m K] \cong [A_m, K]_{\mathbf{sS}}$ to the based homotopy classes of morphisms of simplicial sets. So for every Kan complex K, the induced map $[f_m, X] : [B_m, K] \longrightarrow [A_m, K]$ is bijective, which is equivalent to f_n being a weak equivalence of simplicial sets. Since this holds for all m, the morphism f is a level equivalence.

Now we consider a level equivalence $f : A \longrightarrow B$ with A and B injective; we obtain a homotopy inverse $g : B \longrightarrow A$ by the following standard representability argument. Since $[f, A] : [B, A] \longrightarrow [A, A]$ is bijective there is a morphism $g : B \longrightarrow A$ such that $gf : A \longrightarrow A$ is homotopic to the identity. Since $[f, B] : [B, B] \longrightarrow [B, A]$ is a bijection which takes $fg : B \longrightarrow B$ and Id_B to the homotopy class of f, the morphism fg is homotopic to the identity of B. □

The definition of stable equivalences in the next subsection will use injective Ω-spectra. So we now collect some properties of this class.

Proposition 4.7. *Let X be an injective Ω-spectrum of simplicial sets.*

(i) *For every $n \geq 0$, the adjoint structure map $\tilde{\sigma}_n : X_n \longrightarrow \Omega X_{n+1}$ is a Σ_n-homotopy equivalence.*

(ii) *For every based Σ_m-simplicial set L, the symmetric spectrum $F(L, \mathrm{sh}^m X)^{\Sigma_m}$ is again an injective Ω-spectrum. In particular, the function spectrum X^K for any based simplicial set K, the loop spectrum ΩX and the shifted spectrum $\mathrm{sh}\, X$ are injective Ω-spectra.*

Proof. (i) We show that for every subgroup H of Σ_n the map induced by $\tilde{\sigma}_n$ on H-fixed points is a weak equivalence. Since X is an Ω-spectrum, $\tilde{\sigma}_n$ is a weak equivalence on underlying simplicial sets, so it induces a weak equivalence on H-homotopy fixed points. The simplicial sets X_n and X_{n+1} are strictly Σ_n-fibrant respectively strictly Σ_{n+1}-fibrant by Proposition 4.4 (iii). The property 'strictly fibrant' is preserved under looping and restriction to subgroups, so source and target of $\tilde{\sigma}_n$ are strictly Σ_n-fibrant. Hence the horizontal maps in the commutative square

$$\begin{tikzcd} X_n^H \arrow[r, "\simeq"] \arrow[d, "\tilde{\sigma}_n^H"'] & X_n^{hH} \arrow[d, "\tilde{\sigma}_n^{hH}"] \\ \Omega X_n^H \arrow[r, "\simeq"'] & \Omega X_n^{hH} \end{tikzcd}$$

are weak equivalences. (We note that loops commute with fixed points and homotopy fixed points, so there is no ambiguity in the meaning of the terms ΩX_n^H and ΩX_n^{hH}.) Since the right map is a weak equivalence, so is the left map of H-fixed points. Hence $\tilde{\sigma}_n : X_n \longrightarrow \Omega X_{n+1}$ is a strong Σ_n-weak equivalence between strongly fibrant Σ_n-simplicial sets, hence a Σ_n-homotopy equivalence by [ref to Appendix].

(ii) We know by Proposition 4.4 (iv) that the spectrum $F(L, \mathrm{sh}^m X)^{\Sigma_m}$ is again injective; it remains to show that it is also an Ω-spectrum. The adjoint structure map $\tilde{\sigma}_{m+n} : X_{m+n} \longrightarrow \Omega X_{m+n+1}$ is a Σ_{m+n}-homotopy equivalence by part (i), hence the induced map

$$\mathrm{map}(L, \tilde{\sigma}_{m+n})^{\Sigma_m} \ : \ \mathrm{map}(L, X_{m+n})^{\Sigma_m} \ \longrightarrow \ \mathrm{map}(L, \Omega X_{m+n+1})^{\Sigma_m}$$

is a homotopy equivalence of simplicial sets. But this map is isomorphic to the nth adjoint structure map

$$\tilde{\sigma}_n \ : \ \left(F(L, \mathrm{sh}^m X)^{\Sigma_m}\right)_n \ \longrightarrow \ \Omega\left(F(L, \mathrm{sh}^m X)^{\Sigma_m}\right)_{n+1}$$

of the spectrum $F(L, \mathrm{sh}^m X)^{\Sigma_m}$. So $F(L, \mathrm{sh}^m X)^{\Sigma_m}$ is indeed an Ω-spectrum. □

Now we want to show that, up to level equivalence, the property of being injective is no restriction (see Proposition 4.10 below). This means that we want to force a certain lifting property, and the *small object argument* (see Theorem 1.7 of Appendix A) is the appropriate tool for this purpose. As usual with small object arguments we have to limit the size of objects in order to obtain a set (as opposed to a proper class) of test maps. We call a symmetric spectrum of simplicial sets *countable* if the cardinality of the disjoint union of all simplices in all levels is countable.

Proposition 4.8. *Let B be a symmetric spectrum of simplicial sets and V a symmetric subspectrum of B such that the inclusion $V \longrightarrow B$ is a level equivalence. If $V \neq B$, then there is a countable symmetric subspectrum E of B that is not contained in V and such that the inclusion $E \cap V \longrightarrow E$ is a level equivalence.*

Proof. We let m be the minimum of the numbers n such that $V_n \neq B_n$. For $n < m$ we define $E_n = *$. In level m we choose a simplex v of $B_m - V_m$ (of any dimension) and let C be the simplicial subset of B_m generated by v and the basepoint. Then C is countable, so by Proposition 3.6 of Appendix A there is a countable, Σ_m-invariant simplicial subset E_m of B_m containing v and such that the inclusion $E_m \cap V_m \longrightarrow E_m$ is a weak equivalence. Since the simplex v does not belong to V_m, E_m is not contained in V_m.

Above level m we proceed inductively. Since E_{n-1} is countable, so is $E_{n-1} \wedge S^1$ and its image $\sigma_{n-1}(E_{n-1} \wedge S^1)$ under the structure map $\sigma_n : B_{n-1} \wedge S^1 \longrightarrow B_n$. By Proposition A.3.6 there is a countable, Σ_n-invariant simplicial subset E_n of B_n containing $\sigma_{n-1}(E_{n-1} \wedge S^1)$ and such that the inclusion $E_n \cap V_n \longrightarrow E_n$ is a weak equivalence. By construction, the simplicial sets E_n are closed under the actions of the symmetric groups and the struture maps of B, so they form the desired symmetric subspectrum of B. □

Proposition 4.9. *A symmetric spectrum of simplicial sets X is injective if and only if for every monomorphism $i : A \longrightarrow B$ which is also a level equivalence with B countable and every morphism $f : A \longrightarrow X$ there exists an extension $g : B \longrightarrow X$ with $f = gi$.*

Proof. Suppose that X has the extension property with respect to all injective level equivalences with countable target (and hence source). We consider a general injective level equivalence $i : A \longrightarrow B$, with no restriction on the cardinality of B, and a morphism $f : A \longrightarrow X$ which we want to extend to B.

We denote by $\mathcal{P}$ the set of 'partial extensions': an element of $\mathcal{P}$ is a pair (U, h) consisting of

- a symmetric subspectrum U of B which contains the image of A and such that the inclusion $U \longrightarrow B$ (and hence the morphism $A \longrightarrow U$) is a level equivalence and
- a morphism $h : U \longrightarrow X$ which extends $f : A \longrightarrow X$.

The set $\mathcal{P}$ can be partially ordered by declaring $(U, h) \leq (U', h')$ if U is contained in U' and h' extends h. Then every chain in $\mathcal{P}$ has an upper bound, namely the union of all the subspectra U with the common extension of the morphisms h. We are using here that the inclusion of the union into B is again a level equivalence because homotopy groups commute with such filtered colimits. By Zorn's lemma, the set $\mathcal{P}$ has a maximal element (V, k).

We show that $V = B$, so k provides the required extension of f, showing that the spectrum X is injective. We argue by contradiction and suppose that V is strictly smaller than B. Proposition 4.8 provides a countable subspectrum E of B which is not contained in V and such that the inclusion $E \cap V \longrightarrow E$ is a level equivalence. Since E is countable, the restriction of $k : V \longrightarrow X$ to the intersection $E \cap V$ can be extended to a morphism $g : E \longrightarrow X$. The morphisms g and k together then provide an extension $g \cup k : E \cup V \longrightarrow X$ of k, which contradicts the assumption that (V, k) is a maximal element in the set $\mathcal{P}$ extensions. □

Proposition 4.10. *There is an endofunctor $(-)^{\mathrm{inj}} : \mathcal{S}p_{\mathbf{sS}} \longrightarrow \mathcal{S}p_{\mathbf{sS}}$ on the category of symmetric spectra of simplicial sets and a natural level equivalence $A \longrightarrow A^{\mathrm{inj}}$ such that A^{inj} is an injective spectrum for all symmetric spectra of simplicial sets A.*

Proof. We choose a set K containing one representative of each isomorphism class of countable symmetric spectra of simplicial sets. Then we let I be the set of inclusions of symmetric subspectra $i : A \subseteq B$ for which B is in K and i is a level equivalence.

We apply the small object argument (see Theorem 1.7 of Appendix A) to the unique morphism from a given symmetric spectrum A to the trivial spectrum. We obtain a functor $A \mapsto A^{\mathrm{inj}}$ together with a natural transformation $j : A \longrightarrow A^{\mathrm{inj}}$ which is an I-cell complex. The class of injective level equivalences of symmetric spectra is closed under wedges, cobase change and composition, possibly transfinite. So every I-cell complex is an injective level equivalence. In particular the morphism j is an injective level equivalence. Moreover, the morphism from A^{inj} to the trivial spectrum is I-injective. Proposition 4.9 shows that A^{inj} is an injective spectrum. □

4.2. Stable equivalences. Now we define the important concept of stable equivalences of symmetric spectra.

Definition 4.11. A morphism $f : A \longrightarrow B$ of symmetric spectra of simplicial sets is a *stable equivalence* if for every injective Ω-spectrum X the induced map

$$[f, X] \; : \; [B, X] \; \longrightarrow \; [A, X]$$

of homotopy classes of spectrum morphisms is bijective.

A morphism f of symmetric spectra of topological spaces is a *stable equivalence* if the singular complex $\mathcal{S}(f) : \mathcal{S}(A) \longrightarrow \mathcal{S}(B)$ is a stable equivalence in the previous sense.

Proposition 4.6 immediately implies that every level equivalence of symmetric spectra is a stable equivalence. Theorem 4.23 below shows that more generally every $\hat{\pi}_*$-isomorphism is a stable equivalence.

Lemma 4.12. (i) *Let A be a symmetric spectrum of simplicial sets, Y a symmetric spectrum of spaces and $h : |A| \longrightarrow Y$ a morphism. Then h is a stable equivalence if and only if its adjoint $\hat{h} : A \longrightarrow \mathcal{S}(Y)$ is a stable equivalence.*

(ii) *A morphism of symmetric spectra of simplicial sets is a stable equivalence if and only if its geometric realization is a stable equivalence of symmetric spectra of spaces.*

PROOF. (i) The composite of the adjunction unit $A \longrightarrow \mathcal{S}|A|$ and $\mathcal{S}(h) : \mathcal{S}|A| \longrightarrow \mathcal{S}(Y)$ is the adjoint $\hat{h}$. The adjunction unit is a level equivalence, hence stable equivalence, so $\hat{h}$ is a stable equivalence if and only if $\mathcal{S}(h)$ is. But the latter is equivalent, by definition, to h being a stable equivalence.

(ii) A morphism $g : A \longrightarrow B$ of symmetric spectra of simplicial sets is a stable equivalence if and only if its composite with the adjunction unit $\eta : B \longrightarrow \mathcal{S}|B|$ is a stable equivalence (since η is a level, hence stable, equivalence). By (i), the composite $\eta g : A \longrightarrow \mathcal{S}|B|$ is a stable equivalence if and only if its adjoint $\widehat{\eta g} = |g| : |A| \longrightarrow |B|$ is. □

Proposition 4.13. *Every stable equivalence between Ω-spectra is a level equivalence. In the context of symmetric spectra of simplicial sets, every stable equivalence between injective Ω-spectra is a homotopy equivalence.*

PROOF. A spectrum of spaces is an Ω-spectrum if and only if its singular complex is an Ω-spectrum of simplicial sets. So the case of spaces is a direct consequence of the case of simplicial sets.

For symmetric spectra of simplicial sets we argue as follows. Let $f : X \longrightarrow Y$ be a stable equivalence between Ω-spectra. We suppose first that X and Y are also injective. Since $[f, X] : [Y, X] \longrightarrow [X, X]$ is bijective there exists a morphism $g : Y \longrightarrow X$ such that gf is homotopic to the identity of X. Since fgf is homotopic to f and $[f, Y] : [Y, Y] \longrightarrow [X, Y]$ is bijective, we conclude that fg is homotopic to the identity of Y. So f is a homotopy equivalence.

If X and Y are Ω-spectra (but not necessarily injective), we consider the injective replacement $f^{\mathrm{inj}} : X^{\mathrm{inj}} \longrightarrow Y^{\mathrm{inj}}$ as in Proposition 4.10. Then f^{inj} is a stable equivalence between injective Ω-spectra, hence a homotopy equivalence by the above. In particular, f^{inj} is a level equivalence, hence so is the original morphism f. □

Proposition 4.14. *For every based Σ_m-simplicial set L, the twisted smash product functor $L \triangleright_m -$ preserves stable equivalences between symmetric spectra of simplicial sets.*

PROOF. We let $f : A \longrightarrow B$ be a stable equivalence between symmetric spectra of simplicial sets and X any injective Ω-spectrum. In the commutative square

$$\begin{tikzcd}
{[L \triangleright_m B, X]} \arrow[r, "{[L \triangleright_m f, X]}"] \arrow[d, "\cong"'] & {[L \triangleright_m A, X]} \arrow[d, "\cong"] \\
{[B, F(L, \mathrm{sh}^m X)^{\Sigma_m}]} \arrow[r, "{[f, F(L, \mathrm{sh}^m X)^{\Sigma_m}]}"'] & {[A, F(L, \mathrm{sh}^m X)^{\Sigma_m}]}
\end{tikzcd}$$

the vertical bijections are induced by the adjunction. The lower horizontal map is bijective since f is a stable equivalence and $F(L, \mathrm{sh}^m X)^{\Sigma_m}$ is an injective Ω-spectrum, (by Proposition 4.7 (ii)). So the upper horizontal map is bijective and hence $L \triangleright_m f$ is a stable equivalence. □

Remark 4.15. Proposition 4.14 has an analogue for symmetric spectra of spaces: if L is a cofibrant based Σ_m-space (i.e., a retract of a based Σ_m-CW-complex), then the twisted smash product functor $L \triangleright_m -$ preserves stable equivalences between symmetric spectra of spaces. Indeed, the functor $L \triangleright_m -$ is isomorphic to smashing with with the flat semifree spectrum $G_m L$ (see Proposition 5.13 below) and hence preserves stable equivalences by Proposition 5.50 (iii).

We establish some useful criteria for stable equivalences in the context of simplicial sets. We call a symmetric spectrum *stably contractible* if the unique morphism from the trivial spectrum to it is a stable equivalence, or –equivalently– if the unique morphism from it to the trivial spectrum is a stable equivalence. In the next proof we need the natural morphism

$$\tilde{\lambda}_X \;:\; X \;\longrightarrow\; \Omega(\mathrm{sh}\, X) \tag{4.16}$$

adjoint to the morphism $\lambda_X : S^1 \wedge X \longrightarrow \mathrm{sh}\, X$ defined in (3.12). In level n the morphism $\tilde{\lambda}_X$ is the composite

$$X_n \xrightarrow{\tilde{\sigma}_n} \Omega(X_{n+1}) \xrightarrow{\Omega(\chi_{n,1})} \Omega(X_{1+n}) = (\Omega(\mathrm{sh}\, X))_n \ .$$

Since the second map $\Omega(\chi_{n,1})$ is an isomorphism, $(\tilde{\lambda}_X)_n$ is a weak equivalence if and only if $\tilde{\sigma}_n$ is; so $\tilde{\lambda}_X$ is a level equivalence if and only if X is an Ω-spectrum.

The conditions (i), (iv) and (v) in the following proposition are also equivalent in the context of symmetric spectra of spaces, compare Proposition 4.29 below.

Proposition 4.17. *For every morphism $f : A \longrightarrow B$ of symmetric spectra of simplicial sets the following are equivalent:*

(i) *the morphism f is a stable equivalence;*
(ii) *for every injective Ω-spectrum X the induced map* $\mathrm{map}(f, X) : \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$ *is a homotopy equivalence of simplicial sets;*
(iii) *for every injective Ω-spectrum X the induced map* $\mathrm{Hom}(f, X) : \mathrm{Hom}(B, X) \longrightarrow \mathrm{Hom}(A, X)$ *is a level equivalence of symmetric spectra;*
(iv) *the mapping cone $C(f)$ of f is stably contractible;*
(v) *the suspension $S^1 \wedge f : S^1 \wedge A \longrightarrow S^1 \wedge B$ is a stable equivalence.*

Proof. (i)⇒(ii) For every simplicial set K and every injective Ω-spectrum X the function spectrum X^K is again injective by Proposition 4.4 (iv) and an Ω-spectrum by Example 3.6. We have an adjunction bijection $[K, \mathrm{map}(A, X)] \cong [A, X^K]$ where the left hand side means homotopy classes of morphisms of simplicial sets. We recall that $\mathrm{map}(A, X)$ and $\mathrm{map}(B, X)$ are Kan simplicial sets by part (ii) of Proposition 4.4. So if f is a stable equivalence, then $[f, X^K]$ is bijective, hence $[K, \mathrm{map}(f, X)] : [K, \mathrm{map}(B, X)] \longrightarrow [K, \mathrm{map}(A, X)]$ is bijective. Since this holds for all simplicial sets K, $\mathrm{map}(f, X)$ is a homotopy equivalence.

(ii)⇒(iii) For every injective Ω-spectrum X and $n \geq 0$ the n-fold shifted spectrum $\mathrm{sh}^n X$ is again injective (Proposition 4.4 (iv)) and an Ω-spectrum. So if $f : A \longrightarrow B$ satisfies (ii), it also satisfies (iii) since the nth level of the spectrum $\mathrm{Hom}(A, X)$ is defined as $\mathrm{map}(A, \mathrm{sh}^n X)$.

(iii)⇒(iv) Let X be an injective Ω-spectrum. The simplicial set $\mathrm{map}(C(f), X)$ is isomorphic to the homotopy fibre of the morphism $\mathrm{Hom}(f, X)_0 : \mathrm{Hom}(B, X)_0 \longrightarrow \mathrm{Hom}(A, X)_0$ between Kan simplicial sets. So if condition (iii) holds, then $\mathrm{map}(C(f), X)$ is contractible. In particular, the set $[C(f), X] = \pi_0\, (\mathrm{map}(C(f), X))$ contains only one element, so the morphism from $C(f)$ to the trivial spectrum is a stable equivalence.

(iv)⇒(v) Let X be an injective Ω-spectrum. The simplicial set $\mathrm{map}(C(f), X)$ is (isomorphic to) the homotopy fiber of the map $\mathrm{map}(f, X) : \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$. By the already established implication '(i)⇒(ii)' the simplicial set $\mathrm{map}(C(f), X)$ is contractible, so $\mathrm{map}(f, X) : \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$ induces a bijection on fundamental groups. Since the simplicial set $\mathrm{map}(A, X)$ is Kan, its fundamenal group is in natural bijection with $[S^1, \mathrm{map}(A, X)]$, the set of homotopy classes of morphisms from the simplicial circle. Since $[S^1, \mathrm{map}(A, X)]$ moreover bijects naturally with the set of $[S^1 \wedge A, X]$, this shows that the map $[S^1 \wedge f, X] : [S^1 \wedge B, X] \longrightarrow [S^1 \wedge A, X]$ is bijective. So the suspension of f is a stable equivalence.

(v)⇒(i) Suppose that the suspension $S^1 \wedge f$ is a stable equivalence and let X be an injective Ω-spectrum. By Proposition 4.7 (ii) the shift $\mathrm{sh}\, X$ is then again an injective Ω-spectrum, so the map $[S^1 \wedge f, \mathrm{sh}\, X] : [S^1 \wedge B, \mathrm{sh}\, X] \longrightarrow [S^1 \wedge A, \mathrm{sh}\, X]$ is bijective. By adjointness, the lower horizontal map in the commutative square

$$\begin{tikzcd}
[B,X] \arrow[r, "{[f,X]}"] \arrow[d, "{[B,\lambda_X]}"'] & [A,X] \arrow[d, "{[A,\lambda_X]}"] \\
{[B,\Omega(\operatorname{sh} X)]} \arrow[r, "{[f,\Omega(\operatorname{sh} X)]}"'] & {[B,\Omega(\operatorname{sh} X)]}
\end{tikzcd}$$

is also bijective. The morphism $\tilde{\lambda}_X : X \longrightarrow \Omega(\operatorname{sh} X)$ is a level equivalence between injective Ω-spectra, and hence a homotopy equivalence by Proposition 4.6. So the two vertical morphisms in the previous commutative square are bijective, and so is the upper horizontal map $[f, X]$. This shows that f is a stable equivalence. □

Our next aim is to show that every morphism of symmetric spectra that induces isomorphisms of all naive homotopy groups is a stable equivalence. For this purpose we introduce a functor called '$\Omega^\infty\operatorname{sh}^\infty$' which attempts to turn a symmetric spectrum into an Ω-spectrum while keeping the naive homotopy groups; however, the attempt is not always successful. We let $\Omega^\infty\operatorname{sh}^\infty X$ be the mapping telescope (see Example 2.21) of the sequence

$$X \xrightarrow{\tilde{\lambda}_X} \Omega \operatorname{sh} X \xrightarrow{\Omega(\tilde{\lambda}_{\operatorname{sh} X})} \cdots \longrightarrow \Omega^m \operatorname{sh}^m X \xrightarrow{\Omega^m(\tilde{\lambda}_{\operatorname{sh}^m X})} \Omega^{m+1} \operatorname{sh}^{m+1} X \longrightarrow \cdots . \tag{4.18}$$

Here $\tilde{\lambda}_X : X \longrightarrow \Omega \operatorname{sh} X$ was defined in (4.16) and is adjoint to the morphism $\lambda_X : S^1 \wedge X \longrightarrow \operatorname{sh} X$ from (3.12). This construction comes with a canonical natural morphism $\lambda_X^\infty : X \longrightarrow \Omega^\infty\operatorname{sh}^\infty X$, the embedding of the initial term into the mapping telescope.

Proposition 4.19. *Let X be a symmetric Ω-spectrum. Then the morphism $\lambda_X^\infty : X \longrightarrow \Omega^\infty\operatorname{sh}^\infty X$ is a level equivalence.*

Proof. The n-th level of the morphism $\tilde{\lambda}_X$ is the composite of the adjoint structure map $\tilde{\sigma}_n : X_n \longrightarrow \Omega X_{n+1}$ and an isomorphism. So if X is an Ω-spectrum, then the morphism $\tilde{\lambda}_X$ is a level equivalence. Since the shift and loop functors preserve level equivalences (between spectra levelwise Kan spectra in the simplicial context), all the morphisms $\Omega^n(\tilde{\lambda}_{\operatorname{sh}^n X})$ are then level equivalences. Hence the inclusion into the mapping telescope $\lambda_X^\infty : X \longrightarrow \Omega^\infty\operatorname{sh}^\infty X$ is also a level equivalence. □

For every triple of integers k, n, m with $n, m \geq 0$, $k + n \geq 0$ and $k + m \geq 2$ and every symmetric spectrum X we now define an *intertwining isomorphism*

$$\tau_{m,n} \; : \pi_{k+m}(\Omega^n \operatorname{sh}^n X)_m = \pi_{k+m}(\Omega^n X_{n+m}) \;\longrightarrow\; \pi_{k+n}(\Omega^m X_{m+n}) = \pi_{k+n}(\Omega^m \operatorname{sh}^m X)_n \; .$$

Note that in the source and target of $\tau_{m,n}$, the indices m and n have changed places twice; the idea is to let the intertwiner $\tau_{m,n}$ apply the shuffle permutations which are suggested by the two swaps of m and n in the notation. In the source, however, the shuffle permutation $\chi_{n,m}$ may not make sense when k is negative, so we use its sign instead. The intertwiner $\tau_{m,n}$ thus takes the class of a map $f : S^{k+m} \longrightarrow \Omega^n X_{n+m}$ to the class

$$\tau_{m,n}[f] \;=\; (-1)^{mn} \cdot [\Omega^n(\chi_{n,m}) \circ f] \; .$$

Proposition 4.20. *Let X be a symmetric spectrum which is levelwise Kan when in the context of simplicial sets. The intertwining isomorphism interchanges the effects of the stabilization map and the morphism $\Omega^m(\tilde{\lambda}_{\operatorname{sh}^m X})$, i.e., the square*

$$\begin{tikzcd}
\pi_{k+m}(\Omega^n \operatorname{sh}^n X)_m \arrow[r, "\tau_{m,n}"] \arrow[d, "\iota"'] & \pi_{k+n}(\Omega^m \operatorname{sh}^m X)_n \arrow[d, "\pi_{k+n}(\Omega^m \tilde{\lambda}_{\operatorname{sh}^m X})_n"] \\
\pi_{k+m+1}(\Omega^n \operatorname{sh}^n X)_{m+1} \arrow[r, "\tau_{m+1,n}"'] & \pi_{k+n}(\Omega^{m+1} \operatorname{sh}^{m+1} X)_n
\end{tikzcd} \tag{4.21}$$

commutes for all integers k, m, n such that $m, n \geq 0$, $k + n \geq 0$ and $k + m \geq 2$. So for all $n \geq 0$ and all integers k such that $k + n \geq 0$, passage to colimits over m yields a natural bijection:

$$\tau_n \ : \ \hat{\pi}_k(\Omega^n \operatorname{sh}^n X) \xrightarrow{\cong} \pi_{k+n}(\Omega^\infty \operatorname{sh}^\infty X)_n$$

Moreover, the square

$$\begin{tikzcd}
\hat{\pi}_k(\Omega^n \operatorname{sh}^n X) \arrow[r, "\tau_n"] \arrow[d, "\hat{\pi}_k(\Omega^n \tilde{\lambda}_{\operatorname{sh}^n X})"'] & \pi_{k+n}(\Omega^\infty \operatorname{sh}^\infty X)_n \arrow[d, "\iota"] \\
\hat{\pi}_k(\Omega^{n+1} \operatorname{sh}^{n+1} X) \arrow[r, "\tau_{n+1}"'] & \pi_{k+n+1}(\Omega^\infty \operatorname{sh}^\infty X)_{n+1}
\end{tikzcd} \tag{4.22}$$

commutes.

Proof. The commutativity of the square (4.21) is straightforward check from the definitions. Since this square commutes we can pass to colimits over m in the vertical direction. The intertwining isomorphisms then induce a natural isomorphism

$$\hat{\pi}_k(\Omega^n \operatorname{sh}^n X) = \operatorname{colim}_m \pi_{k+m}(\Omega^n \operatorname{sh}^n X)_m \ \longrightarrow \ \operatorname{colim}_m \pi_{k+n}(\Omega^m \operatorname{sh}^m X)_n \ .$$

The homotopy groups of a mapping telescope are naturally isomorphic to the colimit of the homotopy groups of the terms [ref]. So the target of the previous isomorphism identifies with the group $\pi_{k+n}(\Omega^\infty \operatorname{sh}^\infty X)_n$, and together we have constructed the isomorphism τ_n.

It remains to show the relation $\tau_{n+1} \circ \hat{\pi}_k(\Omega^n \tilde{\lambda}_{\operatorname{sh}^n X}) = \iota \circ \tau_n$ as maps from the stable homotopy group $\hat{\pi}_k(\Omega^n \operatorname{sh}^n X)$ to the unstable homotopy group $\pi_{k+n+1}(\Omega^\infty \operatorname{sh}^\infty X)_{n+1}$. Since the source is defined as a colimit, it suffices to check this relation after restriction to $\pi_{k+m}(\Omega^n \operatorname{sh}^n X)_m$ for every $m \geq 0$. But the restriction of τ_n to $\pi_{k+m}(\Omega^n \operatorname{sh}^n X)_m$ is the composite of $\tau_{m,n}$ and the canonical map $\pi_{k+n}(\Omega^m \operatorname{sh}^m X)_n \longrightarrow \pi_{k+n}(\Omega^\infty \operatorname{sh}^\infty X)_n$, so the desired relation follows again from the commutativity of the square (4.21) (but this time with roles of m and n interchanged). □

The previous proposition identifies the (unstable) homotopy groups of the levels of the spectrum $\Omega^\infty \operatorname{sh}^\infty X$ with the (stable) naive homotopy groups of X. A word of warning: while the proposition shows that the groups $\pi_{k+n}(\Omega^\infty \operatorname{sh}^\infty X)_n$ and $\pi_{k+n+1}(\Omega^\infty \operatorname{sh}^\infty X)_{n+1}$ are isomorphic, the spectrum $\Omega^\infty \operatorname{sh}^\infty X$ is *not* in general an Ω-spectrum since the isomorphism obtained from the proposition need not coincide with the stabilization map of the spectrum $\Omega^\infty \operatorname{sh}^\infty X$. In fact, $\Omega^\infty \operatorname{sh}^\infty X$ is an Ω-spectrum if and only if X is semistable, see Proposition 4.24 and Theorem 8.25 below.

Theorem 4.23. *Every $\hat{\pi}_*$-isomorphism of symmetric spectra is a stable equivalence.*

Proof. We treat the context of spectra of simplicial sets first. We start with an observation for an injective Ω-spectrum X. Proposition 4.19 shows that the morphism $\lambda_X^\infty : X \longrightarrow \Omega^\infty \operatorname{sh}^\infty X$ is a level equivalence. Since X is an injective spectrum the map $[\lambda_X^\infty, X] : [\Omega^\infty \operatorname{sh}^\infty X, X] \longrightarrow [X, X]$ is bijective by Proposition 4.6. So there exists a morphism $r : \Omega^\infty \operatorname{sh}^\infty X \longrightarrow X$ such that the composite $r\lambda_X^\infty$ is homotopic to the identity of X (the other composite need not be homotopic to the identity of $\Omega^\infty \operatorname{sh}^\infty X$).

Now we establish the following special case of the theorem: let C be a symmetric spectrum of simplicial sets which is levelwise Kan and all of whose naive homotopy groups vanish. We show that then C is stably contractible. Since looping and shifting shift naive homotopy groups, the hypotheses on C guarantee that all naive homotopy groups of the spectrum $\Omega^n \operatorname{sh}^n C$ are trivial for all $n \geq 0$. By Proposition 4.20 the unstable homotopy group $\pi_l(\Omega^\infty \operatorname{sh}^\infty C)_n$ is isomorphic to the naive stable homotopy group $\hat{\pi}_{l-n}(\Omega^n \operatorname{sh}^n C)$, hence trivial, for all $l \geq 0$. Since all homotopy groups of the simplicial set $(\Omega^\infty \operatorname{sh}^\infty C)_n$ are trivial, the spectrum $\Omega^\infty \operatorname{sh}^\infty C$ is levelwise weakly contractible. Then by Proposition 4.6 the set $[\Omega^\infty \operatorname{sh}^\infty C, X]$ has just one element for every injective Ω-spectrum X.

Shifting, looping and taking mapping telescopes are constructions which preserve the homotopy relation, hence so does the functor $\Omega^\infty \operatorname{sh}^\infty$. So we can define a map

$$[C, X] \ \longrightarrow \ [\Omega^\infty \operatorname{sh}^\infty C, X] \quad \text{ by } \quad [\varphi] \mapsto [r \circ \Omega^\infty \operatorname{sh}^\infty \varphi] \ .$$

A map $[\lambda_C^\infty, X] : [\Omega^\infty \mathrm{sh}^\infty C, X] \longrightarrow [C, X]$ in the other direction is given by precomposition with $\lambda_C^\infty : C \longrightarrow \Omega^\infty \mathrm{sh}^\infty C$. Since r is a retraction (up to homotopy) to λ_X^∞, the composite of the two natural maps is the identity on $[C, X]$. Since the set $[\Omega^\infty \mathrm{sh}^\infty C, X]$ has only one element, the same is true for the set $[C, X]$. Since X is an arbitrary injective Ω-spectrum, C is indeed stably contractible.

Now we consider a symmetric spectrum C of simplicial sets, not necessarily levelwise Kan, such that all naive homotopy groups of C vanish. We show that then C is stably contractible. We apply the functors 'geometric realization' and 'singular complex' to replace C by the level equivalent spectrum $\mathcal{S}(|C|)$ which is levelwise Kan and has trivial naive homotopy groups. So $\mathcal{S}(|C|)$ is stably contractible by the case above; since level equivalences are stable equivalences, C is stably contractible.

Now we prove the theorem. Let $f : A \longrightarrow B$ be a $\hat{\pi}_*$-isomorphism and let $C(f)$ be its mapping cone. By the long exact of naive homotopy groups (see Proposition 2.12; taking mapping cones commutes with realization) the naive homotopy groups of $C(f)$ are trivial. By the previous paragraph, $C(f)$ is stably contractible, so f is a stable equivalence by the criterion (iv) of Proposition 4.17.

Now suppose that $f : A \longrightarrow B$ is a $\hat{\pi}_*$-isomorphism between symmetric spectra of spaces. Then its singular complex $\mathcal{S}(f) : \mathcal{S}(A) \longrightarrow \mathcal{S}(B)$ is also a $\hat{\pi}_*$-isomorphism, hence a stable equivalence by the above. So f is a stable equivalence by definition. □

Proposition 4.24. *Let X be a semistable symmetric spectrum. In the context of simplicial sets, suppose also that X is levelwise Kan. Then the morphism $\lambda_X^\infty : X \longrightarrow \Omega^\infty \mathrm{sh}^\infty X$ is a $\hat{\pi}_*$-isomorphism and $\Omega^\infty \mathrm{sh}^\infty X$ is a symmetric Ω-spectrum.*

Proof. Since X is semistable the morphism $\lambda_X : S^1 \wedge X \longrightarrow \mathrm{sh}\, X$ is a $\hat{\pi}_*$-isomorphism, by definition. So its adjoint $\tilde{\lambda}_X : X \longrightarrow \Omega\, \mathrm{sh}\, X$ is a $\hat{\pi}_*$-isomorphism by Proposition 3.8. If X is semistable, then so is $\mathrm{sh}^n X$, hence the morphism $\tilde{\lambda}_{\mathrm{sh}^n X}$ is a $\hat{\pi}_*$-isomorphism for every $n \geq 0$. Since looping shifts homotopy groups, the morphism $\Omega^n(\tilde{\lambda}_{\mathrm{sh}^n X})$ is a $\hat{\pi}_*$-isomorphism for every n; so the canonical morphism λ_X^∞ from X to the mapping telesope $\Omega^\infty \mathrm{sh}^\infty X$ is a $\hat{\pi}_*$-isomorphism.

Since $\Omega^n(\tilde{\lambda}_{\mathrm{sh}^n X})$ is a $\hat{\pi}_*$-isomorphism, the commutative square (4.22) shows that the adjoint structure map $\tilde{\sigma}_n : (\Omega^\infty \mathrm{sh}^\infty X)_n \longrightarrow \Omega(\Omega^\infty \mathrm{sh}^\infty X)_{n+1}$ induces a bijection on all homotopy groups, taken with respect to the given basepoints. This almost shows that $\tilde{\sigma}_n$ is a weak equivalence.

We let K be a finite based CW-complex. Since X is semistable, so is the function spectrum $\mathrm{map}(K, -)$ (compare Proposition 3.16 (iv)). The square

$$\begin{tikzcd}
\pi_0(\Omega^\infty \mathrm{sh}^\infty \mathrm{map}(K, X))_n \arrow[r, "(\tilde{\sigma}_n)_*"] \arrow[d, "\cong"'] & \pi_0(\Omega(\Omega^\infty \mathrm{sh}^\infty \mathrm{map}(K, X))_{n+1}) \arrow[d, "\cong"] \\
{[K, (\Omega^\infty \mathrm{sh}^\infty X)_n]} \arrow[r, "{[K, \tilde{\sigma}_n]}"'] & {[K, \Omega(\Omega^\infty \mathrm{sh}^\infty X)_{n+1}]}
\end{tikzcd}$$

commutes and the upper horizontal map is bijective by the above. Since the lower map is bijective for all finite based CW-complexes K, the adjoint structure map $\tilde{\sigma}_n : (\Omega^\infty \mathrm{sh}^\infty X)_n \longrightarrow \Omega(\Omega^\infty \mathrm{sh}^\infty X)_{n+1}$ is a weak equivalence. Thus $\Omega^\infty \mathrm{sh}^\infty X$ is an Ω-spectrum. □

For every pair of symmetric spectra A and B the canonical morphism $A \vee B \longrightarrow A \times B$ is a $\hat{\pi}_*$-isomorphism by Proposition 2.19. So Theorem 4.23 immediately implies

Corollary 4.25. *For every pair of symmetric spectra A and B the canonical morphism $A \vee B \longrightarrow A \times B$ is a stable equivalence.*

Now we give an important example of a stable equivalence which is *not* a $\hat{\pi}_*$-isomorphism,

Example 4.26. Let $\lambda : F_1S^1 \longrightarrow F_0S^0 = \mathbb{S}$ denote the morphism which is adjoint to the identity in level 1. For any injective Ω-spectrum X we consider the commutative square

$$\begin{array}{ccc} [F_0S^0, X] & \xrightarrow{[\lambda,X]} & [F_1S^1, X] \\ {\scriptstyle \mathrm{ev}_0}\big\downarrow\cong & & \cong\big\downarrow{\scriptstyle \mathrm{ev}_1} \\ \pi_0(X_0) & \xrightarrow[\iota]{} & \pi_1(X_1) \end{array}$$

The vertical maps given by evaluation at levels 0 respectively 1 are adjunction bijections (the 'freeness property' in Example 3.20). Since X is an Ω-spectrum, the stabilization map $\iota : \pi_0(X_0) \longrightarrow \pi_1(X_1)$ is bijective. So the map $[\lambda, X]$ is bijective and λ is a stable equivalence.

In Example 3.20 we determined the 0-th naive homotopy group of F_1S^1 as an infinitely generated free abelian group, whereas $\hat{\pi}_0\mathbb{S} = \pi_0^{\mathrm{s}}$ is free abelian of rank 1. Thus the morphism λ is not a $\hat{\pi}_*$-isomorphism.

The stable equivalence $\lambda : F_1S^1 \longrightarrow \mathbb{S}$ of the previous example is only one special case of a whole class of stable equivalences (which are typically not $\hat{\pi}_*$-isomorphisms either). For every symmetric spectrum A we introduced a morphism $\lambda_A : S^1 \wedge A \longrightarrow \mathrm{sh}\, A$ in (3.12) and its adjoint $\tilde{\lambda}_A : A \longrightarrow \Omega(\mathrm{sh}\, A) = \mathrm{sh}(\Omega A)$ in (4.16). One should beware that even though sources and target of these two morphisms have abstractly isomorphic naive homotopy groups, λ_A and $\tilde{\lambda}_A$ are not in general $\hat{\pi}_*$-isomorphisms.

Using the adjunction $(\triangleright, \mathrm{sh})$ we can adjoin $\tilde{\lambda}_A$ two more times and product two more natural morphisms

$$\bar{\lambda}_A \ : \ \triangleright A \ \longrightarrow \ \Omega A \qquad \text{and} \qquad \hat{\lambda}_A \ : \ S^1 \wedge \triangleright A \ \longrightarrow \ A \ . \tag{4.27}$$

More precisely, we define these two morphisms as the composite

$$\triangleright A \ \xrightarrow{\triangleright \tilde{\lambda}_A} \ \triangleright(\mathrm{sh}(\Omega A)) \ \xrightarrow{\epsilon_{\Omega A}} \ \Omega A$$

respectively the composite

$$S^1 \wedge \triangleright A \ \xrightarrow{S^1 \wedge \bar{\lambda}_A} \ S^1 \wedge \Omega A \ \xrightarrow{\mathrm{ev}} \ A \ .$$

If we use the identification $F_1S^1 \cong S^1 \wedge \triangleright\mathbb{S}$, then for the sphere spectrum, the morphism $\hat{\lambda}_{\mathbb{S}}$ specializes to the morphism $\lambda : F_1S^1 \longrightarrow \mathbb{S}$.

Proposition 4.28. *For every symmetric spectrum A the morphism $\hat{\lambda}_A : S^1 \wedge \triangleright A \longrightarrow A$ is a stable equivalence. In the context of topological spaces, or if A is levelwise Kan, then the morphism $\bar{\lambda}_A : \triangleright A \longrightarrow \Omega A$ is a stable equivalence.*

Proof. We first treat the case of symmetric spectra of simplicial sets. For every symmetric spectrum X the adjunction bijections for the adjoint pairs $(\triangleright, \mathrm{sh})$ and $(S^1 \wedge -, \Omega)$, applied to morphisms and homotopies, provide natural bijections

$$[S^1 \wedge \triangleright A, X] \ \cong \ [A, \mathrm{sh}\, \Omega X] \ .$$

Under this correspondence the map $[\hat{\lambda}_A, X] : [A, X] \longrightarrow [S^1 \wedge \triangleright A, X]$ becomes the map $[A, \tilde{\lambda}_X] : [A, X] \longrightarrow [A, \mathrm{sh}\, \Omega X]$. If X is an injective Ω-spectrum, then $\tilde{\lambda}_X : X \longrightarrow \mathrm{sh}\, \Omega X$ is a level equivalence between injective spectra, hence a homotopy equivalence by Proposition 4.6. So $[A, \tilde{\lambda}_X]$, and consequently also $[\hat{\lambda}_A, X]$, is bijective, which proves that $\hat{\lambda}_A$ is a stable equivalence. If the symmetric spectrum A is also levelwise Kan, then the evaluation morphism $\mathrm{ev} : S^1 \wedge \Omega A \longrightarrow A$ is a $\hat{\pi}_*$-isomorphism, hence a stable equivalence. Since $\hat{\lambda}_A = \mathrm{ev} \circ (S^1 \wedge \bar{\lambda}_A)$ is also a stable equivalence, the suspension of $\bar{\lambda}_A$ is a stable equivalence. Proposition 4.17 shows that then $\bar{\lambda}_A$ is a stable equivalence.

Now we treat symmetric spectra of spaces. We exploit that $S^1 \wedge \triangleright A$ is naturally isomorphic to the twisted smash product $S^1 \triangleright_1 A$. Under this isomorphism, the composite

$$S^1 \triangleright_1 \mathcal{S}(A) \ \xrightarrow{\epsilon \triangleright_1 \mathcal{S}(A)} \ \mathcal{S}(S^1) \triangleright_1 \mathcal{S}(A) \ \longrightarrow \ \mathcal{S}(S^1 \triangleright_1 A) \ \xrightarrow{\mathcal{S}(\hat{\lambda}_A)} \ \mathcal{S}(A)$$

corresponds to the morphism $\hat{\lambda}_{\mathcal{S}(A)}$ which we just recognized as a stable equivalence. The first map $\epsilon \triangleright_1 \mathcal{S}(A)$ is a level equivalence since $\epsilon : S^1 \longrightarrow \mathcal{S}(S^1)$ is a weak equivalence of simplicial sets. The second map is a $\hat{\pi}_*$-isomorphism, hence stable equivalence, by Proposition 3.31. So the third map $\mathcal{S}(\hat{\lambda}_A)$ is also a stable equivalence; thus $\hat{\lambda}_A$ is a stable equivalence, by definition.

To treat the map $\bar{\lambda}_A : \triangleright A \longrightarrow \Omega A$ we consider the commutative square:

$$\begin{tikzcd} \triangleright\mathcal{S}(A) \arrow[r] \arrow[d, "\bar{\lambda}_{\mathcal{S}(A)}"'] & \mathcal{S}(\triangleright A) \arrow[d, "\mathcal{S}(\bar{\lambda}_A)"] \\ \Omega\mathcal{S}(A) \arrow[r] & \mathcal{S}(\Omega A) \end{tikzcd}$$

The upper horizontal map is the $\hat{\pi}_*$-isomorphism of Proposition 3.31 for $m = 1$ and $L = S^0$. The lower horizontal map is a level equivalence. By the first paragraph, the left vertical map is a stable equivalence, hence so is the right vertical map $\mathcal{S}(\bar{\lambda}_A)$. So $\bar{\lambda}_A$ is a stable equivalence, by definition. □

Our next task is to develop various equivalent characterizations for stable equivalences. Some parts of this have already been shown in the previous propositions, but they are repeated here for easier reference.

Proposition 4.29. *For every morphism $f : A \longrightarrow B$ of symmetric spectra the following are equivalent:*

(i) *f is a stable equivalence;*
(ii) *the mapping cone $C(f)$ of f is stably contractible;*
(iii) *the suspension $S^1 \wedge f : S^1 \wedge A \longrightarrow S^1 \wedge B$ is a stable equivalence;*
(iv) *the induction $\triangleright f : \triangleright A \longrightarrow \triangleright B$ is a stable equivalence.*

In the context of spaces, or if A and B are levelwise Kan, then conditions (i)-(iv) *are also equivalent to the following two conditions:*

(v) *the homotopy fiber $F(f)$ of f is stably contractible;*
(vi) *the loop $\Omega f : \Omega A \longrightarrow \Omega B$ is a stable equivalence.*

Proof. We start by showing the equivalence of conditions (i) through (iv) for symmetric spectra of simplicial sets. The equivalence of conditions (i), (ii) and (iii) is contained in Proposition 4.17. The implication (i)⇒(iv) is the special case of Proposition 4.14 for $m = 1$ and $L = S^0$. Suppose conversely that $\triangleright f$ is a stable equivalence. Then so is the suspension $S^1 \wedge \triangleright f$ by Proposition 4.14. The commutative square

$$\begin{tikzcd} S^1 \wedge \triangleright A \arrow[r, "S^1 \wedge \triangleright f"] \arrow[d, "\hat{\lambda}_A"'] & S^1 \wedge \triangleright B \arrow[d, "\hat{\lambda}_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

(whose vertical morphisms are stable equivalences by Proposition 4.28) shows that f is a stable equivalence.

Now we show that also for symmetric spectra of spaces, conditions (i) through (iv) are equivalent.

(i)⇔(ii) By the definitions of 'stable equivalence' in the topological context we need to show that the singular complex $\mathcal{S}(f) : \mathcal{S}(A) \longrightarrow \mathcal{S}(B)$ is a stable equivalence if and only if $\mathcal{S}(C(f))$, the singular complex of the mapping cone, is stably contractible in the simplicial world. This follows from the equivalence of conditions (i) and (ii) in the simplicial context and the fact that the natural map $C(\mathcal{S}(f)) \longrightarrow \mathcal{S}(C(f))$ from the mapping cone of the singular complex to the singular complex of the mapping cone is a $\hat{\pi}_*$-isomorphism (see Proposition 3.4), hence a stable equivalence.

(i)⇔(iii) This is the same argument as in the last pararaph, just that now we exploit that the natural map $S^1 \wedge \mathcal{S}(A) \longrightarrow \mathcal{S}(S^1 \wedge A)$ from the suspension of the singular complex of a spectrum A to the singular complex of the suspension is a $\hat{\pi}_*$-isomorphism, hence a stable equivalence (see Proposition 3.4)

(i)⇔(iv) Again we reduce to the case of symmetric spectra of simplicial sets. This time we exploit that the natural map $\triangleright\mathcal{S}(A) \longrightarrow \mathcal{S}(\triangleright A)$ is a $\hat{\pi}_*$-isomorphism, hence a stable equivalence, by Proposition 3.31 for $m = 1$ and $L = S^0$.

For the rest of the proof we assume that we are in the context of spaces or A and B are levelwise Kan. Conditions (iv) and (vi) are then equivalent because the natural morphism $\bar{\lambda}_A : \triangleright A \longrightarrow \Omega A$ is a stable equivalence by Proposition 4.28.

(ii)$\Longleftrightarrow$(v) By the already established equivalence between conditions (ii) and (iii), the homotopy fiber $F(f)$ is stably contractible if and only if its suspension $S^1 \wedge F(f)$ is stably contractible. By Proposition 2.17 the spectrum $S^1 \wedge F(f)$ is $\hat{\pi}_*$-isomorphic, hence stable equivalent, to the mapping $C(f)$. Altogether this proves that $F(f)$ is stably contractible if and only if $C(f)$ is. □

Remark 4.30. As far as I can see, the suspension functor $S^1 \wedge -$ does not in general preserve weak equivalences of spaces, hence not level equivalences of symmetric spectra of spaces [give example]. However, $S^1 \wedge -$ preserves $\hat{\pi}_*$-isomorphisms and stable equivalences in the topological context.

For the mapping telescope and the diagonal of a sequence of symmetric spectra, see Example 2.21.

Proposition 4.31. (i) *A wedge of stable equivalences is a stable equivalence.*

(ii) *A finite product of stable equivalences is a stable equivalence.*

(iii) *Consider a commutative square of symmetric spectra*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "j"'] & D \end{tikzcd} \tag{4.32}$$

and let $h = C(f) \cup g : C(i) \longrightarrow C(j)$ be the map induced by f and g on mapping cones. Then if two of the three morphisms f, g and h are stable equivalences, so is the third. [same for homotopy fibers]

(iv) *Consider a commutative square (4.32) of symmetric spectra and suppose that in the context of simplicial sets all four spectra are levelwise Kan. Let $e : F(i) \longrightarrow F(j)$ be the map induced by f and g on homotopy fibers. Then if two of the three morphisms e, f and g are stable equivalences, so is the third.*

(v) *Consider a commutative square (4.32) of symmetric spectra for which one of the following conditions holds:*

(a) *the square is a pushout and i or f is a level cofibration.*

(b) *the square is a pullback and j or g is a level fibration.*

Then f is a stable equivalence if and only if g is.

(vi) *Let L be a cofibrant Σ_m-space respectively or any Σ_m-simplicial set. Then the twisted smash product functor $L \triangleright_m -$ preserves stable equivalences. In particular, the levelwise smash product $K \wedge -$ with a cofibrant based space or a based simplicial set preserves stable equivalences.*

(vii) *Let K be a finite based CW-complex respectively a finite based simplicial set, and $f : A \longrightarrow B$ a stable equivalence. Suppose that A and B are levelwise Kan complexes when in the simplicial context. Then the morphism $\mathrm{map}(K, f) : \mathrm{map}(K, A) \longrightarrow \mathrm{map}(K, B)$ is a stable equivalence.*

(viii) *We consider a commutative diagram of symmetric spectra*

$$\begin{tikzcd} A^0 \arrow[r, "f^0"] \arrow[d, "\varphi^0"'] & A^1 \arrow[r, "f^1"] \arrow[d, "\varphi^1"'] & A^2 \arrow[r, "f^2"] \arrow[d, "\varphi^2"] & A^3 \arrow[r, "f^3"] \arrow[d, "\varphi^3"] & \cdots \\ B^0 \arrow[r, "g^0"'] & B^1 \arrow[r, "g^1"'] & B^2 \arrow[r, "g^2"'] & B^3 \arrow[r, "g^3"'] & \cdots \end{tikzcd}$$

in which all vertical morphisms $\varphi^n : A^n \longrightarrow B^n$ are stable equivalences. Then the map $\mathrm{tel}_n \varphi^n : \mathrm{tel}_n A^n \longrightarrow \mathrm{tel}_n B^n$ induced on mapping telescopes and the map $\mathrm{diag}_n \varphi^n : \mathrm{diag}_n A^n \longrightarrow \mathrm{diag}_n B^n$ induced on the diagonal symmetric spectrum are also stable equivalences. In the context of simplicial sets, the map $\mathrm{colim}_{n \geq 0} \varphi^n : \mathrm{colim}_{n\geq 0} A^n \longrightarrow \mathrm{colim}_{n\geq 0} B^n$ induced on colimits is also a stable equivalence.

(ix) *Let $f^n : A^n \longrightarrow A^{n+1}$ for $n \geq 0$ be a sequence of composable stable equivalences of symmetric spectra. Then canonical morphism $A^0 \longrightarrow \mathrm{tel}_n A^n$ to the mapping telescope is a stable equivalence. In* *the context of simplicial sets, the canonical morphism* $A^0 \longrightarrow \mathrm{colim}_{n\geq 0} A^n$ *to the colimit is a stable equivalence.*

(x) *Let* I *be a filtered category and let* $A, B : I \longrightarrow \mathcal{S}p$ *be functors which take all morphisms in* I *to monomorphisms of symmetric spectra. If* $\tau : A \longrightarrow B$ *is a natural transformation such that* $\tau(i) : A(i) \longrightarrow B(i)$ *is a stable equivalence for every object* i *of* I*, then the induced morphism* $\mathrm{colim}_I \tau : \mathrm{colim}_I A \longrightarrow \mathrm{colim}_I B$ *on colimits is a stable equivalence.*

(xi) *Let* G *be a group and* $f : A \longrightarrow B$ *an* G*-equivariant morphism of* G*-symmetric spectra. If the underlying morphism of symmetric spectra is a stable equivalence, then so is the induced morphism* $f_{hG} : A_{hG} \longrightarrow B_{hG}$ *on homotopy orbit spectra.*

Proof. (i) For symmetric spectra of simplicial sets, we argue from the definition: for every family $\{A_i\}_{i\in I}$ of symmetric spectra and every injective Ω-spectrum X the natural map

$$[\bigvee_{i\in I} A_i, X] \longrightarrow \prod_{i\in I} [A_i, X]$$

is bijective by the universal property of the wedge, applied to morphisms and homotopies. A wedge of stable equivalences is a stable equivalence.

For symmetric spectra of spaces we note that for every family $\{A_i\}_{i\in I}$ of symmetric spectra the canonical map $\bigvee_{i\in I} \mathcal{S}(A^i) \longrightarrow \mathcal{S}(\bigvee_{i\in I} A^i)$ is a $\hat{\pi}_*$-isomorphism because naive homotopy groups take wedges to sums (Proposition 2.19) and taking singular complex does not chance the naive homotopy groups. This reduces the claim for spaces to the claim for simplicial sets.

(ii) Finite products are stably equivalent to finite products by Corollary 4.25, so (ii) follows from part (i).

(iii) We start with the special case where C and D are trivial spectra. In other words, we show first that given any morphism of symmetric spectra $i : A \longrightarrow B$, then if two of the spectra A, B and the mapping cone $C(i)$ are stably contractible, so is the third. If the cone $C(i)$ is stably contractible, then i is a stable equivalence (Proposition 4.29), hence A is stably contractible if and only if B is. If A and B are stably contractible, then i is a stable equivalence, so $C(i)$ is stably contrctible, again by Proposition 4.29. This completes the special case.

In the general case, we exploit that the order in which we take iterated mapping cones in a commutative square does not matter. More precisely, the mapping cone of the morphism $h = C(f) \cup g : C(i) \longrightarrow C(j)$ is isomorphic to the mapping cone of the morphism $k = C(i) \cup j : C(f) \longrightarrow C(g)$ induced by i and j. By Proposition 4.29, a morphism is a stable equivalence if and only if its mapping cone is stably contractible. So the general case follows by applying the special case to the morphism $k : C(f) \longrightarrow C(g)$.

(iv) The suspension of the homotopy fiber $F(i)$ is naturally π_*-isomorphic, hence stably equivalent, to the mapping cone $C(i)$ (compare Proposition 2.17), and similarly for j. Moreover, $e : F(i) \longrightarrow F(j)$ is a stable equivalence if and only its suspensions $S^1 \wedge e : S^1 \wedge F(i) \longrightarrow S^1 \wedge F(j)$ is a stable equivalence. So e is a stable equivalence if and only if the morphism $h : C(i) \longrightarrow C(j)$ on mapping cones is a stable equivalence. Using this, part (iv) follows from part(iii).

(v) We start with case (a) of a pushout square. If f (and hence g) is a level cofibration, then the symmetric spectrum of strict cofibers $C/f(A)$ is level equivalent to the mapping cone $C(f)$, and similarly for g. Since the square is a pushout, the strict cofibers are isomorphic. So the mapping cones $C(f)$ and $C(g)$ are level equivalent, hence stably equivalent. The criterion (ii) of Proposition 4.29 shows that f is a stable equivalence if and only if g is. If i (and hence j) is a level cofibration, then by the same argument as above the morphism $h = C(f) \cup g : C(i) \longrightarrow C(j)$ induced by f and g is a level, hence stable equivalence. Hence part (iii) shows that f is a stable equivalence if and only if g is.

The case (b) of a pullback square is strictly dual. If g (and hence f) is a level fibration, then the symmetric spectrum of strict fibers is level equivalent to the homotopy fiber $F(g)$, and similarly for f. Since the square is a pullback, the strict fibers are isomorphic. So the homotopy fibers $F(g)$ and $F(f)$ are level equivalent, hence stably equivalent. The criterion (v) of Proposition 4.29 shows that g is a stable equivalence if and only if f is. If j (and hence i) is a level fibration, we use 'properness' to reduce to the previous case. We choose a factorization $g = g'\varphi$ where $\varphi : B \longrightarrow B'$ is a level equivalence and $g' : B' \longrightarrow D$ is a level fibration. The morphism g' is then a stable equivalence if and only if g is, so by the above the pullback $j^*(g') : C \times_D B' \longrightarrow C$ is a stable equivalence if and only if g is. Now f factors as the composite of the level equivalence (by properness) $(f, \varphi i) : A \longrightarrow C \times_D B'$ and the morphism $j^*(g')$, so f is a stable equivalence if and only if g is.

(vi) The case of simplicial sets has already been taken care of in Proposition 4.14. In the context of space we contemplate the commutative diagram

$$\begin{tikzcd}
\mathcal{S}(L) \rhd_m \mathcal{S}(A) \arrow[r, "\mathcal{S}(L)\rhd_m\mathcal{S}(f)"] \arrow[d] & \mathcal{S}(L) \rhd_m \mathcal{S}(B) \arrow[d] \\
\mathcal{S}(L \rhd_m A) \arrow[r, "\mathcal{S}(L\rhd_m f)"'] & \mathcal{S}(L \rhd_m B)
\end{tikzcd}$$

in which the vertical maps are $\hat{\pi}_*$-isomorphisms, hence stable equivalences, by Proposition 3.31. The upper map is a stable equivalence by the case of simplicial sets, hence so is the lower map. This shows that $L \rhd_m f$ is a stable equivalence. Levelwise smash product with a based space (or simplicial set) K is the special case of twisted smash product in level 0.

(vii) We start by showing that if A is stably contractible (and levelwise Kan when in the simplicial context), then $\mathrm{map}(K, A)$ is stably contractible. We prove this by induction over the number of cells (respectively the number of non-degenerate simplices) of K. If K consists of only one point, then $\mathrm{map}(K, A)$ is a trivial spectrum, hence stably contractible. Now suppose that the claim is true for K and K' is obtained from K by attaching an n-cell (respectively K' contains K and has exactly one additional non-degenerate simplex of dimension n). Then the mapping cone of the inclusion $i : K \longrightarrow K'$ is homotopy equivalent (respectively weakly equivalent) to S^n, and so $\mathrm{map}(C(i), A)$ is level equivalent to $\Omega^n A$, hence stably contractible by Proposition 4.29. On the other hand, $\mathrm{map}(C(i), A)$ is isomorphic to the homotopy fiber of the map $\mathrm{map}(i, A) : \mathrm{map}(K', A) \longrightarrow \mathrm{map}(K, A)$. Since this homotopy fiber is stably contractible, $\mathrm{map}(i, A)$ is a stable equivalence, again by Proposition 4.29. So $\mathrm{map}(K', A)$ is stably contractible because $\mathrm{map}(K, A)$ is.

Now we consider a stable equivalence $f : A \longrightarrow B$ (with A and B levelwise Kan complexes in the simplicial context). Then the homotopy fiber $F(f)$ is stably contractible by Proposition 4.29, so the symmetric spectrum $\mathrm{map}(A, F(f))$ is stably contractible by the above. But the symmetric spectrum $\mathrm{map}(A, F(f))$ is isomorphic to the homotopy fiber of the morphism $\mathrm{map}(K, f) : \mathrm{map}(K, A) \longrightarrow \mathrm{map}(K, B)$, so $\mathrm{map}(K, f)$ is a stable equivalence, again by Proposition 4.29.

(viii) We treat the simplicial context first. For every injective Ω-spectrum X the simplicial set $\mathrm{map}(\mathrm{tel}_n A^n, X)$ is isomorphic to the 'mapping microscope' of the tower of simplicial sets $\mathrm{map}(f^n, X) : \mathrm{map}(A^{n+1}, X) \longrightarrow \mathrm{map}(A^n, X)$. All the simplicial sets $\mathrm{map}(A^n, X)$ are Kan complexes by Proposition 4.4 (ii) and the microscope construction takes sequences of weak equivalences between Kan simplicial sets to weak equivalences. So the map $\mathrm{map}(\mathrm{tel}_n \varphi^n, X)$ is a weak equivalence, and so $\mathrm{tel}_n \varphi^n$ is a stable equivalence by Proposition 4.17. Lemma 2.23 relates the mapping telescope $\mathrm{tel}_n A^n$ to the diagonal spectrum $\mathrm{diag}_n A^n$ through a chain of two natural $\hat{\pi}_*$-isomorphisms. Since every $\hat{\pi}_*$-isomorphism is also a stable equivalence, the result for mapping telescopes implies the one for diagonals. [case of spaces]

In the simplicial world, the colimit of a sequence is level equivalent to the mapping telescope [ref], hence also stably equivalent.

(ix) We can reduce statement (viii) to (vi) by comparing the given sequence with the constant sequence consist of the spectra A^0 and its identity map. The canonical map $A^0 \longrightarrow \mathrm{tel}_n A^0$ is a level equivalence.

(x) For every injective Ω-spectrum X the simplicial set $\mathrm{map}(\mathrm{colim}_I A, X)$ is isomorphic to the inverse limit of the functor $\mathrm{map}(A, X) : I^{op} \longrightarrow \mathbf{sS}$, and similarly for the functor B. Since A and B consists of injective morphisms, all morphisms in the inverse systems $\mathrm{map}(A, X)$ and $\mathrm{map}(B, X)$ are Kan fibrations (by Proposition 4.4 (i)). Filtered inverse limits of weak equivalences along Kan fibrations are again weak equivalences, so the map $\mathrm{map}(\mathrm{colim}_I B, X) \longrightarrow \mathrm{map}(\mathrm{colim}_I A, X)$ is a weak equivalence of simplicial set, which means that $\mathrm{colim}_I A \longrightarrow \mathrm{colim}_I B$ is a stable equivalence. [topological case]

(xi) Use the isomorphism

$$\mathrm{map}(A_{hG}, X) \ \cong\ \mathrm{map}^G(EG_+, \mathrm{map}(A, X)) \ =\ \mathrm{map}(A, X)^{hG}$$

and the fact that homotopy fixed points takes underlying weak equivalences between Kan simplicial G-sets to weak equivalences. [topological case] □

Example 4.33. We give an example which shows that stable equivalences are in general badly behaved with respect to infinite products: we present a stably contractible symmetric spectrum $X = H(A/A_\bullet)$ such that the countably infinite product of copies of X is not stably contractible. This shows that the restriction to *finite* products in part (ii) of Proposition 4.31 is essential. Since the product $\prod_{n\geq 1} X$ is the same as the mapping spectrum from an infinite discreted space (or an infinite constant simplicial set) to X, the example also shows that the restriction to *finite* CW-complexes (respectively simplicial sets) is essential in part (iv) of Proposition 4.31.

For the example we pick an abelian group A and an exhaustive filtration $0 = A_0 \subset A_1 \subset A_2 \subset \dots$ such that A_n is a proper subgroup of A_{n+1} for all n. We define a modified Eilenberg-Mac Lane spectrum by

$$X_n \ =\ (A/A_n)[S^n]\ ,$$

the linearization, with coefficient group A/A_n, of the n-sphere (either topological or simplicial). The symmetric group acts on X_n through the permutation action on the sphere. The structure map is the composite

$$(A/A_n)[S^n]\wedge S^1 \ \longrightarrow\ (A/A_n)[S^{n+1}] \ \longrightarrow\ (A/A_{n+1})[S^n]\ ,$$

where the first is the structure map of the Eilenberg-Mac Lane spectrum for the group A/A_n (see Example 1.14) and the second map is induced by the quotient map $A/A_n \longrightarrow A/A_{n+1}$ of coefficient groups. (This is also an example of the more general construction of the Eilenberg-Mac Lane spectrum associated to an **I**-functor, compare Exercise E.I.52 (iv)).

As we argued in Example 1.14, the space X_n is an Eilenberg-Mac Lane space of type $(A/A_n, n)$; however, since the filtration is not constant, the spectrum X is not an Ω-spectrum. The naive homotopy groups of X are trivial in all non-zero dimensions. The naive homotopy group $\hat{\pi}_0 X$ is isomorphic to the colimit of the sequence of projection maps

$$A = A/A_0 \ \longrightarrow\ A/A_1 \ \longrightarrow\ A/A_2 \ \longrightarrow\ \cdots$$

so $\hat{\pi}_0 X$ is trivial since the groups A_n exhaust A. Thus X is stably contractible by Theorem 4.23.

Now we calculate the naive homotopy groups of the infinite product $X^{\mathbb{N}}$ of copies of X. Again, the space (or simplicial set) in level n is an Eilenberg-Mac Lane space of dimension n, so the naive homotopy groups are trivial in all non-zero dimensions. The naive homotopy group $\hat{\pi}_0(X^{\mathbb{N}})$ is isomorphic to the colimit of the sequence maps

$$A^{\mathbb{N}} = (A/A_0)^{\mathbb{N}} \ \longrightarrow\ (A/A_1)^{\mathbb{N}} \ \longrightarrow\ (A/A_2)^{\mathbb{N}} \ \longrightarrow\ \cdots$$

each of which is an infinite product of projection maps. If we choose a sequence of elements $a_n \in A_n - A_{n-1}$, for $n \geq 1$, then the tuple $(a_n)_n \in A^{\mathbb{N}} = \pi_0 X_0^{\mathbb{N}}$ does not become zero at any finite stage of the colimit system, hence it represents a non-trivial element in $\hat{\pi}_0(X^{\mathbb{N}})$. As a semistable symmetric spectrum [see below] with a non-trivial homotopy group, the product $X^{\mathbb{N}}$ is not stably contractible.

Example 4.34. We have mentioned many constructions which preserves stable equivalences, and now we also mention one which does not, namely shifting; this should be contrasted with the fact that shifting does preserve $\hat{\pi}_*$-isomorphisms because $\hat{\pi}_{k+1}(\mathrm{sh}\, X)$ equals $\hat{\pi}_k X$ as abelian groups. An example is the fundamental stable equivalence $\lambda : F_1 S^1 \longrightarrow \mathbb{S}$ of Example 3.20 which is adjoint to the identity of S^1. The symmetric spectrum $\mathrm{sh}(F_1 S^1)$ is isomorphic to the wedge of $F_0 S^1 = \Sigma^\infty S^1$ and $F_1 S^2$, while $\mathrm{sh}\,\mathbb{S} \cong F_0 S^1$; the map $\mathrm{sh}\,\lambda : \mathrm{sh}(F_1 S^1) \longrightarrow \mathrm{sh}\,\mathbb{S}$ is the projection to the wedge summand. The complementary summand $F_1 S^2 \cong S^1 \wedge F_1 S^1$ is stably equivalent, via the suspension of λ, to $S^1 \wedge \mathbb{S} \cong \Sigma^\infty S^1$, and is thus not stably contractible (for example since $\pi_1(\Sigma^\infty S^1) \cong \pi_0 \mathbb{S}$ is free abelian of rank 1).

Example 4.35 (Stable homotopy type of free spectra)**.** We show that the free symmetric spectrum $F_m K$ generated by a based space K in level m is stably equivalent to the m-fold loop of the suspension spectrum of K. We start from the adjunction unit $K \longrightarrow \Omega^m(K \wedge S^m) = \Omega^m(\Sigma^\infty K)_m$ and claim that the adjoint morphism

$$\varphi^m \ : \ F_m K \ \longrightarrow \ \Omega^m(\Sigma^\infty K)$$

is a stable equivalence.

To prove our claim we argue by induction on m, the case $m = 0$ being clear since φ^0 is an isomorphism. In general, the square

$$\begin{tikzcd} \triangleright(F_m K) \arrow[r, "\hat\lambda_{F_m K}"] \arrow[d, "\cong"'] & \Omega(F_m K) \arrow[d, "\Omega(\varphi^m)"] \\ F_{1+m} K \arrow[r, "\varphi^{1+m}"'] & \Omega^{1+m}(\Sigma^\infty K) \end{tikzcd}$$

commutes, where the left isomorphism was constructed in (3.21). The morphism $\hat\lambda_{F_m K}$ is a stable equivalence by Proposition 4.28, the morphism $\Omega(\varphi^m)$ is a stable equivalence by induction and Proposition 4.29. So φ^{1+m} is a stable equivalence.

For later reference we establish another property of the stable equivalence φ^m, namely that it is equivariant with respect to the two right actions of the symmetric group Σ_m. The right Σ_m-action on the source is on the 'free coordinates' as defined in (3.22); the right action on the target is obtained from the left action on the m loop coordinates. The equivariance property is not completely obvious; however, if we unravel the definition of φ^m, its $(m+n)$th level

$$(\varphi^m)_{m+n} \ : \ \Sigma^+_{m+n} \wedge_{1\times\Sigma_n} K \wedge S^n \ \longrightarrow \ \mathrm{map}(S^m, K \wedge S^{m+n})$$

turns out to be given by

$$(\varphi^m)_{m+n}[\gamma \wedge k \wedge t](s) \ = \ \gamma \cdot (k \wedge s \wedge t) \ ,$$

where $\gamma \in \Sigma_{m+n}$, $k \in K$, $t \in S^n$ and $s \in S^m$. For any permutation $\sigma \in \Sigma_m$, the relations

$$\begin{aligned} (\varphi^m)_{m+n}\left([\gamma \wedge k \wedge t] \cdot \sigma\right)(s) \ &= \ (\varphi^m)_{m+n}[\gamma(\sigma + 1_n) \wedge k \wedge t](s) \ = \ \gamma(\sigma + 1_n) \cdot (k \wedge s \wedge t) \\ &= \ \gamma \cdot (k \wedge \sigma s \wedge t) \ = \ (\varphi^m)_{m+n}[\gamma \wedge k \wedge t](\sigma s) \end{aligned}$$

show that indeed $\varphi^m(z \cdot \sigma) = \varphi(z) \cdot \sigma$.

Example 4.36 (Stable homotopy type of semifree spectra)**.** We show that the semifree symmetric spectrum $G_m L$ generated by a based Σ_m-space L in level m is stably equivalent to a Σ_m-homotopy orbit spectrum of the m-fold loop of the suspension spectrum of L. To produce a stable equivalence we start from the map $\varphi^m : F_m L \longrightarrow \Omega^m(\Sigma^\infty L)$ of the previous Example 4.35. As we showed above, this map is equivariant for the two right actions of the symmetric group Σ_m (on the 'free coordinates' in the source (3.22) respectively the loop coordinates in the target). Since φ^m is also natural in L, it is also equivariant for the two left Σ_m actions in source and target induced from the left action on L.

Now we coequalize the left and right Σ_m-actions on source and target homotopically; more precisely we take homotopy orbits of the diagonal Σ_m-action (i.e., homotopy orbits of the equivalence relation $z \sim \sigma z \sigma^{-1}$) on both sides of the map φ^m. Proposition 4.31 (x) allows us to deduce a stable equivalence of homotopy orbit spectra

$$\varphi^m_{h\Sigma_m} \ : \ (F_m L)_{h\Sigma_m} \ \longrightarrow \ (\Omega^m(\Sigma^\infty L))_{h\Sigma_m} \ .$$

We claim that the diagonal action of Σ_m on the free spectrum $F_m L$ is levelwise free away from the basepoint. If the left action on L is trivial, this was shown in Example 3.20. In our more general situation, the argument is similar: on

$$(F_m L)_{m+n} \ = \ \Sigma^+_{m+n} \wedge_{1\times\Sigma_n} L \wedge S^n$$

a permutation $\sigma \in \Sigma_m$ acts diagonally by

$$\sigma \cdot [\gamma \wedge l \wedge t] \ = \ [\gamma(\sigma^{-1} + 1_n) \wedge \sigma l \wedge t] \ .$$

The symmetric group Σ_{m+n} is free as a right $\Sigma_m \times \Sigma_n$-set. So the underlying space of $(F_mL)_{m+n}$ is isomorphic to a wedge of $(m+n)!/n!$ copies of $L \wedge S^n$; in this decomposition, the diagonal action freely permutes the wedge summands (and acts on L). Hence the diagonal Σ_m-action on F_mL is levelwise free away from the basepoint, as claimed. Because the diagonal action is free, the natural map

$$(F_mL)_{h\Sigma_m} \longrightarrow (F_mL)/\Sigma_m$$

from homotopy orbits to strict orbits is a level equivalence. The strict orbit spectrum $(F_mL)/\Sigma_m$ (i.e., the result of coequalizing the left and right Σ_m-actions), finally, is isomorphic to the semifree spectrum G_mL, compare (3.24). So altogether we have obtained a chain of two natural stable equivalences

$$G_mL \longleftarrow (F_mL)_{h\Sigma_m} \xrightarrow{\varphi^m_{h\Sigma_m}} (\Omega^m(\Sigma^\infty L))_{h\Sigma_m}$$

linking the semifree spectrum G_mL to the homotopy orbit spectrum $(\Omega^m(\Sigma^\infty L))_{h\Sigma_m}$.

Example 4.37 (Stable homotopy type of twisted smash products)**.** We show that the twisted smash product$L \triangleright_m X$ of a based Σ_m-space L with a symmetric spectrum X is stably equivalent to a Σ_m-homotopy orbit spectrum of the m-fold loop of the (untwisted) smash product $L \wedge X$.

We start from the map

$$\bar{\lambda}_{K\wedge X} \;:\; \triangleright(K \wedge X) \longrightarrow \Omega(K \wedge X)$$

where K is a (non-equivariant) based space. This map is a stable equivalence by Proposition 4.28. We iterate this m times and arrive at the stable equivalence

$$\lambda^{[m]} \;:\; (\Sigma_m^+ \wedge K) \triangleright_m X \;=\; \underbrace{\triangleright(\cdots(\triangleright}_{m}(K \wedge X)\cdots) \longrightarrow \Omega^m(K \wedge X) \;.$$

We claim that $\lambda^{[m]}$ is equivariant for the two right actions of the symmetric group Σ_m, coming from the right translation action of Σ_m on itself in the source (3.22) and the left action on the loop coordinates in the target. [show...]

Now we consider the map $\lambda^{[m]}$ where we replace K by a based Σ_m-space L. As we saw above, $\lambda^{[m]}$ is equivariant for the right Σ_m-actions. Since $\lambda^{[m]}$ is natural in L, it is also equivariant for the two left Σ_m actions in source and target induced from the left action on L.

Now we coequalize the left and right Σ_m-actions on source and target homotopically; more precisely we take homotopy orbits of the diagonal Σ_m-action (i.e., homotopy orbits of the equivalence relation $z \sim \sigma z \sigma^{-1}$) on both sides of $\lambda^{[m]}$. Proposition 4.31 (x) allows us to deduce a stable equivalence of homotopy orbit spectra

$$\lambda^{[m]}_{h\Sigma_m} \;:\; ((\Sigma_m^+ \wedge L) \triangleright_m X)_{h\Sigma_m} \longrightarrow (\Omega^m(L \wedge X))_{h\Sigma_m} \;.$$

We claim that the diagonal action of Σ_m on the twisted smash product $(\Sigma_m^+ \wedge L) \triangleright_m X$ is levelwise free away from the basepoint. Indeed, on

$$(\Sigma_m^+ \wedge L) \triangleright_m X_{m+n} \;\cong\; \Sigma_{m+n}^+ \wedge_{1\times\Sigma_n} L \wedge X_n$$

a permutation $\sigma \in \Sigma_m$ acts diagonally by

$$\sigma \cdot [\gamma \wedge l \wedge x] \;=\; [\gamma(\sigma^{-1} + 1_n) \wedge \sigma l \wedge x] \;.$$

The symmetric group Σ_{m+n} is free as a right $\Sigma_m \times \Sigma_n$-set. So the underlying space of $(\Sigma_m^+ \wedge L) \triangleright_m X_{m+n}$ is isomorphic to a wedge of $(m+n)!/n!$ copies of $L \wedge X_n$; in this decomposition, the diagonal action freely permutes the wedge summands (and acts on L). Hence the diagonal Σ_m-action on $(\Sigma_m^+ \wedge L) \triangleright_m X_{m+n}$ is levelwise free away from the basepoint, as claimed. Because the diagonal action is free, the natural map

$$((\Sigma_m^+ \wedge L) \triangleright_m X)_{h\Sigma_m} \longrightarrow ((\Sigma_m^+ \wedge L) \triangleright_m X)/\Sigma_m$$

from homotopy orbits to strict orbits is a level equivalence. The strict orbit spectrum $((\Sigma_m^+ \wedge L) \triangleright_m X)/\Sigma_m$ (i.e., the result of coequalizing the left and right Σ_m-actions), finally, is isomorphic to the twisted smash product $L \triangleright_m X$. So altogether we have obtained a chain of two natural stable equivalences

$$L \triangleright_m X \longleftarrow ((\Sigma_m^+ \wedge L) \triangleright_m X)_{h\Sigma_m} \xrightarrow{\lambda^{[m]}_{h\Sigma_m}} (\Omega^m(L \wedge X))_{h\Sigma_m}$$

linking the twisted smash product $L \triangleright_m X$ to the homotopy orbit spectrum $(\Omega^m(L \wedge X))_{h\Sigma_m}$.

We close this section with a construction which shows that up to stable equivalence, every symmetric spectrum can be replaced by an Ω-spectrum. We construct a functor $Q : \mathcal{S}p \longrightarrow \mathcal{S}p$ with values in Ω-spectra together with a natural stable equivalence $\eta_A : A \longrightarrow QA$. The main construction is with symmetric spectra of spaces.

First we let $K_{m,n}$ be the mapping cone of the morphism $\lambda_m^n : F_{m+1}S^{n+1} \longrightarrow F_m S^n$ that is adjoint to the wedge summand inclusion $S^{n+1} \longrightarrow (F_m S^n)_{m+1} = \Sigma_{m+1}^+ \wedge S^n \wedge S^1$ indexed by the identity element. A morphism $f : K_{m,n} \longrightarrow X$ then corresponds to a morphism $\varphi : F_m S^n \longrightarrow X$ and a null-homotopy $H : [0,1] \wedge F_{m+1}S^{n+1} \longrightarrow X$ of the composite $\varphi \circ \lambda_m^n$. By the freeness property and adjointness of loop and suspension this data corresponds bijectively to a based map $\hat{\varphi} : S^n \longrightarrow X_m$ and a null-homotopy $\hat{H} : [0,1] \wedge S^n \longrightarrow \Omega X_{m+1}$ of the composite $\tilde{\sigma}_m \circ \hat{\varphi}$. If we also adjoin the interval coordinate, then the data altogether correponds to a based map $\hat{f} : S^n \longrightarrow F(\tilde{\sigma}_m : X_m \longrightarrow \Omega X_{m+1})$ to the homotopy fiber of the adjoint structure map. The same reasoning applies to homotopies between morphisms out of $K_{m,n}$, so we conclude that the map

$$[K_{m,n}, X] \longrightarrow \pi_n F(\tilde{\sigma}_m) \quad [f] \longmapsto [\hat{f}] \tag{4.38}$$

is a bijection from the set of homotopy classes of morphisms from $K_{m,n}$ to X to the n-th homotopy group of the homotopy fiber $F(\tilde{\sigma}_m)$ of the adjoint structure map $\tilde{\sigma}_m : X_m \longrightarrow \Omega X_{m+1}$.

Given a symmetric spectrum X we define GX by coning off all morphisms from $K_{m,n}$ to X for all $m, n \geq 0$:

$$GX \;=\; C(\mathrm{ev} \;:\; \bigvee_{m,n\geq 0} \; \bigvee_{f : K_{m,n} \longrightarrow X} K_{m,n} \;\longrightarrow\; X) \;.$$

We let $i_X : X \longrightarrow GX$ denote the inclusion into the cone. We then iterate this construction and let $G^\infty X$ denote the colimit

$$G^\infty X \;=\; \operatorname{colim}\left(X \xrightarrow{i_X} GX \xrightarrow{i_{GX}} G^2 X \xrightarrow{i_{G^2X}} \cdots \right) \;.$$

Finally, we set $QX = \Omega\,\mathrm{sh}(G^\infty X)$. Then Q is a functor that comes with a natural transformation $\eta_X : X \longrightarrow QX$ defined as the composite

$$X \;\longrightarrow\; G^\infty X \xrightarrow{\tilde{\lambda}_{G^\infty X}} \Omega\,\mathrm{sh}(G^\infty X) \;=\; QX \;,$$

where the first is the canonical map from X to the colimit.

To get a 'Q'-functor for symmetric spectra of simplicial sets we set $QX = \mathcal{S}(Q(|X|))$ when X is a symmetric spectrum of simplicial sets. The natural stable equivalence is then obtained as the composite

$$\eta_X \;:\; X \xrightarrow{\text{unit}} \mathcal{S}|X| \xrightarrow{\mathcal{S}(\eta_{|X|})} \mathcal{S}(Q(|X|)) \;.$$

Proposition 4.39. *For every symmetric spectrum X the spectrum QX is an Ω-spectrum and the morphism $\eta_X : X \longrightarrow QX$ is a stable equivalence.*

Proof. We start with the case of spectra of spaces. We observe first that every morphism $f : K_{m,n} \longrightarrow G^\infty X$ for any $m, n \geq 0$, is null-homotopic. Indeed, such a morphism factors through $G^k X$ for some finite number k, and the lift $\tilde{f} : K_{m,n} \longrightarrow G^k X$ is among the morphisms that are coned off to build $G^{k+1}X$. So the composite of $i_{G^kX} \circ \tilde{f} : K_{m,n} \longrightarrow G^{k+1}X$ is null-homotopic, hence so is the original map f.

Since every morphism from $K_{m,n}$ to $G^\infty X$ is null-homotopic, the bijection (4.38) shows that the homotopy group $\pi_n F(\tilde{\sigma}_m)$ of the homotopy fiber of the adjoint structure map $\tilde{\sigma}_m : (G^\infty X)_m \longrightarrow \Omega(G^\infty X)_{m+1}$ vanishes. This implies that the adjoint structure map $\tilde{\sigma}_m$ induces an injection on path components and an isomorphism of all homotopy groups of positive dimensions. This does not yet mean that $\tilde{\sigma}_m$ is a weak equivalence, but it implies that $\Omega\tilde{\sigma}_m : \Omega((G^\infty X)_m) \longrightarrow \Omega^2((G^\infty X)_{m+1})$ as a weak equivalence. Since $\Omega\tilde{\sigma}_m$ is the $(m-1)$-th adjoint structure map of the spectrum $\Omega\,\mathrm{sh}(G^\infty X)$, we have shown that $QX = \Omega\,\mathrm{sh}(G^\infty X)$ is an Ω-spectrum.

The morphism $\lambda^n_m : F_{m+1}S^{n+1} \longrightarrow F_mS^n$ is a stable equivalence [ref] so its mapping cone $K_{m,n}$ is stably contractible. Hence every wedge of copies of $K_{m,n}$, for varying m and n, is stably contractible. But then for every symmetric spectrum X the morphism $X \longrightarrow GX$ is a stable equivalence. All the morphisms in the colimits system defining $G^\infty X$ are simultaneously h-cofibrations and stable equivalences, so the canonical morphism $X \longrightarrow G^\infty X$ is a stable equivalence by [ref]. The morphism $\tilde{\lambda}_{G^\infty X}$ isa $\hat{\pi}_*$-isomorphism, thus a stable equivalence. So η_X is a stable equivalence.

The singular complex of the Ω-spectrum of spaces $Q|X|$ is an Ω-spectrum of simplicial sets. Every level equivalence is a stable equivalence and the singular complex functor preserves stable equivalences (by definition). Hence the case of simplicial sets follows from the case of spaces. □

Corollary 4.40. *There is an endofunctor $(Q-)^{\mathrm{inj}} : \mathcal{S}p_{\mathbf{sS}} \longrightarrow \mathcal{S}p_{\mathbf{sS}}$ on the category of symmetric spectra of simplicial sets and a natural stable equivalence $A \longrightarrow A^{\mathrm{inj}}$ such that $(QA)^{\mathrm{inj}}$ is an injective Ω-spectrum for all symmetric spectra of simplicial sets A.*

Proof. We define $(QA)^{\mathrm{inj}}$ as the composite of the functor Q of Proposition 4.39 and the injective replacement of Proposition 4.10. The morphism from A to QA is the composite

$$A \xrightarrow{\eta_A} QA \ \to \ (QA)^{\mathrm{inj}}$$

of the stable equivalence of Proposition 4.39 with the level equivalence (for the spectrum QA) of Proposition 4.10. □

5. Smash product

5.1. Construction of the smash product. One of the main features which distinguishes *symmetric* spectra from the more classical spectra without symmetric group actions is the internal smash product. The smash product of symmetric spectra is very much like the tensor product of modules over a commutative ring. To stress that analogy, we recall three different ways to look at the classical tensor product and then give analogies involving the smash product of symmetric spectra.

In the following, R is a commutative ring and M, N and W are right R-modules.

(A) Tensor product via bilinear maps. A bilinear map from M and N to another right R-module W is a map $b : M \times N \longrightarrow W$ such that for each $m \in M$ the map $b(m,-) : N \longrightarrow W$ is R-linear and for each $n \in N$ the map $b(-,n) : M \longrightarrow W$ is R-linear. The tensor product $M \otimes_R N$ is the universal example of a right R-module together with a bilinear map from $M \times N$. In other words, there is a specified bilinear map $i : M \times N \longrightarrow M \otimes_R N$ such that for every R-module W the map

$$\mathrm{Hom}_R(M \otimes_R N, W) \ \longrightarrow \ \mathrm{Bilin}_R(M \times N, W) \ , \quad f \ \mapsto \ f \circ i$$

is bijective. As usual, the universal property characterizes the pair $(M \otimes_R N, i)$ uniquely up to preferred isomorphism.

(B) Tensor product as an adjoint to internal Hom. The category of right R-modules has 'internal Hom-objects': the set $\mathrm{Hom}_R(N, W)$ of R-linear maps between two right R-modules N and W is naturally an R-module by pointwise addition and scalar multiplication. For fixed right R-modules M and N, the functor $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N,-)) : \text{mod-}R \longrightarrow \text{mod-}R$ is representable and tensor product $M \otimes_R N$ can be defined as a representing R-module. This point of view is closely related to the first approach since the R-modules $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, W))$ and $\mathrm{Bilin}_R(M \times N, W)$ are naturally isomorphic.

(C) Tensor product as a construction. Often the tensor product $M \otimes_R N$ is introduced as a specific construction, usually the following: $M \otimes_R N$ is the free R-module generated by symbols of the form $m \otimes n$ for all $m \in M$ and $n \in N$ subject to the following set of relations

- $(m + m') \otimes n \ = \ m \otimes n + m' \otimes n \ , \quad m \otimes (n + n') \ = \ m \otimes n + m \otimes n'$
- $(mr) \otimes n \ = \ (m \otimes n) \cdot r \ = \ m \otimes (nr)$

for all $m, m' \in M$, $n, n' \in N$ and $r \in R$. Since this is a minimal set of relations which make the map $M \times N \longrightarrow M \otimes_R N$ given by $(m, n) \mapsto m \otimes n$ into a bilinear map, the tensor product is constructed as to have the universal property (A).

Now we introduce the smash product of symmetric spectra in three ways, analogous to the ones in the algebraic context.

(A) Smash product via bilinear maps. We define a *bimorphism* $b : (X, Y) \longrightarrow Z$ from a pair of symmetric spectra (X, Y) to a symmetric spectrum Z as a collection of $\Sigma_p \times \Sigma_q$-equivariant maps of pointed spaces or simplicial sets, depending on the context,

$$b_{p,q} \ : \ X_p \wedge Y_q \ \longrightarrow \ Z_{p+q}$$

for $p, q \geq 0$, such that the 'bilinearity diagram'

$$\begin{tikzcd}
& X_p \wedge Y_q \wedge S^1 \arrow[r, "X_p\wedge \mathrm{twist}"] \arrow[dl, "X_p\wedge\sigma_q"'] \arrow[d, "b_{p,q}\wedge S^1"] & X_p \wedge S^1 \wedge Y_q \arrow[d, "\sigma_p\wedge Y_q"] \\
X_p \wedge Y_{q+1} \arrow[dr, "b_{p,q+1}"'] & Z_{p+q} \wedge S^1 \arrow[d, "\sigma_{p+q}"] & X_{p+1} \wedge Y_q \arrow[d, "b_{p+1,q}"] \\
& Z_{p+q+1} & Z_{p+1+q} \arrow[l, "1\times\chi_{1,q}"]
\end{tikzcd} \tag{5.1}$$

commutes for all $p, q \geq 0$. In Exercise E.I.15 we give a justification for the name 'bimorphism'.

We can then define a smash product of X and Y as a universal example of a symmetric spectrum with a bimorphism from X and Y. More precisely, a smash product for X and Y is a pair $(X \wedge Y, i)$ consisting of a symmetric spectrum $X \wedge Y$ and a universal bimorphism $i : (X, Y) \longrightarrow X \wedge Y$, i.e., a bimorphism such that for every symmetric spectrum Z the map

$$\mathcal{S}p(X \wedge Y, Z) \ \longrightarrow \ \mathrm{Bimor}((X, Y), Z) \ , \quad f \longmapsto fi = \{f_{p+q} \circ i_{p,q}\}_{p,q} \tag{5.2}$$

is bijective.

Example 5.3. As an example, and for later reference, we define a a bimorphism $j : (G_m L, Y) \longrightarrow L \triangleright_m Y$ out of a semifree symmetric spectrum $G_m L$ and an arbitrary symmetric spectrum Y to the twisted smash product and then show that this is in fact a universal bimorphism. The component of j of bidegree (p, q) is necessarily trivial for $p < m$. For $p = m + n$ the respective component

$$\begin{aligned} j_{m+n,q} \ : \ (G_m L)_{m+n} \wedge Y_q \ &= \ (\Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge S^n) \wedge Y_q \\ &\longrightarrow \ \Sigma^+_{m+n+q} \wedge_{\Sigma_m \times \Sigma_{n+q}} L \wedge Y_{n+q} \ = \ (L \triangleright_m Y)_{m+n+q} \end{aligned} \tag{5.4}$$

is the $(\Sigma_{m+n} \times \Sigma_q)$-equivariant extension of the $(\Sigma_m \times \Sigma_n \times \Sigma_q)$-equivariant composite

$$\begin{aligned} (L \wedge S^n) \wedge Y_q \ &\xrightarrow{L\wedge \mathrm{twist}} \ L \wedge Y_q \wedge S^n \ \xrightarrow{L\wedge\sigma^q} \ L \wedge Y_{q+n} \\ &\xrightarrow{L\wedge\chi_{q,n}} \ L \wedge Y_{n+q} \ \xrightarrow{[1\wedge -]} \ \Sigma^+_{m+n+q} \wedge_{\Sigma_m \times \Sigma_{n+q}} L \wedge Y_{n+q} \ . \end{aligned}$$

We omit the straightforward verification that the maps $j_{p,q}$ indeed form a bimorphism.

Proposition 5.5. *The bimorphism $j : (G_m L, Y) \longrightarrow L \triangleright_m Y$ is universal. Hence the pair $(L \triangleright_m Y, j)$ is a smash product of the semifree symmetric spectrum $G_m L$ and Y.*

Proof. We have to show that for every bimorphism $b : (G_m L, Y) \longrightarrow Z$ there is a unique homomorphism $f : L \triangleright_m Y \longrightarrow Z$ such that $b = f \circ j$. For uniqueness we note that f is necessarily trivial below level m and in level $m+q$ the map $f_{m+q} : \Sigma^+_{m+q} \wedge_{\Sigma_m \times \Sigma_q} L \wedge Y_q \longrightarrow Z_{m+q}$ is Σ_{m+q}-equivariant and hence determined by the composite with

$$j_{m,q} = [1 \wedge -] \ : \ L \wedge Y_q \ \longrightarrow \ \Sigma^+_{m+q} \wedge_{\Sigma_m \times \Sigma_q} L \wedge Y_q \ .$$

So there is at most one morphism $f : L \rhd_m Y \longrightarrow Z$ with $b = f \circ j$.

Conversely, we can construct a morphism $f : L \rhd_m Y \longrightarrow Z$ from a bimorphism $b : (G_m L, Y) \longrightarrow Z$ as follows. Again f is necessarily trivial below level m. For $q \geq 0$ we define

$$f_{m+q} \ : \ (L \rhd_m Z)_{m+q} \ = \ \Sigma^+_{m+q} \wedge_{\Sigma_m \times \Sigma_q} L \wedge Y_q \ \longrightarrow \ Z_{m+q}$$

as the Σ_{m+q}-equivariant extension of the is the $(\Sigma_m \times \Sigma_q)$-equivariant map $b_{m,q} : L \wedge Y_q \longrightarrow Z_{m+q}$. We omit the verification, which uses the bilinearity of b, that these maps really define a homomorphism of symmetric spectra. [$b = f \circ j$...] □

(B) Smash product as an adjoint to internal Hom. In Example 3.38 we introduced 'internal Hom objects' in the category of symmetric spectra. For every pair of symmetric spectra Y, Z we defined another symmetric spectrum $\mathrm{Hom}(Y, Z)$ such that the morphism from Y to Z are (in natural bijection with) the points respectively vertices of the 0th level of $\mathrm{Hom}(Y, Z)$. We claim that for fixed symmetric spectra X and Y, the set-valued functor $\mathcal{S}p(X, \mathrm{Hom}(Y, -))$ is representable; the smash product $X \wedge Y$ can then be defined as a representing symmetric spectrum. This point of view can be reduced to perspective (A) since the sets $\mathcal{S}p(X, \mathrm{Hom}(Y, Z))$ and $\mathrm{Bimor}((X, Y), Z)$ are in natural bijection (see Exercise E.I.15). In particular, since the functor $\mathrm{Bimor}((X, Y), -)$ is representable, so is the functor $\mathcal{S}p(X, \mathrm{Hom}(Y, -))$.

(C) Smash product as a construction. Now we construct a symmetric spectrum $X \wedge Y$ from two given symmetric spectra X and Y. We want $X \wedge Y$ to be the universal recipient of a bimorphism from (X, Y), and this pretty much tells us what we have to do. For $n \geq 0$ we define the nth level $(X \wedge Y)_n$ as the coequalizer, in the category of pointed Σ_n-spaces or pointed Σ_n-simplicial sets (depending on the context), of two maps

$$\alpha_X, \, \alpha_Y \ : \ \bigvee_{p+1+q=n} \Sigma^+_n \wedge_{\Sigma_p \times \Sigma_1 \times \Sigma_q} X_p \wedge S^1 \wedge Y_q \ \longrightarrow \ \bigvee_{p+q=n} \Sigma^+_n \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \ .$$

The wedges run over all non-negative values of p and q which satisfy the indicated relations. The map α_X takes the wedge summand indexed by $(p, 1, q)$ to the wedge summand indexed by $(p+1, q)$ using the map

$$\sigma^X_p \wedge \mathrm{Id} \ : \ X_p \wedge S^1 \wedge Y_q \ \longrightarrow \ X_{p+1} \wedge Y_q$$

and inducing up. The other map α_Y takes the wedge summand indexed by $(p, 1, q)$ to the wedge summand indexed by $(p, 1+q)$ using the composite

$$X_p \wedge S^1 \wedge Y_q \xrightarrow{\mathrm{Id} \wedge \mathrm{twist}} X_p \wedge Y_q \wedge S^1 \xrightarrow{\mathrm{Id} \wedge \sigma^Y_q} X_p \wedge Y_{q+1} \xrightarrow{\mathrm{Id} \wedge \chi_{q,1}} X_p \wedge Y_{1+q}$$

and inducing up.

The structure map $(X \wedge Y)_n \wedge S^1 \longrightarrow (X \wedge Y)_{n+1}$ is induced on coequalizers by the wedge of the maps

$$\Sigma^+_n \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \wedge S^1 \ \longrightarrow \ \Sigma^+_{n+1} \wedge_{\Sigma_p \times \Sigma_{q+1}} X_p \wedge Y_{q+1}$$

induced from $\mathrm{Id} \wedge \sigma^Y_q : X_p \wedge Y_q \wedge S^1 \longrightarrow X_p \wedge Y_{q+1}$. One should check that this indeed passes to a well-defined map on coequalizers. Equivalently we could have defined the structure map by moving the circle past Y_q, using the structure map of X (instead of that of Y) and then shuffling back with the permutation $\chi_{1,q}$; the definition of $(X \wedge Y)_{n+1}$ as a coequalizer precisely ensures that these two possible structure maps coincide, and that the collection of maps

$$X_p \wedge Y_q \xrightarrow{x \wedge y \mapsto 1 \wedge x \wedge y} \bigvee_{p+q=n} \Sigma^+_n \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \xrightarrow{\text{projection}} (X \wedge Y)_{p+q}$$

forms a bimorphism – and in fact a universal one.

Construction 5.6. We define the smash product $X \wedge Y$ as the universal example of a symmetric spectrum with a bimorphism from X and Y. More precisely, for every pair of symmetric spectra (X, Y) we *choose* a symmetric spectrum $X \wedge Y$ and a universal bimorphism $i : (X, Y) \longrightarrow X \wedge Y$, i.e., a bimorphism such that for every symmetric spectrum Z the map

$$\mathcal{S}p(X \wedge Y, Z) \longrightarrow \mathrm{Bimor}((X, Y), Z) , \quad f \longmapsto fi = \{f_{p+q} \circ i_{p,q}\}_{p,q}$$

is bijective. Such a universal pair $(X \wedge Y, i)$ always exists, as the coequalizer construction in (C) above shows.

It will be convenient later to make the sphere spectrum $\mathbb{S}$ into a strict unit for the smash product (as opposed to a unit up to coherent isomorphisms). So we make the following **conventions:**

- (Right unit) We choose $X \wedge \mathbb{S} = X$ with universal bimorphism $i : (X, \mathbb{S}) \longrightarrow X$ given by the iterated structure map,
$$i_{p,q} = \sigma^q \ : \ X_p \wedge S^q \longrightarrow X_{p+q} .$$
- (Left unit) We choose $\mathbb{S} \wedge Y = Y$ with universal bimorphism $i : (\mathbb{S}, Y) \longrightarrow Y$ given by the composite
$$i_{p,q} \ : \ S^p \wedge Y_q \xrightarrow{\text{twist}} Y_q \wedge S^p \xrightarrow{\sigma^p} Y_{q+p} \xrightarrow{\chi_{q,p}} Y_{p+q} .$$

We have to observe a couple of things for this to make sense. First, the iterated structure maps and the composites [...] are indeed universal bimorphisms [...] Second, the two conventions are consistent for the sphere spectrum because in both definitions, the universal bimorphism has the canonical isomorphism $S^n \wedge S^m \cong S^{n+m}$ as its (n, m)-component.

Very often only the object $X \wedge Y$ will be referred to as the smash product, but one should keep in mind that it comes equipped with a specific, universal bimorphism. We will often refer to the bijection (5.2) as the *universal property* of the smash product of symmetric spectra.

The smash product $X \wedge Y$ is a functor in both variables. This is fairly evident from the construction (C), but since we defined the smash product via the universal property (A), we have to explain functoriality in this context. If we use the universal property (A) the contravariant functoriality of the set $\mathrm{Bimor}((X, Y), Z)$ in X and Y turns into functoriality of the representing objects. In more detail, if $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are morphisms of symmetric spectra, then the collection of pointed maps

$$\left\{ X_p \wedge \ Y_q \xrightarrow{f_p \wedge g_q} X'_p \wedge \ Y'_q \xrightarrow{i'_{p,q}} (X' \wedge Y')_{p+q} \right\}_{p,q \geq 0}$$

forms a bimorphism $(X, Y) \longrightarrow X' \wedge Y'$. So there is a unique morphism of symmetric spectra $f \wedge g : X \wedge Y \longrightarrow X' \wedge Y'$ such that $(f \wedge g)_{p+q} \circ i_{p,q} = i'_{p,q} \circ (f_p \wedge g_q)$ for all $p, q \geq 0$. The uniqueness part of the universal property implies that this is compatible with identities and composition in both variables.

If we define the smash product as a representing object for the functor $\mathrm{Hom}(X, \mathrm{Hom}(Y, -))$, then functoriality in X and Y follows from functoriality of the latter functor in X and Y.

5.2. Coherence isomorphisms. Now that we have constructed a smash product functor we can investigate its formal and homotopical properties. We start with the formal properties now; the homotopical analysis will be taken up in Section 5.5.

The first thing to show is that the smash product is symmetric monoidal. Since 'symmetric monoidal' is extra data, and not a property, we are obliged to construct associativity isomorphisms

$$\alpha_{X,Y,Z} \ : \ (X \wedge Y) \wedge Z \longrightarrow X \wedge (Y \wedge Z) ,$$

symmetry isomorphisms

$$\tau_{X,Y} \ : \ X \wedge Y \longrightarrow Y \wedge X .$$

We have arranged things so that the sphere spectrum $\mathbb{S}$ is a strict unit, so the left and right unit isomorphisms which are part of a symmetric monoidal structure are the identity maps and don't have to be explicitly mentioned.

First construction. We can obtain all the isomorphisms of the symmetric monoidal structure just from the universal property. In the construction of the smash product, we had chosen, for each pair of symmetric spectra (X, Y), a smash product $X \wedge Y$ and a universal bimorphism $i = \{i_{p,q}\} : (X, Y) \longrightarrow X \wedge Y$.

For construction the associativity isomorphism we notice that the family

$$\left\{ X_p \wedge\ Y_q \wedge Z_r \xrightarrow{i_{p,q} \wedge Z_r} (X \wedge Y)_{p+q} \wedge Z_r \xrightarrow{i_{p+q,r}} ((X \wedge Y) \wedge Z)_{p+q+r} \right\}_{p,q,r \geq 0}$$

and the family

$$\left\{ X_p \wedge\ Y_q \wedge Z_r \xrightarrow{X_p \wedge i_{q,r}} X_p \wedge (Y \wedge Z)_{q+r} \xrightarrow{i_{p,q+r}} (X \wedge (Y \wedge Z))_{p+q+r} \right\}_{p,q,r \geq 0}$$

both have the universal property of a *tri*morphism (whose definition is hopefully clear) out of X, Y and Z. The uniqueness of representing objects gives a unique isomorphism of symmetric spectra

$$\alpha_{X,Y,Z} \ : \ (X \wedge Y) \wedge Z \ \cong X \wedge (Y \wedge Z)$$

such that $(\alpha_{X,Y,Z})_{p+q+r} \circ i_{p+q,r} \circ (i_{p,q} \wedge Z_r) = i_{p,q+r} \circ (X_p \wedge i_{q,r})$.

The symmetry isomorphism $\tau_{X,Y} : X \wedge Y \longrightarrow Y \wedge X$ corresponds to the bimorphism

$$\left\{ X_p \wedge\ Y_q \xrightarrow{\text{twist}} Y_q \wedge\ X_p \xrightarrow{i_{q,p}} (Y \wedge X)_{q+p} \xrightarrow{\chi_{q,p}} (Y \wedge X)_{p+q} \right\}_{p,q \geq 0} . \tag{5.7}$$

The block permutation $\chi_{q,p}$ is crucial here: without it the diagram (5.1) would not commute and we would not have a bimorphism. If we restrict the composite $\tau_{Y,X} \circ \tau_{X,Y}$ in level $p + q$ along the map $i_{p,q} : X_p \wedge Y_q \longrightarrow (X \wedge Y)_{p+q}$ we get $i_{p,q}$ again. Thus $\tau_{Y,X} \circ \tau_{X,Y} = \mathrm{Id}_{X \wedge Y}$ and $\tau_{Y,X}$ is inverse to $\tau_{X,Y}$.

Second construction. The coherence isomorphisms can also be obtained from the construction of the smash product in (C) above, as opposed to the universal property. In level n the spectra $(X \wedge Y) \wedge Z$ and $X \wedge (Y \wedge Z)$ are quotients of the spaces

$$\bigvee_{p+q+r=n} \Sigma_n^+ \wedge_{\Sigma_{p+q} \times \Sigma_r} \left(\Sigma_{p+q}^+ \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \right) \wedge Z_r$$

respectively

$$\bigvee_{p+q+r=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_{q+r}} X_p \wedge \left(\Sigma_{q+r}^+ \wedge_{\Sigma_q \times \Sigma_r} Y_q \wedge Z_r \right) \ .$$

The wedges run over all non-negative values of p, q and r which sum up to n. We get a well-defined map between these two wedges by wedging over the maps

$$\begin{array}{ccc} \Sigma_n^+ \wedge_{\Sigma_{p+q} \times \Sigma_r} \left(\Sigma_{p+q}^+ \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \right) \wedge Z_r & \longleftrightarrow & \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_{q+r}} X_p \wedge \left(\Sigma_{q+r}^+ \wedge_{\Sigma_q \times \Sigma_r} Y_q \wedge Z_r \right) \\ \sigma \wedge ((\tau \wedge x \wedge y) \wedge z) & \longmapsto & (\sigma(\tau \times 1)) \wedge (x \wedge (1 \wedge y \wedge z)) \\ \sigma(1 \times \gamma) \wedge ((1 \wedge x \wedge y) \wedge z) & \longleftarrow & \sigma \wedge (x \wedge (\gamma \wedge y \wedge z)) \end{array}$$

where $\sigma \in \Sigma_n$, $\tau \in \Sigma_{p+q}$, $\gamma \in \Sigma_{q+r}$, $x \in X_p$, $y \in Y_q$ and $z \in Z_r$.

The symmetry isomorphism $\tau_{X,Y} : X \wedge X \longrightarrow Y \wedge X$ is obtained by wedging over the maps

$$\begin{array}{ccc} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q & \longrightarrow & \Sigma_n^+ \wedge_{\Sigma_q \times \Sigma_p} Y_q \wedge X_p \\ \sigma \wedge x \wedge y & \longmapsto & (\sigma \chi_{q,p}) \wedge y \wedge x \end{array} \tag{5.8}$$

where $\sigma \in \Sigma_n$, $x \in X_p$ and $y \in Y_q$ and passing to quotient spaces. The shuffle permutation $\chi_{q,p}$ is needed to make this map well-defined on quotients.

Remark 5.9. We'll try to motivate the shuffle permutation $\chi_{q,p}$ in the definition (5.8) of the twist isomorphism; this is more natural in the 'coordinate free' description of the smash product $X \wedge Y$ (we refer to Exercise E.I.5 to a discussion of coordinate free symmetric spectra). If A is any finite set, then the value of $X \wedge Y$ at A is the appropriate coequalizer of wedges of terms

$$\mathrm{Bij}(\mathbf{p}+\mathbf{q}, A)^+ \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q$$

where $\mathrm{Bij}(\mathbf{p}+\mathbf{q}, A)$ is the set of bijections from the disjoint union of $\mathbf{p}$ and $\mathbf{q}$ to A. The set $\mathrm{Bij}(\mathbf{p}+\mathbf{q}, A)$ is naturally acted upon by $\Sigma_A = \mathrm{Bij}(A, A)$ from the left and by $\Sigma_p \times \Sigma_q$ from the right. Similarly, $(Y \wedge X)_A$ arises from

$$\mathrm{Bij}(\mathbf{q}+\mathbf{p}, A)^+ \wedge_{\Sigma_q \times \Sigma_p} Y_q \wedge X_p \ .$$

Given a bijection $\sigma : \mathbf{p}+\mathbf{q} \longrightarrow A$, we obtain a bijection $\sigma\chi_{q,p} : \mathbf{q}+\mathbf{p} \longrightarrow A$ by precomposing σ with the symmetry isomorphism $\chi_{q,p} : \mathbf{q}+\mathbf{p} \longrightarrow \mathbf{p}+\mathbf{q}$; this explains the assignment (5.8).

Theorem 5.10. *The associativity and symmetry isomorphisms make the smash product of symmetric spectra into a symmetric monoidal product with the sphere spectrum $\mathbb{S}$ as a strict unit object. This product is* closed *symmetric monoidal in the sense that the smash product is adjoint to the internal Hom spectrum, i.e., there is an adjunction isomorphism*

$$\mathrm{Hom}(X \wedge Y, Z) \ \cong \ \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) \ .$$

Proof. We have to verify that several coherence diagrams commute. We start with the pentagon condition for associativity. Given a fourth symmetric spectrum W we consider the pentagon

$$\begin{tikzcd}
 & ((W \wedge X) \wedge Y) \wedge Z \arrow[dl, "\alpha_{W,X,Y} \wedge Z"'] \arrow[dr, "\alpha_{W \wedge X,Y,Z}"] & \\
(W \wedge (X \wedge Y)) \wedge Z \arrow[d, "\alpha_{W,X \wedge Y,Z}"'] & & (W \wedge X) \wedge (Y \wedge Z) \arrow[d, "\alpha_{W,X,Y \wedge Z}"] \\
W \wedge ((X \wedge Y) \wedge Z) \arrow[rr, "W \wedge \alpha_{X,Y,Z}"'] & & W \wedge (X \wedge (Y \wedge Z))
\end{tikzcd}$$

If we evaluate either composite at level $o+p+q+r$ and precompose with

$$\begin{aligned} W_o \wedge X_p \wedge Y_q \wedge Z_r \ &\xrightarrow{i_{o,p} \wedge Y_q \wedge Z_r} (W \wedge X)_{o+p} \wedge Y_q \wedge Z_r \\ &\xrightarrow{i_{o+p,q} \wedge Z_r} ((W \wedge X) \wedge Y)_{o+p+q} \wedge Z_r \ \xrightarrow{i_{o+p+q,r}} (((W \wedge X) \wedge Y) \wedge Z)_{o+p+q+r} \end{aligned}$$

then both ways around the pentagon yield the composite

$$\begin{aligned} W_o \wedge X_p \wedge Y_q \wedge Z_r \ &\xrightarrow{W_o \wedge X_p \wedge i_{q,r}} W_o \wedge X_p \wedge (Y \wedge Z)_{q+r} \\ &\xrightarrow{W_0 \wedge i_{p,q+r}} W_o \wedge (X \wedge (Y \wedge Z))_{p+q+r} \ \xrightarrow{i_{o,p+q+r}} (W \wedge (X \wedge (Y \wedge Z)))_{o+p+q+r} \ . \end{aligned}$$

So the uniqueness part of the universal property shows that the pentagon commutes.

Coherence between associativity and symmetry isomorphisms means that the two composites from the upper left to the lower right corner of the diagram

(5.11)
$$\begin{tikzcd}
 & (X \wedge Y) \wedge Z \arrow[dl, "\tau_{X,Y} \wedge Z"'] \arrow[dr, "\alpha_{X,Y,Z}"] & \\
(Y \wedge X) \wedge Z \arrow[d, "\alpha_{Y,X,Z}"'] & & X \wedge (Y \wedge Z) \arrow[d, "\tau_{X,Y \wedge Z}"] \\
Y \wedge (X \wedge Z) \arrow[dr, "Y \wedge \tau_{X,Z}"'] & & (Y \wedge Z) \wedge X \arrow[dl, "\alpha_{Y,Z,X}"] \\
 & Y \wedge (Z \wedge X) &
\end{tikzcd}$$

should be equal, and the same kind of argument as for the pentagon relation for associativity works.

If we have a non-strict unit object with left and right unit isomorphisms, there would be more coherence conditions relating associativity and symmetry isomorphisms to the unit morphisms. Since we made the sphere spectrum a strict unit, these conditions are replaced by the following observations. The associativity isomorphisms

$$\begin{aligned}
\alpha_{\mathbb{S},Y,Z} &: Y \wedge Z = (\mathbb{S} \wedge Y) \wedge Z \longrightarrow \mathbb{S} \wedge (Y \wedge Z) = Y \wedge Z \ , \\
\alpha_{X,\mathbb{S},Z} &: X \wedge Z = (X \wedge \mathbb{S}) \wedge Z \longrightarrow X \wedge (\mathbb{S} \wedge Z) = X \wedge Z \quad \text{and} \\
\alpha_{X,Y,\mathbb{S}} &: X \wedge Y = (X \wedge Y) \wedge \mathbb{S} \longrightarrow X \wedge (Y \wedge \mathbb{S}) = X \wedge Y
\end{aligned}$$

are the identity morphisms. The symmetry isomorphisms

$$\tau_{\mathbb{S},Y} \ : \ Y = \mathbb{S} \wedge Y \longrightarrow Y \wedge \mathbb{S} = Y \qquad \text{and} \qquad \tau_{X,\mathbb{S}} \ : \ X = X \wedge \mathbb{S} \longrightarrow \mathbb{S} \wedge X = X$$

are the identity morphisms. [justify] [closed monoidal...] □

Remark 5.12. (Permutation action of Σ_n) In every symmetric monoidal category the symmetric group Σ_n acts naturally on the n-th smash power of any object by 'permuting the factors'. Suppose first that the monoidal product $\Box$ happens to be strictly associative, i.e., $(X\Box Y)\Box Z = X\Box(Y\Box Z)$ as functors in three variables, and the associativity isomorphism $\alpha_{X,Y,Z}$ is the identity. Then we can omit parentheses altogether and define an action of Σ_n on $X^{(n)} = X\Box\cdots\Box X$ (n factors) by letting the transposition $(i, i+1)$ act as the automorphism $t_i = X^{(i-1)}\Box\tau_{X,X}\Box X^{(n-1-i)}$ (for $1 \leq i \leq n-1$). Then t_i^2 is the identity for all i (since $\tau_{X,X} = \tau_{X,X}^{-1}$), and t_i and t_j commute for $|i-j| \geq 2$ (since the monoidal product is a functor in two variables). The hexagon relation (5.11) specializes to

$$t_{i+1}t_i \ = \ X^{(i-1)}\Box\tau_{X,X\Box X}\Box X^{(n-2-i)} \ ,$$

and from this we deduce

$$\begin{aligned}
t_i t_{i+1} t_i \ &= \ (X^{(i-1)}\Box\tau_{X,X}\Box X^{(n-1-i)}) \circ (X^{(i-1)}\Box\tau_{X,X\Box X}\Box X^{(n-2-i)}) \\
&= \ X^{(i-1)}\Box((\tau_{X,X}\Box X) \circ \tau_{X,X\Box X})\Box X^{(n-2-i)} \\
&= \ X^{(i-1)}\Box(\tau_{X,X\Box X} \circ (X\Box\tau_{X,X}))\Box X^{(n-2-i)} \\
&= \ (X^{(i-1)}\Box\tau_{X,X\Box X}\Box X^{(n-2-i)}) \circ (X^{(i)}\Box\tau_{X,X}\Box X^{(n-2-i)}) \\
&= t_{i+1}t_it_{i+1} \ .
\end{aligned}$$

The third equation is the naturality of the symmetry isomorphism. In general, however, the monoidal product is not strictly associative, and if we want to be completely honest, we have to choose a way of parenthesising the factors in $X^{(n)}$. One can still define an action of Σ_n on $X^{(n)}$ in a way that generalizes the above construction, but we delegate the details to Exercise E.I.19.

5.3. Smash product and various other constructions. Now we identify the smash product of certain kinds of symmetric spectra and relate it by natural maps to other constructions. We start by comparing the smash product with a semifree spectrum $G_m L$ with the twisted smash product $L \triangleright_m -$, where L is a based Σ_m-space (or Σ_m-simplicial set). In (5.4) we introduced a bimorphism $j : (G_m L, Y) \longrightarrow L \triangleright_m Y$ and showed in Proposition 5.5 that this bimorphism is universal. If $(G_m L \wedge Y, i)$ is the chosen smash product of $G_m L$ and Y, then universality of i provides a unique morphism $\kappa : G_m L \wedge Y \longrightarrow L \triangleright_m Y$ of symmetric spectra such that $\kappa i = j$. Since i and j are both universal bimorphisms out of $(G_m L, Y)$, the morphism κ induces a bijection on morphisms into any symmetric spectrum. This shows

Proposition 5.13. *Let L be a pointed Σ_m-space (or Σ_m-simplicial set) for some $m \geq 0$ and Y a symmetric spectrum. Then the unique morphism of symmetric spectra*

$$\kappa \ : \ G_m L \wedge Y \ \longrightarrow \ L \triangleright_m Y$$

which satisfies $\kappa i = j$ is a natural isomorphism.

We specialize the previous proposition in several steps. The special case $m = 0$ provides a natural isomorphism

$$K \wedge X \ = \ K \triangleright_0 X \ \cong \ (\Sigma^\infty K) \wedge X$$

for pointed spaces (or simplicial sets) K and symmetric spectra X. We can also consider a Σ_m-space L and a Σ_n-space L'. If we spell out all definitions we see that $L \triangleright_m (G_n L')$ is isomorphic to the semifree symmetric spectrum $G_{m+n}(\Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge L')$. So Proposition 5.13 specializes to a natural isomorphism

$$G_{m+n}(\Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge L') \ \cong \ G_m L \wedge G_n L' \ . \tag{5.14}$$

The isomorphism is adjoint to the Σ_{m+n}-equivariant map

$$\Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge L' \ \longrightarrow \ (G_m L \wedge G_n L')_{m+n} \ ,$$

which in turn is adjoint to the $\Sigma_m \times \Sigma_n$-equivariant map

$$L \wedge L' = (G_m L)_m \wedge (G_n L')_n \ \xrightarrow{i_{m,n}} \ (G_m L \wedge G_n L')_{m+n}$$

given by the universal bimorphism. So the isomorphism (5.14) rephrases the fact that a bimorphism from $(G_m L, G_n L')$ to Z is uniquely determined by its (m,n)-component, which can be any $\Sigma_m \times \Sigma_n$-equivariant map $L \wedge L' \longrightarrow Z_{m+n}$. The isomorphism (5.14), and the ones which follow below, are suitably associative, commutative and unital.

As a special case we can consider smash products of free symmetric spectra. If K and K' are pointed spaces or simplicial sets then we have $F_m K = G_m(\Sigma^+_m \wedge K)$ and $F_n K' = G_n(\Sigma^+_n \wedge K')$, so the isomorphism (5.14) specializes to an associative, commutative and unital isomorphism

$$F_{m+n}(K \wedge K') \ \cong \ F_m K \wedge F_n K' \ .$$

As the even more special case for $m = n = 0$ we obtain a natural isomorphism of suspension spectra

$$(\Sigma^\infty K) \ \wedge \ (\Sigma^\infty L) \ \cong \ \Sigma^\infty (K \wedge L)$$

for all pairs of pointed spaces (or pointed simplicial sets) K and L. [$\Sigma_m \times \Sigma_n$-equivariant!]

We define a natural, $\Sigma_n \times \Sigma_m$-equivariant morphism

$$\xi \ : \ \mathrm{sh}^n X \ \wedge \ \mathrm{sh}^m Y \ \longrightarrow \ \mathrm{sh}^{n+m}(X \wedge Y) \tag{5.15}$$

which interrelates smash product and shifts of symmetric spectra. By definition, the morphism ξ corresponds, under the universal property of the smash product, to the bimorphism with (p,q)-component the composite

$$\begin{aligned} (\mathrm{sh}^n X)_p \wedge (\mathrm{sh}^m Y)_q \ &= \ X_{n+p} \wedge Y_{m+q} \ \xrightarrow{i_{n+p,m+q}} \ (X \wedge Y)_{n+p+m+q} \\ &\xrightarrow{1_n + \chi_{p,m} + 1_q} \ (X \wedge Y)_{n+m+p+q} \ = \ (\mathrm{sh}^{n+m}(X \wedge Y))_{p+q} \ . \end{aligned}$$

If we want to emphasize the number of shifts or the spectra involved, we may decorate the map ξ with indices, as in $\xi^{n,m}_{X,Y}$. The homomorphism ξ is unital and compatible with the associativity and commutativity isomorphisms of the smash product, in the sense that a couple of diagrams commute. For example, unitality refers to the commuting diagrams:

$$\begin{tikzcd}
\mathrm{sh}^n X \wedge \mathrm{sh}^m \mathbb{S} \arrow[r, "\cong"] \arrow[d, "\xi_{X,\mathbb{S}}"'] & \mathrm{sh}^n X \wedge (S^m \wedge \mathbb{S}) \arrow[r, "\cong"] & S^m \wedge \mathrm{sh}^n X \arrow[d, "\lambda^{(m)}_{\mathrm{sh}^n X}"] \\
\mathrm{sh}^{n+m}(X \wedge \mathbb{S}) \arrow[r, equal] & \mathrm{sh}^{n+m} X \arrow[r, equal] & \mathrm{sh}^m(\mathrm{sh}^n X)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{sh}^n \mathbb{S} \wedge \mathrm{sh}^m Y \arrow[r, "\cong"] \arrow[d, "\xi_{\mathbb{S},Y}"'] & (S^n \wedge \mathbb{S}) \wedge \mathrm{sh}^m Y \arrow[r, "\cong"] & S^n \wedge \mathrm{sh}^m Y \arrow[d, "\lambda^{(n)}_{\mathrm{sh}^m Y}"] \\
\mathrm{sh}^{n+m}(\mathbb{S} \wedge Y) \arrow[r, equal] & \mathrm{sh}^{n+m} Y \arrow[r, "\chi_{n,m}"'] & \mathrm{sh}^n(\mathrm{sh}^m Y)
\end{tikzcd}$$

Here we used the identification $\mathrm{sh}^n \mathbb{S} \cong S^n \wedge \mathbb{S}$ via the canonical isomorphisms $S^n \wedge S^m \cong S^{n+m}$. The diagrams may be interpreted as saying that, up to the indicated identifications, the maps $\xi_{X,\mathbb{S}}$ and $\xi_{\mathbb{S},Y}$ 'are' the morphism $\lambda^{(m)}_X : S^m \wedge X \longrightarrow \mathrm{sh}^m X$ respectively $\lambda_Y : S^1 \wedge Y \longrightarrow \mathrm{sh}\, Y$.

The associativity property is expressed in the commuting diagram:

$$\begin{tikzcd}
(\mathrm{sh}^n X \wedge \mathrm{sh}^m Y) \wedge \mathrm{sh}^k Z \arrow[r, "\alpha_{\mathrm{sh}^n X, \mathrm{sh}^m Y, \mathrm{sh}^k Z}"] \arrow[d, "\xi_{X,Y} \wedge \mathrm{Id}"'] & \mathrm{sh}^n X \wedge (\mathrm{sh}^m Y \wedge \mathrm{sh}^k Z) \arrow[d, "\mathrm{Id} \wedge \xi_{Y,Z}"] \\
\mathrm{sh}^{n+m}(X \wedge Y) \wedge \mathrm{sh}^k Z \arrow[d, "\xi_{X \wedge Y, Z}"'] & \mathrm{sh}^n X \wedge \mathrm{sh}^{m+k}(Y \wedge Z) \arrow[d, "\xi_{X, Y \wedge Z}"] \\
\mathrm{sh}^{n+m+k}((X \wedge Y) \wedge Z) \arrow[r, "\mathrm{sh}^{n+m+k}(\alpha_{X,Y,Z})"'] & \mathrm{sh}^{n+m+k}(X \wedge (Y \wedge Z))
\end{tikzcd}$$

Commutativity refers to the commuting diagrams:

$$\begin{tikzcd}
\mathrm{sh}^n X \wedge \mathrm{sh}^m Y \arrow[rr, "\tau_{\mathrm{sh}^n X, \mathrm{sh}^m Y}"] \arrow[d, "\xi^{n,m}_{X,Y}"'] & & \mathrm{sh}^m Y \wedge \mathrm{sh}^n Y \arrow[d, "\xi^{m,n}_{Y,X}"] \\
\mathrm{sh}^{n+m}(X \wedge Y) \arrow[r, "\mathrm{sh}^{n+m}(\tau_{X,Y})"'] & \mathrm{sh}^{n+m}(Y \wedge X) \arrow[r, "\chi_{n,m}"'] & \mathrm{sh}^{m+n}(Y \wedge X)
\end{tikzcd}$$

As a special case of the above, or by direct verification from the definitions, we obtain that the square

$$\begin{tikzcd}
(S^n \wedge X) \wedge (S^m \wedge Y) \arrow[r, "\cong"] \arrow[d, "\lambda^{(n)}_X \wedge \lambda^{(m)}_Y"'] & S^{n+m} \wedge (X \wedge Y) \arrow[d, "\lambda^{(n+m)}_{X \wedge Y}"] \\
\mathrm{sh}^n X \wedge \mathrm{sh}^m Y \arrow[r, "\xi_{X,Y}"'] & \mathrm{sh}^{n+m}(X \wedge Y)
\end{tikzcd} \tag{5.16}$$

commutes. There is yet another commuting diagram, saying that the map ξ itself is in a sense associative:

$$\begin{tikzcd}
\mathrm{sh}^k(\mathrm{sh}^n X) \wedge \mathrm{sh}^l(\mathrm{sh}^m Y) \arrow[r, "\xi_{\mathrm{sh}^n X, \wedge^m Y}"] \arrow[d, equal] & \mathrm{sh}^{k+l}(\mathrm{sh}^n X \wedge \mathrm{sh}^m Y) \arrow[r, "\mathrm{sh}^{k+l} \xi_{X,Y}"] & \mathrm{sh}^{k+l}(\mathrm{sh}^{n+m}(X \wedge Y)) \arrow[d, equal] \\
\mathrm{sh}^{n+k} X \wedge \mathrm{sh}^{m+l} Y \arrow[r, "\xi_{X,Y}"'] & \mathrm{sh}^{n+k+m+l}(X \wedge Y) \arrow[r, "1_n + \chi_{k,m} + 1_l"'] & \mathrm{sh}^{n+m+k+l}(X \wedge Y)
\end{tikzcd}$$

We can use the morphisms $\xi^{1,0}_{X,Y}$ and $\xi^{0,1}_{X,Y}$ to express the shift of a smash product $X \wedge Y$ as an amalgamated union of the spectra $(\operatorname{sh} X) \wedge Y$ and $X \wedge (\operatorname{sh} Y)$. By direct verification from the definitions we see that the diagram

$$\begin{tikzcd}
(S^1 \wedge X) \wedge Y \arrow[r, "\text{twist}"] \arrow[d, "\lambda_X \wedge \mathrm{Id}"'] & X \wedge (S^1 \wedge Y) \arrow[r, "\mathrm{Id} \wedge \lambda_Y"] & X \wedge (\operatorname{sh} Y) \arrow[d, "\xi^{0,1}_{X,Y}"] \\
(\operatorname{sh} X) \wedge Y \arrow[rr, "\xi^{1,0}_{X,Y}"'] & & \operatorname{sh}(X \wedge Y)
\end{tikzcd} \tag{5.17}$$

commutes. Moreover, up to the associativity isomorphism, the composite morphism from the initial to the terminal vertex agrees with the morphism $\lambda_{X \wedge Y} : S^1 \wedge (X \wedge Y) \longrightarrow \operatorname{sh}(X \wedge Y)$.

Proposition 5.18. *For every pair of symmetric spectra X and Y the square* (5.17) *is a pushout.*

Proof. We fix Y and prove the proposition for successively more general spectra X. Let us suppose first that $X = \Sigma^\infty K$ is the suspension spectrum of a based space (or simplicial set) K. Then $\lambda_{\Sigma^\infty K} : S^1 \wedge \Sigma^\infty K \longrightarrow \operatorname{sh}(\Sigma^\infty K)$ is an isomorphism, hence so is the left vertical map $\lambda_{\Sigma^\infty K} \wedge Y$. Moreover, smashing with the suspension spectrum $\Sigma^\infty K$ is isomorphic to smashing with the space K, which is done levelwise and commutes with shifting. So for $X = \Sigma^\infty K$ the right vertical map $\xi^{0,1}_{\Sigma^\infty K, Y}$ is also an isomorphism. Since both vertical maps are isomorphisms, the square is a pushout.

Now we prove that the proposition holds for the induced spectrum $\triangleright X$ if it holds for X. Since the induction functor $\triangleright$ is isomorphic to smashing with the free spectrum F_1, the two upper spectra $(S^1 \wedge \triangleright X) \wedge Y$ and $(\triangleright X) \wedge (\operatorname{sh} Y)$ are isomorphic to $\triangleright(S^1 \wedge X \wedge Y)$ respectively $\triangleright(X \wedge (\operatorname{sh} Y))$. We exploit the splitting (3.19) of $\operatorname{sh}(\triangleright A)$ as $A \vee \triangleright(\operatorname{sh} A)$ for $A = X$ and $A = X \wedge Y$ and distribute smash product over wedge to rewrite the two lower corners of the square (5.17) as

$$(\operatorname{sh}(\triangleright X)) \wedge Y \;\cong\; (X \wedge Y) \;\vee\; \triangleright((\operatorname{sh} X) \wedge Y)$$

respectively

$$\operatorname{sh}((\triangleright X) \wedge Y) \;\cong\; (X \wedge Y) \;\vee\; \triangleright(\operatorname{sh}(X \wedge Y)) \; .$$

Making all these substitutions in the square (5.17) for $\triangleright X$ we arrive at the isomorphic commutative square:

$$\begin{tikzcd}
\triangleright(S^1 \wedge X \wedge Y) \arrow[r, "\triangleright((X \wedge \lambda_X) \circ \text{twist})"] \arrow[d, "\triangleright(\lambda_X \wedge Y)"'] & \triangleright(X \wedge (\operatorname{sh} Y)) \arrow[d, "\triangleright \xi^{0,1}_{X,Y}"] \\
(X \wedge Y) \;\vee\; \triangleright((\operatorname{sh} X) \wedge Y) \arrow[r, "\mathrm{Id} \vee \triangleright \xi^{1,0}_{X,Y}"'] & (X \wedge Y) \;\vee\; \triangleright(\operatorname{sh}(X \wedge Y))
\end{tikzcd}$$

The two summands of $X \wedge Y$ in the lower row map across by the identity and the vertical maps have images in the summands complementary to the two copies of $X \wedge Y$. The rest of the square is obtained from the square (5.17) for X by applying induction $\triangleright$. Since $\triangleright$ is a left adjoint it preserves pushouts, which proves that the last square is a pushout.

Now we have shown that the square (5.17) is a pushout when X is a suspension spectrum, and that the proposition for X implies the proposition for $\triangleright X$. Since $F_0 K \cong \Sigma^\infty K$ and $F_{1+m} K \cong \triangleright F_m K$, induction on m proves the proposition whenever $X = F_m K$ is a free symmetric spectrum.

All four corners of the square as well as the pushout commute with colimits. Every semifree symmetric spectrum is a coequalizer of a free symmetric spectrum, compare (3.24), so the proposition follows whenever $X = G_m L$ is a semifree symmmetric spectrum. Since every symmetric spectrum is naturally a coequalizer of two morphisms between wedges of semifree spectra, compare (3.25), the proposition finally follows for an arbitrary symmetric spectrum X. □

Example 5.19. We can use Proposition 5.18 to make the smash product more explicit, at least in low levels. For example, if we evaluate the pushout (5.17) in level 0, we obtain a description of $(X \wedge Y)_1$ as a pushout of the maps

$$X_1 \wedge Y_0 \xleftarrow{\sigma_0 \wedge Y_0} X_0 \wedge S^1 \wedge Y_0 \cong X_0 \wedge Y_0 \wedge S^1 \xrightarrow{X_0 \wedge \sigma_0} X_0 \wedge Y_1 \ .$$

There are natural composition morphisms

$$\circ \ : \ \mathrm{Hom}(Y, Z) \wedge \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(X, Z) \tag{5.20}$$

which are associative and unital with respect to a unit map $\mathbb{S} \longrightarrow \mathrm{Hom}(X, X)$ adjoint to the identity of X (which is a vertex in level 0 of the spectrum $\mathrm{Hom}(X, X)$). The composition morphism is obtained, by the universal property of the smash product, from the bimorphism consisting of the maps

$$\begin{aligned} \mathrm{map}(Y, \mathrm{sh}^n Z) \wedge \mathrm{map}(X, \mathrm{sh}^m Y) &\xrightarrow{\mathrm{sh}^m \wedge \mathrm{Id}} \mathrm{map}(\mathrm{sh}^m Y, \mathrm{sh}^m(\mathrm{sh}^n Z)) \wedge \mathrm{map}(X, \mathrm{sh}^m Y) \\ &= \mathrm{map}(\mathrm{sh}^m Y, \mathrm{sh}^{n+m} Z) \wedge \mathrm{map}(X, \mathrm{sh}^m Y) \xrightarrow{\ \circ\ } \mathrm{map}(X, \mathrm{sh}^{n+m} Z) \end{aligned}$$

where the second map is the composition pairing of Example 3.36. We refer to (3.10) for why it is 'right' to identify $\mathrm{sh}^m(\mathrm{sh}^n Z)$ with $\mathrm{sh}^{n+m} Z$ (note the orders in which m and n occur), so that no shuffle permutation is needed. If we specialize to $X = Y = Z$, we recover the multiplication of the endomorphism ring spectrum as defined in Example 3.41.

If X and Y are symmetric spectra we can also define natural coherent morphisms

$$X^K \wedge Y^L \longrightarrow (X \wedge Y)^{K \wedge L}$$

for pointed spaces (simplicial sets) K and L and morphisms

$$\begin{aligned} \wedge \ &: \ \mathrm{map}(A, X) \wedge \mathrm{map}(B, Y) \longrightarrow \mathrm{map}(A \wedge B, X \wedge Y) \quad \text{and} \\ \wedge \ &: \ \mathrm{Hom}(A, X) \wedge \mathrm{Hom}(B, Y) \longrightarrow \mathrm{Hom}(A \wedge B, X \wedge Y) \end{aligned} \tag{5.21}$$

for symmetric spectra A and B. The last morphism is obtained from the bimorphism whose (n, m)-component is the composite

$$\begin{aligned} \mathrm{Hom}(A, X)_n \wedge \mathrm{Hom}(B, Y)_m &= \mathrm{map}(A, \mathrm{sh}^n X) \wedge \mathrm{map}(B, \mathrm{sh}^m Y) \xrightarrow{\ \wedge\ } \mathrm{map}(A \wedge B, \mathrm{sh}^n X \wedge \mathrm{sh}^m Y) \\ &\xrightarrow{\mathrm{map}(A \wedge B, \xi_{X,Y})} \mathrm{map}(A \wedge B, \mathrm{sh}^{n+m}(X \wedge Y)) = \mathrm{Hom}(A \wedge B, X \wedge Y)_{n+m} \ , \end{aligned}$$

where the morphism $\xi_{X,Y} : \mathrm{sh}^n X \wedge \mathrm{sh}^m Y \longrightarrow \mathrm{sh}^{n+m}(X \wedge Y)$ was defined in (5.15).

The functors of geometric realization and singular complex (compare Section 3.1) which relate symmetric spectra of spaces respectively simplicial sets are nicely compatible with the smash products for symmetric spectra. As for the unstable objects themselves, geometric realization is strong symmetric monoidal, i.e., commutes with the smash product up to coherent isomorphism. The singular complex is at least lax symmetric monoidal, i.e., allows an associative and commutative natural transformation. Here are some more details. We let Y and Y' be symmetric spectra of simplicial sets. Then $|Y|$ is the geometric realization, a symmetric spectrum of topological spaces, and similarly for $|Y'|$. We can consider the composite maps

$$|Y_p| \wedge |Y'_q| \xrightarrow{r_{Y_p, Y'_q}} |Y_p \wedge Y'_q| \xrightarrow{|i_{p,q}|} |(Y \wedge Y')_{p+q}|$$

where the isomorphism $r_{A,B} : |A| \wedge |B| \cong |A \wedge B|$ was discussed in (3.2) and $i_{p,q}$ is a component of the universal bimorphism. As p and q vary the collection of these maps constitute a bimorphism from $(|Y|, |Y'|)$ to $|Y \wedge Y'|$, which gives rise to a preferred morphism of symmetric spectra of spaces

$$r_{Y,Y'} \ : \ |Y| \wedge |Y'| \longrightarrow |Y \wedge Y'| \ . \tag{5.22}$$

This morphism is a natural isomorphism, and associative, unital and commutative.

Now we let X and X' be two symmetric spectra of topological spaces. Similarly as above, the composite maps

$$\mathcal{S}(X_p) \wedge \mathcal{S}(X'_q) \ \to \ \mathcal{S}(X_p \wedge X'_q) \xrightarrow{\mathcal{S}(i_{p,q})} \mathcal{S}((X \wedge X')_{p+q})$$

constitute a bimorphism from $(\mathcal{S}(X), \mathcal{S}(X'))$ to $\mathcal{S}(X \wedge X')$, which gives rise to a preferred morphism of symmetric spectra of simplicial sets

$$\mathcal{S}(X) \wedge \mathcal{S}(X') \longrightarrow \mathcal{S}(X \wedge X') \ . \tag{5.23}$$

This natural morphism is also associative, unital and commutative, but in general *not* an isomorphism.

Now we can make precise the idea that symmetric ring spectra are the same as monoid objects in the symmetric monoidal category of symmetric spectra with respect to the smash product.

Construction 5.24. Let us define an *implicit symmetric ring spectrum* as a symmetric spectrum R together with morphisms $\mu : R \wedge R \longrightarrow R$ and $\iota : \mathbb{S} \longrightarrow R$ which are associative and unital in the sense that the following diagrams commute:

$$\begin{tikzcd}
(R \wedge R) \wedge R \arrow[r, "\alpha_{R,R,R}"] \arrow[d, "\mu \wedge R"'] & R \wedge (R \wedge R) \arrow[r, "R \wedge \mu"] & R \wedge R \arrow[d, "\mu"] \\
R \wedge R \arrow[rr, "\mu"'] & & R
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S} \wedge R \arrow[r, "\iota \wedge R"] \arrow[dr, equal] & R \wedge R \arrow[d, "\mu"] & R \wedge \mathbb{S} \arrow[l, "R \wedge \iota"'] \arrow[dl, equal] \\
& R &
\end{tikzcd}$$

We say that the implicit symmetric ring spectrum (R, μ, ι) is *commutative* if the multiplication is unchanged when composed with the symmetric isomorphism, i.e., if the relation $\mu \circ \tau_{R,R} = \mu$ holds.

Given an implicit symmetric ring spectrum (R, μ, ι) we can make the collection of Σ_n-spaces R_n into a symmetric ring spectrum in the sense of the original Definition 1.3 as follows. As unit maps we simply take the components of $\iota : \mathbb{S} \longrightarrow R$ in levels 0 and 1. We define the multiplication map $\mu_{n,m} : R_n \wedge R_m \longrightarrow R_{n+m}$ as the composite

$$R_n \wedge R_m \xrightarrow{i_{n,m}} (R \wedge R)_{n+m} \xrightarrow{\mu_{n+m}} R_{n+m} \ .$$

Then the associativity condition for μ above directly translates into the associativity condition of Definition 1.3 for the maps $\mu_{n,m}$. Evaluating the two commuting unit triangles in level 0 gives the unit condition of Definition 1.3. The condition $\mu(R \wedge \iota) = \mathrm{Id}_R$ in level $n+1$ composed with $i_{n,1} : R_n \wedge S^1 \longrightarrow (R \wedge \mathbb{S})_{n+1} = R_{n+1}$ shows that $\mu_{n,1} \circ (R \wedge \iota_1)$ equals the structure map $\sigma_n : R_n \wedge S^1 \longrightarrow R_{n+1}$ of the underlying symmetric spectrum of R. So the conceivably different meaning of 'underlying symmetric spectrum' in the sense of Remark 1.6 (iii) in fact coincides with the underlying spectrum R. We recall from Construction 5.6 that the $(1, n)$-component of the universal bimorphism $i : (\mathbb{S}, R) \longrightarrow \mathbb{S} \wedge R = R$ is the composite

$$S^1 \wedge R_n \xrightarrow{\text{twist}} R_n \wedge S^1 \xrightarrow{\sigma_n} R_{n+1} \xrightarrow{\chi_{n,1}} R_{1+n} \ .$$

So the condition $\mu(\iota \wedge R) = \mathrm{Id}_R$ in level $1 + n$, composing with the map $i_{1,n} : S^1 \wedge R_n \longrightarrow R_{1+n}$ gives the centrality condition of Definition 1.3. If the implicit multiplication is commutative, then in the diagram

$$\begin{tikzcd}
R_n \wedge R_m \arrow[r, "i_{n,m}"] \arrow[dd, "\text{twist}"'] \arrow[rr, bend left, "\mu_{n,m}"] & (R \wedge R)_{n+m} \arrow[d, "(\tau_{R,R})_{n+m}"'] \arrow[dr, "\mu_{n+m}"] & \\
& (R \wedge R)_{n+m} \arrow[r, "\mu_{n+m}"'] \arrow[d, "\chi_{n,m}"'] & R_{n+m} \arrow[d, "\chi_{n,m}"] \\
R_m \wedge R_n \arrow[r, "i_{m,n}"'] \arrow[rr, bend right, "\mu_{m,n}"'] & (R \wedge R)_{m+n} \arrow[r, "\mu_{m+n}"'] & R_{m+n}
\end{tikzcd}$$

the upper right triangle commutes for all $n, m \geq 0$. The left square commutes by the definition (5.7) of the symmetry isomorphism $\tau_{R,R}$, and the lower right square because μ is a homomorphism of symmetric spectra. So the entire diagram commutes, and that means that the explicit multiplication is commutative.

Altogether this proves:

Theorem 5.25. *The construction 5.24 which turns an implicit symmetric ring spectrum into a symmetric ring spectrum in sense of the original Definition 1.3 is an isomorphism between the category of implicit symmetric ring spectra and the category of symmetric ring spectra. The functor restricts to an isomorphism from the category of commutative implicit symmetric ring spectra to the category of commutative symmetric ring spectra.*

Now that we have carefully stated and proved Theorem 5.25 we will start to systematically blur the distinction between implicit and explicit symmetric ring spectra. Whenever convenient we use the isomorphism of categories to go back and forth between the two notions without further mentioning.

Example 5.26 (Smash product of symmetric ring spectra)**.** Here is a construction of a new symmetric ring spectrum from old ones for which the possibility to define ring spectra 'implicitly' is crucial. If R and S are symmetric ring spectra, then the smash product $R \wedge S$ has a natural structure as symmetric ring spectrum as follows. The unit map is defined from the unit maps of R and S

$$\iota^{R\wedge S} = \iota^R \wedge \iota^S \ : \ \mathbb{S} \ = \ \mathbb{S} \wedge \mathbb{S} \ \longrightarrow \ R \wedge S \ .$$

The multiplication map of $R \wedge S$ is defined from the multiplications of R and S as the composite

$$(R \wedge S) \wedge (R \wedge S) \ \xrightarrow{\mathrm{Id} \wedge \tau_{S,R} \wedge \mathrm{Id}} \ (R \wedge R) \wedge (S \wedge S) \ \xrightarrow{\mu \wedge \mu} \ R \wedge S \ ,$$

where we have suppressed some associativity isomorphisms. It is a good exercise to insert these associativity isomorphisms and observe how the hexagon condition for associativity and symmetry isomorphisms enters the verification that the product of $R \wedge S$ is in fact associative.

Example 5.27 (Tensor and symmetric algebra)**.** Another class of examples which can only be given as implicit symmetric ring spectra are symmetric ring spectra 'freely generated' by a symmetric spectrum. These come in two flavors, an associative and a commutative (and associative) version.

Given a symmetric spectrum X we define the *tensor algebra* as the symmetric spectrum

$$TX \ = \ \bigvee_{n \geq 0} \underbrace{X \wedge \cdots \wedge X}_{n}$$

with the convention that a 0-fold smash product is the unit object $\mathbb{S}$. The unit morphism $\iota : \mathbb{S} \longrightarrow TX$ is the inclusion of the wedge summand for $n = 0$. The multiplication is given by 'concatenation', i.e., the restriction of $\mu : TX \wedge TX \longrightarrow TX$ to the (n, m) wedge summand is the canonical isomorphism

$$X^{\wedge n} \wedge X^{\wedge m} \ \xrightarrow{\cong} \ X^{\wedge (n+m)}$$

followed by the inclusion of the wedge summand indexed by $n+m$. In order to be completely honest here we should throw in several associativity isomorphisms; strictly speaking already the definition of TX requires choices of how to associate expressions such as $X \wedge X \wedge X$ and higher smash powers. However, all of this is taken care of by the coherence conditions of the associativity (and later the symmetry) isomorphisms, and we will not belabor this point any further.

Given any symmetric ring spectrum R and a morphism of symmetric spectra $f : X \longrightarrow R$ we can define a new morphism $\hat{f} : TX \longrightarrow R$ which on the nth wedge summand is the composite

$$X^{\wedge n} \ \xrightarrow{f^{\wedge n}} \ R^{\wedge n} \ \xrightarrow{\mu_n} \ R \ .$$

Here μ_n is the iterated multiplication map, which for $n = 0$ has to be interpreted as the unit morphism $\iota : \mathbb{S} \longrightarrow R$. This extension $\hat{f} : TX \longrightarrow R$ is in fact a homomorphism of (implicit) symmetric ring spectra. Moreover, if $g : TX \longrightarrow R$ is any homomorphism of symmetric ring spectra then $g = \hat{g}_1$ for $g_1 : X \longrightarrow R$ the restriction of g to the wedge summand indexed by 1. Another way to say this is that

$$\mathrm{Hom}_{\text{ring spectra}}(TX, R) \ \longrightarrow \ \mathcal{S}p(X, R) \ , \quad g \ \mapsto g_1$$

is a natural bijection. In fact, this bijection makes the tensor algebra functor into a left adjoint of the forgetful functor from symmetric ring spectra to symmetric spectra.

The construction has a commutative variant. We define the *symmetric algebra* generated by a symmetric spectrum X as

$$PX \;=\; \bigvee_{n\geq 0} \, (X^{\wedge n})/\Sigma_n \;.$$

Here Σ_n permutes the smash factors of $X^{\wedge n}$ using the symmetry isomorphisms (in the way made precise in Remark 5.12 and Exercise E.I.19), and we take the quotient symmetric spectrum. This symmetric spectrum has unique unit and multiplication maps such that the quotient morphism $TX \longrightarrow PX$ becomes a homomorphism of symmetric ring spectra. So the unit morphism $\iota : \mathbb{S} \longrightarrow PX$ is again the inclusion of the wedge summand for $n = 0$ and the multiplication is the wedge of the morphisms

$$(X^{\wedge n})/\Sigma_n \wedge (X^{\wedge m})/\Sigma_m \;\longrightarrow\; (X^{\wedge (n+m)})/\Sigma_{n+m}$$

induced on quotients by $X^{\wedge n} \wedge X^{\wedge m} \cong X^{\wedge (n+m)}$.

Example 5.28. For two abelian groups A and B, a natural morphism of symmetric spectra

$$m_{A,B} \;:\; HA \wedge HB \;\longrightarrow\; H(A \otimes B)$$

is obtained, by the universal property (5.2), from the bilinear morphism

$$(HA)_n \wedge (HB)_m \;=\; A[S^n] \,\wedge\, B[S^m] \;\longrightarrow\; (A \otimes B)[S^{n+m}] \;=\; (H(A \otimes B))_{n+m}$$

given by

$$\left(\sum_i a_i \cdot x_i\right) \wedge \left(\sum_j b_j \cdot y_j\right) \;\longmapsto\; \sum_{i,j} (a_i \otimes b_j) \cdot \,(x_i \wedge y_j) \;.$$

A unit map $\mathbb{S} \longrightarrow H\mathbb{Z}$ is given by the inclusion of generators. With respect to these maps, H becomes a lax symmetric monoidal functor from the category of abelian groups (under tensor product) to the category of symmetric spectra (under smash product). As a formal consequence, H turns a ring A into a symmetric ring spectrum with multiplication map

$$HA \wedge HA \;\xrightarrow{m_{A,A}}\; H(A \otimes A) \;\xrightarrow{H\mu}\; HA \;,$$

where $\mu : A \otimes A \longrightarrow A$ is the multiplication in A, i.e., $\mu(a \otimes b) = ab$. This is the 'implicit' construction of an Eilenberg-Mac Lane ring spectrum whose explicit variant appeared in Example 1.14. Similarly, an A-module structure on B gives rise to an HA-module structure on HB.

The definition of the symmetric spectrum HA makes just as much sense when A is a *simplicial* abelian group; thus the Eilenberg-Mac Lane functor makes simplicial rings into symmetric ring spectra, respecting possible commutativity of the multiplications.

5.4. Skeleta and latching spaces. There is a functorial way to write a symmetric spectrum as a sequential colimit of spectra which are made from the information below a fixed level. This is somewhat analogous to the skeleton filtration of a simplicial set, which ultimately arises from filtering the category $\mathbf{\Delta}$ of finite ordered sets by cardinality. We thus refer to this as the *skeleton filtration* of a symmetric spectrum. The word 'filtration' should maybe be set in quotes here because in the context of symmetric spectra the maps from the skeleta to the symmetric spectrum need not be injective.

Construction 5.29. For every symmetric spectrum X and $k \geq 0$ we define the following data by induction on k:

- a based Σ_k-space (or Σ_k-simplicial set) $L_k X$, the *k-th latching space* of X, equipped with a natural map of pointed Σ_k-spaces (resp. Σ_k-simplicial sets) $\nu_k : L_k X \longrightarrow X_k$.
- a symmetric spectrum $F^k X$, the *k-skeleton* of X, equipped with a natural morphism $i_k : F^k X \longrightarrow X$,
- a natural morphism $j_k : F^{k-1} X \longrightarrow F^k X$ which satisfies $i_k j_k = i_{k-1}$.

We start with $F^{-1}X = *$, the trivial spectrum. For $k \geq 0$ we define the latching space by

$$L_k X \;=\; (F^{k-1}X)_k \;, \tag{5.30}$$

the k-th level of the $(k-1)$-skeleton, and the morphism $\nu_k : L_k X = (F^{k-1}X)_k \longrightarrow X_k$ as the k-level of the previously constructed morphism $i_{k-1} : F^{k-1}X \longrightarrow X$. Then we define the k-skeleton $F^k X$ as the pushout

$$\begin{tikzcd} G_k L_k X \arrow[r, "G_k\nu_k"] \arrow[d] & G_k X_k \arrow[d] \\ F^{k-1}X \arrow[r, "j_k"'] & F^k X \end{tikzcd} \tag{5.31}$$

where the left vertical morphism is adjoint to the identity map of $L_k X = (F^{k-1}X)_k$. The morphism $\eta : G_k X_k \longrightarrow X$ which is adjoint to the identity of X_k and $i_{k-1} : F^{k-1}X \longrightarrow X$ restrict to the same morphism on $G_k L_k X$. So the universal property of the pushout provides a unique morphism $i_k : F^k X \longrightarrow X$ which satisfied $i_k j_k = i_{k-1}$ and whose restriction to $G_k X_k$ is η.

We want to be a bit more specific about the pushout above in 'low levels'. Colimits in general, and pushouts in particular, are not completely well-defined, but only up to preferred isomorphism. In our situation, we can – and will – choose the pushout (5.31) so that

$$(F^k X)_n \;=\; X_n \qquad \text{for } n \leq k \tag{5.32}$$

and so that the morphisms $j_{k+1} : F^k X \longrightarrow F^{k+1}X$ and $i_k : F^k X \longrightarrow X$ are the identity maps in level k and below. This convention is convenient because it will later make some maps equalities which would otherwise merely be isomorphisms. We note that this convention also forces the structure maps of the symmetric spectrum $F^k X$ to coincide with those of X up to level k.

We have to justify that this choice is possible, and we do this by induction on k. For $k = -1$ there is nothing to show, so we may assume $k \geq 0$. The semifree spectra $G_k L_k X$ and $G_k X_k$ are trivial in level k and below, so the morphism $G_{k+1}\nu_{k+1} : G_{k+1}L_{k+1}X \longrightarrow G_{k+1}X_{k+1}$ is an isomorphism in level k and below. Hence its cobase change $j_k : F^{k-1}X \longrightarrow F^k X$ is an isomorphism in level $k-1$ and below. This means that we can take $(F^k X)_n = (F^{k-1}X)_n$ and $(j_k)_n = \mathrm{Id}$ for $n < k$. By induction, $(F^{k-1}X)_n = X_n$ below level k, so this justifies our choice (5.32) except in level k. In the pushout square (5.31) the left vertical morphism is an isomorphism in level k since $(G_k L_k X)_k = L_k X = (F^{k-1}X)_k$; so in level k, the right vertical morphism in (5.31) is an isomorphism $X_k = (G_k X)_k \longrightarrow (F^k X)_k$. The composite of this morphism with the k-th level of $i_k : F^k X \longrightarrow X$ is the identity of X_k, so $(i_k)_k$ is an isomorphism, and we can choose $(F^k X)_k = X_k$ and have $(i_k)_k$ be the identity morphism. The relation $i_k = j_k i_{k-1}$ lets us deduce inductively that i_k is an isomorphism below level k.

The sequence of skeleta $F^k X$ stabilizes to X in a very strong sense. In every given level m, there is a point from which on all the spaces $(F^k X)_m$ are equal to X_m and the morphisms i_k and j_k are identity maps in level m. In particular, X_m is the colimit with respect to the morphisms $(i_k)_m$, of the sequence of maps $(j_k)_m$, Since colimits in the category of symmetric spectra are created levelwise, we deduce that the spectrum X is a colimit, with respect to the morphisms i_k, of the sequence of morphisms j_k.

In low dimensions, latching spaces and skeleta can be described explicitly as follows. The spectrum $F^{-1}X$ is trivial, so the 0-th latching space $L_0 X$ is a point and the pushout (5.31) for the 0-skeleton reduces to an isomorphism $F^0 X \cong G_0 X_0 \cong \Sigma^\infty X_0$, the suspension spectrum of level 0. The first latching space is thus given by $L_1 X = (F^0 X)_1 = X_0 \wedge S^1$ and the first latching morphism $\nu_1 : L_1 X \longrightarrow X_1$ equals the structure map $\sigma_0 : X_0 \wedge S^1 \longrightarrow X_1$.

One step further we obtain the 1-skeleton $F^1 X$ as a pushout of the diagram

$$\Sigma^\infty X_0 \;\longleftarrow\; G_1(X_0 \wedge S^1) \;\xrightarrow{G_1(\sigma_0)}\; G_1 X_1$$

where the left morphism is adjoint to the identity of $X_0 \wedge S^1$. Taking level 2 of this exhibits the second latching space L_2X as the pushout of the diagram

$$X_0 \wedge S^2 \xleftarrow{\text{act on } S^2} \Sigma_2^+ \wedge X_0 \wedge S^2 \xrightarrow{\mathrm{Id}\wedge\sigma_0\wedge\mathrm{Id}} \Sigma_2^+ \wedge X_1 \wedge S^1 \ .$$

Thus L_2X is the quotient of $\Sigma_2^+ \wedge X_1 \wedge S^1$ by the equivalence relation generated by

$$\gamma \wedge \sigma_0(a \wedge x) \wedge y \quad \sim \quad (\gamma(1,2)) \wedge \sigma_0(a \wedge y) \wedge x$$

for $\gamma \in \Sigma_2$, $a \in X_0$ and $x, y \in S^1$. In Proposition 5.39 below we will recognize the latching spaces as the spaces in the symmetric spectrum $X \wedge \bar{\mathbb{S}}$, the smash product of X with the truncated sphere spectrum. This shows that in general L_kX is a quotient space (or simplicial set) of $\Sigma_k^+ \wedge_{\Sigma_{k-1}} X_{k-1} \wedge S^1$.

There are other ways to introduce the skeleton functors: in Exercise E.I.26 we reinterpret the skeleton functor F^k as a left adjoint to a truncation functor from the category of symmetric spectra to a category of 'k-truncated symmetric spectra'; in Exercise E.I.22 we show that the k-skeleton F^kX can be defined by truncating the construction (3.25) which expresses X as a coequalizer of semifree spectra; in Exercise E.I.25 we relate F^kX to the smash products of X with certain truncated sphere spectra.

A word of warning: although the functor F^k behaves in many ways like a skeleton of a simplicial set, the morphism $i_k : F^kX \longrightarrow X$ is not generally injective! An example is provided by the *truncated sphere spectrum* $\bar{\mathbb{S}}$, which we define now.

Example 5.33. The truncated sphere spectrum $\bar{\mathbb{S}}$ is the symmetric subspectrum of $\mathbb{S}$ with levels

$$\bar{\mathbb{S}}_n \;=\; \begin{cases} * & \text{for } n = 0 \\ S^n & \text{for } n \geq 1. \end{cases} \tag{5.34}$$

In other words, $\bar{\mathbb{S}}$ differs from the sphere spectrum $\mathbb{S}$ only in one missing point in level 0. Since $\bar{\mathbb{S}}$ is trivial in level 0, we have $F^1\bar{\mathbb{S}} = G_1\bar{\mathbb{S}}_1 = G_1S^1$; this 1-skeleton of $\bar{\mathbb{S}}$ is too big for the morphism $i_1 : F^1\bar{\mathbb{S}} \longrightarrow \bar{\mathbb{S}}$ to be injective. In particular the latching map

$$\nu_2 = (i_1)_2 \;:\; (F^1\bar{\mathbb{S}})_2 = (G_1\bar{\mathbb{S}}_1)_2 = \Sigma_2^+ \wedge S^2 \;\longrightarrow\; S^2 = \bar{\mathbb{S}}_2$$

is the action map of Σ_2 on S^2, and is not injective.

Example 5.35. As another example we calculate the skeleton filtration and the latching spaces of a semifree spectrum G_mL, where L is a based Σ_m-space (or Σ_m-simplicial set). Since any semifree spectrum is trivial below its defining level, the symmetric spectrum $F^k(G_mL)$ is trivial for $k < m$ and the latching space $L_k(G_mL)$ is trivial for $k \leq m$.

Now we claim that for $k \geq m$ the morphism

$$i_k \;:\; F^k(G_mL) \;\longrightarrow\; G_mL$$

is the identity and that for $k > m$ the latching morphism

$$\nu_k \;:\; L_k(G_mL) \;\longrightarrow\; (G_mL)_k$$

is the identity. In this sense the semifree spectrum G_mL is 'purely m-dimensional'. Indeed, for $k = m$ the latching space $L_m(G_mL)$ is trivial and we have $(G_mL)_m = L$; so the right vertical morphism in the defining pushout (5.31) is an isomorphism from G_mL to $F^m(G_mL)$. Hence the morphism $i_m : F^m(G_mL) \longrightarrow G_mL$ is also an isomorphism. For $k > m$ we can now argue inductively: since i_{k-1} is an isomorphism, so is its k-th level, the latching morphism $\nu_k : L_k(G_mL) \longrightarrow (G_mL)_k$. Since ν_k is an isomorphism, so is the upper horizontal morphism $G_k\nu_k$ in the defining pushout (5.31) for $F^{k+1}(G_mL)$. Hence the cobase change $j_{k+1} : F^k(G_mL) \longrightarrow F^{k+1}(G_mL)$ is an isomorphism. Finally, by the relation $i_{k+1}j_{k+1} = i_k$, the morphism i_{k+1} is an isomorphism since the other two are.

Example 5.36. We identify the latching spaces of twisted smash products. Let K be a pointed Σ_m-space (or simplicial set) for some $m \geq 0$ and X a symmetric spectrum. The twisted smash product $K \triangleright_m X$ is trivial below level m, hence the skeleton $F^k(K \triangleright_m X)$ is trivial for $i < m$ and the latching object

$L_k(K \triangleright_m X)$ is trivial for $k \leq m$. To describe skeleta and latching objects for larger values we construct a natural isomorphism of symmetric spectra

$$q \ : \ F^{m+k}(K \triangleright_m X) \ \longrightarrow \ K \triangleright_m (F^k X) \tag{5.37}$$

which is the identity in level $k+m$ and below. Moreover, the diagrams

$$\begin{tikzcd} F^{m+k-1}(K \triangleright_m X) \arrow[r, "q"] \arrow[d, "j_{m+k}"'] & K \triangleright_m (F^{k-1}X) \arrow[d, "K \triangleright_k j_k"] \\ F^{m+k}(K \triangleright_m X) \arrow[r, "q"'] & K \triangleright_m (F^k X) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F^{m+k}(K \triangleright_m X) \arrow[r, "q"] \arrow[dr, "i_{m+k}"'] & K \triangleright_m (F^k X) \arrow[d, "K \triangleright_m i_k"] \\ & K \triangleright_m X \end{tikzcd}$$

commute for all $k \geq 0$. In particular, the level $m+k$ component of q for $m+k-1$ is a Σ_{m+k}-equivariant isomorphism

$$\tilde{q} \ : \ L_{m+k}(K \triangleright_m X) \ \xrightarrow{\cong} \ \Sigma^+_{m+k} \wedge_{\Sigma_m \times \Sigma_k} (K \wedge L_k X) \ . \tag{5.38}$$

For the construction of the morphism (5.37), we recall the convention (5.32), substitute the definition of twisted smash products and thereby arrive at

$$\begin{aligned} (F^{m+k}(K \triangleright_m X))_n \ &= \ (K \triangleright_m X)_n \ = \ \Sigma^+_n \wedge_{\Sigma_m \times \Sigma_{n-m}} K \wedge X_{n-m} \\ &= \ \Sigma^+_n \wedge_{\Sigma_m \times \Sigma_{n-m}} K \wedge (F^k X)_{n-m} \ = \ (K \triangleright_m (F^k X))_n \ , \end{aligned}$$

whenever $n \leq m+k$ (and hence $n-m \leq k$); in other words, source and target of q are *equal* in level $m+k$ and below (where both sides are a point when n is less than m).

We construct q by induction on k. For $k=-1$ there is nothing to show since source and target of q are both trivial spectra. Since the source of q is defined as a pushout (5.31), it suffices to define morphisms

$$q_1 : F^{m+k-1}(K \triangleright_m X) \longrightarrow K \triangleright_m (F^k X) \qquad \text{and} \qquad q_2 : G_{m+k}(K \triangleright_m X)_{m+k} \longrightarrow K \triangleright_m (F^k X)$$

which 'restrict' to the same morphism on $G_{m+k}L_{m+k}(K \triangleright_m X)$. We define q_1 as the composite

$$F^{m+k-1}(K \triangleright_m X) \ \xrightarrow{q} \ K \triangleright_m (F^k X) \ \xrightarrow{K \triangleright_m j_k} \ K \triangleright_m (F^k X)$$

where the first map q is from the previous induction step. With this choice of q_1, we have automatically arranged that the square above commutes. We define q_2 as the morphism adjoint to the identity of $(K \triangleright_m (F^k X))_{m+k} = (K \triangleright_m X)_{m+k}$. The two 'restrictions' of q_1 respectively q_2 to morphisms $G_{m+k}L_{m+k}(K \triangleright_m X) \longrightarrow K \triangleright_m (F^k X)$ are then adjoint to the latching morphism

$$\nu_{m+k} \ : \ L_{m+k}(K \triangleright_m X) \ \longrightarrow \ (K \triangleright_m X)_{m+k} = (K \triangleright_m (F^k X))_{m+k} \ .$$

Therefore q_1 and q_2 'glue' to a unique morphism q defined on the pushout $F^{m+k}(K \triangleright_m X)$. Naturality of q in K and X are clear. [q is identity in levels $k+m$ and below; q is an isomorphism]

It remains to show that the triangle above commutes. However, both maps i_{m+k} and $(K \triangleright_m i_k) \circ q$ are the identity in levels $k+m$ and below by our convention (5.32); since the source is an $(m+k)$-skeleton, i_{m+k} and $(K \triangleright_m i_k) \circ q$ have to coincide [ref].

In the next proposition we reinterpret the latching space using the smash product with a truncated sphere spectrum $\bar{\mathbb{S}}$, compare Example 5.33.

Before we construct this isomorphism we recall a description of the space $(X \wedge \bar{\mathbb{S}})_k$ (and hence, a posteriori, of the latching space $L_k X$) as a coequalizer. We obtain this presentation by making the construction (C) of the smash product of symmetric spectra explicit when the second factor is the truncated sphere spectrum $\bar{\mathbb{S}}$. If we unravel the definitions and specialize to the appropriate values we obtain $(X \wedge \bar{\mathbb{S}})_k$ as a coequalizer, in the category of pointed Σ_k-spaces (resp. Σ_k-simplicial sets), of two maps

$$\bigvee_{n=0}^{k-2} \Sigma^+_k \wedge_{\Sigma_n \times \Sigma_1 \times \Sigma_{k-n-1}} X_n \wedge S^1 \wedge S^{k-n-1} \rightrightarrows \bigvee_{n=0}^{k-1} \Sigma^+_k \wedge_{\Sigma_n \times \Sigma_{k-n}} X_n \wedge S^{k-n}$$

(we have already simplified the coequalizer by omitting trivial wedge summands which contain the one point space $\bar{\mathbb{S}}_0$). One of the maps takes the wedge summand indexed by n to the wedge summand indexed by $n+1$ using the map

$$\sigma_n \wedge S^{k-n-1} \ : \ X_n \wedge S^1 \wedge S^{k-n-1} \ \longrightarrow \ X_{n+1} \wedge S^{k-n-1}$$

and inducing up. The other map takes the wedge summand indexed by n to the wedge summand indexed by n using the canonical isomorphism

$$X_n \wedge S^1 \wedge S^{k-n-1} \ \xrightarrow{\cong} \ X_n \wedge S^{k-n}$$

and inducing up.

Proposition 5.39. *For every symmetric spectrum X and $k \geq 0$ there is a unique morphism of based Σ_k-spaces (or simplicial sets)*

$$i_k \ : \ (X \wedge \bar{\mathbb{S}})_k \ \longrightarrow \ L_k X$$

whose composite with the component $i_{k-1,1} : X_{k-1} \wedge S^1 \longrightarrow (X \wedge \bar{\mathbb{S}})_k$ of the universal bimorphism equals the structure map $\sigma_{k-1} : X_{k-1} \wedge S^1 \longrightarrow L_k X$ of the spectrum $F^{k-1}X$. Moreover, the map i_k is an isomorphism, natural in X and the composite of i_k with the latching map $\nu_k : L_k X \longrightarrow X_k$ equals the k-th level of the morphism

$$X \wedge \text{incl.} \ : \ X \wedge \bar{\mathbb{S}} \ \longrightarrow \ X \wedge \mathbb{S} \ = \ X \ .$$

Proof. The map $i_{k-1,1} : X_{k-1} \wedge S^1 \longrightarrow (X \wedge \bar{\mathbb{S}})_k$ is a surjection [ref], so there can be at most one map i_k satisfying $i_k i_{k-1,1} = \sigma_{k-1}$. For the construction of i_k we recall the convention (5.32) that the spaces and structure maps of the skeleton $F^{k-1}X$ agree with those of X up to level $k-1$. So for $i \leq k-1$ the iterated structure maps

$$X_i \wedge S^{k-i} \ = \ (F^{k-1}X)_i \wedge S^{k-i} \ \xrightarrow{\sigma^m} \ (F^{k-1}X)_k \ = \ L_k X$$

of the $(k-1)$-skeleton are $\Sigma_i \times \Sigma_{k-i}$ equivariant maps which we can induce up and assemble into a Σ_k-equivariant map

$$\bigvee_{i=0}^{k-1} \Sigma_k^+ \wedge_{\Sigma_i \times \Sigma_k} X_i \wedge S^{k-i} \ \longrightarrow \ L_k X \ .$$

The space $(X \wedge \bar{\mathbb{S}})_k$ is a coequalizer of two maps with target the previous wedge; since the map above was assembled from structure maps of the symmetric spectrum $F^{k-1}X$, it coequalizes the two maps and so factors over a unique morphism of Σ_k-spaces

$$i_k \ : \ (X \wedge \bar{\mathbb{S}})_k \ \longrightarrow \ L_k X \ .$$

Now we verify the relation $\nu_k i_k = (r_X \circ (\mathrm{Id} \wedge \text{incl.}))_k$. After precomposition with $i_{k-1,1}$, both sides of the equation become the strucure map $\sigma_{k-1} : X_{k-1} \wedge S^1 \longrightarrow X_k$ of X. Since $i_{k-1,1} : X_{k-1} \wedge S^1 \longrightarrow (X \wedge \bar{\mathbb{S}})_k$ is a surjection, this proves the relation.

Finally, we show that i_k is an isomorphism. We first check the special case $X = G_m L$ of a semifree symmetric spectrum generated by based Σ_m-space (or simplicial set) L. If $k \leq m$, then $L_k(G_m L)$ and $(G_m L \wedge \bar{\mathbb{S}})_k \cong (L \triangleright_m \bar{\mathbb{S}})_k$ are trivial, so i_k is an isomorphism. If $k > m$ then $\mathrm{Id} \wedge \text{incl.} : G_m L \wedge \bar{\mathbb{S}} \longrightarrow G_m L \wedge \mathbb{S}$ is an isomorphism in level k since $G_m L \wedge A \cong L \triangleright_m A$ which in level k depends only on L and A_{n-k}. The latching map ν_k (by Example 5.35) and the unit isomorphism are also isomorphisms, hence so is i_k.

For general X we exploit that every symmetric spectrum can be written as a coequalizer (3.25) of a map between wedges of semifree spectra. Since the map i_k is natural and its source and target commute with colimits, the special case of semifree spectra implies the general case. □

A generalization of the previous proposition to the 'generalized latching spaces' $(F^i X)_m$ for $i < m$ is given in Exercise E.I.25.

Example 5.40. If we specialize the pushout square (5.17) for $\mathrm{sh}(X \wedge Y)$ in the special case $Y = \bar{\mathbb{S}}$ of the truncated sphere spectrum, we obtain an inductive description of the latching objects. Indeed, since L_k is isomorphic to to the k-th level of $X \wedge \bar{\mathbb{S}}$, the k-the level of the square (5.17) is the pushout:

$$\begin{tikzcd} L_k(S^1 \wedge X) \arrow[r, "\nu_k^{S^1 \wedge X}"] \arrow[d, "L_k \lambda_X"'] & S^1 \wedge X_k \arrow[d, "\sigma_k"] \\ L_k(\mathrm{sh}\, X) \arrow[r] & L_{1+k} X \end{tikzcd}$$

In the upper right corner we have used that $X \wedge \mathrm{sh}\, \bar{\mathbb{S}}$ is isomorphic to $S^1 \wedge X$.

5.5. Flat symmetric spectra. We recall that a morphism of pointed simplicial sets is a *cofibration* if it is a categorical monomorphism, i.e., dimensionwise injective. A morphism of pointed spaces is a *cofibration* if it is a retract of a pointed cell complex. These are the cofibrations in the standard Quillen model structures on pointed simplicial sets respectively pointed spaces.

Definition 5.41. A morphism $f : X \longrightarrow Y$ of symmetric spectra is a *level cofibration* if for every $n \geq 0$ the morphism $f_n : X_n \longrightarrow Y_n$ is a cofibration of the underlying pointed spaces or pointed simplicial sets (depending on the context). A symmetric spectrum A is *flat* if the functor $A \wedge -$ preserves level cofibrations.

To motivate the terminology we recall that a module over a commutative ring is called flat if tensoring with it preserves monomorphisms. Level cofibrations of symmetric spectra of simplicial sets are just the categorical monomorphisms, i.e., those morphisms which are injective in every spectrum level and every simplicial dimension. So in the context of simplicial sets, a symmetric spectrum A is flat if and only if $A \wedge -$ preserves monomorphisms. In the context of symmetric spectra of topological spaces, simply requiring that $A \wedge -$ preserves monomorphisms is not the right condition, so the analogy with flatness in algebra is less tight.

We will show in Chapter III that flat symmetric spectra are the cofibrant objects in various 'flat model structures'.

Example 5.42. Let L be a pointed Σ_m-space (or Σ_m-simplicial set), for some $m \geq 0$, whose underlying pointed space is cofibrant (this is automatic in the context of simplicial sets). Then smashing with the semifree symmetric spectrum preserves level cofibrations and level acyclic cofibrations. In particular, the semifree spectrum $G_m L$ is flat. As special cases, this applies to free symmetric spectra $F_n K$ and suspension spectra $\Sigma^\infty K$, for every based cofibrant space K (respectively every based simplicial set).

Indeed, if X is another symmetric spectrum then $G_m L \wedge X$ is isomorphic to the twisted smash product $L \triangleright_m X$ (see Proposition 5.13) and so it is trivial in levels below m and we have a natural isomorphism

$$(G_m L \wedge X)_{m+n} \;\cong\; \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge X_n \tag{5.43}$$

for $n \geq 0$. If $f : X \longrightarrow Y$ is a level cofibration respectively a level acyclic cofibration then $\mathrm{Id} \wedge f_n : L \wedge X_n \longrightarrow L \wedge Y_n$ is a cofibration respectively acyclic cofibration by the pushout product property [ref]. As a pointed space (simplicial set), the right hand side of (5.43) is a wedge of $\binom{m+n}{m}$ copies of $L \wedge X_n$. Hence the morphism $\mathrm{Id} \wedge f : G_m L \wedge X \longrightarrow G_m L \wedge Y$ is levelwise a cofibration respectively acyclic cofibration.

An example of a non-flat symmetric spectrum is the truncated sphere spectrum $\bar{\mathbb{S}}$, the subspectrum of the sphere spectrum given by $\bar{\mathbb{S}}_0 = *$ and $\bar{\mathbb{S}}_n = S^n$ for $n \geq 1$. So the difference between $\bar{\mathbb{S}}$ and $\mathbb{S}$ is only one missing point in level 0, but that point makes a huge difference with respect to flatness. Indeed, in Example 5.33 we identified the latching map

$$\nu_2 \;:\; L_2\bar{\mathbb{S}} = \Sigma_2^+ \wedge S^2 \;\longrightarrow\; S^2 = \bar{\mathbb{S}}_2$$

and concluded that ν_2 is not injective. Hence $\bar{\mathbb{S}}$ is not flat by the criterion of Proposition 5.47 below.

Some other properties of flat spectra are fairly straightforward from the definition:

Proposition 5.44. (i) *A wedge of flat symmetric spectra is flat.*

(ii) *For symmetric spectra of simplicial sets, a filtered colimit of flat symmetric spectra is flat.*
(iii) *The smash product of two flat symmetric spectra is flat.*
(iv) *If A is a flat symmetric spectrum and K a cofibrant space (respectively simplicial set), then $K \wedge A$ is flat.*

Proof. Properties (i) and (ii) follow from the two facts that the smash product commutes with colimits and that a wedge and (in the case of simplicial sets) a filtered colimit of level cofibrations is a level cofibration.

(iii) Let A and B be flat symmetric spectra. If $f : X \longrightarrow Y$ is a level cofibration, then so is $B \wedge f : B \wedge X \longrightarrow B \wedge Y$ since B is flat; then $A \wedge B \wedge f : A \wedge B \wedge X \longrightarrow A \wedge B \wedge Y$ is also a level cofibration since A is flat (where we have implicitly used the associativity isomorphisms). Thus $A \wedge B$ is flat.

(iv) The spectrum $K \wedge A$ is isomorphic to the smash product $(\Sigma^\infty K) \wedge A$. Since A is flat and the suspension spectrum is flat by Example 5.42, (iv) follows from (iii). □

There are more construction that preserve flatness. As a special case of Proposition III.2.9 below we shall see that for every flat symmetric spectrum A the morphism $\lambda_A : S^1 \wedge A \longrightarrow \operatorname{sh} A$ is a level cofibration (even a 'flat cofibration') and the shifted spectrum $\operatorname{sh} A$ is again flat. This implies that all structure maps $\sigma_n : A_n \wedge S^1 \longrightarrow A_{n+1}$ are cofibrations.

In [...] below we shall prove that the product $A \times B$ of two flat symmetric spectra A and B is again flat.

A morphism $f : X \longrightarrow Y$ of symmetric spectra is a *level acyclic cofibration* if it is simultaneously a level cofibration and a level equivalence.

Proposition 5.45. *Let A be a symmetric spectrum such that for every $k \geq 0$ the latching morphism $\nu_k : L_k A \longrightarrow A_k$ is cofibration of underlying pointed spaces (respectively simplicial sets). Then the functor $A \wedge -$ preserves level cofibrations and level acyclic cofibrations.*

Proof. Let $f : X \longrightarrow Y$ be a level cofibration (respectively level acyclic cofibration) of symmetric spectra. We use the skeleton filtration of A by the spectra $F^k A$ (see Construction 5.29) and show inductively that the map $\mathrm{Id} \wedge f : F^k A \wedge X \longrightarrow F^k A \wedge Y$ is a level cofibration (resp. level acyclic cofibration). Since $F^k A$ agrees with A up to level k, the smash product $F^k A \wedge X$ agrees with $A \wedge X$ up to level k, and similarly for Y. Since k can be arbitrarily large, this proves the claim.

We can start the induction with $k = -1$, where there is nothing to show. For the inductive step we use the pushout square (5.31) which defines $F^k A$. Since smashing is a left adjoint the spectrum $F^k A \wedge X$ is a pushout of the upper row in the commutative diagram

$$(5.46)\qquad \begin{tikzcd} F^{k-1}A \wedge X \arrow[d, "\mathrm{Id} \wedge f"'] & L_k A \rhd_k X \arrow[l] \arrow[r, "\nu_k \rhd_k \mathrm{Id}"] \arrow[d, "\mathrm{Id} \rhd_k f"] & A_k \rhd_k X \arrow[d, "\mathrm{Id} \rhd_k f"] \\ F^{k-1}A \wedge Y & L_k A \rhd_k Y \arrow[l] \arrow[r, "\nu_k \rhd_k \mathrm{Id}"'] & A_k \rhd_k Y \end{tikzcd}$$

and $F^k A \wedge Y$ is a pushout of the lower row; we have used the identification $G_k L \wedge X \cong L \rhd_k X$ provided by Proposition 5.13. The left vertical morphism is a level cofibration (resp. acyclic cofibration) by induction hypothesis, and we claim that in addition the pushout product map of the right square in (5.46)

$$\nu_k \rhd_k f \ : \ L_k A \rhd_k Y \cup_{L_k A \rhd_k X} A_k \rhd_k X \ \longrightarrow \ A_k \rhd_k Y$$

is a level cofibration (resp. acyclic cofibration). It is then a general model category fact (see Lemma A.1.10) that induced map on pushouts is levelwise a cofibration (resp. acyclic cofibration).

It remains to justify the claim that $\nu_k \rhd_k f$ is a level cofibration (resp. acyclic cofibration). There is nothing to show below level k since both sides are trivial. For $n \geq 0$ a we have

$$(L \rhd_k X)_{k+n} \ = \ \Sigma^+_{k+n} \wedge_{\Sigma_k \times \Sigma_n} L \wedge X_n \ ,$$

which non-equivariantly is a wedge of $\binom{n+k}{n}$ copies of $L \wedge X_n$. Since $\nu_k : L_k A \longrightarrow A_k$ is a cofibration and $f_n : X_n \longrightarrow Y_n$ is a cofibration (resp. acyclic cofibration), the pushout product property [ref] shows that the map

$$\nu_k \wedge f_n \ : \ L_kA \wedge Y_n \cup_{L_kA\wedge X_n} A_k \wedge X_n \ \longrightarrow \ A_k \wedge Y_n$$

is a cofibration (resp. acyclic cofibration). So $(\nu_k \rhd_k f)_{k+n}$ is a cofibration (resp. acyclic cofibration). □

Proposition 5.47. *The following are equivalent for a symmetric spectrum A:*

(i) *The symmetric spectrum A is flat.*
(ii) *For every $k \geq 0$ the latching morphism $\nu_k : L_kA \longrightarrow A_k$ is a cofibration of underlying pointed spaces, respectively simplicial sets.*
(iii) *For every $k \geq 0$ the morphism $j_k : F^{k-1}A \longrightarrow F^kA$ is a level cofibration of symmetric spectra.*
(iv) *For every $k \geq -1$ the morphism $i_k : F^kA \longrightarrow A$ is a level cofibration of symmetric spectra.*

Proof. (i)$\Longrightarrow$(ii) If A is flat, then $\mathrm{Id}\wedge i : A \wedge \bar{\mathbb{S}} \longrightarrow A \wedge \mathbb{S}$ is a level cofibration because the inclusion $i : \bar{\mathbb{S}} \longrightarrow \mathbb{S}$ of the truncated sphere spectrum (5.34) is a level cofibration. By Proposition 5.39 the m-th level of this morphism is isomorphic to the latching map $\nu_k : L_kA \longrightarrow A_k$, which is thus a cofibration.

(ii)$\Longrightarrow$(i) If the latching maps for A are cofibrations, then $A \wedge -$ preserves level cofibrations by Proposition 5.45, so A is flat.

(ii)$\Longrightarrow$(iii) If the latching map $\nu_k : L_kA \longrightarrow A_k$ is a cofibration, then the morphism $G_k\nu_k : G_kL_kA \longrightarrow G_kA_k$ is a level cofibration of symmetric spectra. Cofibrations are stable under pushouts, so the defining pushout for F^kA (5.31) shows that $j_k : F^{k-1}A \longrightarrow F^kA$ is a level cofibration.

The morphism $i_k : F^kA \longrightarrow A$ is the countable composite of the morphisms $j_i : F^{i-1}A \longrightarrow F^iA$ for $i - 1 \geq k$. Cofibrations are closed under countable composition, so (iv) follows from (iii).

Condition (iv) implies condition (ii) because the latching morphism $\nu_k : L_kA \longrightarrow A_k$ was defined as the k-th level of the morphism i_{k-1}. □

Corollary 5.48. *Let A be a flat symmetric spectrum of spaces. Then for every symmetric spectrum of spaces C and every $k \geq 0$ the morphism $j_k \wedge C : F^{k-1}A \wedge C \longrightarrow F^kA \wedge C$ is levelwise an h-cofibration.*

Proof. Since A is flat the latching morphisms $\nu_k : L_kA \longrightarrow A_k$ are cofibrations by Proposition 5.47. The class of h-cofibrations contains the cofibrations and is closed under wedges and smash product with any based space. For every $m \geq 0$ the map

$$(\nu_k \rhd_k C)_{k+m} = \Sigma^+_{k+m} \wedge_{\Sigma_k\times\Sigma_m} \nu_k \wedge C_m \ : \ \Sigma^+_{k+m} \wedge_{\Sigma_k\times\Sigma_m} L_kA \wedge C_m \ \longrightarrow \ \Sigma^+_{k+m} \wedge_{\Sigma_k\times\Sigma_m} A_k \wedge C_m$$

is thus an h-cofibration. So the morphism $\nu_k \rhd_k C : L_kA \rhd_k C \longrightarrow A_k \rhd_k C$ is levelwise an h-cofibration. Smashing the pushout square (5.31) with the spectrum C gives a pushout square

$$\begin{tikzcd} L_kA \rhd_k C \arrow[r, "\nu_k \rhd_k C"] \arrow[d] & A_k \rhd_k C \arrow[d] \\ F^{k-1}A \wedge C \arrow[r, "j_k \wedge C"'] & F^kA \wedge C \end{tikzcd}$$

(where we have used the natural isomorphism $G_kA_k \wedge C \cong A_k \rhd_k C$). The class of h-cofibrations is also closed under cobase change, so this proves the claum. □

Corollary 5.49. *For every flat symmetric spectrum of simplicial sets A the geometric realization $|A|$ is a flat symmetric spectrum of topological spaces.*

Proof. The latching space $L_m|A|$ is homeomorphic to $|L_mA|$ in a way compatible with the maps $\nu_m^{|A|}$ and $|\nu_m^A|$ to $|A_m|$. [add this to the section on the filtration] Since geometric realization takes cofibrations of simplicial sets to cofibrations of spaces, the claim follows from the flatness criterion of Proposition 5.47. □

Now we prove a key result about flat spectra, namely that smashing with them preserves various kinds of equivalences. We recall that every based simplicial set is cofibrant, hence every symmetric spectrum of simplicial sets is level cofibrant. So the condition 'level cofibrant' in part (i) of the next proposition is vacuous in the context of simplicial sets. In the context of spaces, part (iv) of Proposition 5.47 in particular says that for every flat symmetric spectrum A the morphism $* = F^{-1}A \longrightarrow A$ is a level cofibration; in other words: flat symmetric spectra of spaces are level cofibrant.

Proposition 5.50. *Let A be a flat symmetric spectrum. Then the functor $A \wedge -$ preserves level equivalences between level cofibrant spectra, $\hat{\pi}_*$-isomorphisms and stable equivalences.*

Proof. We start with the first claim and let $f : X \longrightarrow Y$ be a level equivalence between level cofibrant symmetric spectra. The morphism f factors as a composite

$$X \xrightarrow{i_X} Z(f) \xrightarrow{p} Y$$

where $Z(f) = [0,1]^+ \wedge X \cup_f Y$ is the mapping cylinder of f. Since the projection $p : Z(f) \longrightarrow Y$ is a homotopy equivalence and f is a level equivalence, the 'front inclusion' $i_X : X \longrightarrow Z(f)$ is a level equivalence. Since X and Y are level cofibrant, the front inclusion i_X is also a level cofibration, hence a level acyclic cofibration. Since A is flat the latching morphisms $\nu_k : L_k A \longrightarrow A_k$ are cofibrations, so $A \wedge i_X$ is a level acyclic cofibration by Proposition 5.45. Smashing with any spectrum preserves homotopies, so $A \wedge p$ is again a homotopy equivalence, thus a level equivalence. So we conclude that $A \wedge f = (A \wedge p) \circ (A \wedge i_X)$ is a level equivalence.

Now we prove a special case of the second and third claim: we let C be a symmetric spectrum with trivial naive homotopy groups (respectively such that C is stably contractible) and we show that then $A \wedge C$ also has trivial naive homotopy groups (respectively is stably contractible). We start in the context of spaces. We first show by induction on k that $F^k A \wedge C$ has trivial homotopy groups (respectively is stably contractible) for all $k \geq -1$. The induction starts for $k = -1$ using that $F^{-1}A$, and hence $F^{-1}A \wedge C$ are trivial spectra. In the inductive step we use the pushout square (5.31) that defines $F^k A$ and smash it with C to obtain another pushout square:

$$\begin{tikzcd}
G_k L_k A \wedge C \arrow[r, "G_k \nu_k \wedge C"] \arrow[d] & G_k A_k \wedge C \arrow[d] \\
F^{k-1} A \wedge C \arrow[r, "j_k \wedge C"'] & F^k A \wedge C
\end{tikzcd}$$

As left adjoints, both G_k and $- \wedge C$ commute with colimits, so the cokernel of the upper horizontal morphism $G_k \nu_k \wedge C$ is isomorphic to $G_k(A_k/L_k A) \wedge C$. Since A is flat, $\nu_k : L_k A \longrightarrow A_k$ is a cofibration and the quotient $A_k/L_k A$ is cofibrant. The spectrum $G_k(A_k/L_k A) \wedge C$ is isomorphic to $(A_k/L_k A) \triangleright_k C$, so it has trivial naive homotopy groups by Proposition 3.31 (respectively, it is stably contractible by Proposition 4.31 (vi)).

Since the square above is a pushout, the cokernel of $j_k \wedge C : F^{k-1} A \wedge C \longrightarrow F^k A \wedge C$ is isomorphic to the cokernel of $G_k \nu_k \wedge C$ and hence has trivial naive homotopy groups (respectively is stably contractible). Since A is flat the morphism $j_k \wedge C$ is levelwise an h-cofibration by Corollary 5.48. For the second claim we can now appeal to the long exact homotopy group sequence (2.13) and conclude by the inductive hypothesis that all naive homotopy groups of the spectrum $F^k A \wedge C$ vanish. For the third claim we use [...] and the inductive hypothesis to deduce that the spectrum $F^k A \wedge C$ is stably contractible. The spectrum $A \wedge C$ is a sequential colimit of the spectra $F^k A \wedge C$ over the sequence of morphisms $j_k \wedge C$ that are levelwise h-cofibrations. For the second claim claim we exploit that homotopy groups commute with such colimits [ref], so the groups $\hat{\pi}_*(A \wedge C)$ vanish as the colimit of trivial groups. For the third claim we use [...] instead, showing that $A \wedge C$ is again stably contractible.

Now we let A and C be symmetric spectra of simplicial sets such that C has trivial naive homotopy groups (respectively such that C is stably contractible). By the previous case, $|A| \wedge |C|$ then has trivial naive homotopy groups (respectively is stably contractible). Hence the same holds for the isomorphic spectrum $|A \wedge C|$, and this shows that also $A \wedge C$ has trivial naive homotopy groups (respectively is stably contractible).

Now we consider a general $\hat{\pi}_*$-isomorphism (respective stable equivalence) f. The mapping cone $C(f)$ then has trivial naive homotopy groups by the long exact sequence of Proposition 2.12 (respective, $C(f)$ is stably contractible by the criterion of Proposition 4.29 (ii)). By the above, the spectrum $A \wedge C(f)$ has trivial naive homotopy groups (respective is stably contractible). Smash product with A commutes with the mapping cone construction, so the symmetic spectrum $C(A \wedge f)$ has trivial naive homotopy groups (respective is stably contractible). This again is equivalence to $A \wedge f$ being a $\hat{\pi}_*$-isomorphism (respectively stable equivalence). $\square$

Corollary 5.51. *Let A be a flat symmetric spectrum of simplicial sets. For every injective spectrum X the internal function spectrum* $\mathrm{Hom}(A, X)$ *is injective. If X is an injective Ω-spectrum, then so is* $\mathrm{Hom}(A, X)$.

Proof. Given a level acyclic cofibration of symmetric spectra $i : B \longrightarrow B'$ and a morphism $g : B \longrightarrow \mathrm{Hom}(A, X)$, we have to produce an extension $\bar{g} : B' \longrightarrow \mathrm{Hom}(A, X)$ satisfying $\bar{g} \circ f = g$. The latching morphisms $\nu_k : L_k A \longrightarrow A_k$ are cofibrations by by Proposition 5.47. So the morphism $A \wedge i : A \wedge B \longrightarrow A \wedge B'$ is again a level acyclic cofibration by Proposition 5.45. Since X is injective, the adjoint $G : B \wedge A \longrightarrow X$ of g has an extension $\bar{G} : B' \wedge A \longrightarrow X$ satisfying $\bar{G}(f \wedge \mathrm{Id}) = G$. The adjoint $L \longrightarrow \mathrm{Hom}(A, X)$ of $\bar{G}$ is then the required extension of g.

Now suppose that X is an injective Ω-spectrum. Then the morphism $\tilde{\lambda}_X : X \longrightarrow \Omega(\mathrm{sh}\, X)$ is a level equivalence between injective spectra, hence a homotopy equivalence. So the morphism

$$\tilde{\lambda}_{\mathrm{Hom}(A,X)} \;=\; \mathrm{Hom}(A, \tilde{\lambda}_X) \;:\; \mathrm{Hom}(A, X) \;\longrightarrow\; \Omega(\mathrm{sh}\, \mathrm{Hom}(A, X)) \;=\; \mathrm{Hom}(A, \Omega(\mathrm{sh}\, X))$$

is a homotopy equivalence. So the symmetric function spectrum $\mathrm{Hom}(A, X)$ is again an Ω-spectrum. $\square$

Example 5.52. Here is an example which shows that smashing with an arbitrary symmetric spectrum does not preserve level equivalences. Let X be the symmetric spectrum with $X_0 = S^0$, $X_1 = CS^1$ and $X_n = *$ for $n \geq 2$. Here $CS^1 = [0, 1] \wedge S^1$ is the cone on S^1, where the unit internal $[0, 1]$ is pointed by 0. The only nontrivial structure map $\sigma_0 : X_0 \wedge S^1 \longrightarrow X_1$ is the cone inclusion $S^1 \longrightarrow CS^1$. Let Y be the symmetric spectrum with $Y_0 = S^0$ and $Y_n = *$ for $n \geq 1$. Then the unique morphism $f : X \longrightarrow Y$ which is the identity in level 0 is a level equivalence, but we claim that $f \wedge \bar{\mathbb{S}} : X \wedge \bar{\mathbb{S}} \longrightarrow Y \wedge \bar{\mathbb{S}}$ is not a level equivalence. Indeed, in level 2 we have

$$(X \wedge \bar{\mathbb{S}})_2 \;=\; L_2 X \;\cong\; \mathrm{pushout}(S^2 \xleftarrow{\mathrm{act}} \Sigma_2^+ \wedge S^2 \xrightarrow{i \wedge S^1} \Sigma_2^+ \wedge (CS^1) \wedge S^1)$$

which is the suspension of the double cone on S^1, i.e., a 3-sphere. In contrast,

$$(Y \wedge \bar{\mathbb{S}})_2 \;=\; L_2 Y \;\cong\; \mathrm{pushout}(S^2 \xleftarrow{\mathrm{act}} \Sigma_2^+ \wedge S^2 \longrightarrow *)$$

is a point, so $f \wedge \bar{\mathbb{S}}$ is not a weak equivalence in level 2. [give an example where $\hat{\pi}_*$-isos or stable equivalences are not preserved]

Construction 5.53 (Flat resolution)**.** Now we construct a functorial 'flat resolution', i.e., a functor $(-)^\flat : \mathcal{S}p \longrightarrow \mathcal{S}p$ with values in flat symmetric spectra and a natural level equivalence $r_A : A^\flat \longrightarrow A$.

We consider the case of symmetric spectra of simplicial sets first. Given a symmetric spectrum A we construct $A^\flat$ and the level equivalence r_A level by level, starting with $A^\flat_0 = A_0$ and $r_0 = \mathrm{Id}$. Suppose now that $A^\flat$ and r have been constructed up to level $n-1$. The definition of the nth latching only involves the data of a symmetric spectrum in levels strictly smaller than n. So we have a latching object $L_n A^\flat$ and the partial morphism r induces a Σ_n-equivariant map $L_n r : L_n A^\flat \longrightarrow L_n A$. We define $A^\flat_n$ as the reduced mapping cylinder of the composite map

$$L_n A^\flat \xrightarrow{L_n r} L_n A \xrightarrow{\nu_n} A_n \;.$$

This inherits a Σ_n-action from the actions on $L_n A^\flat$ and A_n, and the trivial action on the cylinder coordinate. The structure map

$$\sigma_{n-1} \;:\; A^\flat_{n-1} \wedge S^1 \;\longrightarrow\; A^\flat_n = Z(\nu_n \circ L_n r : L_n A^\flat \longrightarrow A_n)$$

is the composite of the map $A^\flat_{n-1} \wedge S^1 \longrightarrow L_n A^\flat$ that comes with the latching object and the inclusion into the mapping cylinder. The nth level of the morphism r is the projection $A^\flat_n = Z(\nu_n \circ L_n r) \longrightarrow A_n$ of the mapping cylinder onto the target; this is a homotopy equivalence so in particular a weak equivalence.

After the dust settles we have constructed a symmetric spectrum $A^\flat$ and a morphism $r : A^\flat \longrightarrow A$ which is levelwise a simplicial homotopy equivalence, so altogether a level equivalence (but in general *not* a homotopy equivalence of symmetric spectra). The construction is made so that the map $\nu_n : L_n A^\flat \longrightarrow A^\flat_n$ is the mapping cylinder inclusion, thus injective. So by the criterion of Proposition 5.47 the symmetric spectrum $A^\flat$ is indeed flat.

In the context of spaces we reduce to the previous construction and define the flat resolution by $X^\flat = |\mathcal{S}(X)^\flat|$, which is indeed flat since geometric realization preserves flatness (Corollary 5.49). We use the composite

$$X^\flat = |\mathcal{S}(X)^\flat| \xrightarrow{|r_{\mathcal{S}(X)}|} |\mathcal{S}(X)| \longrightarrow X ,$$

as the level equivalence $r_X : X^\flat \longrightarrow X$, where the second map is the adjunction counit.

Proposition 5.54. *Smashing with a level cofibrant symmetric spectrum preserves level equivalences between flat symmetric spectra. Smashing with any symmetric spectrum preserves $\hat{\pi}_*$-isomorphisms and stable equivalence between flat symmetric spectra.*

Proof. We let $f : X \longrightarrow Y$ be a level equivalence between flat symmetric spectra and we let A be any level cofibrant symmetric spectrum. We contemplate the commutative square

$$\begin{tikzcd} A^\flat \wedge X \arrow[r, "r_A \wedge X"] \arrow[d, "A^\flat \wedge X"'] & A \wedge X \arrow[d, "A \wedge f"] \\ A^\flat \wedge Y \arrow[r, "r_A \wedge Y"'] & A \wedge Y \end{tikzcd} \tag{5.55}$$

where $r_A : A^\flat \longrightarrow A$ is the flat resolution of Construction 5.53. The other three maps apart from $A \wedge f$ are level equivalences by Proposition 5.50, using that flat symmetric spectra are in particular level cofibrant. Hence $A \wedge f$ is a level equivalence as well.

If A is arbitrary and $f : X \longrightarrow Y$ is a $\hat{\pi}_*$-isomorphism (respectively stable equivalence) between flat symmetric spectra, then in the commutative square (5.55) the two horizontal maps are $\hat{\pi}_*$-isomorphisms (and hence also stable equivalences) by Proposition 5.50. Moreover, $A^\flat \wedge f$ is a $\hat{\pi}_*$-isomorphism (respectively stable equivalence) by Proposition 5.50. So $A \wedge f$ is a $\hat{\pi}_*$-isomorphism (respectively a stable equivalence). □

In (5.15) we defined a natural map $\xi^{1,0}_{X,Y} : (\operatorname{sh} X) \wedge Y \longrightarrow \operatorname{sh}(X \wedge Y)$ for symmetric spectra X and Y. This map is not always a $\hat{\pi}_*$-isomorphism; for example if $X = \mathbb{S}$ is the sphere spectrum, then $\operatorname{sh} X = \Sigma^\infty S^1$ and the map $\xi_{\mathbb{S},Y}$ is isomorphic to the map $\lambda_Y : S^1 \wedge Y \longrightarrow \operatorname{sh} Y$, which is a $\hat{\pi}_*$-isomorphism if and only if Y is semistable (by definition). For semistable Y, however, we have:

Proposition 5.56. *Let Y be a flat, semistable symmetric spectrum. Then for every level cofibrant symmetric spectrum X, the map*

$$\xi^{1,0}_{X,Y} \ : \ (\operatorname{sh} X) \wedge Y \ \longrightarrow \ \operatorname{sh}(X \wedge Y)$$

is a $\hat{\pi}_*$-isomorphism.

Proof. We start with the special case where both X and Y are flat. Proposition 5.18 provides a pushout square

$$\begin{tikzcd} S^1 \wedge X \wedge Y \arrow[r, "(X \wedge \lambda_Y) \circ \mathrm{twist}"] \arrow[d, "\lambda_X \wedge Y"'] & X \wedge (\operatorname{sh} Y) \arrow[d, "\xi^{0,1}_{X,Y}"] \\ (\operatorname{sh} X) \wedge Y \arrow[r, "\xi^{1,0}_{X,Y}"'] & \operatorname{sh}(X \wedge Y). \end{tikzcd}$$

Since Y is semistable the morphism $\lambda_Y : S^1 \wedge Y \longrightarrow \operatorname{sh} Y$ is a $\hat{\pi}_*$-isomorphism. Since Y is flat, this morphism is also a level cofibration by [...]. Since X is flat, smashing with it preserves $\hat{\pi}_*$-isomorphism between level cofibrant spectra (by Proposition 5.50) and level cofibrations (by definition [...]). Since both $S^1 \wedge Y$ and $\operatorname{sh} Y$ are flat, thus level cofibrant, the upper horizontal morphism in the pushout square is thus a level cofibration and a $\hat{\pi}_*$-isomorphism. But then the lower horizontal morphism is also a level cofibration, thus $\hat{\pi}_*$-isomorphism by the long exact sequences of homotopy groups and the five lemma.

If X is level cofibrant (but not necessarily flat) we contemplate the commutative square

$$\begin{tikzcd}
(\mathrm{sh}\, X^\flat) \wedge Y \arrow[r, "\xi^{1,0}_{X^\flat, Y}"] \arrow[d, "(\mathrm{sh}\, r)\wedge Y"'] & \mathrm{sh}(X^\flat \wedge Y) \arrow[d, "\mathrm{sh}(r\wedge Y)"] \\
(\mathrm{sh}\, X) \wedge Y \arrow[r, "\xi^{1,0}_{X,Y}"'] & \mathrm{sh}(X \wedge Y)
\end{tikzcd}$$

where $r : X^\flat \longrightarrow X$ is the flat resolution of Construction 5.53. The upper horizontal map $\xi_{X^\flat,Y}$ is a $\hat{\pi}_*$-isomorphism by the above. Since the four spectra $X, X^\flat, \mathrm{sh}\, X$ and $\mathrm{sh}(X^\flat)$ are level cofibrant and Y is flat, both vertical maps are level equivalences by Proposition 5.50. Thus the lower map is a $\hat{\pi}_*$-isomorphism, which finishes the proof. □

Proposition 5.57. *Let X and Y be two semistable spectra one of which is flat and the other level cofibrant. Then the smash product $X \wedge Y$ is semistable.*

Proof. Suppose that Y is flat and X is level cofibrant. The map $\lambda_{X\wedge Y} : S^1 \wedge X \wedge Y \longrightarrow \mathrm{sh}(X \wedge Y)$ factors as the composition

$$S^1 \wedge X \wedge Y \xrightarrow{\lambda_X \wedge Y} (\mathrm{sh}\, X) \wedge Y \xrightarrow{\xi^{1,0}_{X,Y}} \mathrm{sh}(X \wedge Y)$$

(where we suppress an associativity isomorphism). Since X is semistable and level cofibrant, the map λ_X is a $\hat{\pi}_*$-isomorphism between level cofibrant symmetric spectra, and hence $\lambda_X \wedge \mathrm{Id}$ is a $\hat{\pi}_*$-isomorphism by Proposition 5.50. The second map $\xi^{1,0}_{X,Y}$ is a $\hat{\pi}_*$-isomorphism by Proposition 5.56. □

The final result of this section will be an identification (in certain cases) of the naive homotopy groups of a smash product $X \wedge Y$ as a colimit of unstable homotopy groups of the form $\pi_{k+n+n}(X_n \wedge Y_n)$, compare Proposition 5.60 below. For this we will have to restrict to semistable symmetric spectra, and we first

We consider the sequence of symmetric spectra

$$(5.58)\quad \Sigma^\infty|X_0| \xrightarrow{\hat{\sigma}_0} \Omega(\Sigma^\infty|X_1|) \xrightarrow{\Omega\hat{\sigma}_1} \cdots \longrightarrow \Omega^m(\Sigma^\infty|X_m|) \xrightarrow{\Omega^m\hat{\sigma}_m} \Omega^{m+1}(\Sigma^\infty|X_{m+1}|) \longrightarrow \cdots .$$

Here $\hat{\sigma}_m : \Sigma^\infty|X_m| \longrightarrow \Omega(\Sigma^\infty|X_{m+1}|)$ is the morphism freely generated by the adjoint structure map $\tilde{\sigma}_m : |X_m| \longrightarrow \Omega|X_{m+1}|$ of the spectrum $|X|$. We let $\varphi_m : \Sigma^\infty|X_m| \longrightarrow \mathrm{sh}^m |X|$ be the morphism freely generated by the identity of $|X_m|$. Then for every $m \geq 0$ the square

$$\begin{tikzcd}
\Sigma^\infty|X_m| \arrow[r, "\tilde{\sigma}_m"] \arrow[d, "\varphi_m"'] & \Omega(\Sigma^\infty|X_{m+1}|) \arrow[d, "\Omega\varphi_{m+1}"] \\
\mathrm{sh}^m |X| \arrow[r, "\tilde{\lambda}_{\mathrm{sh}^m |X|}"'] & \Omega(\mathrm{sh}^{m+1} |X|)
\end{tikzcd}$$

commutes. So the morphisms $\Omega^m\varphi_m : \Omega^m(\Sigma^\infty|X_m|) \longrightarrow \Omega^m \mathrm{sh}^m |X|$ are compatible morphisms from the sequence (5.58) to the sequence (4.18) whose mapping telescope is the symmetric spectrum $\Omega^\infty \mathrm{sh}^\infty X$. By taking mapping telescopes we obtain a natural morphism

$$\Phi \;:\; \mathrm{tel}_m\, \Omega^m(\Sigma^\infty X_m) \;\longrightarrow\; \mathrm{tel}_m\, \Omega^m\, \mathrm{sh}^m X \;=\; \Omega^\infty \mathrm{sh}^\infty X\ .$$

Proposition 5.59. *For every semistable symmetric spectrum X the morphism $\Phi : \mathrm{tel}_m\, \Omega^m(\Sigma^\infty X_m) \longrightarrow \Omega^\infty \mathrm{sh}^\infty X$ is a level equivalence. Hence the symmetric spectrum $\mathrm{tel}_m\, \Omega^m(\Sigma^\infty X_m)$ is an Ω-spectrum.*

Proof. The key observation is that the n-th level of the morphism Φ, i.e., the map

$$\Phi_n \;:\; (\mathrm{tel}_m\, \Omega^m(\Sigma^\infty X_m)) \;=\; \mathrm{tel}_m\, \Omega^m(X_m \wedge S^n) \;\longrightarrow\; \mathrm{tel}_m\, \Omega^m X_{m+n} \;=\; (\Omega^\infty \mathrm{sh}^\infty X)_n$$

is isomorphic to the 0-th level of the morphism

$$\Omega^\infty \mathrm{sh}^\infty(\lambda^{(n)}_X) : \Omega^\infty \mathrm{sh}^\infty(S^n \wedge X) \;\longrightarrow\; \Omega^\infty \mathrm{sh}^\infty(\mathrm{sh}^n X)\ ,$$

where the morphism $\lambda_X^{(n)} : S^n \wedge X \longrightarrow \mathrm{sh}^n X$ is a certain iteration of the morphisms λ (compare (3.39)).[justify]

Since X is semistable, so are the symmetric spectrum $S^n \wedge X$ and $\mathrm{sh}^n X$. So in the commutative square

$$\begin{tikzcd} S^n \wedge X \arrow[r, "\lambda_X^{(n)}"] \arrow[d, "\lambda^\infty_{S^n\wedge X}"'] & \mathrm{sh}^n X \arrow[d, "\lambda^\infty_{\mathrm{sh}^n X}"] \\ \Omega^\infty \mathrm{sh}^\infty (S^n \wedge X) \arrow[r, "\Omega^\infty \mathrm{sh}^\infty(\lambda_X^{(n)})"'] & \Omega^\infty \mathrm{sh}^\infty (\mathrm{sh}^n X) \end{tikzcd}$$

both vertical morphisms are $\hat{\pi}_*$-isomorphisms and the two symmetric spectra in the bottom row are Ω-spectra (compare Proposition 4.24). Also because X is semistable the morphism $\lambda_X^{(n)}$ is a $\hat{\pi}_*$-isomorphism. So the lower horizontal morphism $\Omega^\infty \mathrm{sh}^\infty(\lambda_X^{(n)})$ is a $\hat{\pi}_*$-isomorphism between Ω-spectra, hence a level equivalence. In particular, the map in level 0 is a weak equivalence of spaces, hence so is the map Φ_n. $\square$

Now we let X and Y be two semistable symmetric spectrum. We define a natural map

$$\mathrm{colim}_n\ \pi_{k+n+n}(X_n \wedge Y_n) \longrightarrow \hat{\pi}_k(X \wedge Y)$$

that will turn out to be an isomorphism in certain cases. Here the colimit is formed over the composite maps

$$\begin{aligned} \pi_{k+n+n}(X_n \wedge Y_n) &\xrightarrow{-\wedge S^1 \wedge S^1} \pi_{k+n+n+1+1}(X_n \wedge Y_n \wedge S^1 \wedge S^1) \\ &\xrightarrow{(-1)^n \cdot (X_n \wedge \mathrm{twist} \wedge S^1)_*} \pi_{k+n+1+n+1}(X_n \wedge S^1 \wedge Y_n \wedge S^1) \\ &\xrightarrow{(\sigma_n \wedge \sigma_n)_*} \pi_{k+n+1+n+1}(X_{n+1} \wedge Y_{n+1}) \end{aligned}$$

We start from the map

$$\pi_{k+n+n}(X_n \wedge Y_n) \xrightarrow{(i_{n,n})_*} \pi_{k+n+n}(X \wedge Y)_{n+n} \longrightarrow \hat{\pi}_k(X \wedge Y)\ ;$$

the first part is the effect of the (n,n)-component of the universal bimorphism for X and Y, and the second map is the canonical one. The bimorphism property implies that the diagram

$$\begin{tikzcd} X_n \wedge Y_n \wedge S^1 \wedge S^1 \arrow[r, "i_{n,n} \wedge S^1 \wedge S^1"] \arrow[d, "X_n \wedge \mathrm{twist} \wedge S^1"'] & (X \wedge Y)_{n+n} \wedge S^1 \wedge S^1 \arrow[d, "\sigma^2"] \\ X_n \wedge S^1 \wedge Y_n \wedge S^1 \arrow[d, "\sigma_n \wedge \sigma_n"'] & (X \wedge Y)_{n+n+1+1} \arrow[d, "n+\chi_{n,1}+1"] \\ X_{n+1} \wedge Y_{n+1} \arrow[r, "i_{n+1,n+1}"'] & (X \wedge Y)_{n+1+n+1} \end{tikzcd}$$

commutes. Since $X \wedge Y$ is again semistable, the effect of the permutation $n+\chi_{n,1}+1$ on the homotopy groups of $(X \wedge Y)_{n+1+n+1}$ becomes the sign $(-1)^n$ in the colimit. So as n increases, the maps $\pi_{k+n+n}(X_n \wedge Y_n) \longrightarrow \hat{\pi}_k(X \wedge Y)$ are compatible with the colimit system.

Proposition 5.60. *Let X and Y be semistable symmetric spectra. Suppose also that X or Y is flat and the other spectrum is level cofibrant. Then the morphism*

$$\mathrm{colim}_n\ \pi_{k+n+n}(X_n \wedge Y_n) \longrightarrow \hat{\pi}_k(X \wedge Y)$$

is an isomorphism for every integer k.

PROOF. The square

$$\begin{tikzcd}
\mathrm{colim}_n\ \pi_{k+n+n}(X_n \wedge Y_n) \arrow[r] \arrow[d] & \hat\pi_k(X\wedge Y) \arrow[d, "(\lambda^\infty_{X\wedge Y})_*"] \\
\hat\pi_k(\mathrm{tel}_m\, \Omega^m(X_m \wedge Y)) \arrow[r] & \hat\pi_k(\Omega^\infty \mathrm{sh}^\infty(X\wedge Y))
\end{tikzcd}$$

commutes, where the left vertical map is defined by [...]. The lower horizontal map is obtained on mapping telescopes from the compatible spectrum morphisms

$$\Omega^m i_{m,-} \ :\ \Omega^m(X_m\wedge Y) \ \longrightarrow\ \Omega^m\,\mathrm{sh}^m(X\wedge Y)\ .$$

The morphism $\lambda^\infty_{X\wedge Y} : X\wedge Y \longrightarrow \Omega^\infty\mathrm{sh}^\infty(X\wedge Y)$ is a $\hat\pi_*$-isomorphisms since $X\wedge Y$ is semistable [ref]; The left vertical map is the composite of the bijections

$$\begin{aligned}
\mathrm{colim}_n\ \pi_{k+n+n}(X_n\wedge Y_n) \ &=_{\text{cofinal}}\ \mathrm{colim}_{m,n}\ \pi_{k+m+n}(X_m\wedge Y_n)\\
&=\ \mathrm{colim}_m\ \hat\pi_{k+m}(X_m\wedge Y)\\
&\longrightarrow\ \mathrm{colim}_m\ \hat\pi_k(\Omega^m(X_m\wedge Y))\\
&\longrightarrow\ \hat\pi_k(\mathrm{tel}_m\ \Omega^m(X_m\wedge Y))\ .
\end{aligned}$$

Since the two horizontal maps are isomorphisms, we may show that the lower vertical map is an isomorphism. This map also occurs in the commutative square

$$\begin{tikzcd}
(\mathrm{tel}_m(\Omega^m(\Sigma^\infty X_m)))\wedge Y \arrow[r, "\Phi\wedge Y"] \arrow[d, "\cong"'] & (\mathrm{tel}_m\,\Omega^m\,\mathrm{sh}^m\,X)\wedge Y \arrow[d, "\cong"] \\
\mathrm{tel}_m(\Omega^m(\Sigma^\infty X_m)\wedge Y) \arrow[r] \arrow[d] & \mathrm{tel}_m((\Omega^m\,\mathrm{sh}^m\,X)\wedge Y) \arrow[d] \\
\mathrm{tel}_m\,\Omega^m((\Sigma^\infty X_m)\wedge Y) \arrow[r] \arrow[d, "\cong"'] & \mathrm{tel}_m\,\Omega^m((\mathrm{sh}^m\,X)\wedge Y) \arrow[d, "\mathrm{tel}_m\,\Omega^m(\xi^{m,0})"] \\
\mathrm{tel}_m\,\Omega^m(X_m\wedge Y) \arrow[r] & \mathrm{tel}_m\,\Omega^m\,\mathrm{sh}^m(X\wedge Y)
\end{tikzcd}$$

Since Y is flat the functor $-\wedge Y$ preserves $\hat\pi_*$-isomorphisms, so the morphism $\Phi\wedge Y$ is a $\hat\pi_*$-isomorphism. All vertical maps are also $\hat\pi_*$-isomorphisms, and that finishes the proof. □

6. Homotopy groups

In this section we introduce and discuss another key concept, the true homotopy groups of a symmetric spectrum. The true homotopy groups detect stable equivalences, and are thus more important than the naive homotopy groups; however, but there does not seem to be any way to define the true homotopy groups directly from explicit invariants of the terms of a symmetric spectrum.

6.1. True homotopy groups. In Proposition 4.39 we introduced a functor Q on the category of symmetric spectra (in either flavor) that takes values in Ω-spectra and comes equipped with a natural stable equivalence $\eta_X : X \longrightarrow QX$.

Definition 6.1. Let X be a symmetric spectrum and k an integer. The k-th *true homotopy group* of X is given by

$$\pi_k X \ =\ \hat\pi_k(QX)\ ,$$

the k-th naive homotopy group of the symmetric spectrum QX.

The definition of true homotopy groups is rather indirect, and it is not immediately clear if and how the true homotopy groups of a symmetric spectrum X are determined by the spaces X_n, their homotopy groups and the rest of the available structure. We will later identify $\pi_k X$ with the group of maps from the k-dimensional sphere spectrum $\mathbb{S}^k$ to X in the stable homotopy category (to be introduced in Chapter II).

One of the key properties of the true homotopy groups is that they detect stable equivalences:

Theorem 6.2. *For every morphism $f : A \longrightarrow B$ of symmetric spectra the following are equivalent:*
(i) *the morphism f is a stable equivalence;*
(ii) *the map $\pi_k f : \pi_k A \longrightarrow \pi_k B$ of true homotopy groups is an isomorphism for all integers k.*
Moreover, every $\hat{\pi}_*$-isomorphism induces an isomorphism of true homotopy groups.

PROOF. Proposition 4.39 provides the commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ QA \arrow[r, "Qf"'] & QB \end{tikzcd}$$

in which the vertical morphisms are stable equivalences and the spectra QA and QB are Ω-spectra. So f is a stable equivalence if and only if Qf is a stable equivalence.

In general, every level equivalence is a $\hat{\pi}_*$-isomorphism and every $\hat{\pi}_*$-isomorphism is a stable equivalence. For morphisms between Ω-spectra, every stable equivalence is a level equivalence by Proposition 4.13, so all these three classes of equivalences actually coincide. In particular, f is a stable equivalence if and only if Qf is a $\hat{\pi}_*$-isomorphism. The latter means, by definition, that f induces isomorphisms of all true homotopy groups.

Since every $\hat{\pi}_*$-isomorphism is a stable equivalence (by Theorem 4.23), it also induces an isomorphism of true homotopy groups. $\square$

The true homotopy groups are in general different from the naive homotopy groups; when this happens, the naive groups can be thought of as 'pathological'. The two invariants are related by a natural homomorphism

$$c = (\eta_X)_* : \hat{\pi}_k X \longrightarrow \pi_k X ,$$

the effect of the natural stable equivalence $\eta_X : X \longrightarrow QX$ on naive homotopy groups.

Since the naive homotopy groups are defined in terms of the unstable homotopy groups of the levels of a spectra, they are often more readily accessible than the true homotopy groups. So it will be important to have a criterion for when the naive and true homotopy groups coincide (i.e., when the map c is an isomorphism). It turns out that this happens for the semistable symmetric spectra, i.e., those symmetric spectrum X for which the morphism $\lambda_X : S^1 \wedge X \longrightarrow \mathrm{sh}\, X$ is a $\hat{\pi}_*$-isomorphism. We will later see that this property in fact characterizes semistable symmetric spectra, compare Theorem 8.25.

Proposition 6.3. *For every semistable symmetric spectrum X and integer k the natural map $c : \hat{\pi}_k X \longrightarrow \pi_k X$ from the naive to true homotopy group is an isomorphism. Every stable equivalence between semistable spectra is a $\hat{\pi}_*$-isomorphism.*

PROOF. We consider the commutative square

$$\begin{tikzcd} \hat{\pi}_k X \arrow[r, "c=(\eta_X)_*"] \arrow[d, "\hat{\pi}_k(\lambda_X^\infty)"'] & \hat{\pi}_k(QX) \arrow[r, equal] \arrow[d, "\hat{\pi}_k(Q\lambda_X^\infty)"'] & \pi_k X \arrow[d, "\pi_k(\lambda_X^\infty)"] \\ \hat{\pi}_k(\Omega^\infty \mathrm{sh}^\infty X) \arrow[r, "c=(\eta_{\Omega^\infty \mathrm{sh}^\infty X})_*"'] & \hat{\pi}_k(Q(\Omega^\infty \mathrm{sh}^\infty X)) \arrow[r, equal] & \pi_k(\Omega^\infty \mathrm{sh}^\infty X) \end{tikzcd}$$

where the functor $\Omega^\infty \mathrm{sh}^\infty$ and the natural transformation $\lambda_X^\infty : X \longrightarrow \Omega^\infty \mathrm{sh}^\infty X$ were constructed in (4.18). If X is semistable, then the morphism $\lambda_X^\infty : X \longrightarrow \Omega^\infty \mathrm{sh}^\infty X$ is a $\hat{\pi}_*$-isomorphism (see Proposition 4.24).

So all vertical maps in the square are isomorphisms. Moreover, $\Omega^\infty \mathrm{sh}^\infty X$ is an Ω-spectrum (also Proposition 4.24), so the stable equivalence $\eta_{\Omega^\infty \mathrm{sh}^\infty X}$ is a level equivalence, and the lower horizontal map is an isomorphism. Hence the upper horizontal map is an isomorphism as well.

Now suppose that $f : A \longrightarrow B$ is a stable equivalence between semistable spectra. Then $Qf : QA \longrightarrow QB$ is a stable equivalence between Ω-spectra, thus a level equivalence. In particular Qf is a $\hat{\pi}_*$-isomorphism. Since A and B are semistable, the morphisms η_A and η_B are also $\hat{\pi}_*$-isomorphisms, hence so is f because we have $\eta_B \circ f = Qf \circ \eta_A$. □

In Example 4.26 we considered the stable equivalence $\lambda : F_1 S^1 \longrightarrow \mathbb{S}$ which is adjoint to the identity in level 1. This morphism is not a $\hat{\pi}_*$-isomorphism, but a consequence of Theorem 6.2 is that the stable equivalence λ also induces isomorphisms of true homotopy groups. Since the sphere spectrum $\mathbb{S}$ is semistable, its naive and true homotopy groups 'coincide' and so the homotopy groups of the spectrum $F_1 S^1$ are isomorphic to the stable stems. However, we can also calculate the true homotopy groups $\pi_*(S^n \wedge F_m)$ directly with the tools developed so far.

The topological sphere S^k represents the unstable homotopy group $\pi_k(X, x)$ of a based space X. This has the consequence that natural transformations out of the functor π_k are determined by their value on the 'fundamental class' $[\mathrm{Id}_{S^k}] \in \pi_k(S^k, \infty)$. More precisely: suppose that $F : \mathbf{T} \longrightarrow$ (sets) is a functor from the category of based spaces to sets that takes homotopy equivalences between cofibrant based spaces to bijections. Then for every element $u \in F(S^k)$ there is a unique natural transformation $\tau : \pi_k \longrightarrow F$ such that $\tau_{S^k}([\mathrm{Id}_{S^k}]) = u$. [sketch proof] This fact has an analogue for true homotopy groups of symmetric spectra, where now the dimension k can be any integer. One difference is that there are now many objects that we can reasonbly think of as a 'sphere of dimension k', namely all the free spectra $F_n S^{k+n}$.

We define naive and true *fundamental classes*

$$\hat{\iota}^n_m \in \hat{\pi}_{n-m}(F_m S^n) \qquad \text{and} \qquad \iota^n_m \in \pi_{n-m}(F_m S^n) \tag{6.4}$$

as follows: the naive fundamental class $\hat{\iota}^n_m$ is the class represented by the based map

$$1 \wedge - \;:\; S^n \longrightarrow \Sigma_m^+ \wedge S^n = (F_m S^n)_m \ ,$$

where 1 denotes the identity permutation in Σ_m. The true fundamental class is then defined as the image of $\hat{\iota}^n_m$ under the map from naive to true homotopy groups, i.e.,

$$\iota^n_m \;=\; c(\hat{\iota}^n_m) \ .$$

Proposition 6.5. *Let $G : \mathcal{S}p \longrightarrow$ (sets) be a functor that takes stable equivalences between flat symmetric spectra to isomorphisms.*

(i) *For every element $u \in G(F_m S^n)$ there is a unique natural transformation $\tau : \pi_{n-m} \longrightarrow G$ such that $\tau_{F_m S^n}(\iota^n_m) = u$.*
(ii) *For every integer k and every natural transformation $\psi : \hat{\pi}_k \longrightarrow G$ there is a unique natural transformation $\tau : \pi_k \longrightarrow G$ such that $\tau \circ c = \psi$.*

Proof. (i) We start with a preliminary observation. The first claim is that the functor G takes homotopic morphisms to the same map. Indeed, for every symmetric spectrum X the morphism $c : I^+ \wedge X \longrightarrow X$ that maps all of I to a point is a level equivalence, hence a stable equivalence (here $I = [0, 1]$ in the context of spaces and $I = \Delta[1]$ for spectra of simplicial sets). So $F(c)$ is bijective by hypothesis. The composite with the two end point inclusions $i_0, i_1 : X \longrightarrow I^+ \wedge X$ satisfy $c \circ i_0 = \mathrm{Id}_X = c \circ i_1$, so we have

$$F(c) \circ F(i_0) \;=\; \mathrm{Id}_{F(X)} \;=\; F(c) \circ F(i_1) \ .$$

Since $F(c)$ is an isomorphism, we deduce $F(i_0) = F(i_1)$. If $H : I^+ \wedge X \longrightarrow Y$ is some homotopy from a morphism f to a morphism g, Then we have

$$F(f) \;=\; F(H) \circ F(i_0) \;=\; F(H) \circ F(i_1) \;=\; F(g) \ ,$$

Now we claum that any given true homotopy class $x \in \pi_{n-m}X = \hat{\pi}_{n-m}(QX)$ can be represented by a morphism $F_m S^n \longrightarrow QX$ as follows: since X is an Ω-spectrum, the two maps

$$[F_m S^n, QX] \xrightarrow{\text{adjunction}} [S^n, (QX)_m] \longrightarrow \hat{\pi}_{n-m}(QX) = \pi_{n-m}X$$

are bijective, and the composite sends the homotopy class of a morphism $f : F_m S^n \longrightarrow QX$ to $f_*(\hat{\iota}^n_m)$. So given x, there is a morphism $f : F_m S^n \longrightarrow QX$, unique up to homotopy, such that $f_*(\hat{\iota}^n_m) = x$. By naturality we then have

$$f_*(\iota^n_m) \ = \ f_*(c(\hat{\iota}^n_m)) \ = \ c(f_*(\hat{\iota}^n_m)) \ = \ c(x) \ = \ (\eta_X)_*(x) \ , \tag{6.6}$$

where $\eta_X : X \longrightarrow QX$ is the stable equivalence whose effect on naive homotopy groups defines the map c.

Now we prove the proposition in the special case where G takes all stable equivalences to bijections. Given $x \in \pi_{n-m}X$ we let $f : F_n S^m \longrightarrow QX$ be a morphism such that $f_*(\hat{\iota}^n_m) = x$. If $\tau : \pi_{n-m} \longrightarrow G$ is natural, then the relation (6.6) implies

$$G(\eta_X)(\tau_X(x)) \ = \ \tau_{QX}((\eta_X)_* x) \ = \ \tau_{QX}(f_*(\iota^n_m)) \ = \ G(f)(\tau_{F_m S^n}(\iota^n_m)) \ .$$

Since $G(\eta_X)$ is bijective, this shows that the transformation τ is determined by the image of the fundamental class ι^n_m.

Conversely, the previous paragraph tells us that we have to define

$$\tau_X(x) \ = \ G(\eta_X)^{-1}(G(f)(u))$$

where $f : F_m S^n \longrightarrow QX$ is a morphism satisfying $f_*(\hat{\iota}^n_m) = x$. The functor G sends homotopic maps to the same map, so $\tau_X : \pi_{n-m}X \longrightarrow G(X)$ is well-defined.

As the symmetric spectrum X varies, the maps τ_X form a natural transformation: if $\varphi : X \longrightarrow Y$ is a morphism of symmetric spectra and $f : F_m S^n \longrightarrow QX$ satisfies $f_*(\iota^n_m) = x$, then $(Q\varphi) \circ f : F_m S^n \longrightarrow QY$ satisfies $((Q\varphi) \circ f)_*(\iota^n_m) = \varphi_*(x)$. So we have

$$G(\varphi) \circ G(\eta_X)^{-1} \circ G(f) \ = \ G(\eta_Y)^{-1} \circ G(Q\varphi) \circ G(f) \ = \ G(\eta_Y)^{-1} \circ G(Q\varphi \circ f) \ .$$

Evaluating both sides on the class u gives $G(\varphi)(\psi_X(x)) = \psi_Y(\varphi_*(x))$, i.e., ψ is natural. The map $\eta_{F_m S^n} : F_m S^n \longrightarrow Q(F_m S^n)$ satisfies $(\eta_{F_m S^n})_*(\hat{\iota}^n_m) = c(\hat{\iota}^n_m) = \iota^n_m$, so we have

$$\psi_{F_m S^n}(\iota^n_m) \ = \ G(\eta_{F_m S^n})^{-1}(G(\eta_{F_m S^n})(u)) \ = \ u \ .$$

This completes the construction of τ, and hence the proof of the proposition, in the special case where G takes all stable equivalences to bijections.

Now we treat the general case. We introduce another functor $G^\flat : \mathcal{S}p \longrightarrow (\text{sets})$ as $G^\flat(X) = G(X^\flat)$, where $r : X^\flat \longrightarrow X$ is the functorial flat resolution of Construction 5.53. The functor $G^\flat$ takes all stable equivalences to bijections, so the proposition is true for $G^\flat$ by the special case above.

The morphism $r_{F_m S^n} : (F_m S^n)^\flat \longrightarrow F_m S^n$ is a stable equivalence between flat spectra, so $G(r_{F_m S^n}) : G^\flat(F_m S^n) = G((F_m S^n)^\flat) \longrightarrow G(F_m S^n)$ is bijective. We let $\bar{u} \in G^\flat(F_m S^n)$ denote the preimage of the class $u \in G^\flat(F_m S^n)$. Since the proposition holds for $G^\flat$, there is a natural transformation $\tau^\flat : \pi_{n-m} \longrightarrow G^\flat$ such that $\tau^\flat_{F_m S^n}(\iota^n_m) = \bar{u}$. The composite transformation

$$\pi_{n-m}X \xrightarrow{\tau^\flat_X} G^\flat(X) \ = \ G(X^\flat) \xrightarrow{G(r_X)} G(X)$$

then satisfies $G(r_{F_m S^n})(\tau^\flat_{F_m S^n}(\iota^n_m)) = G(r_{F_m S^n})(\bar{u}) = u$, so it is the desired natural transformation $\tau : \pi_{n-m} \longrightarrow G$.

If $\psi : \pi_{n-m} \longrightarrow G$ is another transformation that also satisfies $\psi_{F_m S^n}(\iota^n_m)) = u$, then

$$G(r_{F_m S^n})(\psi_{(F_m S^n)^\flat}((r_{F_m S^n})^{-1}_*(\iota^n_m))) \ = \ \psi_{F_m S^n}((r_{F_m S^n})_*(r_{F_m S^n})^{-1}_*(\iota^n_m))) \ = \ \psi_{F_m S^n}(\iota^n_m) \ = \ u \ ;$$

so the composite transformation

$$\pi_{n-m}X \xrightarrow{(r_X)^{-1}_*} \pi_{n-m}(X^\flat) \xrightarrow{\psi_{X^\flat}} G(X^\flat) \ = \ G^\flat(X)$$

sends the fundamental class ι_m^n to $\bar{u}$. By the uniqueness property for the functor $G^\flat$, the composite must be equal to $\tau^\flat$. But then

$$\psi_X \;=\; G(r_X)\circ\psi_{X^\flat}\circ(r_X)_*^{-1} = G(r_X)\circ\tau^\flat_X = \tau_X$$

where the first equation follows from the naturality $\psi_X\circ(r_X)_* = G(r_X)\circ\psi_{X^\flat}$ because $(r_X)_* : \pi_{n-m}(X^\flat) \longrightarrow \pi_{m-n}X$ is invertible.

(ii) We set $m = 0$ if $k \geq 0$ and $m = -k$ if $k \leq 0$. By part (i) there is a natural transformation $\tau : \pi_k \longrightarrow G$ such that $\tau_{F_m S^{k+m}}(\iota_m^{k+m}) = \psi(\hat{\iota}_m^{k+m})$. We claim that then $\tau\circ c = \psi$. For this purpose we consider the difference $\delta = \tau\circ c - \psi$. We have

$$\delta(\hat{\iota}_m^{k+m}) \;=\; \tau(c(\hat{\iota}_m^{k+m})) - \psi(\hat{\iota}_m^{k+m}) \;=\; \tau(\iota_m^{k+m}) - \psi(\hat{\iota}_m^{k+m}) \;=\; 0 \;.$$

For $l \geq 0$ we let $\lambda : F_{m+l}S^{k+m+l} \longrightarrow F_m S^{k+m}$ be the morphism that extends the map

$$1\wedge - \;:\; S^{k+m+l} \;\longrightarrow\; \Sigma^+_{m+l}\wedge_{1\times\Sigma_l} S^{k+m}\wedge S^l = (F_m S^{k+m})_{m+l} \;;$$

then $\lambda_*(\hat{\iota}_{m+l}^{k+m+l}) = \hat{\iota}_m^{k+m}$, and thus

$$G(\lambda)(\delta(\hat{\iota}_{m+l}^{k+m+l})) \;=\; \delta(\lambda_*(\hat{\iota}_{m+l}^{k+m+l})) \;=\; \delta(\hat{\iota}_m^{k+m}) \;=\; 0 \;.$$

Since λ is a stable equivalence between flat symmetric spectra, $G(\lambda)$ is bijective and so $\delta(\hat{\iota}_{m+l}^{k+m+l}) = 0$.

Given any class $x \in \hat{\pi}_k X$, we choose a representative $S^{k+m+l} \longrightarrow X_{m+l}$ for suitable $l \geq 0$; the extension $f : F_{m+l}S^{k+m+l} \longrightarrow X$ then satisfies $f_*(\hat{\iota}_{m+l}^{k+m+l}) = x$. So we have

$$\delta_X(x) \;=\; \delta_X(f_*(\hat{\iota}_{m+l}^{k+m+l})) \;=\; G(f)(\delta_{F_{m+l}S^{k+m+l}}(\hat{\iota}_{m+l}^{k+m+l})) \;=\; 0 \;.$$

So the difference δ is identically zero, and thus $\tau\circ c = \psi$.

To show the uniqueness of the transformation we suppose that $\tau, \tau' : \pi_k \longrightarrow G$ are two natural transformations such that $\tau\circ c = \tau'\circ c$. Then $\tau_{F_m S^{k+m}}(\iota_m^{k+m}) = \tau'_{F_m S^{k+m}}(\iota_m^{k+m})$ because the fundamental class ι_m^{k+m} is in the image of the map $c : \hat{\pi}_k(F_m S^{k+m}) \longrightarrow \pi_k(F_m S^{k+m})$. So $\tau = \tau'$ by the uniqueness statement in part (i). $\square$

The true homotopy groups share many of the formal properties of naive homotopy groups: they shift under suspension and looping, takes wedges to sums, preserve finite products, and turn homotopy cofiber and fiber sequences to long exact sequences. We now establish these and other properties.

In the context of symmetric spectra of spaces, the suspension and loop functors preserves stable equivalences by Proposition 4.29; so the functors that take a symmetric spectrum of spaces X to the groups $\pi_{1+k}(S^1\wedge X)$ respective $\pi_k(\Omega X)$ take stable equivalences to isomorphisms. So we can apply Proposition 6.5 (ii) to the natural transformations $c\circ(S^1\wedge -) : \hat{\pi}_k X \longrightarrow \pi_{1+k}X$ and $c\circ\alpha^{-1} : \hat{\pi}_{1+k}X \longrightarrow \pi_k(\Omega X)$. The proposition yields natural homomorphisms

$$S^1\wedge - \;:\; \pi_k X \;\longrightarrow\; \pi_{1+k}(S^1\wedge X) \qquad \text{and} \qquad \alpha^{-1} \;:\; \pi_{1+k}X \;\longrightarrow\; \pi_k(\Omega X)$$

that are uniquely determined by the property that the squares

$$\begin{tikzcd} \hat{\pi}_k X \arrow[r, "c"] \arrow[d, "S^1\wedge -"'] & \pi_k X \arrow[d, "S^1\wedge -"] \\ \hat{\pi}_{1+k}(S^1\wedge X) \arrow[r, "c"'] & \pi_{1+k}(S^1\wedge X) \end{tikzcd} \qquad \begin{tikzcd} \hat{\pi}_{1+k} X \arrow[r, "c"] \arrow[d, "\alpha^{-1}"'] & \pi_{1+k} X \arrow[d, "\alpha^{-1}"] \\ \hat{\pi}_k(\Omega X) \arrow[r, "c"'] & \pi_k(\Omega X) \end{tikzcd}$$

commute. At this point, the expression 'α^{-1}' is just a name for a natural transformation; however, we shall now show that α^{-1} is invertible, and then we will denote its inverse by 'α'.

Proposition 6.7. *Let X be a symmetric spectrum X of spaces and k an integer. Then the suspension and loop homomorphisms*

$$S^1\wedge - \;:\; \pi_k X \;\longrightarrow\; \pi_{1+k}(S^1\wedge X) \qquad \textit{and} \qquad \alpha^{-1} \;:\; \pi_{1+k}X \;\longrightarrow\; \pi_k(\Omega X)$$

are isomorphisms of true homotopy groups. Moreover, the triangles

$$\begin{tikzcd} \pi_k(\Omega X) \arrow[rr, "\alpha"] \arrow[dr, "S^1\wedge -"'] & & \pi_{1+k}X \\ & \pi_{1+k}(S^1\wedge(\Omega X)) \arrow[ur, "\pi_k\epsilon"'] & \end{tikzcd} \qquad \begin{tikzcd} \pi_k X \arrow[rr, "S^1\wedge -"] \arrow[dr, "\pi_k\eta"'] & & \pi_{1+k}(S^1\wedge X) \\ & \pi_k(\Omega(S^1\wedge X)) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

commute where $\eta : X \longrightarrow \Omega(S^1 \wedge X)$ and $\epsilon : S^1 \wedge (\Omega X) \longrightarrow X$ are the unit respectively counit of the adjunction and α denotes the inverse of α^{-1}.

Proof. The natural transformation $\pi_k(\epsilon)\circ(S^1\wedge -)\circ\alpha^{-1}$ from the functor π_{1+k} to itself satisfies

$$\begin{aligned}\pi_k(\epsilon)\circ(S^1\wedge -)\circ\alpha^{-1}\circ c \;&=\; \pi_k(\epsilon)\circ(S^1\wedge -)\circ c\circ\alpha^{-1} \;=\; \pi_k(\epsilon)\circ c\circ(S^1\wedge -)\circ\alpha^{-1}\\ &=\; c\circ\hat\pi_k(\epsilon)\circ(S^1\wedge -)\circ\alpha^{-1} \;=\; c\end{aligned}$$

where the third equation is the commutativity of the triangle (2.5). By Proposition 6.5 (ii) any natural transformation $\pi_k \longrightarrow \pi_k$ is determined by its precomposition with $c : \hat\pi_k \longrightarrow \pi_k$, so we deduce that $\pi_k(\epsilon)\circ(S^1\wedge -)\circ\alpha^{-1}$ is the identity transformation of the functor $\pi_{1+k}X$. The adjunction counit is a $\hat\pi_*$-isomorphism by Proposition 2.6, and hence a stable equivalence. So $\pi_k(\epsilon)$ is bijective, α^{-1} is injective and $S^1\wedge -$ is surjective.

Similarly, we have

$$\alpha^{-1}\circ(S^1\wedge -)\circ c \;=\; \alpha^{-1}\circ c\circ(S^1\wedge -) \;=\; c\circ\alpha^{-1}\circ(S^1\wedge -) \;=\; c\circ\hat\pi_k(\eta) \;=\; \pi_k(\eta)\circ c$$

as transformations from $\hat\pi_k$ to $\pi_k(\Omega(S^1\wedge -))$, using again (2.5). This implies $\alpha^{-1}\circ(S^1\wedge -) = \pi_k(\eta)$. The adjunction unit is also a $\hat\pi_*$-isomorphism by Proposition 2.6, and hence $\pi_k(\eta)$ is bijective. So α^{-1} is surjective and $S^1\wedge -$ is injective. Altogether, both transformations $S^1\wedge -$ and α^{-1} are bijective.

Finally, composing the relations $\pi_k(\epsilon)\circ(S^1\wedge -)\circ\alpha^{-1} = \mathrm{Id}$ and $\alpha^{-1}\circ(S^1\wedge -) = \pi_k(\eta)$ with α from the appropriate side, we yields the desired relations $\pi_k(\epsilon)\circ(S^1\wedge -) = \alpha$ respectively $(S^1\wedge -) = \alpha\circ\pi_k(\eta)$. □

In Example 1.11 we discussed a right action of the stable stems π_*^{s} (also known as the naive homotopy groups of the sphere spectrum) on the *naive* homotopy groups $\hat\pi_* X$ of a symmetric spectrum X. Since the true homotopy groups X are just the naive homotopy groups of the symmetric spectrum QX, we also obtain an action of the stable stems on the *true* homotopy groups $\pi_* X$. Moreover, the natural transformation $c : \hat\pi_k X \longrightarrow \pi_k X$ is π_*^{s}-linear because it is induced by a homomorphism of symmetric spectra.

Proposition 6.8. *The true homotopy groups $\pi_*(F_m S^n)$ are freely generated as a graded π_*^{s}-module by the fundamental class ι_m^n.*

Proof. The spectrum $F_0 S^0$ is isomorphic to the sphere spectrum and thus semistable, so

$$\pi_k^{\mathrm{s}} = \hat\pi_k\mathbb{S} \xrightarrow{\;c\;} \pi_k\mathbb{S} \cong \pi_k(F_0 S^0)$$

is bijective by Proposition 6.3. Since the map c is π_*^{s}-linear and takes the unit $1 \in \hat\pi_0\mathbb{S}$ to ι_0^0, this proves the claim in the case $n = m = 0$.

The canonical isomorphism $a : S^1\wedge F_0 S^n \longrightarrow F_0 S^{1+n}$ satisfies

$$a_*(S^1\wedge\hat\iota_0^n) \;=\; \hat\iota_0^{1+n} \quad\text{and}\quad a_*(S^1\wedge\iota_0^n) \;=\; \iota_0^{1+n}\ .$$

Moreover, the composite

$$\pi_k(F_0 S^n) \xrightarrow{\;S^1\wedge -\;} \pi_{1+k}(S^1\wedge(F_0 S^n)) \xrightarrow{\;a_*\;} \pi_{1+k}(F_0 S^{1+n})$$

is a π_*^s-linear isomorphism, so induction on n proves the proposition for $m = 0$ and any n.

We let $\lambda : F_{m+1}S^{n+1} \longrightarrow F_m S^n$ be the morphism adjoint to the map

$$1\wedge - \;:\; S^{n+1} \longrightarrow \Sigma_{m+1}^+\wedge S^n\wedge S^1 = (F_m S^n)_{m+1}\ .$$

This morphism is a stable equivalence [...], so it induces a π_*^{s}-linear isomorphism of true homotopy groups. Moreover, $\lambda_*(\hat\iota_{m+1}^{n+1}) = \hat\iota_m^n$ and $\lambda_*(\iota_{m+1}^{n+1}) = \iota_m^n$. So the proof finishes by induction on m. □

Given a morphism of symmetric spectra $f : X \longrightarrow Y$, the mapping cone $C(f)$ and homotopy fiber $F(f)$ were defined in (2.8) respectively (2.14). We can mimick the definitions of the connecting homomorphisms for naive homotopy groups (compare (2.11) respectively (2.15)) with true homotopy groups. We then obtain long exact sequences of true homotopy groups as follows. We define a *connecting homomorphism* $\delta : \pi_{1+k}C(f) \longrightarrow \pi_k X$ as the composite

$$(6.9) \qquad \pi_{1+k}C(f) \xrightarrow{\pi_{1+k}(p)} \pi_{1+k}(S^1 \wedge X) \cong \pi_k X ,$$

where the first map is the effect of the projection $p : C(f) \longrightarrow S^1 \wedge X$ on true homotopy groups, and the second map is the inverse of the suspension isomorphism $S^1 \wedge - : \pi_k X \longrightarrow \pi_{1+k}(S^1 \wedge X)$. We define a *connecting homomorphism* $\delta : \pi_{1+k}Y \longrightarrow \pi_k F(f)$ as the composite

$$(6.10) \qquad \pi_{1+k}Y \xrightarrow{\alpha^{-1}} \pi_k(\Omega Y) \xrightarrow{\pi_k(i)} \pi_k F(f) ,$$

where $\alpha : \pi_k(\Omega Y) \longrightarrow \pi_{1+k}Y$ is the loop isomorphism and $i : \Omega X \longrightarrow F(f)$ the injection of the loop spectrum into the homotopy fiber. Since loop and suspension isomorphisms for naive and true homotopy groups are compatible with the tautological map $c : \hat{\pi}_k \longrightarrow \pi_k$, both connecting homomorphisms are compatible with the tautological map $c : \hat{\pi}_k \longrightarrow \pi_k$.

Proposition 6.11. *Let $f : X \longrightarrow Y$ be a morphism of symmetric spectra.*

(i) *The long sequence of true homotopy groups*

$$\cdots \longrightarrow \pi_k X \xrightarrow{f_*} \pi_k Y \xrightarrow{i_*} \pi_k C(f) \xrightarrow{\delta} \pi_{k-1} X \longrightarrow \cdots$$

is exact.

(ii) *In the simplicial context, suppose also that X and Y are levelwise Kan complexes. Then the long sequence of true homotopy groups*

$$\cdots \longrightarrow \pi_k X \xrightarrow{f_*} \pi_k Y \xrightarrow{\delta} \pi_{k-1} F(f) \xrightarrow{p_*} \pi_{k-1} X \longrightarrow \cdots$$

is exact.

(iii) *Suppose that f is an h-cofibration of symmetric spectra of topological spaces or an injective morphism of symmetric spectra of simplicial sets. Denote by $q : Y \longrightarrow Y/X$ the quotient map. Then the natural sequence of true homotopy groups*

$$\cdots \longrightarrow \pi_k X \xrightarrow{f_*} \pi_k Y \xrightarrow{q_*} \pi_k(Y/X) \xrightarrow{\delta} \pi_{k-1} X \longrightarrow \cdots$$

is exact, where the connecting map δ is the composite of the inverse of the isomorphism $\pi_k C(f) \longrightarrow \pi_k(Y/X)$ induced by the level equivalence $C(f) \longrightarrow Y/X$ which collapses the cone of X and the connecting homomorphism $\pi_k C(f) \longrightarrow \pi_{k-1} X$ defined in (6.9)*.*

(iv) *Suppose that f is levelwise a Serre fibration of spaces respectively Kan fibration of simplicial sets. Denote by $i : F \longrightarrow X$ the inclusion of the fiber over the basepoint. Then the natural sequence of true homotopy groups*

$$\cdots \longrightarrow \pi_k F \xrightarrow{i_*} \pi_k X \xrightarrow{f_*} \pi_k Y \xrightarrow{\delta} \pi_{k-1} F \longrightarrow \cdots$$

is exact, where the connecting map δ is the composite of the connecting homomorphism $\pi_k Y \longrightarrow \pi_{k-1} F(f)$ defined in (6.10) *and the inverse of the isomorphism $\pi_k F(f) \longrightarrow \pi_k F$ induced by the level equivalence $F \longrightarrow F(f)$ which send $x \in F$ to* (const_*, x)*.*

Proof. (i)

(ii) The stable equivalences $\eta_X : X \longrightarrow QX$ and $\eta_Y : Y \longrightarrow QY$ and the morphism $Qf : QX \longrightarrow QY$ satisfy $Qf \circ \eta_X = \eta_Y \circ f$. By Proposition 4.31 (iii) the morphism $\bar{\eta} : F(f) \longrightarrow F(Qf)$ induced by η_X and η_Y on homotopy fibers is then again a stable equivalence. Moreover, $F(Qf)$ is an Ω-spectrum since source and target of Qf are [ref]. In the commutative square

$$\begin{tikzcd} F(f) \arrow[r, "\bar{\eta}"] \arrow[d, "\eta_{F(f)}"'] & F(Qf) \arrow[d, "\eta_{F(Qf)}"] \\ QF(f) \arrow[r, "Q(\bar{\eta})"'] & QF(Qf) \end{tikzcd}$$

the morphism $Q(\bar{\eta})$ is then a stable equivalence since the other three maps are. Moreover, the three spectra $QF(f)$, $QF(Qf)$ and $F(Qf)$ are Ω-spectra, so the stable equivalences $Q(\bar{\eta})$ and $\eta_{F(Qf)}$ are $\hat{\pi}_*$-isomorphisms. We can thus form the composite

$$(\eta_{F(Qf)})_*^{-1} \circ Q(\bar{\eta})_* \;:\; \pi_k F(f) = \hat{\pi}_k(QF(f)) \;\longrightarrow\; \hat{\pi}_k F(Qf) \ .$$

We can now compare the sequence in question with the long homotopy sequence of naive homotopy groups for $Qf : QX \longrightarrow QY$ via the diagram:

$$\begin{tikzcd} \pi_{k+1}X \arrow[r, "f_*"] \arrow[d, equal] & \pi_{k+1}Y \arrow[r, "\delta"] \arrow[d, equal] & \pi_k F(f) \arrow[r, "i_*"] \arrow[d, "(\eta_{F(Qf)})_*^{-1}\circ Q(\bar{\eta})_*"] & \pi_k X \arrow[r, "f_*"] \arrow[d, equal] & \pi_k Y \arrow[d, equal] \\ \hat{\pi}_{k+1}QX \arrow[r, "(Qf)_*"'] & \hat{\pi}_{k+1}QY \arrow[r, "\delta"'] & \hat{\pi}_k F(Qf) \arrow[r, "i_*"'] & \hat{\pi}_k QX \arrow[r, "(Qf)_*"'] & \hat{\pi}_k QY \end{tikzcd}$$

The two middle squares commute [...] The lower row of naive homotopy groups is exact by Proposition 2.17, applied to $Qf : QX \longrightarrow QY$. Since all vertical maps are bijective, the upper row is exact.

(i) The natural morphism $h : S^1 \wedge F(f) \longrightarrow C(f)$ defined in (2.16) is a $\hat{\pi}_*$-isomorphism (and hence a stable equivalence) by Proposition 2.17. Hence h induces an isomorphism of true homotopy groups, and so does the composite $h_* \circ (S^1 \wedge -) : \pi_k F(f) \longrightarrow \pi_{1+k} C(f)$. We can now compare the sequence in question with the exact sequence of part (ii). We claim that the diagram the diagram

$$\begin{tikzcd} \pi_{1+k}X \arrow[r, "f_*"] \arrow[d, "(-1)\cdot"'] & \pi_{1+k}Y \arrow[r, "\delta"] \arrow[d, "(-1)\cdot"'] & \pi_k F(f) \arrow[r, "p_*"] \arrow[d, "h_*\circ(S^1\wedge -)"'] & \pi_k X \arrow[r, "f_*"] \arrow[d, equal] & \pi_k Y \arrow[d, equal] \\ \pi_{1+k}X \arrow[r, "f_*"'] & \pi_{1+k}Y \arrow[r, "i_*"'] & \pi_{1+k}C(f) \arrow[r, "\delta"'] & \pi_k X \arrow[r, "f_*"'] & \pi_k Y \end{tikzcd}$$

commutes. This is clear for the two outer squares. By the uniqueness part of Proposition **??** it suffices to show this after precomposition with the transformation $c : \hat{\pi}_{1+k}Y \longrightarrow \pi_{1+k}Y$ (for the left square) respectively the transformation $c : \hat{\pi}_k F(f) \longrightarrow \pi_k F(f)$ (for the right square). [no: this data depends on the morphism f!]

The upper row of the commutative diagram is exact by part (ii), and all vertical evaluation maps are bijective; so the lower row is exact.

(iii) Since f is an h-cofibration (in the topological context) respectively levelwise injective (in the simplicial context), the collaps morphism $C(f) \longrightarrow Y/X$ is a level equivalence, hence stable equivalence.

(iv) If $f : X \longrightarrow Y$ is a morphism of symmetric spectra which is levelwise a Serre fibration of spaces respectively Kan fibration of simplicial sets, the strict fiber F is level equivalent to the homotopy fiber. □

Proposition 2.19 says that naive homotopy groups commute with finite products and arbitrary coproducts. Now we show that analogous result for true homotopy groups.

Proposition 6.12. (i) *For every family of symmetric spectra $\{A^i\}_{i\in I}$ and every integer k the canonical map*

$$\bigoplus_{i\in I} \pi_k A^i \;\longrightarrow\; \pi_k \left(\bigvee_{i\in I} A^i \right)$$

is an isomorphism of abelian groups.

(ii) *For every finite indexing set I, every family $\{A^i\}_{i\in I}$ of symmetric spectra and every integer k the canonical map*
$$\pi_k\left(\prod_{i\in I} A^i\right) \longrightarrow \prod_{i\in I} \pi_k A^i$$
is an isomorphism of abelian groups.

(iii) *True homotopy groups commute with filtered colimits over closed embeddings.*

(iv) *Let $f_n : X_n \longrightarrow X_{n+1}$ be morphisms of symmetric spectra of simplicial sets for $n \geq 0$. Then the natural map*
$$\mathrm{colim}_n\ \pi_k(X_n) \longrightarrow \pi_k(\mathrm{colim}_n X_n)$$
is an isomorphism.

Proof. (i) We use the same proof as for naive homotopy groups in Proposition 2.19. In the special case of two summands A and B Proposition 6.11 (i) provides a long exact true homotopy group sequence associated to the wedge inclusion $i_A : A \longrightarrow A \vee B$. Since i_A has a retraction, the sequence splits into short exact sequences
$$0 \longrightarrow \pi_k A \xrightarrow{(i_A)_*} \pi_k(A \vee B) \xrightarrow{i_*} \pi_k(C(i_A)) \longrightarrow 0\ .$$
The mapping cone $C(i_A)$ is homotopy equivalent to B and we can replace $\pi_k(C(i_A))$ by $\pi_k B$ and to conclude that $\pi_k(A \vee B)$ splits as the sum of $\pi_k A$ and $\pi_k B$, via the canonical map. The case of a finite indexing set I now follows by induction, and the general case follows since homotopy groups of symmetric spectra commute with filtered colimits [more precisely, the image of every compact space in an infinite wedge lands in a finite wedge].

(ii) For finite indexing sets I the canonical map $\bigvee_I A^i \longrightarrow \prod_I A^i$ from the wedge to the product is a stable equivalence by Corollary 4.25. Hence this map induced isomorphisms of true homotopy groups, and the claim follows from part (i) and the fact that finite sums of abelian groups are also products.

(iii)

(iv) □

Now we calculate the true homotopy groups in some examples. Most of the time this 'calculation' will consist in a reduction of the problem to the calculation of *naive* homotopy groups of other symmetric spectra.

Example 6.13. The true homotopy groups of a free symmetric spectrum $F_m K$ generated by a based space (or simplicial set) K in level m are isomorphic to the stable homotopy groups of K, shifted m dimensions. Indeed, in Example 4.35 we proved that the morphism
$$\varphi^m\ :\ F_m K \longrightarrow \Omega^m(\Sigma^\infty K)$$
is a stable equivalence, where φ^m is adjoint to the adjunction unit $K \longrightarrow \Omega^m(K \wedge S^m) = \Omega^m(\Sigma^\infty K)_m$. Hence the adjoint
$$\hat{\varphi}^m\ :\ S^m \wedge F_m K \longrightarrow \Sigma^\infty K$$
is a stable equivalence as well (and this actually holds in the context of spaces and simplicial set). So the composite
$$\pi_k(F_m K) \xrightarrow{S^m \wedge -} \pi_{m+k}(S^m \wedge F_m K) \xrightarrow{\hat{\varphi}^m_*} \pi_{m+k}(\Sigma^\infty K) = \pi^{\mathrm{s}}_{m+k} K$$
is an isomorphism.

For a suspension spectrum the morphism $\lambda_{\Sigma^\infty K} : S^1 \wedge \Sigma^\infty K \longrightarrow \mathrm{sh}(\Sigma^\infty K)$ is an isomorphism So suspension spectra are in particular semistable. Hence true and naive homotopy groups coincide for suspension spectra by Proposition 6.3.

As we showed in Example 4.35, the stable equivalence φ^m is equivariant with respect to the two right actions of the symmetric group Σ_m (on the 'free coordinates' in the source, and via the left action on the m loop coordinates in the target). Hence the isomorphisms
$$\pi_{m+k}(F_m K) \xrightarrow{\varphi^m_*} \pi_{m+k}(\Omega^m(\Sigma^\infty K)) \xleftarrow{c} \hat{\pi}_{m+k}(\Omega^m(\Sigma^\infty K))$$

are Σ_m-equivariant. Permuting loop coordinates clearly acts by sign on naive homotopy groups; altogether this shows that the right Σ_m-action on the free coordinates of $F_m K$ induces the sign action on true homotopy groups.

6.2. Products on homotopy groups. The true homotopy groups are nicely compatible with the multiplicative structure given by the smash product, as we explain in this section. We construct a biadditive pairing of true homotopy groups

$$\cdot \ : \ \pi_k X \times \pi_l Y \ \longrightarrow \ \pi_{k+l}(X \wedge Y)$$

for pairs of integers k, l, with certain desirable properties which are listed in the next theorem. The theorem can be summarized in fancy language by saying that the pairing makes the graded true homotopy groups into a lax symmetric monoidal functor from the category of symmetric spectra (under smash product) to the category of graded abelian groups (under graded tensor product, with Koszul sign convention for the symmetry isomorphism). Later we show that if X is $(k-1)$-connected, Y is $(l-1)$-connected and at least one of them is flat, then $X \wedge Y$ is $(k+l-1)$-connected and the map (6.17) is an isomorphism, compare Remark II.5.23.

A formal consequence of the compatibility properties is that the true homotopy groups of a symmetric ring spectrum naturally form a graded ring, which is commutative in the graded sense if the ring spectrum is, compare Proposition 6.25.

The *fundamental class* $1 \in \pi_0\mathbb{S}$ of the sphere spectrum is defined as the image, under the map $c : \hat{\pi}_0\mathbb{S} \longrightarrow \pi_0\mathbb{S}$, of the naive homotopy class represented by the identity of $S^0 = \mathbb{S}_0$. The class 1 is essentially the class ι_0^0 of (6.4). More precisely, there is a unique isomorphism $F_0 S^0 \cong \mathbb{S}$, the adjoint of the identity of $S^0 = \mathbb{S}_0$, and the induces isomorphism from $\pi_0(F_0 S^0)$ to $\pi_0\mathbb{S}$ takes ι_0^0 to 1. We recall from Construction 5.6 that the sphere spectrum is a strict unit for the smash product of symmetric spectra.

Corollary 6.14. *Let $\{\Phi_k\}_{k\in\mathbb{Z}}$ be a family of functors $\Phi_k : \mathcal{S}p \longrightarrow$ (sets) equipped with natural isomorphisms $\Sigma : \Phi_k \longrightarrow \Phi_{1+k}(S^1 \wedge -)$. Suppose that every Φ_k takes stable equivalences between flat symmetric spectra to bijections. Then for every element $u \in \Phi_0(\mathbb{S})$ there is a unique collection of natural transformations*

$$\tau^k \ : \ \pi_k \ \longrightarrow \ \Phi_k \ ,$$

for $k \in \mathbb{Z}$, satisfying $\tau^0_{\mathbb{S}}(1) = u$ and such that the diagrams

$$\begin{tikzcd} \pi_k X \arrow[r, "S^1\wedge -"] \arrow[d, "\tau^k_X"'] & \pi_{1+k}(S^1 \wedge X) \arrow[d, "\tau^{1+k}_{S^1\wedge X}"] \\ \Phi_k(X) \arrow[r, "\Sigma"'] & \Phi_{1+k}(S^1 \wedge X) \end{tikzcd} \tag{6.15}$$

commute for every integer k and every symmetric spectrum X.

Proof. For $k \geq 0$ the isomorphism $j : F_0 S^k \longrightarrow S^k \wedge \mathbb{S}$ adjoint to the canonical isomorphism $S^k \cong S^k \wedge S^0 = (S^k \wedge \mathbb{S})_k$ satisfies $j_*(\iota_0^k) = S^k \wedge 1$ in $\pi_k(S^k \wedge \mathbb{S})$. Via the isomorphism j, Proposition 6.5 (i) translates into the statement that for every element $v \in \Phi_k(S^k \wedge \mathbb{S})$ there is a unique natural transformation $\tau : \pi_k \longrightarrow \Phi_k$ such that $\tau_{S^k\wedge\mathbb{S}}(S^k \wedge 1) = v$.

In particular, for every class $u \in \Phi_0(\mathbb{S})$ there is a unique natural transformation $\tau^0 : \pi_0 \longrightarrow \Phi_0$ such that $\tau_{\mathbb{S}}(1) = u$. To construct τ^{1+k} for $k \geq 0$ we proceed by induction on k. As we just noted, there is a unique natural transformation $\tau^{1+k} : \pi_{1+k} \longrightarrow \Phi_{1+k}$ such that $\tau^{1+k}_{S^{1+k}\wedge\mathbb{S}}(S^{1+k} \wedge 1) = \Sigma(\tau^k_{S^k\wedge\mathbb{S}}(S^k \wedge 1))$. Since the two natural transformations

$$\Sigma \circ \tau^k \ , \ \tau^{1+k}_{S^1\wedge -} \circ (S^1 \wedge -) \ : \pi_k X \ \longrightarrow \ \Phi_{1+k}(S^1 \wedge X)$$

agree on the class $S^k \wedge 1$, they agree altogether by the uniqueness clause above. In other words, the compatibility diagram (6.15) commutes for $k \geq 0$, and there only one family of transformations with that property.

For negative k the compatibility condition (6.15) forces us to define τ^k by downward induction as

$$\tau^k_X \;=\; \Sigma^{-1} \circ \tau^{1+k}_{S^1\wedge X} \circ (S^1 \wedge -)$$

and this definition automatically makes the square (6.15) commute in negative degrees. □

Now we can state and prove the main result of this section, the construction and properties of the pairing of true homotopy groups.

One of the key properties of the pairing refers to the isomorphisms

$$(S^1 \wedge X) \wedge Y \xrightarrow{\;a_1\;} S^1 \wedge (X \wedge Y) \xleftarrow{\;a_2\;} X \wedge (S^1 \wedge Y)$$

[recall/define] The two isomorphisms are related by $a_1 \circ \tau_{X,S^1\wedge Y} = (S^1 \wedge \tau_{X,Y}) \circ a_2$.

Theorem 6.16. *There is a unique family of natural pairings*

$$\cdot \;:\; \pi_k X \times \pi_l Y \;\longrightarrow\; \pi_{k+l}(X \wedge Y) \tag{6.17}$$

of true homotopy groups for $k, l \in \mathbb{Z}$ subject to the following two conditions.

(Normalization) *We have $1 \cdot 1 = 1$ in $\pi_0(\mathbb{S} \wedge \mathbb{S}) = \pi_0\mathbb{S}$ where $1 \in \pi_0\mathbb{S}$ is the fundamental class.*

(Suspension) *The pairing is compatible with the suspensions isomorphisms in the following sense: the maps induced by the isomorphisms*

$$(S^1 \wedge X) \wedge Y \xrightarrow{\;a_1\;} S^1 \wedge (X \wedge Y) \xleftarrow{\;a_2\;} X \wedge (S^1 \wedge Y)$$

on true homotopy groups satisfy

$$(a_1)_*((S^1 \wedge x) \cdot y) \;=\; S^1 \wedge (x \cdot y) \;=\; (-1)^k \cdot (a_2)_*(x \cdot (S^1 \wedge y)) \tag{6.18}$$

for all integers k, l and true homotopy classes $x \in \pi_k X$ and $y \in \pi_l Y$.

Moreover, the pairing (6.17) is biadditive and has the following properties, for all symmetric spectra X, Y and Z, integers k, l, j and true homotopy classes $x \in \pi_k X$, $y \in \pi_l Y$ and $z \in \pi_j Z$:

(Unitality) *We have*

$$x \cdot 1 \;=\; x \;=\; 1 \cdot x$$

in $\pi_k(X \wedge \mathbb{S}) = \pi_k X = \pi_k(\mathbb{S} \wedge X)$.

(Commutativity) *The map induced by the commutativity isomorphism*

$$\tau_{X,Y} \;:\; X \wedge Y \;\longrightarrow\; Y \wedge X$$

on true homotopy groups takes $x \cdot y$ to $(-1)^{kl} \cdot y \cdot x$.

(Associativity) *The map induced by the associativity isomorphism*

$$\alpha_{X,Y,Z} \;:\; (X \wedge Y) \wedge Z \;\longrightarrow\; X \wedge (Y \wedge Z)$$

on true homotopy groups takes $(x \cdot y) \cdot z$ to $x \cdot (y \cdot z)$.

For diagramatically inclined readers we rewrite the various properties of the smash product pairing in that form. The suspension relation (6.18) can be rephrased as saying that the diagram

$$\begin{tikzcd}
 & \pi_k X \times \pi_l Y \arrow[dl, "(S^1\wedge -)\times \mathrm{Id}"'] \arrow[d, "\cdot"] \arrow[dr, "\mathrm{Id}\times(S^1\wedge -)"] & \\
\pi_{1+k}(S^1 \wedge X) \times \pi_l Y \arrow[d, "\cdot"'] & \pi_{k+l}(X \wedge Y) \arrow[d, "S^1\wedge -"] & \pi_k X \times \pi_{1+l} Y \arrow[d, "\cdot"] \\
\pi_{1+k+l}((S^1 \wedge X) \wedge Y) \arrow[r, "(a_1)_*"'] & \pi_{1+k+l}(S^1 \wedge (X \wedge Y)) & \pi_{k+1+l}(X \wedge (S^1 \wedge Y)) \arrow[l, "(-1)^k\cdot(a_2)_*"]
\end{tikzcd}$$

commutes. The associativity condition is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
\pi_k X \times \pi_l Y \times \pi_j Z \arrow[rr, "\mathrm{Id}\times\cdot"] \arrow[d, "\cdot\times\mathrm{Id}"'] & & \pi_k X \times \pi_{l+j}(Y \wedge Z) \arrow[d, "\cdot"] \\
\pi_{k+l}(X \wedge Y) \times \pi_j Z \arrow[r, "\cdot"'] & \pi_{k+l+j}((X \wedge Y) \wedge Z) \arrow[r, "(\alpha_{X,Y,Z})_*"'] & \pi_{k+l+j}(X \wedge (Y \wedge Z))
\end{tikzcd}$$

commutes. The commutativity condition is equivalent to the commutativity, up to the sign $(-1)^{kl}$, of the diagram:

$$\begin{tikzcd}
\pi_k X \times \pi_l Y \arrow[r, "\cdot"] \arrow[d, "\mathrm{twist}"'] & \pi_{k+l}(X \wedge Y) \arrow[d, "(\tau_{X,Y})_*"] \\
\pi_l Y \times \pi_k X \arrow[r, "\cdot"'] & \pi_{l+k}(Y \wedge X)
\end{tikzcd}$$

Unitality means that the two maps

$$\pi_k X \xrightarrow{\cdot 1} \pi_k(X \wedge \mathbb{S}) = \pi_k X \qquad \text{and} \qquad \pi_k X \xrightarrow{1\cdot} \pi_k(\mathbb{S} \wedge X) = \pi_k X$$

are identities.

Now we can give the

Proof of Theorem 6.16. In the next proposition we fix a symmetric spectrum Y and an integer l and consider families of natural transformations $\tau_k : \pi_k \longrightarrow \pi_{k+l}(- \wedge Y)$ for $k \in \mathbb{Z}$. We call a collection of such natural transformations *stable* if the squares

$$\begin{tikzcd}
\pi_k X \arrow[rr, "S^1\wedge -"] \arrow[d, "\tau_k^X"'] & & \pi_{1+k}(S^1 \wedge X) \arrow[d, "\tau_{1+k}^{S^1\wedge X}"] \\
\pi_{k+l}(X \wedge Y) \arrow[r, "S^1\wedge -"'] & \pi_{1+k+l}(S^1 \wedge (X \wedge Y)) & \pi_{1+k+l}((S^1 \wedge X) \wedge Y) \arrow[l, "(a_1)_*"']
\end{tikzcd} \tag{6.19}$$

commute for every integer k and every symmetric spectrum X. Let Y be any symmetric spectrum and $y \in \pi_l Y$ a true homotopy class. Then there is a unique stable collection of natural transformations

$$\cdot y \ : \ \pi_k \ \longrightarrow \ \pi_{k+l}(- \wedge Y) \ ,$$

for $k \in \mathbb{Z}$, such that

$$1 \cdot y \ = \ y \quad \text{in} \quad \pi_l(\mathbb{S} \wedge Y) = \pi_l Y \ .$$

We start with the definition of the pairing. For every symmetric spectrum Y and integer l we can consider the family of functors $\pi_{k+l}(- \wedge Y)$ as k varies. We connect these functors by the natural isomorphisms

$$\pi_{k+l}(X \wedge Y) \xrightarrow{S^1\wedge -} \pi_{1+k+l}(S^1 \wedge (X \wedge Y)) \xrightarrow{a_1^{-1}} \pi_{1+k+l}((S^1 \wedge X) \wedge Y) \ .$$

For every homotopy class $y \in \pi_l Y$, Corollary 6.14 provides a unique collection of compatible natural transformations $\tau_l : \pi_k \longrightarrow \pi_{k+l}(- \wedge Y)$ characterized by the property $\tau_0^{\mathbb{S}}(1) = y$ in $\pi_l(\mathbb{S} \wedge Y) = \pi_l Y$. For $x \in \pi_k X$ we can then define the pairing by

$$x \cdot y \ = \ \tau_k(x) \ \in \ \pi_{k+l}(X \wedge Y) \ .$$

With our choice of connecting isomorphism, the commutativitiy of the diagram (6.15) becomes the relation $(a_1)_*((S^1 \wedge x) \cdot y) = S^1 \wedge (x \cdot y)$.

The assignment $\pi_k X \times \pi_l Y \longrightarrow \pi_{k+l}(X \wedge Y)$ that sends (x, y) to $x \cdot y$ is natural in X by construction, but naturality in Y needs justification. If $f : Y \longrightarrow Y'$ is a morphism of symmetric spectra, then we have a collection of natural transformations $\{\tau_k : \pi_k \longrightarrow \pi_{k+l}(- \wedge Y')\}_{k\in\mathbb{Z}}$ that sends a class $x \in \pi_k X$ to

$\tau_k^X(x) = (X \wedge f)_*(x \cdot y)$. This collection of transformations is compatible in the sense of diagram (6.15) because

$$\begin{aligned} a_1(\tau_{1+k}^{S^1\wedge X}(S^1 \wedge x)) &= (a_1((S^1 \wedge X) \wedge f))_*((S^1 \wedge x) \cdot y) \\ &=_{\text{nat}} ((S^1 \wedge (X \wedge f))a_1)_*((S^1 \wedge x) \cdot y) \\ &=_{\text{stable}} (S^1 \wedge (X \wedge f))_*((S^1 \wedge (x \cdot y)) \\ &=_{\text{nat}} S^1 \wedge ((X \wedge f)_*(x \cdot y)) \quad = \quad S^1 \wedge \tau_k^X(x) \ . \end{aligned}$$

Moreover, $\tau_0^{\mathbb{S}}(1) = (\mathbb{S} \wedge f)_*(1 \cdot y) = f_*(y)$ because $\mathbb{S} \wedge f = f$ by the strict unit property. So the natural transformations have the properties that characterize $- \cdot f_*(y)$. So the uniqueness clause in Corollary 6.14 forces $x \cdot f_*(y) = \tau_k^X(x) = (X \wedge f)_*(x \cdot y)$ for all $x \in \pi_k X$. In other words, the pairing $x \cdot y$ is also natural in Y.

Now we have a well-defined, normalized and natural product. It remains to establish the additional properties of the pairing. Most of the arguments follow the same pattern: we formulate the respective relation as the claim that some collection of natural transformations $\pi_0 \longrightarrow \pi_k(- \wedge Y)$ agrees with $\cdot y$ for suitable $y \in \pi_l$. Here the spectrum Y, the homotopy class y and the transformations are suitably chosen for the particular property in question, as summarized in the following table:

property	spectrum Y	class y	transformation
naturality	Y'	$f_*(y)$	$(X \wedge f)_*(- \cdot y)$
suspension	$S^1 \wedge Y$	$S^1 \wedge y$	$(-1)^k \cdot (a_2)_*^{-1}(S^1 \wedge (- \cdot y))$
right unitality	$\mathbb{S}$	1	$(r_X)_*^{-1}(-)$
commutativity	Y	y	$(-1)^{kl} \cdot (\tau_{Y,X})_*(y \cdot -)$
right additivity	Y	$y + y'$	$- \cdot y + - \cdot y'$
associativity	$Y \wedge Z$	$y \cdot z$	$(\alpha_{X,Y,Z})_*((- \cdot y) \cdot z)$
uniqueness, I	$\mathbb{S}$	1	$(\tau_{\mathbb{S},Y})_*(1 * -)$
uniqueness, II	Y	y	$- * y$

We then verify that the natural transformations in different dimensions are compatible and that in the special case of the sphere spectrum the transformation sends $1 \in \pi_0\mathbb{S}$ to the desired element y. The uniqueness clause of Corollary 6.14 then shows that the transformation yields the class $x \cdot y$ in general.

(Suspension property) We consider the collection of natural transformations $\{\tau_k : \pi_k \longrightarrow \pi_{k+l}(- \wedge (S^1 \wedge Y))\}_{k \in \mathbb{Z}}$ given by $\tau_k^X(x) = (-1)^k \cdot (a_2)_*^{-1}(S^1 \wedge (x \cdot y))$. We claim that these transformations are compatible. Indeed, we have

$$\begin{aligned} S^1 \wedge \tau_k^X(x) &= (-1)^k \cdot S^1 \wedge (a_2)_*^{-1}(S^1 \wedge (x \cdot y)) \\ &=_{\text{nat}} (-1)^k \cdot (S^1 \wedge a_2)_*^{-1}(S^1 \wedge (S^1 \wedge (x \cdot y))) \\ &= (-1)^{1+k} \cdot (a_1 a_2^{-1}(S^1 \wedge a_1)^{-1})_*(S^1 \wedge (S^1 \wedge (x \cdot y))) \\ &=_{\text{nat}} (-1)^{1+k} \cdot (a_1 a_2^{-1})_*(S^1 \wedge a_1^{-1}(S^1 \wedge (x \cdot y))) \\ &=_{\text{susp}} (-1)^{1+k} \cdot (a_1)_*((a_2^{-1})_*(S^1 \wedge (S^1 \wedge x) \cdot y)) \\ &= (a_1)_*(\tau_{1+k}^{S^1\wedge X}(S^1 \wedge x)) \ . \end{aligned}$$

The second equation uses that the pentagon

$$\begin{tikzcd}[column sep=small]
& (S^1 \wedge X) \wedge (S^1 \wedge Y) \arrow[dl, "a_1^{X,S^1\wedge Y}"'] \arrow[dr, "a_2^{S^1\wedge X,Y}"] & \\
S^1 \wedge (X \wedge (S^1 \wedge Y)) \arrow[d, "S^1\wedge a_2^{X,Y}"'] & & S^1 \wedge ((S^1 \wedge X) \wedge Y) \arrow[d, "S^1\wedge a_1^{X,Y}"] \\
S^1 \wedge S^1 \wedge (X \wedge Y) \arrow[rr, "\mathrm{twist}\wedge\alpha_{X,Y}"'] & & S^1 \wedge S^1 \wedge (X \wedge Y)
\end{tikzcd}$$

commutes and that the transposition of the two circles induced multiplication by -1 on homotopy groups. For the sphere spectrum $X = \mathbb{S}$ we have

$$\tau_0^{\mathbb{S}}(1) \;=\; (a_2)_*^{-1}(S^1 \wedge (1 \cdot y)) \;=\; S^1 \wedge y$$

bcause $1 \cdot y = y$ and the isomorphism

$$S^1 \wedge Y = \mathbb{S} \wedge (S^1 \wedge Y) \xrightarrow{\;a_2\;} S^1 \wedge (\mathbb{S} \wedge Y) \;=\; S^1 \wedge Y$$

is the identity. So the natural transformations have the properties that characterize $- \cdot (S^1 \wedge y)$ and the uniqueness clause in Corollary 6.14 forces $(-1)^k \cdot (a_2)_*^{-1}(S^1 \wedge (x \cdot y)) = \tau_k^X(x) = x \cdot (S^1 \wedge y)$ for all $x \in \pi_k X$.

(Unitality) The relation $1 \cdot y = y$ holds by definition of the pairing; the other relation is obtained as follows. The collection of identity natural transformations $\{\pi_k \longrightarrow \pi_k = \pi_k(- \wedge \mathbb{S})\}_{k\in\mathbb{Z}}$ are compatible because the isomorphism

$$S^1 \wedge X = (S^1 \wedge X) \wedge \mathbb{S} \xrightarrow{\;a_1\;} S^1 \wedge (X \wedge \mathbb{S}) \;=\; S^1 \wedge X$$

is the identity. Moreover, for $X = \mathbb{S}$ the unit $1 \in \pi_0\mathbb{S}$ is sent to itself. So the identity natural transformations have the properties that characterize $- \cdot 1$ and the uniqueness clause in Corollary 6.14 forces $x = x \cdot 1$ for all $x \in \pi_k X$.

(Commutativity) Given any symmetric spectrum Y and any element $y \in \pi_l Y$, the natural transformations $\{\tau_k : \pi_k \longrightarrow \pi_{k+l}(- \wedge Y)\}$ with $\tau_k^X(x) = (-1)^{kl}(\tau_{Y,X})_*(y \cdot x)$ are compatible:

$$\begin{aligned}
S^1 \wedge \tau_k^X(x) \;&=\; (-1)^{kl} \cdot S^1 \wedge (\tau_{Y,X}(y \cdot x)) \\
&=_{\mathrm{nat}} \; (-1)^{kl} \cdot (S^1 \wedge \tau_{Y,X})(S^1 \wedge (y \cdot x)) \\
&=_{\mathrm{susp}} \; (-1)^{kl}(-1)^l \cdot ((S^1 \wedge \tau_{Y,X})a_2)(y \cdot (S^1 \wedge x)) \\
&=\; (-1)^{(1+k)l} \cdot (a_1 \tau_{Y,S^1\wedge X}))(y \cdot (S^1 \wedge x)) \;=\; a_1(\tau_{1+k}^{S^1\wedge X}(S^1 \wedge x))
\end{aligned}$$

The fourth equation uses the commutative square:

$$\begin{tikzcd}
Y \wedge (S^1 \wedge X) \arrow[r, "a_2"] \arrow[d, "\tau_{Y,S^1\wedge X}"'] & S^1 \wedge (Y \wedge X) \arrow[d, "S^1\wedge\tau_{X,Y}"] \\
(S^1 \wedge X) \wedge Y \arrow[r, "a_1"'] & S^1 \wedge (X \wedge Y)
\end{tikzcd}$$

Moreover, we have $\tau_0^{\mathbb{S}}(1) = (\tau_{Y,\mathbb{S}})_*(y \cdot 1) = y \cdot 1 = y$ by right unitality and because the symmetry isomorphism $\tau_{Y,\mathbb{S}} : Y = Y \wedge \mathbb{S} \longrightarrow \mathbb{S} \wedge Y = Y$ is the identity. So the natural transformations have the properties that characterize $- \cdot y$ and the uniqueness clause in Corollary 6.14 proves $(-1)^{kl} \cdot (\tau_{Y,X})_*(y \cdot x) = \tau_k^X(x) = x \cdot y$ for all classes $x \in \pi_k X$.

(Biadditivity) We first establish additivity in the second variable. We fix a symmetric spectrum Y and true homotopy classes $y, y' \in \pi_l Y$. The natural transformations $\pi_k \longrightarrow \pi_{k+l}(- \wedge Y)$ which sends $x \in \pi_0 X$ to $x \cdot y + x \cdot y'$ is pointwise the sum of two stable transformations. So $x \mapsto x \cdot y + x \cdot y'$ is also a stable natural transformation. On the sphere spectrum we have $1 \cdot y + 1 \cdot y' = y + y'$. So the natural transformations have the properties that characterize $- \cdot (y + y')$ and the uniqueness clause Corollary 6.14

proves $x \cdot y + x \cdot y' = x \cdot (y + y')$ for all classes $x \in \pi_k X$. Additivity in x can be reduced to additivity in y by exploiting commutativity:

$$\begin{aligned}(x + x') \cdot y &= (-1)^{kl} \cdot \tau_{Y,X}(y \cdot (x + x')) = (-1)^{kl} \cdot \tau_{Y,X}(y \cdot x + y \cdot x') \\ &= (-1)^{kl} \cdot \tau_{Y,X}(y \cdot x) + (-1)^{kl} \cdot \tau_{Y,X}(y \cdot x') = x \cdot y' + x' \cdot y\end{aligned}$$

(Associativity) We fix $y \in \pi_l Y$ and $z \in \pi_j Z$ and consider the family of transformations $\{\tau_k : \pi_k \longrightarrow \pi_{k+l+j}(- \wedge (Y \wedge Z))\}$ given by $\tau_k^X(x) = (\alpha_{X,Y,Z})_*((x \cdot y) \cdot z)$. These transformations are compatible:

$$\begin{aligned}S^1 \wedge \tau_k^X(x) &=_{\text{nat}} S^1 \wedge (\alpha_{X,Y,Z})_*((x \cdot y) \cdot z) \\ &=_{\text{nat}} (S^1 \wedge \alpha_{X,Y,Z})_*(S^1 \wedge ((x \cdot y) \cdot z)) =_{\text{susp}} ((S^1 \wedge \alpha_{X,Y,Z})a_1)_*((S^1 \wedge (x \cdot y)) \cdot z) \\ &=_{\text{susp}} ((S^1 \wedge \alpha_{X,Y,Z})a_1(a_1 \wedge Z))_*(((S^1 \wedge x) \cdot y) \cdot z) \\ &= (a_1 \alpha_{S^1 \wedge X,Y,Z})_*(((S^1 \wedge x) \cdot y) \cdot z) = a_1(\tau_{1+k}^{S^1 \wedge X}(S^1 \wedge x)) .\end{aligned}$$

The fifh equation is the commutativity of the diagram:

$$\begin{tikzcd}((S^1 \wedge X) \wedge Y) \wedge Z \arrow[r, "a_1 \wedge Z"] \arrow[d, "\alpha_{S^1 \wedge X,Y,Z}"'] & (S^1 \wedge (X \wedge Y)) \wedge Z \arrow[r, "a_1"] & S^1 \wedge ((X \wedge Y) \wedge Z) \arrow[d, "S^1 \wedge \alpha_{X,Y,Z}"] \\ (S^1 \wedge X) \wedge (Y \wedge Z) \arrow[rr, "a_1"'] & & S^1 \wedge (X \wedge (Y \wedge Z))\end{tikzcd}$$

We have $\tau_0^{\mathbb{S}}(1) = (\alpha_{\mathbb{S},Y,Z})_*((1 \cdot y) \cdot z)) = y \cdot z$ because $1 \cdot y = y$ and the associativity isomorphism

$$\alpha_{\mathbb{S},Y,Z} : Y \wedge Z = (\mathbb{S} \wedge Y) \wedge Z \longrightarrow \mathbb{S} \wedge (Y \wedge Z) = Y \wedge Z$$

is the identity. So the natural transformations have the properties that characterize $- \cdot (y \cdot z)$ and the uniqueness clause in Corollary 6.14 proves $(\alpha_{X,Y,Z})_*((x \cdot y) \cdot z) = \tau_k^X(x) = x \cdot (y \cdot z)$ for all classes $x \in \pi_k X$.

It remains to show that the normalization and suspension conditions uniquely characterize the homotopy group pairing. So we consider an arbitrary family of natural pairings

$$(6.20) \qquad * : \pi_k X \times \pi_l Y \longrightarrow \pi_{k+l}(X \wedge Y)$$

which are normalized and satisfy the analog of the suspension condition (6.18). In a first step we consider the family of natural transformations $\pi_l \longrightarrow \pi_l(\mathbb{S} \wedge -) = \pi_l(- \wedge \mathbb{S})$ given by sending $y \in \pi_l Y$ to $1 * y$. This family of natural transformations is stable; indeed, we have

$$S^1 \wedge (1 * y) =_{\text{susp}} (a_2)_*(1 * (S^1 \wedge y)) = 1 * (S^1 \wedge y) = (a_1)_*(1 * (S^1 \wedge y))$$

using the second suspension condition for $*$ and that both isomorphism

$$S^1 \wedge Y = \mathbb{S} \wedge (S^1 \wedge Y) \xrightarrow{a_2} S^1 \wedge (\mathbb{S} \wedge Y) = S^1 \wedge Y$$
$$S^1 \wedge Y = (S^1 \wedge Y) \wedge \mathbb{S} \xrightarrow{a_1} S^1 \wedge (Y \wedge \mathbb{S}) = S^1 \wedge Y$$

are the identity. For $X = \mathbb{S}$ we have $1 * 1 = 1$ by normalization. So the natural transformations have the properties that characterize $- \cdot 1$ and the uniqueness clause in Corollary 6.14 proves $1 * y = y \cdot 1 = y$.

Now we fix $y \in \pi_l Y$. The first suspension condition for $*$ is equivalent to the property that the family of natural transformations $\pi_k \longrightarrow \pi_k(- \wedge Y)$ given by sending x to $x * y$ is stable. For $X = \mathbb{S}$ we have just shown that $1 * y = y$. So the natural transformations have the properties that characterize $- \cdot y$ and the uniqueness clause in Corollary 6.14 proves $x * y = x \cdot y$. □

The construction of the pairing (6.17) was about as abstract as the definition of true homotopy groups. We now give a more concrete description of the product for classes in the image of the natural map $c : \hat{\pi}_* X \longrightarrow \pi_* X$ from naive to true homotopy groups. For semistable spectra this map is bijective, so then we get a complete description of the true homotopy group pairing. One should beware, though, that in general the map $c : \hat{\pi}_k X \longrightarrow \pi_k X$ need not be surjective; so even though the next proposition is very helpful for calculating this product in certain situations, it does not in general determine the product on true homotopy groups.

Proposition 6.21. *Let X and Y be symmetric spectra, and let $f : S^{k+n} \longrightarrow X_n$ and $g : S^{l+m} \longrightarrow Y_m$ be based maps. Define $f \cdot g$ as the composite*

$$S^{k+n+l+m} \xrightarrow{f \wedge g} X_n \wedge Y_m \xrightarrow{i_{n,m}} (X \wedge Y)_{n+m} \ . \tag{6.22}$$

Then the relation

$$c[f] \ \cdot \ c[g] \ = \ (-1)^{nl} \cdot c[f \cdot g]$$

holds in the group $\pi_{k+l}(X \wedge Y)$, where $[f] \in \hat{\pi}_k X$ respectively $[g] \in \hat{\pi}_l Y$ are the naive homotopy classes represented by f and g and where c is the map from naive to true homotopy groups.

In the situation of the previous proposition, one could hope for the stronger statement that the naive homotopy class $(-1)^{nl} \cdot [f \cdot g]$ in $\hat{\pi}_{k+l}(X \wedge Y)$ only depends on the classes $[f] \in \hat{\pi}_k X$ and $[g] \in \hat{\pi}_l Y$, and that the dot operation passes to a well-defined pairing from $\hat{\pi}_k X \times \hat{\pi}_l Y$ to $\hat{\pi}_{k+l}(X \wedge Y)$. However, this does not generally work! If the spectra involved are not semistable, then the situation is more subtle, and we investigate pairings of naive homotopy groups in more detail later.

Proof of Proposition 6.21. For sufficiently large m we consider the map

$$f \star - \ : \ \pi_{l+m} Y_m \ \longrightarrow \ \pi_{k+n+l+m}(X \wedge Y)_{n+m} \ , \quad [g] \ \longmapsto \ (-1)^{nl} \cdot [f \cdot g] \ .$$

As m increases, these maps are compatible with the two stabilization systems for the spectra Y and $X \wedge Y$. So we obtain an induced map

$$f \star - \ : \ \hat{\pi}_l Y \ \longrightarrow \ \hat{\pi}_{k+l}(X \wedge Y)$$

on colimits over m. As Y varies, these maps form a natural transformation of functors $f \star - : \hat{\pi}_l \longrightarrow \hat{\pi}_{k+l}(X \wedge -)$. Moreover, the diagram

$$\begin{tikzcd}
\hat{\pi}_l Y \arrow[rr, "f\star -"] \arrow[d, "S^1\wedge -"'] & & \hat{\pi}_{k+l}(X \wedge Y) \arrow[d, "S^1\wedge -"] \\
\hat{\pi}_{1+l}(S^1 \wedge Y) \arrow[r, "f\star -"'] & \hat{\pi}_{k+1+l}(X \wedge (S^1 \wedge Y)) \arrow[r, "(-1)^k \cdot a_2"'] & \hat{\pi}_{1+k+l}(S^1 \wedge (X \wedge Y))
\end{tikzcd} \tag{6.23}$$

commutes [justify]. We can apply Proposition 6.5 (ii) to the functor $\pi_{k+l}(X \wedge -)$ and the composite natural transformation

$$\hat{\pi}_l Y \ \xrightarrow{f \star -} \ \hat{\pi}_{k+l}(X \wedge Y) \ \xrightarrow{c} \ \pi_{k+l}(X \wedge Y) \ .$$

We obtain a natural transformation $\tau_l : \pi_l \longrightarrow \pi_{k+l}(X \wedge -)$ that is uniquely determined by the property $\tau_l(c(u)) = c(f \star u)$ for all $u \in \hat{\pi}_l Y$.

We claim that the collection of transformations $\{\tau_l\}_{l \in \mathbb{Z}}$ is compatible in the sense of (6.15), where we connect the target functors $\pi_{k+l}(X \wedge -)$ by the natural isomorphisms

$$\pi_{k+l}(X \wedge -) \ \xrightarrow{S^1 \wedge -} \ \pi_{1+k+l}(S^1 \wedge (X \wedge -)) \ \xrightarrow{(-1)^k \cdot a_2^{-1}} \ \pi_{k+1+l}(X \wedge (S^1 \wedge -)) \ .$$

Indeed, we have

$$\begin{aligned}
a_2(\tau_{1+l}(S^1 \wedge c(u))) \ &= \ a_2(\tau_{1+l}(c(S^1 \wedge u))) \ =_{\text{def}} \ a_2(c(f \star (S^1 \wedge u))) \\
&= \ c(a_2(f \star (S^1 \wedge u))) \ = \ (-1)^k \cdot c(S^1 \wedge (f \star u)) \\
&=_{(6.23)} \ (-1)^k \cdot S^1 \wedge c(f \star u) \ =_{\text{def}} \ (-1)^k \cdot S^1 \wedge (\tau_l(c(u))) \ .
\end{aligned}$$

We have exploited that the transformation c from naive to true homotopy groups is natural and compatible with the suspension isomorphisms.

By Proposition 6.5 (ii), two natural transformations from π_l to $\pi_{k+1+l}(X \wedge (S^1 \wedge -))$ agree if they coincide after precomposition with $c : \hat{\pi}_l \longrightarrow \pi_l$. So the previous relation implies

$$(a_2)_*(\tau_{1+l}(S^1 \wedge -)) \ = \ (-1)^k \cdot S^1 \wedge \tau_l$$

as transformations $\pi_l Y \longrightarrow \pi_{k+1+l}(X \wedge (S^1 \wedge Y))$. This completes the proof that the collection of transformations $\{\tau_l\}_{l \in \mathbb{Z}}$ makes the diagrams (6.15) commute.

Now we evaluate τ_0 at the fundamental class $1 \in \pi_0\mathbb{S}$. We obtain

$$\tau_0(1) \;=\; \tau_0(c(\iota)) \;=\; c(f \star \iota) \;=\; c[f]\;,$$

because $\iota \in \hat{\pi}_0\mathbb{S}$ is represented by the identity of S^0. However, left multiplication by the class $c[f]$ is another stable natural transformation $c[f]\cdot - : \pi_l Y \longrightarrow \pi_{l+k}(X \wedge Y)$; by the uniqueness part of Corollary 6.14, the two stable transformation coincides. If we spell this out we obtain

$$c[f] \;\cdot\; c[g] \;=\; \tau_l(c[g]) \;=\; c(f \star [g]) \;=\; (-1)^{nl}\cdot c[f\cdot g]\;.$$

□

Example 6.24. In Example 1.11 we discussed an action of the stable stems π_*^{s} (also known as the naive homotopy groups of the sphere spectrum) on the *naive* homotopy groups $\hat{\pi}_*X$ of a symmetric spectrum X. Since the true homotopy groups X are just the naive homotopy groups of the symmetric spectrum QX, we also obtain an action of the stable stems on the *true* homotopy groups π_*X. Moreover, the natural transformation $c : \hat{\pi}_k X \longrightarrow \pi_k X$ is π_*^{s}-linear because it is induced by a homomorphism of symmetric spectra.

We claim that the action of π_*^{s} on $\hat{\pi}_*(QX)$ coincides with the action of $\pi_*\mathbb{S}$ on π_*X via the true homotopy groups pairing of Theorem 6.16 under the natural map $c : \pi_*^{\mathrm{s}} = \hat{\pi}_*\mathbb{S} \longrightarrow \pi_*\mathbb{S}$. In other words: the diagram

$$\begin{tikzcd}
\hat{\pi}_k(QX)\times\hat{\pi}_l\mathbb{S} \arrow[r, "(1.12)", "\cdot"'] \arrow[d, "\mathrm{Id}\times c"'] & \hat{\pi}_k(QX) \arrow[d, equal] \\
\pi_k X\times\pi_l\mathbb{S} \arrow[r, "\cdot", "(6.17)"'] & \pi_k X
\end{tikzcd}$$

commutes.

Indeed, for $Y = \mathbb{S}$ the universal bimorphism component $i_{n,m} : X_n \wedge Y_m \longrightarrow (X \wedge Y)_{n+m}$ is the iterated structure map $\sigma^m : X_n \wedge S^m \longrightarrow X_{n+m}$, so the pairing $f \cdot g$ as in (6.22) specializes to the construction in (1.12). We have $c = \eta_* : \hat{\pi}_{k+l}(QX) \longrightarrow \pi_{k+l}(QX) = \hat{\pi}_{k+l}(QQX)$ by definition of the transformation c, where $\eta = \eta_{QX} : QX \longrightarrow QQX$ is the stable equivalence. So if $x \in \hat{\pi}_k(QX) = \pi_k X$ is represented by $f : S^{k+n} \longrightarrow (QX)_n$, then we get

$$\begin{aligned}\eta_*(x\cdot c[g]) \;&=_{\mathrm{nat}}\; \eta_*(x)\cdot c[g] \;=\; c[f]\cdot c[g] \;=_{6.21}\; (-1)^{nl}\cdot c[f\cdot g]\\ &=_{(1.12)}\; c([f]\cdot[g]) \;=\; \eta_*(x\cdot[g])\;.\end{aligned}$$

Since η_* is an isomorphism we can conclude that $x \cdot c[g] = x \cdot [g]$, as claimed.

As a formal consequences of Theorem 6.16 and Proposition 6.21 we obtain:

Proposition 6.25. *Let R be a symmetric ring spectrum with multiplication (in internal form) $\mu : R\wedge R \longrightarrow R$ and unit $\eta : \mathbb{S} \longrightarrow R$.*

(i) *The composite maps*

$$\pi_k R\times\pi_l R \;\xrightarrow{\;\cdot\;}\; \pi_{k+l}(R\wedge R) \;\xrightarrow{\mu_*}\; \pi_{k+l}R$$

make the true homotopy groups of R into a graded ring with identity element $\eta_*(1) \in \pi_0 R$. If R is commutative, then this product on π_*R is graded commutative.

(ii) *The true homotopy groups of a right R-module M naturally form a graded right module over the graded ring π_*R via the composite map*

$$\pi_k M\times\pi_l R \;\xrightarrow{\;\cdot\;}\; \pi_{k+l}(M\wedge R) \;\xrightarrow{a_*}\; \pi_{k+l}M\;,$$

where $a : M \wedge R \longrightarrow M$ is the action morphism in internal form.

(iii) *Let $f : S^{k+n} \longrightarrow M_n$ and $g : S^{l+m} \longrightarrow R_m$ be unstable representatives for naive homotopy classes $[f] \in \hat{\pi}_k M$ respectively $[g] \in \hat{\pi}_l R$. Define $f \cdot g \in \pi_{k+l+n+m}M_{n+m}$ as the composite*

$$S^{k+n+l+m} \xrightarrow{f\wedge g} M_n\wedge R_m \xrightarrow{i_{n,m}} (M\wedge R)_{n+m} \xrightarrow{a_{n+m}} M_{n+m}\;. \tag{6.26}$$

Then the relation
$$c[f] \ \cdot\ c[g] \ = \ (-1)^{nl} \cdot c[f \cdot g]$$
holds in the group $\pi_{k+l}M$, where c is the map from naive to true homotopy groups.

(iv) *For every R-module M the suspension and induction isomorphisms*
$$S^1 \wedge - \ : \ \pi_k M \ \longrightarrow \ \pi_{1+k}(S^1 \wedge M) \quad \textit{respectively} \quad \triangleright \ : \ \pi_k M \ \longrightarrow \ \pi_{1+k}(\triangleright M)$$
are $\pi_* R$-linear.

(v) *Let $f : M \longrightarrow N$ be a homomorphism of right R-modules and let $C(f)$ be the mapping cone of f, endowed with the natural R-action. Then the connecting homomorphism $\delta : \pi_{1+k}C(f) \longrightarrow \pi_k M$ defined in* (6.9) *is $\pi_* R$-linear. Hence the long exact sequence of true homotopy groups of Proposition 6.11* (i) *is $\pi_* R$-linear.*

Proof. We prove (i) and (ii) together. We have to show that the product of $\pi_* R$ and the action on $\pi_* M$ are associative and unital, and that the product of $\pi_* R$ is graded commutative if R is commutative. We show only the associativity property; this should suffice to indicate how these properties are direct consequences of the properties of the homotopy groups pairing of Theorem 6.16.

We consider homotopy classes $x \in \pi_k M$, $y \in \pi_l R$ and $z \in \pi_j R$. Then
$$\begin{aligned} x(yz) \ &=_{\text{def}} \ a_*(x \cdot \mu_*(y \cdot z)) \ =_{\text{nat}} \ (a(M \wedge \mu))_*(x \cdot (y \cdot z)) \\ &= \ (a(M \wedge \mu)\alpha_{M,R,R})_*((x \cdot y) \cdot z)) \ = \ (a(a \wedge R))_*((x \cdot y) \cdot z) \\ &=_{\text{nat}} \ a_*(a_*(x \cdot y) \cdot z) \ =_{\text{def}} \ (xy)z \ . \end{aligned}$$
The third equation is the associativity property of the homotopy group pairing, and the fourth equation is the associativity relation
$$a \circ (M \wedge \mu) \circ \alpha_{M,R,R} \ = \ a \circ (a \wedge R) \ : \ (M \wedge R) \wedge R \ \longrightarrow \ M$$
of the action of R on M.

(iii) This part follows from Proposition 6.21 and naturality.

(iv) The R-action on $S^1 \wedge M$ respectively $\triangleright M$ are defined as the composite maps
$$(S^1 \wedge M) \wedge R \xrightarrow{a_1} S^1 \wedge (M \wedge R) \xrightarrow{S^1 \wedge \alpha} S^1 \wedge M \quad \text{respectively} \quad (\triangleright M) \wedge R \xrightarrow{b_1} \triangleright(M \wedge R) \xrightarrow{\triangleright \alpha} \triangleright M \ .$$
So the suspension relation for the dot product gives
$$\begin{aligned} (S^1 \wedge x)r \ &= \ \alpha_{S^1 \wedge M}((S^1 \wedge x) \cdot r) \ = \ ((S^1 \wedge \alpha)a_1)((S^1 \wedge x) \cdot r) \ = \ (S^1 \wedge \alpha)(S^1 \wedge (x \cdot r)) \\ &= \ S^1 \wedge \alpha(x \cdot r) \ = \ S^1 \wedge xr \ , \end{aligned}$$
and similarly for induction.

(v) The connecting morphism $\delta = (S^{-1} \wedge -)p_*$ is a composite of two maps. One ingredient is the map $p_* : \pi_{1+k}C(f) \longrightarrow \pi_{1+k}M$ is induced by the R-linear projection $p : C(f) \longrightarrow M$, which is thus $\pi_* R$-linear. The other ingredient is the inverse of the suspension isomorphism $S^1 \wedge - : \pi_k M \longrightarrow \pi_{1+k}(S^1 \wedge M)$, which is $\pi_* R$-linear by part (iii). So the connecting morphism is $\pi_* R$-linear. □

Of course, there are analogous statements for left modules and bimodules.

For a general symmetric ring spectrum R which is not semistable, the naive homotopy groups $\hat{\pi}_* R$ should be regarded as pathological, and then the true homotopy groups are what one really cares about. In this situation, the naive homotopy groups do not have a preferred structure of graded ring (while the true homotopy groups do, compare Proposition 6.25). Instead, the natural algebraic structure present in $\hat{\pi}_* R$ is that of an algebra over the *injection operad*. While this is an interesting piece of algebra, it is not relevant for stable homotopy theory, and so we defer this discussion to the next section and to Exercise E.I.69.

Example 6.27 (Eilenberg-Mac Lane ring spectra)**.** In Example 1.14 we associates to every abelian group A a semistable symmetric Eilenberg-Mac Lane spectrum HA (even an Ω-spectrum) whose homotopy groups (naive and true) are concentrated in dimension 0, where they are isomorphic to the group A. We explained how an additional ring structure on A can be used to make HA into a symmetric ring spectrum. Now we do the necessary reality check: for an abelian group A, an isomorphism $j_A : A \longrightarrow \hat{\pi}_0(HA)$ is given by the composite $A = \pi_0(HA_0) \longrightarrow \hat{\pi}_0(HA)$; in other words, j_A sends an element $a \in A$ to the class represented by the based map $S^0 \longrightarrow A = (HA)_0$ which sends the non-basepoint element of S^0 to a.

As we explained in Example 5.28, the multiplication of HA is composed of the a special case of the monoidal transformation $m_{A,B} : HA \wedge HB \longrightarrow H(A \otimes B)$ and the multiplication map $A \otimes A \longrightarrow A$ of the ring A. So multiplicativity of the isomorphism j_A is a formal consequence of the fact that for all abelian groups A and B the composite

$$A \otimes B \xrightarrow{j_A \otimes j_B} \pi_0 HA \otimes \pi_0 HB \xrightarrow{\cdot} \pi_0(HA \wedge HB) \xrightarrow{\pi_0(m_{A,B})} H(A \otimes B)$$

equals $j_{A \otimes B}$. This in turn is straightforward from the definitions.

Example 6.28 (Monoid ring spectra)**.** Here we will eventually revisit Example 3.43 and a construct natural isomorphism of graded rings

$$\pi_*(RM) \cong (\pi_* R)M$$

for any symmetric ring spectrum R and discrete monoid M.

Example 6.29 (Matrix ring spectra)**.** Here we will eventually revisit Example 3.44 and a construct natural isomorphism of graded rings

$$\pi_* M_k(R) \cong M_k(\pi_* R)$$

for any symmetric ring spectrum R.

Example 6.30 (Opposite ring spectrum)**.** For every symmetric ring spectrum R we can define the *opposite* ring spectrum R^{op} by keeping the same spaces (or simplicial sets), symmetric group actions and unit maps, but with new multiplication $\mu^{\mathrm{op}}_{n,m}$ on R^{op} given by the composite

$$R^{\mathrm{op}}_n \wedge R^{\mathrm{op}}_m = R_n \wedge R_m \xrightarrow{\text{twist}} R_m \wedge R_n \xrightarrow{\mu_{m,n}} R_{m+n} \xrightarrow{\chi_{m,n}} R_{n+m} = R^{\mathrm{op}}_{n+m} .$$

As a consequence of centrality of ι_1, the higher unit maps for R^{op} agree with the higher unit maps for R. By definition, a symmetric ring spectrum R is commutative if and only if $R^{\mathrm{op}} = R$. In the internal form, the multiplication μ^{op} is obtained from the multiplication $\mu : R \wedge R \longrightarrow R$ as the composite

$$R \wedge R \xrightarrow{\tau_{R,R}} R \wedge R \xrightarrow{\mu} R .$$

For example, we have $(HA)^{\mathrm{op}} = H(A^{\mathrm{op}})$ for the Eilenberg-Mac Lane ring spectra (Example 1.14) of an ordinary ring A and its opposite, and $(RM)^{\mathrm{op}} = (R^{\mathrm{op}})M^{\mathrm{op}}$ for the monoid ring spectra (Example 3.42) of a simplicial or topological monoid M and its opposite.

By the centrality of the unit, the underlying symmetric spectra of R and R^{op} are equal (not just isomorphic), hence R and R^{op} have the same (not just isomorphic) naive and true homotopy groups. The graded commutativity of the external product implies that we have

$$\pi_*(R^{\mathrm{op}}) = (\pi_* R)^{\mathrm{op}}$$

(again equality) as graded rings, where the right hand side is the graded-opposite ring, i.e., the graded abelian group $\pi_* R$ with new product $x \cdot_{\mathrm{op}} y = (-1)^{kl} \cdot y \cdot x$ for $x \in \pi_k R$ and $y \in \pi_l R$.

Let X be a symmetric spectrum and let $x : S^{k+n} \longrightarrow X_n$ be a based map. We can define a natural morphism

$$\lambda_x : S^{k+n} \wedge Y \longrightarrow \mathrm{sh}^n(X \wedge Y)$$

which we call *left multiplication* by x, where Y is any symmetric spectrum. In level m, the morphism λ_x is the composite

$$S^{k+n} \wedge Y_m \xrightarrow{x \wedge Y_m} X_n \wedge Y_m \xrightarrow{i_{n,m}} (X \wedge Y)_{n+m} = (\mathrm{sh}^n(X \wedge Y))_m . \tag{6.31}$$

On naive homotopy groups this produces a map $x \cdot : \hat{\pi}_l Y \longrightarrow \hat{\pi}_{k+l}(X \wedge Y)$ as the composite

$$\hat{\pi}_l Y \xrightarrow{S^{k+n} \wedge -} \hat{\pi}_{k+n+l}(S^{k+n} \wedge Y) \xrightarrow{(\lambda_x)_*} \hat{\pi}_{k+n+l}(\mathrm{sh}^n(X \wedge Y)) = \hat{\pi}_{k+l}(X \wedge Y) .$$

The map $x\cdot : \hat\pi_l Y \longrightarrow \hat\pi_{k+l}(X\wedge Y)$ depends on the homotopy class of x in the *unstable* group $\pi_{k+n}X_n$, and not just on the stable class $[x]$ in $\hat\pi_k X$. In other words, the maps $x\cdot : \hat\pi_l Y \longrightarrow \hat\pi_{k+l}(X\wedge Y)$ and $\iota(x)\cdot : \hat\pi_l Y \longrightarrow \hat\pi_{k+l}(X\wedge Y)$ are typically *not* the same maps, where $\iota(x) = \sigma_n(x\wedge S^1) : S^{k+n+1} \longrightarrow X_{n+1}$ is the stabilization of x. Here is a specific example:

Example 6.32. We consider the left multiplication maps in the case $X = \mathbb{S}$ of the sphere spectrum. The unit element $1 \in \hat\pi_0\mathbb{S}$ is represented by the identity $\mathrm{Id}_{\mathbb{S}^n} : S^n \longrightarrow \mathbb{S}_n$ for every n; however, the maps $\mathrm{Id}_{S^n}\cdot : \hat\pi_l Y \longrightarrow \mathrm{sh}^n\,\hat\pi_{k+l}(\mathbb{S}\wedge Y) = \mathrm{sh}^n\,\hat\pi_{k+l}Y$ are all different (except when Y is semistable). For example, $\lambda_{\mathrm{Id}_{S^0}} : S^0\wedge Y \longrightarrow \mathrm{sh}^0\, Y = Y$ is the unique natural isomorphism $S^0\wedge Y \cong Y$ and $\mathrm{Id}_{S^0}\cdot : \hat\pi_l Y \longrightarrow \hat\pi_l Y$ is the identity. However, $\lambda_{\mathrm{Id}_{S^1}} : S^1\wedge Y \longrightarrow \mathrm{sh}\, Y$ specializes to the map $\lambda_Y : S^1\wedge Y \longrightarrow \mathrm{sh}\, Y$. By Lemma 8.6 the map $\mathrm{Id}_{S^1}\cdot : \hat\pi_l Y \longrightarrow \hat\pi_l Y$ is thus given by $(-1)^k\cdot d$, where $d\in\mathcal{M}$ is the shift operator.

Later we will investigate more closely how the map $x\cdot : \hat\pi_l Y \longrightarrow \mathrm{sh}^n\,\hat\pi_{k+l}(X\wedge Y)$ changes when we stabilize x. It will turn out that $x\cdot$ and $\iota(x)\cdot$ differ by the action of the injection monoid. This action is coequalized by the tautological map from naive to true homotopy groups, so after passage to true homotopy groups, the map $x\cdot$ should only depend on the stable class of x. Indeed, this is the case:

Proposition 6.33. *Let X be a symmetric spectrum and $x : S^{k+n} \longrightarrow X_n$ a based map. We denote by $\langle x\rangle = c[x]$ the true homotopy class represented by x in $\pi_k X$. Then for every symmetric spectrum Y the square*

$$\begin{tikzcd} \hat\pi_l Y \arrow[r, "x\cdot"] \arrow[d, "c"'] & \hat\pi_{k+l}(X\wedge Y) \arrow[d, "c"] \\ \pi_l Y \arrow[r, "\langle x\rangle\cdot"'] & \pi_{k+l}(X\wedge Y) \end{tikzcd}$$

commutes up to the sign $(-1)^{nl}$.

Proof. Let $[y]\in\hat\pi_l Y$ be represented by the based map $y : S^{l+m} \longrightarrow Y_m$. Then by the very definitions, $x\cdot[y]$ is represented by the map $x\cdot y = i_{n,m}(x\wedge y) : S^{k+n+l+m} \longrightarrow (X\wedge Y)_{n+m}$. So by Proposition 6.21 we get

$$c(x\cdot[y]) \;=\; c[x\cdot y] \;=\; (-1)^{nl}\cdot c[x]\cdot c[y] \;=\; (-1)^{nl}\cdot\langle x\rangle\cdot c[y]\ .$$

$\square$

Now we consider a symmetric ring spectrum R and a based map $x : S^{k+n} \longrightarrow R_n$. For every left R-module M we can then form the map $\lambda_x : S^{k+n}\wedge M \longrightarrow \mathrm{sh}^n M$ defined as the composite

$$S^{k+n}\wedge M \xrightarrow{\lambda_x} \mathrm{sh}^n(R\wedge M) \xrightarrow{\mathrm{sh}^n a} \mathrm{sh}^n M \tag{6.34}$$

where $a : R\wedge M \longrightarrow M$ is the action morphism. [abuse of notation...] We refer to λ_x as the *left multiplication* by x. So in level m, the map λ_x is the composite

$$S^{k+n}\wedge M_m \xrightarrow{x\wedge M_m} R_n\wedge M_m \xrightarrow{a_{n,m}} M_{n+m} \;=\; (\mathrm{sh}^n M)_m\ .$$

We let $\tilde x : F_n S^{k+n} \longrightarrow R$ denote the morphism of symmetric spectra which is adjoint to $x : S^{k+n} \longrightarrow R_n$. With this definition, the induced map $\tilde x_* : \hat\pi_k(F_n S^{k+n}) \longrightarrow \hat\pi_k R$ takes the naive fundamental class $\hat\iota_n^{k+n}$ (see (6.4)) to the class $[x]$ represented by the map x. By naturality we get the relation

$$\tilde x_*(\iota_n^{k+n}) \;=\; \langle x\rangle \quad \in \pi_k R \tag{6.35}$$

where $\iota_n^{k+n} = c(\hat\iota_n^{k+n}) \in \pi_k(F_n S^{k+n})$ is the true fundamental class and $\langle x\rangle = c[x]$.

For every left R-module M we can define a morphism of symmetric spectra

$$\hat\lambda_x \;:\; F_n S^{k+n}\wedge M \;\longrightarrow\; M \tag{6.36}$$

as the composite

$$F_n S^{k+n}\wedge M \xrightarrow{\tilde x\wedge M} R\wedge M \xrightarrow{a} M$$

where a is the action of R on M (in external form). Again the name $\hat{\lambda}_x$ stands for 'left multiplication by x'. For ordinary modules over an ordinary ring, left multiplication by a product xy agrees with the composite of left multiplications by y, followed by left multiplication by x. We leave it to Exercise E.I.44 to establish a suitably analogue which relates the composite of $\hat{\lambda}_y$ and $\hat{\lambda}_x$ to $\hat{\lambda}_{x\cdot y}$.

While left multiplication with a ring element on a module is always an additive map, it is typically *not* a module homomorphism unless the element which acts is central. This classical fact has an analog for ring spectra, where the notion of central ring element has to be adapted as follows.

Definition 6.37. Let R be a symmetric ring spectrum and $x : K \longrightarrow R_n$ a based map of spaces (or simplicial sets). The map x is *central* if the diagram

$$\begin{tikzcd} K \wedge R_m \arrow[r, "x\wedge R_m"] \arrow[d, "\text{twist}"'] & R_n \wedge R_m \arrow[r, "\mu_{n,m}"] & R_{n+m} \arrow[d, "\chi_{n,m}"] \\ R_m \wedge K \arrow[r, "R_m\wedge x"'] & R_m \wedge R_n \arrow[r, "\mu_{m,n}"'] & R_{m+n} \end{tikzcd}$$

commutes for all $m \geq 0$.

We will mostly be interested in central maps whose source is a sphere. The unit maps $\iota_n : S^n \longrightarrow R_n$ of any symmetric ring spectrum are examples of central maps, compare Remark 1.6. If R is commutative, then any map to R_n is central.

Proposition 6.38. *Let R be a symmetric ring spectrum, M a left R-module and $x : S^{k+n} \longrightarrow R_n$ a based map. We denote by $\langle x\rangle = c[x]$ the true homotopy class in $\pi_k R$ represented by x.*

(i) *The morphism $\hat{\lambda}_x : F_n S^{k+n} \wedge M \longrightarrow M$ realizes left multiplication by the class $\langle x\rangle$ in homotopy. More precisely, the composite*

$$\pi_l M \xrightarrow[\cong]{\iota_n^{k+n}\cdot} \pi_{k+l}(F_n S^{k+n} \wedge M) \xrightarrow{(\hat{\lambda}_x)_*} \pi_{k+l} M$$

equals left multiplication by $\langle x\rangle$, where $\iota_n^{k+n} \in \pi_k(F_n S^{k+n})$ is the fundamental class.

(ii) *If the map x is central, then the class $\langle x\rangle$ is central, in the graded sense, in the graded true homotopy ring of R; in other words, for every true homotopy class $y \in \pi_l R$ the relation*

$$\langle x\rangle \cdot y \;=\; (-1)^{kl}\, y \cdot \langle x\rangle$$

holds in $\pi_{k+l} R$.

(iii) *If the map x is central, then the morphism of symmetric spectra $\hat{\lambda}_x : F_n S^{k+n} \wedge M \longrightarrow M$ is a homomorphism of left R-modules, where the R-action on the source is the composite*

$$R \wedge F_n S^{k+n} \wedge M \xrightarrow{\tau_{R,F_n S^{k+n}} \wedge M} F_n S^{k+n} \wedge R \wedge M \xrightarrow{F_m S^{l+m} \wedge x} F_n S^{k+n} \wedge M \ .$$

Proof. Part (i) is a simple calculation:

$$\begin{aligned} (\hat{\lambda}_x)_*(\iota_n^{k+n} \cdot y) \;&=_{\text{def}}\; (a(\tilde{x} \wedge M))_*(\iota_n^{k+n} \cdot y) \;=_{\text{nat}}\; a_*(\tilde{x}_*(\iota_n^{k+n}) \cdot y) \\ &=_{(6.35)}\; a_*(\langle x\rangle \cdot y) \;=_{\text{def}}\; \langle x\rangle \cdot y \ . \end{aligned}$$

(ii) As before we let $\tilde{x} : F_n S^{k+n} \longrightarrow R$ denote the morphism of symmetric spectra adjoint to the based map x. Under the adjunction, the centrality property of x translates into the commutative diagram

$$\begin{tikzcd} R \wedge F_n S^{k+n} \arrow[r, "R\wedge\tilde{x}"] \arrow[dd, "\tau_{R,F_n S^{k+n}}"'] & R \wedge R \arrow[dr, "\mu"] & \\ & & R \\ F_n S^{k+n} \wedge R \arrow[r, "\tilde{x}\wedge R"'] & R \wedge R \arrow[ur, "\mu"'] & \end{tikzcd}$$

Moreover, we have $\tilde{x}_*(\iota_n^{k+n}) = \langle x\rangle$, hence we obtain

$$\begin{aligned}\langle x\rangle \cdot y \;&=_{\text{def}}\; \mu_*(\langle x\rangle \cdot y) \;=_{\text{nat}}\; (\mu(\tilde{x}\wedge R))_*(\iota_n^{k+n}\cdot y) \;=_{\text{com}}\; (-1)^{kl}\cdot(\mu(\tilde{x}\wedge R)\tau_{R,F_nS^{k+n}})_*(y\cdot\iota_n^{k+n})\\ &=\; (-1)^{kl}\cdot(\mu(R\wedge\tilde{x}))_*(y\cdot\iota_n^{k+n}) \;=_{\text{nat}}\; (-1)^{kl}\cdot\mu_*(y\cdot\langle x\rangle) \;=_{\text{def}}\; (-1)^{kl}\cdot y\cdot\langle x\rangle\end{aligned}$$

The third equation is the commutativity property of the homotopy group pairing.

(iii)

□

In fact, part (ii) of Proposition 6.38 does not really need that the map $x : S^{l+m} \longrightarrow R_m$ is central on the pointset level; it suffices that the centrality diagram of Definition 6.37 commutes up to based homotopy. There is no converse of this lemma: if R is a semistable symmetric ring spectrum and $y \in \hat{\pi}_l R$ a central naive homotopy class, then in general y *cannot* be represented by a central map $x : S^{l+m} \longrightarrow R_m$ for any m (not even so that the centrality diagram commutes up to based homotopy).

Example 6.39 (Killing a homotopy class)**.** We describe a construction that can be used to 'kill' the action of a homotopy class in a ring spectrum on a given module. We consider a symmetric ring spectrum R, a left R-module M and a central map $x : S^{k+n} \longrightarrow R_n$. We let $M/x = C(\hat{\lambda}_x)$ denote the mapping cone of the morphism $\hat{\lambda}_x : F_nS^{k+n}\wedge M \longrightarrow M$ (see (6.36)). The long exact true homotopy group sequence of a mapping cone (Proposition 6.11 (i)) is π_*R-linear by Proposition 6.25 (iv). Since the morphism $\hat{\lambda}_x$ realizes multiplication by $\langle x\rangle \in \pi_k R$ on true homotopy groups (Proposition 6.38 (i)), this long exact sequence breaks up into a short exact sequence of π_*R-modules

$$0 \longrightarrow \pi_*M/\langle x\rangle(\pi_{*-k}M) \longrightarrow \pi_*(M/x) \longrightarrow \{\pi_{*-k-1}M\}_{\langle x\rangle} \longrightarrow 0$$

where the first map is induced by the mapping cone inclusion $M \longrightarrow M/x$ and $\{-\}_{\langle x\rangle}$ denotes the π_*R-submodule of homotopy classes annihilated by $\langle x\rangle$. If $\langle x\rangle$ acts injectively on the true homotopy groups of M, then we can conclude that the morphism $M \longrightarrow M/x$ realizes the quotient map $\pi_*M \longrightarrow \pi_*M/\langle x\rangle(\pi_{*-k}M)$ on true homotopy groups.

Remark 6.40. Example 6.39 allows us to 'kill' homotopy classes which are in the image of the map c from naive to true homotopy groups. For semistable symmetric spectra this is no restriction since the map c is bijective.

There is a variation of the construction that kills arbitrary true homotopy classes $x \in \pi_k R$ for a ring spectrum R, not necessarily in the image of $c : \hat{\pi}_k R \longrightarrow \pi_k R$. We sketch the construction for $k = 0$, but it can be generalized to arbitrary degrees; we hope to get back to this later.

Given any class $x \in \pi_0 R$, there is a flat symmetric spectrum Z, a stable equivalence $f : Z \longrightarrow \mathbb{S}$ and a morphism $g : Z \longrightarrow R$ such that $g_*(f_*^{-1}(1)) = x$ [ref]. Then the map

$$\tilde{g} \;:\; Z\wedge M \xrightarrow{g\wedge M} R\wedge M \xrightarrow{a} M$$

realizes multiplication by x in the following sense. Since Z is flat, the map $f\wedge M : Z\wedge M \longrightarrow \mathbb{S}\wedge M = M$ is again a stable equivalence, and the composite

$$\pi_l M \xrightarrow{(f\wedge M)_*^{-1}} \pi_l(Z\wedge M) \xrightarrow{\tilde{g}_*} \pi_l M$$

equals left multiplication by x. So we can 'kill' the class x on M by forming the mapping cone of the map $\tilde{g}$.

Example 6.41 (Killing a regular sequence)**.** We can iteratively kill homotopy classes as in Example 6.39 and thereby kill the action of certain ideals in the homotopy groups of a symmetric ring spectrum. We just saw that we can only control the homotopy groups of M/x if the homotopy class $\langle x\rangle$ which is killed is not a zero divisor on π_*M. So iterating the construction naturally leads us to consider regular sequences.

Recall that a sequence, finite or countably infinite, of homogeneous elements y_i in a graded commutative ring R_* is a *regular sequence* for a graded R_*-module M_* if y_1 acts injectively on M_* and for all $i \geq 2$ the element y_i acts injectively on $M_*/M_*\cdot(y_1,\ldots,y_{i-1})$. A homogeneous ideal I of R_* is a *regular ideal* for M_* if it can be generated by a regular sequence, finite or countably infinite, for M_*.

To simplify the exposition we now assume that the ring spectrum R we work over is commutative. This guarantees that any map to R is automatically central.

Proposition 6.42. *Let R be a commutative symmetric ring spectrum, M a left R-module and I a homogeneous ideal of $\pi_* R$. If I is a regular ideal for the module $\pi_* M$ then there exists an R-module M/I and a homomorphism $q : M \longrightarrow M/I$ of R-modules such that the induced homomorphism of true homotopy group*

$$\pi_*(q) \ : \ \pi_* M \ \longrightarrow \ \pi_*(M/I)$$

is surjective and has kernel equal to $I \cdot (\pi_* M)$.

Proof. We choose a sequence $x_1, x_2, \ldots$ of homogeneous elements of $\pi_* R$ which generate the ideal I and form a regular sequence for $\pi_* M$. We construct inductively a sequence of R-modules M^i and homomorphisms

$$M = M^0 \xrightarrow{q_1} M^1 \xrightarrow{q_2} M^2 \xrightarrow{q_3} \cdots$$

such that the composite morphism $M \longrightarrow M^i$ is surjective on homotopy groups and has kernel equal to $(x_1, \ldots, x_i) \cdot (\pi_* M)$.

The induction starts with $i = 0$, where there is nothing to show. In the ith step we let k be the dimension of the homotopy class x_i. By induction the true homotopy groups of M^{i-1} realize the $\pi_* R$-module $\pi_* M/(x_1, \ldots, x_{i-1}) \cdot (\pi_* M)$. We realize left multiplication by the class x_i by a homomorphism of R-modules $\hat{\lambda}_{x_i} : Z \wedge M^{i-1} Z \longrightarrow M^{i-1}$. If x_i is in the image of the tautological map $c : \hat{\pi}_k R \longrightarrow \pi_k R$, we can choose $Z = F_n S^{k+n}$ and achieve this by the construction of Proposition 6.38 (i); this is always the case when R is semistable. If x_i is not in the image of the tautological map we have to use the more general construction indicated in Remark 6.40 in the special case $k = 0$.

Since we have a regular sequence for $\pi_* M$ the class x_i acts injectively on the homotopy of M^{i-1}, so the mapping cone inclusion $q_i : M^{i-1} \longrightarrow M^{i-1}/x_i = C(\hat{\lambda}_{x_i})$ realizes the projection $\pi_* M^{i-1} \longrightarrow \pi_* M^{i-1}/x_i \cdot (\pi_* M^{i-1})$. We can thus take $M^i = M^{i-1}/x_i$; then the composite morphism $M \longrightarrow M^i$ is again surjective on homotopy groups and its kernel is

$$(x_1, \ldots, x_{i-1} \cdot)(\pi_* M) \ + \ x_i \cdot (\pi_* M) \ = \ (x_1, \ldots, x_i) \cdot (\pi_* M) \ .$$

This finishes the argument if I is generated by a *finite* regular sequence.

If the generating sequence is countably infinite we define M/I as the mapping telescope (see Example 2.21) of the above sequence of R-modules M^i. Then the natural map

$$\mathrm{colim}_i \ \pi_* M_i \ \longrightarrow \ \pi_*(M/I)$$

is an isomorphism [ref], and the left hand side is isomorphic to

$$\mathrm{colim}_i \ (\pi_* M/(x_1, \ldots, x_i)) \cdot (\pi_* M) \ \cong \ \pi_* M/(x_1, x_2, \ldots) \cdot (\pi_* M) \ = \ \pi_* M/I \cdot (\pi_* M) \ .$$

□

Let R be a symmetric ring spectrum and $x \in \pi_k R$ a zero divisor in the homotopy ring of R. Then we may not be able realize the module $\pi_* R/(\pi_{*-l} R)\langle x \rangle$ as the homotopy of an R-module, and Toda brackets give the first obstructions to such a realization. We will discuss this in more detail in Proposition 2.13 of Chapter III.

Example 6.43 (Periodic cobordism)**.** We define the (unoriented) *periodic cobordism spectrum MOP*, a $\mathbb{Z}$-graded commutative symmetric ring spectrum that behaves like the Laurent series ring spectrum over MO on a generator of dimension 1. For $n \geq 0$ we consider the 'full Grassmannian' $Gr(2n)$ of $\mathbb{R}^{2n}$. A point in $Gr(2n)$ is a sub-vectorspace of $\mathbb{R}^{2n}$ of any dimension, and this space is topologized as the disjoint union of the Grassmannians of k-dimensional subspaces of $\mathbb{R}^{2n}$ for $k = 0, \ldots, 2n$. Over the full Grassmannian $Gr(2n)$ sits a tautological euclidean vector bundle (of non-constant rank!): the total space of this bundle consist of pairs (U, x) where U is a subspace of $\mathbb{R}^{2n}$ and $x \in U$.

We define MOP_n as the Thom space of this tautological vector bundle, i.e., the quotient space of the unit disc bundle by the sphere bundle. The symmetric group Σ_n acts by isometries on $\mathbb{R}^{2n}$ via the monomorphism $\Sigma_n \longrightarrow \Sigma_{2n}$ sending γ to $\gamma+\gamma$. This action induces an action of Σ_n on the Grassmannians, the vector bundles and their Thom spaces via $\gamma \cdot (U,x) = ((\gamma+\gamma)(U), (\gamma+\gamma)(x))$.

The multiplication

$$MOP_n \wedge MOP_m \longrightarrow MOP_{n+m} \tag{6.44}$$

sends $(U,x)\wedge(U',x')$ to $(U+U',(x,x'))$ where $U+U'$ is the image of $U\oplus U'$ under the isometry

$$\mathrm{Id}\wedge\chi_{n,m}\wedge\mathrm{Id} \ : \ \mathbb{R}^{2n}\oplus\mathbb{R}^{2m} \longrightarrow \mathbb{R}^{2(n+m)} \ , \quad (x,x')\oplus(y,y') \longmapsto (x,y,x',y') \ ,$$

where $x,x'\in\mathbb{R}^n$ and $y,y'\in\mathbb{R}^m$.

The unit map $S^n \longrightarrow MOP_n$ sends $x\in\mathbb{R}^n$ to $(\Delta(\mathbb{R}^n),\Delta(x))$ where $\Delta:\mathbb{R}^n\longrightarrow\mathbb{R}^{2n}$ is the diagonal isometry with $\Delta(x)=(x,x)$.

As the name suggests, MOP is a periodic version of the Thom spectrum MO. More precisely, we claim that MOP is $\mathbb{Z}$-graded commutative symmetric ring spectrum whose piece of degree k is $\hat{\pi}_*$-isomorphic to a k-fold suspension of MO. For every integer k we let $Gr^{[k]}(2n)$ be the subspace of $Gr(2n)$ consisting of subspaces of dimension $n+k$. So $Gr(2n)$ is the disjoint union of the spaces $Gr^{[k]}(2n)$ for $k=-n,\dots,n$. We let $MOP_n^{[k]}$ be the Thom space of the tautological $(n+k)$-plane bundle over $Gr^{[k]}(2n)$, so that MOP_n is the one-point union of the Thom spaces $MOP_n^{[k]}$ for $-n\le k\le n$. We note that the unit map $S^n\longrightarrow MOP_n$ has image in the degree 0 summand $MOP_n^{[0]}$. The multiplication map (6.44) is 'graded' in the sense that its retriction to $MOP_n^{[k]}\wedge MOP_m^{[l]}$ has image in $MOP_{n+m}^{[k+l]}$. Together this implies that $MOP^{[k]}$ is a symmetric subspectrum of MOP and that the periodic spectrum decomposes as

$$MOP \ = \ \bigvee_{k\in\mathbb{Z}} MOP^{[k]} \ .$$

It remains to relate the spectrum $MOP^{[0]}$ to MO through $\hat{\pi}_*$-isomorphisms [...].

Now we explain why $MOP^{[k]}$ is, up to $\hat{\pi}_*$-isomorphism, a k-fold suspension of $MOP^{[0]}$. We observe that the Grassmannian $Gr^{[1]}(2)$ has only one point (the entire space $\mathbb{R}^2$), and so $MOP_1^{[1]}=D(\mathbb{R}^2)/S(\mathbb{R}^2)$ which is homeomorphic to the 2-sphere S^2. So a special case of the multiplication map is

$$S^2\wedge MOP_n^{[k]} \cong MOP_1^{[1]}\wedge MOP_n^{[k]} \longrightarrow MOP_{1+n}^{[1+k]} = (\mathrm{sh}\, MOP^{[1+k]})_n \ .$$

If we let n vary, these maps form a morphism of symmetric spectra

$$S^2\wedge MOP^{[k]} \longrightarrow \mathrm{sh}\, MOP^{[1+k]} \ .$$

Since the map $S^2\wedge MOP_n^{[k]}\longrightarrow MOP_{1+n}^{[1+k]}$ is highly connected [prove] this morphism is in fact a $\hat{\pi}_*$-isomorphism.

It remains to relate the spectrum $MOP^{(0)}$ to MO through $\hat{\pi}_*$-isomorphisms [...]. Now we explain why $MOP^{[k]}$ is, up to $\hat{\pi}_*$-isomorphism, a k-fold suspension of $MOP^{[0]}$. We observe that the Grassmannian $Gr^{[1]}(2)$ has only one point (the entire space $\mathbb{R}^2$), and so $MOP_1^{[1]}=D(\mathbb{R}^2)/S(\mathbb{R}^2)$ which is homeomorphic to the 2-sphere S^2. So a special case of the multiplication map is

$$MOP_n^{[1+k]} \cong MOP_1^{[-1]}\wedge MOP_n^{[1+k]} \longrightarrow MOP_{1+n}^{[k]} \ .$$

If we let n vary, these maps form a morphism of unitary spectra

$$MOP^{[1+k]} \longrightarrow \mathrm{sh}\, MOP^{[k]} \ .$$

Since the map $MOP_n^{[1+k]}\longrightarrow MOP_{1+n}^{[k]}$ is highly connected [prove] the previous morphism of unitary spectra is in fact a $\hat{\pi}_*$-isomorphism.

Corollary 6.45. *We have $2=0$ in $\pi_0 MOP$ and $\pi_0 MO$ and hence all homotopy groups of the Thom spectrum MO are $\mathbb{F}_2$-vector spaces.*

PROOF. The isomorphism $x : S^2 \cong MOP_1^{[1]}$ represents a naive homotopy class in $\hat{\pi}_1 MOP$. The morphism $\hat{\lambda}_x : F_1 S^2\wedge MOP \longrightarrow MOP$ is a stable equivalence and its effect on true homotopy groups of given by multiplication by the true homotopy class $\langle x\rangle \in \pi_1 MOP$, by Proposition 6.38 (i). Hence the class $\langle x\rangle$ is an odd-dimensional unit in the graded commutative homotopy ring $\pi_* MOP$. So we have

$$1 \;=\; \langle x\rangle \cdot \langle x\rangle^{-1} \;=\; -\langle x\rangle^{-1}\cdot\langle x\rangle^{-1} \;=\; -1$$

by graded commutativity. Hence $2 = 0$ holds in $\pi_0 MOP$, and hence also in $\pi_0 MO$. □

Of course, Thom's isomorphism between the homotopy groups of MO and the cobordism classes of smooth closed manifolds provides an easy geometric proof of the fact that $\pi_n MO \cong \mathbf{\Omega}_n^O$ is an $\mathbb{F}_2-$vector space: for every smooth manifold M, the cylinder $M \times [0,1]$ is a null cobordism of $M \amalg M$.

We can try to run the above program with other flavors of Thom spectra. For example, we can consider the Grasmannian $\tilde{Gr}(2n)$ of *oriented* subspaces (of any dimension) of $\mathbb{R}^{2n}$, which supports a tautological oriented euclidean vector bundle. We can define $MSOP_n$ as the Thom space of this tautological bundle. [where is the problem?] In the end we only get a degree-2 periodization of the oriented Thom spectrum MSO.

The spectrum MOP has a straightforward complex analog MUP which we describe in a coordinate free fashion as a unitary ring spectrum in Example 7.8 below. The underlying symmetric ring spectrum of MUP is a degree 2 periodization of the unitary cobordism spectrum MU. Since the units of MUP lie in even dimensions, we cannot derive any simple homotopical consequence as in the case of MOP. [relate MUP to $\Sigma^\infty_+ BU[1/\beta]$]

Construction 6.46 (Inverting a selfmap)**.** Let X be a symmetric spectrum and $f : X \longrightarrow X$ an endomorphism. We can 'invert f' by forming the mapping telescope (see Example 2.21) of the sequence

$$X \xrightarrow{\;f\;} X \xrightarrow{\;f\;} X \xrightarrow{\;f\;} \cdots .$$

By Lemma 2.23 the mapping telescope is $\hat{\pi}_*$-isomorphic to the diagonal of this sequence, which we denote by $f^{-1}X$. Explicitly, this diagonal is given in level n by $(f^{-1}X)_n = X_n$, i.e., the n-th level of the original spectrum with the original Σ_n-action. The structure map $(f^{-1}X)_n \wedge S^1 \longrightarrow (f^{-1}X)_{n+1}$ is the composite around either way in the commutative square

$$\begin{tikzcd} X_n \wedge S^1 \arrow[r, "\sigma_n"] \arrow[d, "f_n\wedge \mathrm{Id}"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ X_n \wedge S^1 \arrow[r, "\sigma_n"'] & X_{n+1} \end{tikzcd}$$

Again by Lemma 2.23 the k-th naive homotopy group of $f^{-1}X$ is given by the colimit of the sequence

$$\hat{\pi}_k X \xrightarrow{\;f_*\;} \hat{\pi}_k X \xrightarrow{\;f_*\;} \hat{\pi}_k X \xrightarrow{\;f_*\;} \cdots$$

which we denote by $f_*^{-1}(\hat{\pi}_k X)$.

Now we generalize this construction to a 'graded selfmap', i.e., one which may shift degrees. More precisely, we start with a symmetric spectrum X and a homomorphism $f : S^k \wedge X \longrightarrow \mathrm{sh}^n X$. If we just wanted a symmetric spectrum whose naive homotopy groups are those of X with 'the effect of f inverted', we could again form a similar mapping telescope above, by iterating the adjoint $\tilde{f} : X \longrightarrow \Omega^k \mathrm{sh}^n X$. However, we introduce a different construction with better multiplicative properties. The advantage is that this construction wil preserve multiplications of ring spectra and actions of ring spectra on module spectra.

For the following construction we start with a symmetric spectrum X and a morphism $f : K \wedge X \longrightarrow \mathrm{sh}^n X$. Later we will mainly be interested in the case where $K = S^k$ is a sphere, but the precise form of the based space (or simplicial set) K is irrelevant for the construction. We define a functor $X(f,-) : \mathbf{I} \longrightarrow \mathcal{S}p$ from the category $\mathbf{I}$ to symmetric spectra. For $\mathbf{p} \in \mathbf{I}$ we set

$$X(f,\mathbf{p}) \;=\; \mathrm{map}(K^{(p)}, \mathrm{sh}^{np} X) \;,$$

where $K^{(p)}$ is the p-fold smash product of copies of K. The symmetric group $\Sigma_p = \mathbf{I}(\mathbf{p},\mathbf{p})$ acts on $K^{(p)}$ by permuting the factors and on $\mathrm{sh}^{np} X$ by restriction along the diagonal embedding

$$\Delta \;:\; \Sigma_p \longrightarrow \Sigma_{np}\ , \quad \Delta(\gamma) \;=\; \mathrm{Id}_{\mathbf{n}} \cdot \gamma$$

(using the notation of Remark 0.4) which unravels to

$$\Delta(\gamma)(j + (i-1)k) \;=\; j + (\gamma(i) - 1)$$

for $i \le j \le n$ and $1 \le i \le p$. The action of Σ_p on the whole symmetric spectrum $X(f,\mathbf{p}) = \mathrm{map}(K^{(p)}, \mathrm{sh}^{np} X)$ is then by conjugation.

To make this assignment into a functor on the category $\mathbf{I}$ we have to prescribe the effect of the inclusion $\iota : \mathbf{p} \longrightarrow \mathbf{p+1}$, and this is where the morphism f enters. We let $X(f,\iota) : X(f,\mathbf{p}) \longrightarrow X(f,\mathbf{p+1})$ be the map

$$\begin{aligned}\mathrm{map}(K^{(p)}, \mathrm{sh}^{np} X) &\xrightarrow{\mathrm{map}(K^{(p)}, \mathrm{sh}^{np} \tilde f)} \mathrm{map}(K^{(p)}, \mathrm{sh}^{np}(\mathrm{map}(K, \mathrm{sh}^n X))) = \mathrm{map}(K^{(p)}, \mathrm{map}(K, \mathrm{sh}^{n+np} X)) \\ &\xrightarrow{\text{adj. iso}} \mathrm{map}(K^{(p+1)}, \mathrm{sh}^{n(1+p)} X) \xrightarrow{\mathrm{map}(K^{(p+1)}, \Delta(\chi_{1,p}))} \mathrm{map}(K^{(p+1)}, \mathrm{sh}^{n(1+p)} X)\ .\end{aligned}$$

To see that this indeed extends to a functor we still have to check that for every $q \ge 1$ the composite

$$X(f,\mathbf{p}) \;\longrightarrow\; X(f,\mathbf{p+1}) \;\longrightarrow\; \cdots \;\longrightarrow\; X(f,\mathbf{p+q})$$

is $\Sigma_p \times \Sigma_q$-equivariant, where Σ_q acts trivially on the source. [justify...]

This finishes the definition of the functor $X(f,-) : \mathbf{I} \longrightarrow \mathcal{S}p$. Now we can define $f^{-1}X$ as

$$f^{-1}X \;=\; \mathrm{diag}\, X(f,-)\ , \tag{6.47}$$

the diagonal, in the sense of Construction 8.33, of the $\mathbf{I}$-functor $X(f,-)$.

Let us make the structure of the symmetric spectrum $f^{-1}X$ explicit. In level p we have

$$(f^{-1}X)_p \;=\; X(f,\mathbf{p})_p \;=\; \mathrm{map}(K^{(p)}, \mathrm{sh}^{np} X)_p \;=\; \mathrm{map}(K^{(p)}, X_{np+p})\ .$$

The symmetric group Σ_p acts by conjugation, on $K^{(p)}$ by permuting the factors and on X_{np+p} by restriction of the original Σ_{np+p}-action along the embedding

$$\Sigma_p \;\longrightarrow\; \Sigma_{np+p}\ , \quad \gamma \;\longmapsto\; \Delta_n(\gamma) + \gamma \;=\; (\mathrm{Id}_{\mathbf{n}} \cdot \gamma) + \gamma\ .$$

One should beware that this homomorphism is *not* the same as the diagonal embedding $\Delta : \Sigma_p \longrightarrow \Sigma_{(n+1)p}$ which sends γ to $\mathrm{Id}_{\mathbf{n+1}} \cdot \gamma$. The structure map $\sigma_p : (f^{-1}X)_p \wedge S^1 \longrightarrow (f^{-1}X)_{p+1}$ is the composite

$$\begin{aligned}\mathrm{map}(K^{(p)}, X_{np+p}) \wedge S^1 &\xrightarrow{\text{assemble}} \mathrm{map}(K^{(p)}, X_{np+p} \wedge S^1) \xrightarrow{\mathrm{map}(K^{(p)}, \sigma_{np+p})} \mathrm{map}(K^{(p)}, X_{np+p+1}) \\ &\xrightarrow{\mathrm{map}(K^{(p)}, \tilde f_{np+p+1})} \mathrm{map}(K^{(p)}, \mathrm{map}(K, X_{n+np+p+1})) \;\cong\; \mathrm{map}(K^{(p+1)}, X_{n(1+p)+p+1}) \\ &\xrightarrow{\mathrm{map}(K^{(p+1)}, \Delta(\chi_{1,p}) + 1_{p+1})} \mathrm{map}(K^{(p+1)}, X_{n(p+1)+p+1})\ .\end{aligned}$$

The construction $f^{-1}X$ has the following straightforward functoriality. Suppose $f : K \wedge X \longrightarrow \mathrm{sk}^n X$ and $g : K \wedge Y \longrightarrow \mathrm{sh}^n Y$ are graded selfmaps and $\varphi : X \longrightarrow Y$ a compatible morphism, i.e., such that the square

$$\begin{tikzcd} K \wedge X \arrow[r, "f"] \arrow[d, "K \wedge \varphi"'] & \mathrm{sh}^n X \arrow[d, "\mathrm{sh}\, \varphi"] \\ K \wedge Y \arrow[r, "g"'] & \mathrm{sh}^n Y \end{tikzcd}$$

commutes. Then the maps

$$\mathrm{map}(K^{(p)}, \mathrm{sh}^{np} \varphi) \;:\; X(f,\mathbf{p}) \;=\; \mathrm{map}(K^{(p)}, \mathrm{sh}^{np} X) \;\longrightarrow\; \mathrm{map}(K^{(p)}, \mathrm{sh}^{np} Y) \;=\; Y(g,\mathbf{p})$$

constitute a natural transformation of $\mathbf{I}$-functors, and so they induce a natural map

$$\varphi_* \;:\; f^{-1}X \;\longrightarrow\; g^{-1}Y$$

on diagonal symmetric spectra.

Construction 6.48. Our next aim is to identify the naive homotopy groups of the spectrum $f^{-1}X$ with an algebraic localization of $\hat{\pi}_*X$ in the case when $K = S^k$ is a sphere. For this purpose we have to explain how to invert a selfmap of a graded $\mathcal{M}$-module which is allowed to change degrees and map into an iterated shift. So we consider a $\mathbb{Z}$-graded $\mathcal{M}$-module $W = \{W_l\}_{l\in\mathbb{Z}}$. For any integer k the kth *degree shift* $W[k]$ is the $\mathbb{Z}$-graded $\mathcal{M}$-module whose degree l component is

$$W[k]_l \ = \ W_{k+l} \ .$$

Now we consider graded-shifted self map $f : W \longrightarrow \operatorname{sh}^n W[k]$, where the $\mathcal{M}$-shift sh^n is applied degreewise. We observe the degree-shift and $\mathcal{M}$-shift commute on the nose, so there is no need for parentheses in $\operatorname{sh}^n W[k]$. So f is consists of a $\mathbb{Z}$-indexed collection of $\mathcal{M}$-linear maps $f_l : W_l \longrightarrow \operatorname{sh}^n W_{k+l}$.

We define a functor $W(f,-) : \mathbf{I} \longrightarrow$ (gr. $\mathcal{M}$-mod) from the category $\mathbf{I}$ to the category of $\mathcal{M}$-modules as follows. We set

$$W(f,\mathbf{p}) \ = \ \operatorname{sh}^{np} W[kp]$$

with Σ_p-action restricted from the Σ_{np}-action on the shifted coordinates along the diagonal embedding $\Delta : \Sigma_p \longrightarrow \Sigma_{np}$, multiplied by the k-th power of the sign representation:

$$\begin{aligned} \gamma \cdot (\operatorname{sh}^{np} w[kp]) \ &= \ \operatorname{sgn}(\gamma)^k \cdot (\Delta(\gamma)\cdot(\operatorname{sh}^{np} w))[kp] \\ &= \ \operatorname{sgn}(\gamma)^{k-n} \cdot \operatorname{sh}^{np}\left((\operatorname{Id}_{\mathbf{n}} \cdot \gamma)\cdot w\right)[kp] \end{aligned}$$

The previous definition has signs for two different reasons. The sign $\operatorname{sgn}(\gamma)^k$ is put in so that Proposition 6.50 (i) comes out without any signs; the reason behind this is that Σ_p really permutes p entities of degree k. The other sign $\operatorname{sgn}(\Delta(\gamma)) = \operatorname{sgn}(\gamma)^n$ is the sign we put into the definition of the iterated algebraic shift of $\mathcal{M}$-modules; this sign is needed to make the relation $\hat{\pi}_{k+m}(\operatorname{sh}^m X) = \operatorname{sh}^m(\hat{\pi}_k X)$ equivariant. The inclusion $\iota : \mathbf{p} \longrightarrow \mathbf{p+1}$ is sent to the morphism

$$\begin{aligned} W(f,\mathbf{p}) \ = \ \operatorname{sh}^{np} W[kp] \ &\xrightarrow{\operatorname{sh}^{np} f[kp]} \ \operatorname{sh}^{np}(\operatorname{sh}^n W[k])[kp] \ = \ \operatorname{sh}^{n+np} W[k+kp] \\ = \ \operatorname{sh}^{n(1+p)} W[k(1+p)] \ &\xrightarrow{(-1)^{kp}\Delta(\chi_{1,p})\cdot} \ \operatorname{sh}^{n(p+1)} W[k(p+1)] \ = \ W(f,\mathbf{p+1}) \ . \end{aligned}$$

[check functoriality] This finishes the definition of the $\mathbf{I}$-$\mathcal{M}$-module $W(f,-)$.

We define the graded $\mathcal{M}$-module $f^{-1}W$ by

$$f^{-1}W \ = \ \operatorname{colim}_{p\in\mathbb{N}} W(f,\mathbf{p}) \ = \ \operatorname{colim}_{p\in\mathbb{N}} \operatorname{sh}^{np} W[kp] \ .$$

as the sequential colimit taken over the inclusions $\iota : \mathbf{p} \longrightarrow \mathbf{p+1}$. If this looks scary, we may want to recall that the algebraic shift of an $\mathcal{M}$-module modifies the $\mathcal{M}$-action, but has no effect on the underlying abelian groups or on morphisms. So the underlying abelian group of $(f^{-1}W)_l$ is the colimit, in the category of $\mathcal{M}$-modules, of the sequence

$$W_l \ \xrightarrow{f_l} \ W_{k+l} \ \xrightarrow{(-1)^k f_{k+l}} \ W_{k2+l} \ \xrightarrow{f_{k2+l}} \ W_{k3+l} \ \xrightarrow{(-1)^k f_{k3+l}} \ \cdots \ .$$

The graded $\mathcal{M}$-module $f^{-1}W$ has extra structure, namely a second $\mathcal{M}$-action which we call the *external* $\mathcal{M}$-action. This action comes from the fact that the sequence whose colimit defines $f^{-1}W$ is underlying a functor from the category $\mathbf{I}$ to the category of $\mathcal{M}$-modules. [ref...]

We make three straightforward observations:

(i) The external $\mathcal{M}$-action on $f^{-1}W$ is degreewise tame.
(ii) If the graded $\mathcal{M}$-module W is degreewise tame, then the internal $\mathcal{M}$-action on $f^{-1}W$ is degreewise tame.
(iii) If the $\mathcal{M}$-action on W is degreewise trivial, then the external and the internal $\mathcal{M}$-action on $f^{-1}W$ are trivial.

Notice the slight twist in the observation (iii) above. If $\mathcal{M}$ acts degreewise trivially on W, then it is *not* generally the case that $\mathcal{M}$ acts degreewise trivially on each $W(f, \mathbf{p}) = \mathrm{sh}^{np} W[kp]$. Indeed, with our definition $\gamma \in \Sigma_p$ acts by the sign $\mathrm{sgn}(\gamma)^k$ on $W(f, \mathbf{p})$. This sign is $+1$ if k is even or γ is an even permutation, but not in general. However, no matter which parity k and $\mathrm{sgn}(\gamma)$ have, since all even permutations act as the identity, the $\mathcal{M}$-action on the colimit $f^{-1}W$ is trivial.

Through the eyes of naive homotopy groups, $S^k \wedge X$ and $\mathrm{sh}^n X$ are degree-shifted versions of X, and f induces a graded selfmaps on naive and true homotopy groups as the composite

$$\hat{\pi}_{l+n} X \xrightarrow{S^k \wedge -} \hat{\pi}_{k+l+n}(S^k \wedge X) \xrightarrow{f_*} \hat{\pi}_{k+l+n}(\mathrm{sh}^n X) \ = \ \mathrm{sh}^n(\hat{\pi}_{k+l} X)$$

respectively

$$\pi_{l+n} X \xrightarrow{S^k \wedge -} \pi_{k+l+n}(S^k \wedge X) \xrightarrow{f_*} \pi_{k+l+n}(\mathrm{sh}^n X) \xrightarrow{\mathrm{sh}_!^n} \pi_{k+l} X \ .$$

The homomorphism $\mathrm{sh}_!^1 = \mathrm{sh}_! : \pi_{k+1}(\mathrm{sh}\, X) \longrightarrow \pi_k X$ was defined in (E.I.35), and $\mathrm{sh}_!^n$ is defined inductively as the composite

$$\pi_{k+n}(\mathrm{sh}^n X) \ = \ \pi_{k+n}(\mathrm{sh}(\mathrm{sh}^{n-1} X)) \xrightarrow{\mathrm{sh}_!} \pi_{k+n-1}(\mathrm{sh}^{n-1} X) \xrightarrow{\mathrm{sh}_!^{n-1}} \pi_k X \ .$$

Now we are ready to calculate the naive homotopy groups of the symmetric spectrum $f^{-1}X$ from the naive homotopy groups of X and the effect of the morphisms f, at least when $K = S^k$ is a sphere. For the identification we need a natural $\mathcal{M}$-linear isomorphism

$$\alpha^{(k,p)} \ : \ \hat{\pi}_l \,\mathrm{map}((S^k)^{(p)}, Y) \xrightarrow{\cong} \hat{\pi}_{kp+l} Y \tag{6.49}$$

which generalizes the loop isomorphism α (see (2.4)) for $k = p = 1$. We define these isomorphisms by induction on p, starting with $\alpha^{(k,1)}$ as the composite

$$\hat{\pi}_l \,\mathrm{map}(S^k, Y) \xrightarrow{S^k \wedge -} \hat{\pi}_{k+l}(S^k \wedge \mathrm{map}(S^k, Y)) \xrightarrow{\mathrm{ev}_*} \hat{\pi}_{k+l} Y$$

where $\mathrm{ev} : S^k \wedge \mathrm{map}(S^k, Y) \longrightarrow Y$ is the evaluation morphism. For $p \geq 1$ we then define $\alpha^{(k,p+1)}$ as the composite

$$\begin{aligned} \hat{\pi}_l \,\mathrm{map}((S^k)^{(p+1)}, Y) &\xrightarrow{\text{adj.}} \hat{\pi}_l \,\mathrm{map}((S^k)^{(p)}, \mathrm{map}(S^k, Y)) \\ &\xrightarrow{\alpha^{(k,p)}} \hat{\pi}_{kp+l} \,\mathrm{map}(S^k, Y) \xrightarrow{\alpha^{(k,1)}} \hat{\pi}_{k+kp+l} Y \ . \end{aligned}$$

Warning: for $p \geq 2$, $\alpha^{(k,p)}$ differs from the composite

$$\hat{\pi}_l \,\mathrm{map}((S^k)^{(p)}, Y) \xrightarrow{(S^k)^{(p)} \wedge -} \hat{\pi}_{kp+l}((S^k)^{(p)} \wedge \mathrm{map}((S^k)^{(p)}, Y)) \xrightarrow{\mathrm{ev}} \hat{\pi}_{kp+l} Y$$

by a permutation [which] in the source spheres, hence possibly by a sign.

Proposition 6.50. *Let X be a symmetric spectrum and $f : S^k \wedge X \longrightarrow \mathrm{sh}^n X$ a graded selfmap. Suppose also that X is levelwise Kan when in the context of spaces.*

(i) *The generalized loop isomorphisms* (6.49)

$$\alpha^{(k,p)} \ : \ \hat{\pi}_l X(f, \mathbf{p}) \ = \ \hat{\pi}_l \,\mathrm{map}((S^k)^{(p)}, \mathrm{sh}^{np} X) \ \longrightarrow \ \hat{\pi}_{kp+l}(\mathrm{sh}^{np} X) \ = \ \mathrm{sh}^{np}(\hat{\pi}_{kp+l-np} X)$$

constitute a natural isomorphism

$$\hat{\pi}_* X(f, -) \xrightarrow{\cong} (\hat{\pi}_* X)(f \cdot, -)$$

of functors from the category $\mathbf{I}$ *to graded* $\mathcal{M}$*-modules.*

(ii) *The generalized loop isomorphisms* (6.49) *induce an isomorphism of graded* $\mathcal{M}$*-modules*

$$\mathrm{diag}\left((f_*^{-1}(\hat{\pi}_* X)\right) \xrightarrow{\cong} \hat{\pi}_*(f^{-1} X) \ .$$

(iii) *If X is semistable, then so is the symmetric spectrum $f^{-1}X$ and the map $\pi_* X \longrightarrow \pi_*(f^{-1}X)$ extends to a natural isomorphism*

$$f_*^{-1}(\pi_* X) \ \longrightarrow \ \pi_*(f^{-1} X)$$

to the true homotopy groups of $f^{-1}X$.

PROOF. (i) We have to check that the isomorphisms $\alpha^{(k,p)}$ are compatible with the Σ_p-actions and the maps induced by the inclusions $\iota : \mathbf{p} \longrightarrow \mathbf{p+1}$. The Σ_p-action on $X(f, \mathbf{p})$ is combined from the permutation action on $(S^k)^{(p)}$ and an diagonal action on the np shifted coordinates. If $\gamma \in \Sigma_p$ permutes the factors of $(S^k)^{(p)}$, this is a map of degree $\operatorname{sgn}(\gamma)^k$, so it becomes multiplication by $\operatorname{sgn}(\gamma)^k$ under the generalized loop isomorphism $\alpha^{(k,p)}$, which matches one of the sign that we built into the Σ_p-action on $(\hat\pi_* X)(f\cdot, \mathbf{p}) = \operatorname{sh}^{np}(\hat\pi_* X)[kp]$. The equality $\hat\pi_{l+m}(\operatorname{sh}^m X) = \operatorname{sh}^m \hat\pi_k X$ is Σ_m-equivariant, compare (8.14). We also need that the generalized loop isomorphism is compatible with the shift identification; more precisely, the diagram

$$\begin{tikzcd}
\hat\pi_{l+m}(\operatorname{sh}^m \operatorname{map}(S^k, Y)) \arrow[r, equal] \arrow[d, equal] & \operatorname{sh}^m \hat\pi_l \operatorname{map}(S^k, Y) \arrow[r, "\operatorname{sh}^m \alpha^{(k,1)}_*"] & \operatorname{sh}^m \hat\pi_{k+l} Y \arrow[d, equal] \\
\hat\pi_{l+m} \operatorname{map}(S^k, \operatorname{sh}^m Y) \arrow[rr, "\alpha^{(k,1)}"'] & & \hat\pi_{k+l+m}(\operatorname{sh}^m Y)
\end{tikzcd}$$

commutes by inspection.

Since the composite $\operatorname{ev} \circ (S^k \wedge \tilde f) : S^k \wedge Y \longrightarrow \operatorname{sh}^n Y$ equals the original map f, the composite

$$\hat\pi_{l+n} X \xrightarrow{\tilde f_*} \hat\pi_{l+n} \operatorname{map}(S^k, \operatorname{sh}^n Y) \xrightarrow{\alpha^{(k,1)}} \hat\pi_{k+l+n}(\operatorname{sh}^n X) \ = \ \operatorname{sh}^n(\hat\pi_{k+l} X)$$

equals $f_* : \hat\pi_{l+n} X \longrightarrow \hat\pi_{k+l} X$ by naturality of the suspension isomorphism. In the diagram

$$\begin{tikzcd}
\hat\pi_l \operatorname{map}((S^k)^{(p)}, \operatorname{sh}^{np} X) \arrow[r, "\alpha^{(k,p)}"] \arrow[d, "\operatorname{map}((S^k)^{(p)}, \operatorname{sh}^{np} \tilde f)_*"] \arrow[dddd, bend right=60, "X(f,\iota)_*"'] & \hat\pi_{kp+l}(\operatorname{sh}^{np} X) \arrow[r, equal] \arrow[d, "(\operatorname{sh}^{np} \tilde f)_*"] & \operatorname{sh}^{np}(\hat\pi_{kp+l-np} X) \arrow[d, "\operatorname{sh}^{np}(\tilde f_*)"'] \arrow[dddd, bend left=60, "(\hat\pi_* X)(f\cdot,\iota)_*"] \\
\hat\pi_l \operatorname{map}((S^k)^{(p)}, \operatorname{sh}^{np} \operatorname{map}(S^k, \operatorname{sh}^n X)) \arrow[r, "\alpha^{(k,p)}"'] \arrow[d, equal] & \hat\pi_{kp+l}(\operatorname{sh}^{np} \operatorname{map}(S^k, \operatorname{sh}^n X)) \arrow[r, equal] \arrow[d, equal] & \operatorname{sh}^{np}(\hat\pi_{kp+l-np} \operatorname{map}(S^k, \operatorname{sh}^n X)) \arrow[d, equal] \\
\hat\pi_l \operatorname{map}((S^k)^{(p)}, \operatorname{map}(S^k, \operatorname{sh}^{n+np} X)) \arrow[r, "\alpha^{(k,p)}"'] \arrow[d, "\text{adj.}"] & \hat\pi_{kp+l} \operatorname{map}(S^k, \operatorname{sh}^{n+np} X) \arrow[r, equal] \arrow[d, "\alpha^{(k,1)}"] & \operatorname{sh}^{n+np}(\hat\pi_{kp+l} \operatorname{map}(S^k, X)) \arrow[d, "\operatorname{sh}^{n(1+p)} \alpha^{(k,1)}"'] \\
\hat\pi_l \operatorname{map}((S^k)^{(p+1)}, \operatorname{sh}^{n(p+1)} X) \arrow[r, "\alpha^{(k,p+1)}"'] \arrow[d, "\operatorname{map}((S^k)^{(p+1)}, \Delta(\chi_{1,p}))_*"] & \hat\pi_{k+kp+l}(\operatorname{sh}^{n(1+p)} X) \arrow[r, equal] \arrow[d, "\Delta(\chi_{1,p})_*"] & \operatorname{sh}^{n(1+p)} \hat\pi_{k+kp+l} X \arrow[d, "\Delta(\chi_{1,p})\cdot"'] \\
\hat\pi_l \operatorname{map}((S^k)^{(p+1)}, \operatorname{sh}^{n(p+1)} X) \arrow[r, "\alpha^{(k,p+1)}"'] & \hat\pi_{k(p+1)+l}(\operatorname{sh}^{n(p+1)} X) \arrow[r, equal] & \operatorname{sh}^{n(p+1)}(\hat\pi_{k(p+1)+l} X)
\end{tikzcd}$$

the upper two squares commute by naturality of the generalized loop isomorphism. The next square below commutes by the inductive definition of $\alpha^{(k,p)}$ [check that $\alpha^{(k,1)} = \operatorname{sh}^{n(1+p)} \alpha^{(k,1)}$] So the generalized loop isomorphisms are compatible with the inclusion, and hence constitute a natural isomorphism of functors.

(ii) This is a combination of two previously established $\mathcal{M}$-linear isomorphisms:

$$\begin{aligned}
\operatorname{diag}\left((f_*^{-1}(\hat\pi_* X)\right) \ = \ \operatorname{diag}\left(\operatorname{colim}_{p\in\mathbb{N}} (\hat\pi_* X)(f\cdot, \mathbf{p})\right) \ &\longrightarrow \ \operatorname{diag}\left(\operatorname{colim}_{p\in\mathbb{N}} \hat\pi_* X(f, \mathbf{p})\right) \\
&\longrightarrow \ \hat\pi_*(\operatorname{diag} X(f,-)) \ = \ \hat\pi_*(f^{-1}X) \ .
\end{aligned}$$

The first isomorphism is obtained from part (i) by taking colimit over $\mathbf{p} \in \mathbb{N}$ and passing to diagonal actions; the second isomorphism is Proposition 8.36 applied to the $\mathbf{I}$-spectrum $X(f,-)$.

(iii) If X is semistable, then the $\mathcal{M}$-action on the naive homotopy groups $\hat\pi_* X$ is degreewise trivial. So both the external and the internal $\mathcal{M}$-action on $f_*^{-1}(\hat\pi_* X)$ are trivial [...]. Hence the diagonal $\mathcal{M}$-action on $f_*^{-1}(\hat\pi_* X)$ is also trivial, and so is the $\mathcal{M}$-action on $\hat\pi_*(f^{-1}X)$ by part (ii). So the symmetric spectrum $f^{-1}X$ is semistable.

Since X and $f^{-1}X$ are semistable, the tautological maps from naive to true homotopy groups are isomorphisms. We claim that the square

$$\begin{tikzcd} \hat\pi_{l+n}X \arrow[r, "f_*"] \arrow[d, "c"'] & \hat\pi_{k+l}X \arrow[d, "c"] \\ \pi_{l+n}X \arrow[r, "f_*"'] & \pi_{k+l}X \end{tikzcd} \tag{6.51}$$

commutes. Since the tautological map is natural and commutes with the suspension isomorphism, it suffices the check, by induction on n, that the square

$$\begin{tikzcd} \hat\pi_{k+l+1}(\mathrm{sh}\, X) \arrow[r, equal] \arrow[d, "c"'] & \mathrm{sh}(\hat\pi_{k+l}X) \arrow[r, equal] & \hat\pi_{k+l}X \arrow[d, "c"] \\ \pi_{k+l+1}(\mathrm{sh}\, X) \arrow[rr, "\mathrm{sh}_!"'] & & \pi_{k+l}X \end{tikzcd}$$

commutes. This follows from Proposition **??** since the injection monoid acts trivially on the naive homotopy groups of X.

Since the diagram (6.51) commutes, the tautological maps assemble into an isomorphism of graded abelian groups

$$f^{-1}c \;:\; f_*^{-1}(\hat\pi_*X) \;\longrightarrow\; f_*^{-1}(\pi_*X)\ .$$

So in the commutative square

$$\begin{tikzcd} f_*^{-1}(\hat\pi_*X) \arrow[r] \arrow[d, "f^{-1}c"'] & \hat\pi_*(f^{-1}X) \arrow[d, "c"] \\ f_*^{-1}(\pi_*X) \arrow[r] & \pi_*(f^{-1}X) \end{tikzcd}$$

the lower horizontal map is bijective since the other three maps are. □

Construction 6.52 (Pairing of localizations)**.** In Example 3.48 we defined a spectrum $f^{-1}X$ from a symmetric spectrum X and a twisted endomorphism $f : K\wedge X \longrightarrow \mathrm{sh}^n X$. Now we discuss the multiplicative properties of the localization construction $f^{-1}X$. We consider three symmetric spectra X, Y and Z and twisted endomorphisms $f : K \wedge X \longrightarrow \mathrm{sh}^n X$, $g : K \wedge Y \longrightarrow \mathrm{sh}^n Y$ and $h : K \wedge Z \longrightarrow \mathrm{sh}^n Z$. Moreover, we consider a morphism $\mu : X \wedge Y \longrightarrow Z$ which is compatible with the twisted endomorphisms in the sense that the diagram

$$\begin{tikzcd} (K\wedge X)\wedge Y \arrow[r, "a_1"] \arrow[d, "f\wedge Y"'] & K\wedge(X\wedge Y) \arrow[d, "K\wedge\mu"] & X\wedge(K\wedge Y) \arrow[l, "a_2"'] \arrow[d, "X\wedge g"] \\ (\mathrm{sh}^n X)\wedge Y \arrow[d, "\xi"'] & K\wedge Z \arrow[d, "h"] & X\wedge(\mathrm{sh}^n Y) \arrow[d, "\xi"] \\ \mathrm{sh}^n(X\wedge Y) \arrow[r, "\mathrm{sh}^n\mu"'] & \mathrm{sh}^n Z & \mathrm{sh}^n(X\wedge Y) \arrow[l, "\mathrm{sh}^n\mu"] \end{tikzcd}$$

commutes. We will mainly be interested in the left multiplication maps arising from a central map $x : S^k \longrightarrow R_n$ in a symmetric ring spectrum R. In that case, we take $X = R$, let $Y = Z = M$ be a left R-module and consider the action map $a : R \wedge M \longrightarrow M$. Then the left multiplication maps make the resulting diagram (see (6.54) below) commute.

From the above data we now define $\Sigma_p \times \Sigma_q$-equivariant action maps

$$\alpha_{p,q} \;:\; X(f,\mathbf{p})\wedge Y(g,\mathbf{q}) \;\longrightarrow\; Z(h,\mathbf{p}+\mathbf{q})$$

as the composite

$$\begin{aligned} \mathrm{map}(K^{(p)}, \mathrm{sh}^{np} X) \wedge \mathrm{map}(K^{(p)}, \mathrm{sh}^{nq} Y) \ &\xrightarrow{\ \wedge\ } \ \mathrm{map}(K^{(p+q)}, \mathrm{sh}^{np} X \wedge \mathrm{sh}^{nq} Y) \\ &\xrightarrow{\mathrm{map}(K^{(p+q)}, (\mathrm{sh}^{n(p+q)} \mu)\xi)} \ \mathrm{map}(K^{(p+q)}, \mathrm{sh}^{n(p+q)} Z) \end{aligned}$$

where $\xi : \mathrm{sh}^{np} X \wedge \mathrm{sh}^{nq} Y \longrightarrow \mathrm{sh}^{np+nq}(X \wedge Y) = \mathrm{sh}^{n(p+q)}(X \wedge Y)$ is the shearing map defined in (5.15). The maps $\alpha_{p,q}$ are $\Sigma_p \times \Sigma_q$-equivariant because the diagonal embeddings $\Delta_p : \Sigma_p \longrightarrow \Sigma_{np}$, $\Delta_q : \Sigma_q \longrightarrow \Sigma_{nq}$ and $\Delta_{p+q} : \Sigma_{p+q} \longrightarrow \Sigma_{n(p+q)}$ satisfy

$$\Delta_p(\gamma) \times \Delta_q(\tau) \ = \ \Delta_{p+q}(\gamma \times \tau) \ .$$

More generally, if $\alpha : \mathbf{p} \longrightarrow \mathbf{p}'$ and $\beta : \mathbf{q} \longrightarrow \mathbf{q}'$ are injective maps, then the square

$$\begin{tikzcd} X(f,\mathbf{p}) \wedge Y(g,\mathbf{q}) \arrow[r, "\alpha_{p,q}"] \arrow[d, "X(f,\alpha)\wedge Y(g,\beta)"'] & Z(h, \mathbf{p}+\mathbf{q}) \arrow[d, "Z(h,\alpha+\beta)"] \\ X(f,\mathbf{p}') \wedge Y(g,\mathbf{q}') \arrow[r, "\alpha_{p',q'}"'] & Z(h, \mathbf{p}'+\mathbf{q}') \end{tikzcd}$$

commutes.[justify]

The action maps are commutative in the sense that the square

$$\begin{tikzcd} X(f,\mathbf{p}) \wedge Y(g,\mathbf{q}) \arrow[r, "\alpha_{p,q}"] \arrow[d, "\tau_{X(f,\mathbf{p}),Y(g,\mathbf{q})}"'] & Z(h, \mathbf{p}+\mathbf{q}) \arrow[d, "Z(h,\chi_{p,q})"] \\ Y(g,\mathbf{q}') \wedge X(f,\mathbf{p}') \arrow[r, "\alpha_{q,p}"'] & Z(h, \mathbf{q}+\mathbf{p}) \end{tikzcd}$$

commutes for all $p, q \geq 0$.[justify]

The action maps are suitably associative, but we do not spell out the precise condition.

Construction 6.53 (Inverting a homotopy class)**.** In Example 3.48 we defined a new symmetric ring spectrum $R[1/x]$ from a given symmetric ring spectrum R and a central map $x : S^1 \longrightarrow R_1$. We now generalize this construction to central maps $x : S^k \longrightarrow R_n$ and also analyze it homotopically.

We let M be any left R-module and recall from (6.34) that the central map x gives rise to a homomorphism of left R-modules

$$\lambda_x \ : \ S^k \wedge M \ \longrightarrow \ \mathrm{sh}^n M$$

which is natural for R-linear maps in M. In level m the morphism λ_x is the composite

$$S^k \wedge M_m \ \xrightarrow{x \wedge M_m} \ R_n \wedge M_m \ \xrightarrow{a_{n,m}} \ M_{n+m} \ .$$

So the construction (6.46) for inverting a graded selfmap produces a functor $M(\lambda_x, -) : \mathbf{I} \longrightarrow R\text{-mod}$ from the category $\mathbf{I}$ to the category of left R-modules, as well as a diagonal left R-module

$$M[1/x] \ = \ \lambda_x^{-1} M \ = \ \mathrm{diag}\, M(\lambda_x, -) \ .$$

The left multiplication maps make the diagram

$$\begin{tikzcd} (S^k \wedge R) \wedge M \arrow[r, "a_1"] \arrow[d, "\lambda_x \wedge M"'] & S^k \wedge (R \wedge M) \arrow[d, "S^k \wedge a"'] & R \wedge (S^k \wedge M) \arrow[l, "a_2"'] \arrow[d, "R \wedge \lambda_x"] \\ (\mathrm{sh}^n R) \wedge M \arrow[d, "\xi"'] & S^k \wedge M \arrow[d, "\lambda_x"'] & R \wedge (\mathrm{sh}^n M) \arrow[d, "\xi"] \\ \mathrm{sh}^n(R \wedge M) \arrow[r, "\mathrm{sh}^n a"'] & \mathrm{sh}^n M & \mathrm{sh}^n(R \wedge M) \arrow[l, "\mathrm{sh}^n a"] \end{tikzcd} \tag{6.54}$$

commute, which uses the centrality condition. So Construction 6.52 provides a pairing of **I**-functors

$$R(\lambda_x, -) \wedge M(\lambda_x, -) \longrightarrow M(\lambda_x, - + -)$$

which is suitably associative and unital. On diagonals this provides maps [spell out]

$$a[1/x] \ : \ R[1/x] \wedge M[1/x] \longrightarrow M[1/x]$$

which are again associative and cover the localization maps in the sense of a commutative square:

$$\begin{tikzcd} R \wedge M \arrow[r, "a"] \arrow[d] & M \arrow[d] \\ R[1/x] \wedge M[1/x] \arrow[r, "{a[1/x]}"'] & M[1/x] \end{tikzcd}$$

A special case of this is $M = R$, where the multiplication morphism $\mu : R \wedge R \longrightarrow R$ provides an associative and unital multiplication map

$$\mu[1/x] \ : \ R[1/x] \wedge R[1/x] \longrightarrow R[1/x]$$

which gives the localization $R[1/x]$ the structure of a symmetric ring spectrum, and $a[1/x]$ makes $M[1/x]$ into a left $R[1/x]$-module.

For easier reference we make the structure of $M[1/x]$ explicit: In level p we have

$$M[1/x]_p \ = \ M(\lambda_x, \mathbf{p})_p \ = \ \mathrm{map}((S^k)^{(p)}, \mathrm{sh}^{mp} M)_p \ = \ \mathrm{map}((S^k)^{(p)}, M_{mp+p}) \ .$$

The symmetric group Σ_p acts on $(S^k)^{(p)}$ by permuting the p factors and on M_{mp+p} by restriction of the original Σ_{mp+p}-action along the embedding

$$\Delta' \ : \ \Sigma_p \longrightarrow \Sigma_{mp+p} \ , \quad \gamma \longmapsto (\mathrm{Id}_{\mathbf{m}} \cdot \gamma) + \gamma \ .$$

Finally, Σ_p acts on the mapping space $M[1/x]_p$ by conjugation. One should beware that the homomorphism Δ' is *not* the same as $\Delta : \Sigma_p \longrightarrow \Sigma_{(m+1)p}$ which sends γ to $\mathrm{Id}_{\mathbf{m+1}} \cdot \gamma$. The structure map $\sigma_p : M[1/x]_p \wedge S^1 \longrightarrow M[1/x]_{p+1}$ is [...]

Example 6.55. As an example of the previous construction we consider $R = \mathbb{S}$, the sphere spectrum, and $x = \mathrm{Id} : S^1 \longrightarrow \mathbb{S}_1$. An $\mathbb{S}$-module is simply a symmetric spectrum, and we have

$$M(\mathrm{Id}_{S^1}, \mathbf{p}) \ = \ \Omega^p \, \mathrm{sh}^p M \ .$$

Since $\lambda_{\mathrm{Id}_{S^1}} : S^1 \wedge M \longrightarrow \mathrm{sh}\, M$ is precisely the map λ_M, the inclusion $\iota : \mathbf{p} \longrightarrow \mathbf{p+1}$ induces

$$M(\mathrm{Id}_{S^1}, \iota) \ = \ \Omega^p(\tilde{\lambda}_{\mathrm{sh}^p M}) \ = \ \Omega^p \, \mathrm{sh}^p M \longrightarrow \Omega^{p+1} \, \mathrm{sh}^{p+1} M \ .$$

[check...] These are exactly the maps whose sequential colimit is the spectrum $\Omega^\infty \mathrm{sh}^\infty M$ as defined in (4.18).

The next proposition is the key property of the localization construction; it says that for *semistable* ring and module spectra, the effect of the localization on the homotopy groups is precisely the algebraic localization, i.e., inverting the powers of the class $\langle x \rangle = c[x] \in \pi_{k-n} R$.

Proposition 6.56. *Let R be a symmetric ring spectrum, $x : S^k \longrightarrow R_n$ a central map and M a semistable left R-module which is levelwise Kan when in the context of spaces.*

(i) *The underlying symmetric spectrum of the $R[1/x]$-module $M[1/x]$ is again semistable.*

(ii) *The central homotopy class $\langle x \rangle \in \pi_{k-n} R$ acts bijectively on the true homotopy groups of $M[1/x]$ and the map*

$$(\pi_* M)[1/\langle x \rangle] \longrightarrow \pi_*(M[1/x])$$

induced by the homomorphism $j : M \longrightarrow M[1/x]$ is an isomorphism of graded $(\pi_* R)[1/\langle x \rangle]$-modules.

(iii) *Suppose that the symmetric ring spectrum R is itself semistable and levelwise Kan when in the context of spaces. Then the symmetric ring spectrum $R[1/x]$ is semistable, the image of the homotopy class $\langle x\rangle$ is a central unit in $\pi_{k-n}(R[1/x])$ and the map*

$$(\pi_* R)[1/\langle x\rangle] \longrightarrow \pi_*(R[1/x])$$

induced by the homomorphism $j : R \longrightarrow R[1/x]$ is an isomorphism of graded rings.

Proof. All the work has already been done. Indeed, by definition, $M[1/x] = \lambda_x^{-1} M$, so part (i) is a special case of Proposition 6.50 (iii). Moreover, since the morphism $\lambda_x : S^k \wedge M \longrightarrow \operatorname{sh}^n M$ induces left multiplication by the class $\langle x\rangle$ on true homotopy groups [ref], part (iii) of Proposition 6.50 also provides an isomorphism

$$(\pi_* M)[1/\langle x\rangle] \;=\; (\lambda_X)_*^{-1}(\pi_* M) \xrightarrow{\cong} \pi_*(M[1/x])$$

which extends the map $j_* : \pi_* M \longrightarrow \pi_*(M[1/x])$.

Part (iii) is just a special case. If R itself is semistable, then so is $R[1/x]$ by part (i) and the central homotopy class $\langle x\rangle$ is a unit by part (ii). The map $j_* : \pi_* R \longrightarrow \pi_*(R[1/x])$ thus extends to a unique homomorphism of graded rings $(\pi_* R)[1/\langle x\rangle] \longrightarrow \pi_*(R[1/x])$ which is bijective by part (ii). □

We note a consequence of Proposition 6.56: if the dimension $l - m$ of x is odd, then $2x^2 = 0$, so $2 = 0$ after inverting x, and hence $2 = 0$ holds in the ring $\pi_0 R$. So inverting a central map of odd degree has the effect that all homotopy groups become $\mathbb{F}_2$-vector spectra.

[Exercise: inverting x and $x \cdot \iota_1$ produces $\hat{\pi}_*$-isomorphic ring spectra; functoriality of localization. Consider the colimit TX of the system

$$X \longrightarrow X[1/\iota_1] \longrightarrow X[1/\iota_1][1/\iota_1] \longrightarrow \dots$$

If X is semistable, this should be a sequence of $\hat{\pi}_*$-isomorphisms whose target is a positive Ω-spectra. The functor is lax symmetric monoidal. Do this with orthogonal spectra to obtain a symmetric monoidal positive Ω-replacement]

Remark 6.57. Let us consider a symmetric ring spectrum R and a central map $x : S^1 \longrightarrow R_1$. Then we can consider an new symmetric ring spectrum $R^{(x)}$ with $R^{(x)}_n = R_n$, i.e., we take the same Σ_n-space (or simplicial set) in each level. Moreover, the multiplication maps $\mu_{n,m}$ of $R^{(x)}$ equal the multiplication maps of R. However, as nth unit map of $R^{(x)}$ we take the composite

$$S^n \xrightarrow{x^{(n)}} R_1 \wedge \dots \wedge R_1 \xrightarrow{\mu_{1,\dots,1}} R_n \ .$$

We can compare the ring spectrum $R^{(x)}$ to the localization $R[1/x]$ as follows. There is a morphism $\varphi : R^{(x)} \longrightarrow R[1/x]$ of symmetric ring spectra which is a $\hat{\pi}_*$-isomorphism whenever R is semistable. [what happens in general?] In that situation, $R[1/x]$ is again semistable by [...], so is $R^{(x)}$.

We define a morphism $\varphi : R^{(x)} \longrightarrow R[1/x]$ in level p as the map $\varphi_p : R_p \longrightarrow \operatorname{map}(S^p, R_{p+p})$ adjoint to the composite

$$R_p \wedge S^p \xrightarrow{R_p \wedge \iota_p} R_p \wedge R_p \xrightarrow{\mu_{p,p}} R_{p+p}$$

where ι_p is the original unit map of R; this map is also the itereated structure map σ^p of the symmetric spectrum which underlies R. [details...]

Example 6.58. Suppose we are given a morphism $h : R \longrightarrow S$ between semistable commutative symmetric ring spectra and a naive homotopy class $y \in \pi_k R$ which becomes a unit in $\pi_* S$, and that $x : S^{k+n} \longrightarrow R_n$ represents y. We claim that then the morphism h 'weakly factors' over the morphisms $j : R \longrightarrow R[1/x]$ that inverts x in the sense of Construction 6.53. More precisely, we can consider the composite $h_n x : S^{k+n} \longrightarrow S_n$

and compare the two localizations via the commutative square

$$\begin{tikzcd} R \arrow[r, "h"] \arrow[d, "j"'] & S \arrow[d, "j", "\simeq"'] \\ R[1/x] \arrow[r, "h{[1/x]}"'] & S[1/h_n x] \end{tikzcd}$$

Since the naive homotopy class $[h_n x] \in \pi_k S$ was assumed to be a unit, the right vertical morphism is a $\hat{\pi}_*$-isomorphism, hence a stable equivalence, by Proposition 6.56. Now let us assume that in addition the map

$$(\hat{\pi}_* R)[1/[x]] \longrightarrow \pi_* S$$

induced by h is an isomorphism. Then the lower vertical morphism $h[1/x] : R[1/x] \longrightarrow S[1/h_n x]$ is also a $\hat{\pi}_*$-isomorphism, again by Proposition 6.56; altogether the symmetric ring spectrum S is $\hat{\pi}_*$-isomorphic, hence stably equivalent, to the localization $R[1/x]$.

Two specific examples of this situation are given by the complex and real topological K-theory spectra, compare Example 1.20. These are semistable commutative symmetric ring spectra which come in connective and periodic versions; these version are related by homomorphisms $ku \longrightarrow KU$ and $ko \longrightarrow KO$. The Bott classes $u \in \pi_2(ku)$ respectively $\beta \in \pi_8(ko)$ become invertible in $\pi_2(KU)$ respectively $\beta \in \pi_8(KO)$, and the homotopy ring of the periodic spectra KU and KO are precisely the algebraic localizations at the powers of u respectively β. So by the previous paragraph, we conclude that KU (respectively KO) can be obtained be inverting the Bott class in ku (respectively ko), in the sense of Construction 6.53, i.e.,

$$KU \simeq ku[1/u] \quad \text{and} \quad KO \simeq ko[1/\beta] \ .$$

Example 6.59 (Complex K-theory from $\mathbb{C}P^\infty$)**.** As an example we give a model for periodic complex topological K-theory (compare Example 1.20) starting from the suspension spectrum of $\mathbb{C}P^\infty$. We start by defining a commutative symmetric ring spectrum $\tilde{\Sigma}^\infty_+ \mathbb{C}P^\infty$ whose underlying symmetric spectrum is $\hat{\pi}_*$-isomorphic to the suspension spectrum of $\mathbb{C}P^\infty$ and that admits a multiplicative map $\tilde{\Sigma}^\infty_+ \mathbb{C}P^\infty \longrightarrow ku$ to the model for connective K-theory of Example 1.20. In level n we define

$$(\tilde{\Sigma}^\infty_+ \mathbb{C}P^\infty)_n \ = \ S^n \wedge P(\mathrm{Sym}(\mathbb{C}^n))_+ \ .$$

Here $\mathrm{Sym}(\mathbb{C}^n)$ is the symmetric algebra, over the complex numbers, generated by the vector space $\mathbb{C}^n$; so $\mathrm{Sym}(\mathbb{C}^n)$ is a complex polynomial algebra in n variables. Moreover, $P(\mathrm{Sym}(\mathbb{C}^n))$ is the projective space of the underlying complex vector space of $\mathrm{Sym}(\mathbb{C}^n)$; so is an infinite dimensional complex projective spaces. While the spaces $\mathrm{Sym}(\mathbb{C}^n)$ are all homeomorphic for $n \geq 1$, it is important for the multiplicative properties that we are using *different* infinite dimensional vector spaces in the various levels.

The symmetric group Σ_n acts on $(\tilde{\Sigma}^\infty_+ \mathbb{C}P^\infty)_n$ by permuting the coordinates of $\mathbb{C}^n$, and then by functoriality of the symmetric algebra and projective space constructions. The multiplication map

$$\mu_{n,m} \ : \ S^n \wedge P(\mathrm{Sym}(\mathbb{C}^n))_+ \wedge S^m \wedge P(\mathrm{Sym}(\mathbb{C}^m))_+ \ \longrightarrow \ S^{n+m} \wedge P(\mathrm{Sym}(\mathbb{C}^{n+m}))_+$$

is given by

$$\mu_{n,m}((x \wedge L) \wedge (y \wedge L')) \ = \ x \wedge y \wedge (L \otimes L') \ ,$$

where $x \in S^n$, $y \in S^m$ and L and L' are complex lines in $\mathrm{Sym}(\mathbb{C}^n)$ respectively $\mathrm{Sym}(\mathbb{C}^m)$. Here $L \otimes L'$ becomes a line in $\mathrm{Sym}(\mathbb{C}^{n+m})$ via the preferred identification $\mathrm{Sym}(\mathbb{C}^n) \otimes \mathrm{Sym}(\mathbb{C}^m) \cong \mathrm{Sym}(\mathbb{C}^{n+m})$. We omit the verification that the multiplication maps are associative and commutative. We define unit maps

$$\iota_n \ : \ S^n \ \longrightarrow \ S^n \wedge P(\mathrm{Sym}(\mathbb{C}^n))_+ \ = \ (\tilde{\Sigma}^\infty_+ \mathbb{C}P^\infty)_n \quad \text{by} \quad \iota_n(x) \ = \ x \wedge \langle 1 \rangle \ ,$$

where 1 denotes the unit element of the symmetric algebra and $\langle 1 \rangle$ the distinguished 'line of constants' in $\mathrm{Sym}(\mathbb{C}^n)$, i.e., the line generated by 1. The line of constants $\langle 1 \rangle$ is Σ_n-invariant and the constants multiply to constants, so the units maps are indeed unital and ι_n is the n-fold product of ι_1. This completes the construction of $\tilde{\Sigma}^\infty_+ \mathbb{C}P^\infty$ as a commutative symmetric ring spectrum.

Now we explain why the symmetric spectrum $\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty$ is $\hat{\pi}_*$-isomorphic to the unreduced suspension spectrum of infinite dimensional complex projective space. The symmetric spectrum $\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty$ is essentially a suspension spectrum from level 1 on, in the sense that for $n \geq 1$ the structure map

$$\sigma_n \ : \ (\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty)_n \wedge S^1 \ \longrightarrow \ (\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty)_{n+1}$$

is a homotopy equivalence [...]. In level 0, however, the unit map $\iota_0 : S^0 \longrightarrow (\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty)_0$ is a homeomorphism since $\mathrm{Sym}(\mathbb{C}^0)$ consists only of the line of constants.

If we take $\mathbb{C}P^\infty = P(\mathrm{Sym}(\mathbb{C}))$ as our model for the infinite dimensional projective space, then as m varies the iterated structure maps

$$(\Sigma^\infty_+\mathbb{C}P^\infty)_{1+m} \ = \ \mathbb{C}P^\infty_+ \wedge S^{1+m} \ \cong \ (\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty)_1 \wedge S^m \ \xrightarrow{\sigma^m} \ (\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty)_{1+m}$$

define a level equivalence of symmetric spectra

$$\tau_1(\Sigma^\infty_+\mathbb{C}P^\infty) \ \longrightarrow \ \tilde{\Sigma}^\infty_+\mathbb{C}P^\infty$$

where τ_1 is the truncation above level 0. So the unreduced suspension spectrum of $\mathbb{C}P^\infty$ and the underlying symmetric spectrum of $\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty$ are related by a chain of two $\hat{\pi}_*$-isomorphisms:

$$\Sigma^\infty_+\mathbb{C}P^\infty \ \longleftarrow \ \tau_1(\Sigma^\infty_+\mathbb{C}P^\infty) \ \longrightarrow \ \tilde{\Sigma}^\infty_+\mathbb{C}P^\infty \ .$$

One can say a little more. The projective space $\mathbb{C}P^\infty$ is an Eilenberg-Mac Lane space of type $(\mathbb{Z}, 2)$, and as such admits a model as a commutative topological group. As a symmetric ring spectrum, $\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty$ is stably equivalent to the spherical group ring (see Example 3.42) of any such commutative group model [exercise...].

A morphism of symmetric ring spectra

$$\gamma \ : \ \tilde{\Sigma}^\infty_+\mathbb{C}P^\infty \ \longrightarrow \ ku \tag{6.60}$$

is given in level n by the map

$$S^n \wedge P(\mathrm{Sym}(\mathbb{C}^n))_+ \ \longrightarrow \ \Lambda(S^n, \mathrm{Sym}(\mathbb{C}^n)) \ , \quad x \wedge L \ \longmapsto \ [x; L] \ .$$

In other words, for $x \in S^n$ and L a line in $\mathrm{Sym}(\mathbb{C}^n)$, the element $x \wedge L$ maps to the configuration consisting of the single point x labeled by L. So the map is an embedding and its image consists of those configurations in which the sum of all labels has dimension at most 1. In level 1, the morphism $\gamma_1 : S^1 \wedge \mathbb{C}P^\infty_+ \longrightarrow ku_1$ represents the tautological line bundle L in $K^0(\mathbb{C}P^\infty)$.

We let e be the idempotent selfmap of $(\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty)_1$ defined as the composite

$$S^1 \wedge P(\mathrm{Sym}(\mathbb{C}))_+ \xrightarrow{\mathrm{proj}} S^1 \xrightarrow{\iota_1} S^1 \wedge P(\mathrm{Sym}(\mathbb{C}))_+ \ .$$

We use the suspension coordinate to form the difference

$$\mathrm{Id} - e \ : \ S^1 \wedge P(\mathrm{Sym}(\mathbb{C}))_+ \ \longrightarrow \ S^1 \wedge P(\mathrm{Sym}(\mathbb{C}))_+ \ .$$

Since e is the identity on the subspace $S^1 \wedge \{\langle 1 \rangle\}_+$, the restriction of the map $\mathrm{Id} - e$ to $S^1 \wedge \{\langle 1 \rangle\}_+$ is homotopic to the constant map with value $\infty \wedge \langle 1 \rangle$. So we can change $\mathrm{Id} - e$ to a homotopic selfmap of $(\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty)_1$ that sends the entire subspace $S^1 \wedge \{\langle 1 \rangle\}_+$ to $\infty \wedge \langle 1 \rangle$. So this latter map factors over a based continuous map

$$\psi \ : \ S^1 \wedge P(\mathrm{Sym}(\mathbb{C})) \ \longrightarrow \ S^1 \wedge P(\mathrm{Sym}(\mathbb{C}))_+ \ .$$

where smash product in the source is formed with respect to the basepoint $\langle 1 \rangle$. We identify the 1-dimensional complex projective space $P(\mathbb{C}\{1, x\})$ with S^2 [how?] and obtain a based map

$$u \ : \ S^3 = S^1 \wedge S^2 \ \cong \ S^1 \wedge P(\mathbb{C}\{1, x\}) \ \xrightarrow{\mathrm{incl.}} \ S^1 \wedge P(\mathrm{Sym}(\mathbb{C})) \ \xrightarrow{\psi} \ S^1 \wedge P(\mathrm{Sym}(\mathbb{C}))_+ = (\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty)_1 \ .$$

[does the class in $\pi_2(\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty)$ represented by the map u depend on the choice of homotopy?] The map u represents a stable homotopy class

$$[u] \ \in \ \hat{\pi}_2(\tilde{\Sigma}^\infty_+\mathbb{C}P^\infty) \ .$$

In fact, the second stable homotopy group of $\mathbb{C}P^\infty_+$ is a sum of a cyclic group of order 2 and an infinite cyclic group, and the class $[u]$ has infinite order.

We can now invert the map u, in the sense of Construction 6.53, and get a commutative symmetric ring spectrum

$$(\tilde{\Sigma}^\infty_+ \mathbb{C}P^\infty)[1/u] ,$$

which has the homotopy type of the mapping telescope of the sequence

$$\Sigma^\infty_+ \mathbb{C}P^\infty \xrightarrow{\ u\ } \Sigma^{\infty-2}_+ \mathbb{C}P^\infty \xrightarrow{\ u\ } \Sigma^{\infty-4}_+ \mathbb{C}P^\infty \xrightarrow{\ u\ } \cdots$$

The morphism (6.60) extends to a homomorphism of commutative symmetric ring spectra

$$\gamma[1/u] \ : \ (\tilde{\Sigma}^\infty_+ \mathbb{C}P^\infty)[1/u] \longrightarrow ku[1/u] \simeq KU \ . \tag{6.61}$$

It is a theorem of Snaith [ref] that the morphism $\gamma[1/u]$ is a π_*-isomorphism:

Theorem 6.62. *The homomorphism* (6.60) *of commutative symmetric ring spectra* $\gamma : \tilde{\Sigma}^\infty_+ \mathbb{C}P^\infty \longrightarrow ku$ *is a rational* $\hat{\pi}_*$*-isomorphism and the induced homorphism*

$$\gamma[1/u] \ : \ (\tilde{\Sigma}^\infty_+ \mathbb{C}P^\infty)[1/u] \longrightarrow KU$$

is a $\hat{\pi}_*$*-isomorphism*

Proof. Here is a sketch of the argument, relying on the a stable splitting result, also due to Snaith [**78**]. First, Snaith constructs an isomorphism in the stable homotopy category between

$$\Sigma^\infty_+ BU \quad \text{and} \quad \bigvee_{n\geq 1} MU(n)$$

and deduces from this an isomorphism

$$\hat{\pi}_*(\Sigma^\infty_+ BU)[1/u] \ \cong \ \hat{\pi}_* MU$$

in non-negative dimensions. Because the stable homotopy groups of MU are free abelian, the groups $(\hat{\pi}_*\Sigma^\infty_+ BU)[1/u]$ are torsion free in non-negative dimensions. The group homomorphism

$$BU(1) \xrightarrow{B\text{incl.}} BU \xrightarrow{B\det} BU(1)$$

induce maps of stable homotopy groups

$$\hat{\pi}_*(\Sigma^\infty_+ BU(1)) \longrightarrow \hat{\pi}_*(\Sigma^\infty_+ BU) \longrightarrow \hat{\pi}_*(\Sigma^\infty_+ BU(1))$$

that are compatible with multiplication by the Bott class u. So the localized groups $\hat{\pi}_*(\Sigma^\infty_+ BU(1)))[1/u]$ are a direct summand in the torsion-free groups $\hat{\pi}_*(\Sigma^\infty_+ BU)[1/u]$, hence torsion-free themselves.

Since the stable homotopy groups of the complex K-theory spectrum KU are also torsion free it suffices to show that the map (6.61) is an isomorphism on rationalized stable homotopy groups. Since $\hat{\pi}_* KU$ is a Laurent polynomial ring generated by u it suffices to show that $\mathbb{Q}\otimes\hat{\pi}_*(\Sigma^\infty_+ BU(1))$ is also a Laurent polynomial ring generated by u. However, rationalized stable homotopy groups are isomorphic to rational homology groups. Since the cohomology Hopf algebra $H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ is polynomial on a 2-dimensional generator, to dual homology Hopf algebra $H_*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ is a divided power algebra on the 2-dimensional generator (the Hurewicz image of the Bott class u). Rationally a divided power algebra is a polynomial algebra, and this finishes the proof. □

Example 6.63 (Brown-Peterson, Johnson-Wilson spectra and Morava K-theory)**.** If we apply the method of 'killing a regular sequence' to the Thom spectrum MU we can construct a whole collection of important spectra. In Example 1.18 we constructed MU as a commutative symmetric ring spectrum, and MU is semistable because it underlies an orthogonal spectrum (compare Proposition 8.26). As input for the following construction we need the knowledge of the homotopy ring of MU. The standard way to perform this calculation is in the following sequence of steps:

- calculate, for each prime p, the mod-p cohomology of the spaces $BU(n)$ and BU,
- use the Thom isomorphism to calculate the mod-p cohomology of the Thom spectrum MU as a module over the mod-p Steenrod algebra,

- use the Adams spectral sequence, which for MU collapses at the E_2-term, to calculate the p-completion of the homotopy groups of MU,
- and finally assemble the p-local calculations into the integral answer.

When the dust settles, the result is that π_*MU is a polynomial algebra generated by infinitely many homogeneous elements x_i of dimension $2i$ for $i \geq 1$. The details of this calculation can be found in [**83**] and [**67**] [check this; other sources?]. A very different geometric approach to this calculation was described by Quillen [**64**], who determines the ring of cobordism classes of stably almost complex manifolds, which by Thom's theorem is isomorphic to π_*MU. (Quillen's argument, however, needs as an input the a priori knowledge that the homotopy groups of MU are finitely generated in each dimension.)

Now fix a prime number p. Using the close connection between the ring spectrum MU and the theory of formal groups laws one can make particular choices for the (p^n-1)-th generator x_{p^n-1}, the so-called *Hazewinkel generator* [these are really p-local...], which is then denoted v_n. Killing all polynomial generators *except* those of the form x_{p^n-1} produces a semistable MU-module $\overline{BP}$ with homotopy groups $\pi_*(\overline{BP}) = \mathbb{Z}[v_1, v_2, v_3, \dots]$ where the degree of v_n is $2p^n-2$. Localizing at p produces a semistable MU-module BP, called the *Brown-Peterson spectrum*, with homotopy groups

$$\pi_*BP \;\cong\; \mathbb{Z}_{(p)}[v_1, v_2, v_3, \dots]\ .$$

The original construction of this spectrum by Brown and Peterson was quite different, and we say more about the history of BP in the 'History and credits' section at the end of this chapter.

Now we can keep going and kill more of the polynomial generators v_i in the homotopy of BP, and possibly also invert another generator. In this way we can produce various MU-modules BP/I and $(BP/I)[v_n^{-1}]$ together with MU-homomorphisms from BP whose underlying stable homotopy types play important roles in stable homotopy theory. Some examples of spectra which we can obtained in this way are given in the following table, along with their homotopy groups:

$BP\langle n\rangle$	$\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n]$	$E(n)$	$\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}]$
$P(n)$	$\mathbb{F}_p[v_n, v_{n+1}, \dots]$	$B(n)$	$\mathbb{F}_p[v_n^{-1}, v_n, v_{n+1}, \dots]$
$k(n)$	$\mathbb{F}_p[v_n]$	$K(n)$	$\mathbb{F}_p[v_n, v_n^{-1}]$

[discuss uniqueness] The spectrum $E(n)$ is referred to as the *Johnson-Wilson spectrum* and $k(n)$ respectively $K(n)$ are the connective and periodic *Morava K-theory spectra*.

We have so far only constructed the spectra above as MU-modules. The way we have presented the homotopy groups of the various spectra above does not only give graded modules over the homotopy ring of MU, but in fact graded commutative *algebras*. This already hints that the spectra have more structure. In fact, all the spectra above can be constructed as MU-algebra spectra, so in particular as symmetric ring spectra. We may or may not get back to this later.

7. Relation to other kinds of spectra

In this section we discuss how symmetric spectra relate to some other kinds of spectra, namely orthogonal spectra $\mathcal{S}p^{\mathbf{O}}$, unitary spectra $\mathcal{S}p^{\mathbf{U}}$, $\mathbf{\Gamma}$-spaces, simplicial and continuous functors. There are also the sequential spectra, which we defined in 2.1 (in the topological version; the simplicial version should be clear). The following diagram of categories and functors provides an overview:

$$(7.1)\qquad \begin{tikzcd}
 & & \mathcal{S}p^{\mathbf{U}} \arrow[d, shift left, "\Phi"] & & \\
\mathbf{\Gamma}\text{-spaces}/\mathbf{T} \arrow[r] \arrow[d, "\mathcal{S}"'] & (\text{continuous functors}) \arrow[r, "\mathrm{ev}_{\mathbb{S}}"'] \arrow[d, "\mathcal{S}"'] & \mathcal{S}p^{\mathbf{O}} \arrow[u, shift left, "\Psi"] \arrow[r, "U"] & \mathcal{S}p_{\mathbf{T}} \arrow[r, "U"] \arrow[d, "\mathcal{S}"] & \mathcal{S}p^{\mathbb{N}}_{\mathbf{T}} \arrow[d, "\mathcal{S}"] \\
\mathbf{\Gamma}\text{-spaces}/\mathbf{sS} \arrow[r] & (\text{simplicial functors}) \arrow[rr, "\mathrm{ev}_{\mathbb{S}}"'] & & \mathcal{S}p_{\mathbf{sS}} \arrow[r, "U"'] & \mathcal{S}p^{\mathbb{N}}_{\mathbf{sS}}
\end{tikzcd}$$

The functors pointing to the right are 'forgetful' or evaluation functors; functors pointing down from the second 'topological' row to the third 'simplicial' row are given by taking singular complexes. [If X is a continuous functor, then $\mathcal{S}X$ is the simplicial functor whose value on a simplicial set K is the simplicial set $\mathcal{S}(X(|K|))$; the left adjoint needs some finiteness assumptions] All functors in the diagram have left adjoints. The lower rectangle commutes up to natural isomorphism. All those symmetric spectra which arise in this way from $\mathbf{\Gamma}$-spaces, continuous or simplicial functors, orthogonal or unitary spectra are semistable.

One can also compare symmetric spectra to the category of S-modules in the sense of Elmendorf, Mandell, May and Kriz [**26**]. However, the S-modules are of a rather different flavor, and we do not need them anywhere else in this book, so we refer to the original paper [**69**] for that comparison. The comparison between S-modules and symmetric spectra factors through orthogonal spectra, and that comparison is discussed in Chapter I of [**52**].

This section differs from the previous sections in several respects. First, we will not use the results in this section in the remainder of the book [check this]. Also, we deviate from our general strategy to be self-contained. We will define everything rigorously, but often replace proofs by references to the literature.

7.1. Orthogonal spectra. As the name suggests, orthogonal spectra are a version of symmetric spectra where symmetric groups are replaced by orthogonal groups. In more detail:

Definition 7.2. An *orthogonal spectrum* consists of the following data:

- a sequence of pointed spaces X_n for $n \geq 0$
- a base-point preserving continuous left action of the orthogonal group $O(n)$ on X_n for each $n \geq 0$
- based maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ for $n \geq 0$.

This data is subject to the following condition: for all $n, m \geq 0$, the iterated structure map

$$\sigma^m \ : \ X_n \wedge S^m \ \longrightarrow \ X_{n+m}$$

(defined as in (1.2)) is $O(n) \times O(m)$-equivariant. The orthogonal group $O(m)$ acts on S^m since this is the one-point compactification of $\mathbb{R}^m$ and $O(n) \times O(m)$ acts on the target by restriction, along orthogonal sum, of the $O(n+m)$-action.

A morphism $f : X \longrightarrow Y$ of orthogonal spectra consists of $O(n)$-equivariant based maps $f_n : X_n \longrightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge \mathrm{Id}_{S^1})$ for all $n \geq 0$. We denote the category of orthogonal spectra by $\mathcal{S}p^{\mathbf{O}}$.

An *orthogonal ring spectrum* R consists of the following data:

- a sequence of pointed spaces R_n for $n \geq 0$
- a base-point preserving continuous left action of the orthogonal group $O(n)$ on R_n for each $n \geq 0$
- $O(n) \times O(m)$-equivariant *multiplication maps* $\mu_{n,m} : R_n \wedge R_m \longrightarrow R_{n+m}$ for $n, m \geq 0$, and
- $O(n)$-equivariant *unit maps* $\iota_n : S^n \longrightarrow R_n$ for all $n \geq 0$.

This data is subject to the same associativity and unit conditions as a symmetric ring spectrum (see Definition 1.3) and a centrality condition for every unit map ι_n. In the unit condition, permutations such as $\chi_{n,m} \in \Sigma_{n+m}$ have to be interpreted as permutation matrices in $O(n+m)$. An orthogonal ring spectrum R is *commutative* if for all $n, m \geq 0$ the relation $\chi_{n,m} \circ \mu_{n,m} = \mu_{m,n} \circ \mathrm{twist}$ holds as maps $R_n \wedge R_m \longrightarrow R_{m+n}$.

A *morphism* $f : R \longrightarrow S$ of orthogonal ring spectra consists of $O(n)$-equivariant based maps $f_n : R_n \longrightarrow S_n$ for $n \geq 0$, which are compatible with the multiplication and unit maps (in the same sense as for symmetric ring spectra).

Orthogonal spectra are 'symmetric spectra with extra symmetry' in the sense that every orthogonal spectrum X has an *underlying symmetric spectrum* UX. Here $(UX)_n = X_n$ and the symmetric group acts by restriction along the monomorphism $\Sigma_n \longrightarrow O(n)$ given by permutation matrices. The structure maps of UX are the structure maps of X.

Many symmetric spectra that we have discussed in this chapter have the 'extra symmetry', i.e., they are underlying orthogonal spectra. Examples are the sphere spectrum, suspension spectra or the various Thom spectra such as MO and MU arise from orthogonal spectra by forgetting symmetry. Free symmetric spectra $F_m K$ or semifree symmetric spectra $G_m K$ do not arise this way (unless $m = 0$ or $K = *$).

The symmetric spectra which underly orthogonal spectra always semistable (compare Proposition 3.16 (vi)) so for orthogonal spectra, the naive and true homotopy groups coincide.

For $m \geq 0$ the *free orthogonal spectrum* $F_m^{\mathbf{O}}$ generated in level m is the orthgonal spectrum corresponding to the representable functor $\mathbf{O}(m,-)$. Explicitly, $F_m^{\mathbf{O}}$ is trivial below level m and is otherwise given by

$$(F_m^{\mathbf{O}})_{m+n} \;=\; \mathbf{O}(m, m+n) \;=\; O(m+n)^+ \wedge_{1\times O(n)} S^n \ .$$

[Structure maps] By the enriched Yoneda lemma (Proposition 7.43) morphisms from $F_m^{\mathbf{O}}$ to an orthogonal spectrum X are in bijective correspondence with elements of X_m.

A morphism of symmetric spectra $F_m^{\mathbf{\Sigma}} \longrightarrow U(F_m^{\mathbf{O}})$ is given by the adjoint of $S^0 \longrightarrow (F_m^{\mathbf{O}})_m = O(m)^+ \wedge S^0$ which sends the non-basepoint of S^0 to the neutral element of $O(m)$, smashed with the non-basepoint of S^0. In the special case $n = 0$ the morphism $\mathbb{S} \cong F_0^{\mathbf{\Sigma}} \longrightarrow U(F_0^{\mathbf{O}})$ is in fact an isomorphism.

The naive homotopy groups of the free symmetric spectra $F_m S^m$ are 'too big' in the sense that the morphism $\lambda : F_m S^m \longrightarrow \mathbb{S}$ is not injective on naive homotopy groups as soon as $m \geq 1$. Replacing the free *symmetric* spectrum by the free *orthogonal* spectra fixes this, as the following proposition shows.

Proposition 7.3. (i) *The morphism of orthogonal spectra $\lambda^{\mathbf{O}} : S^m \wedge F_m^{\mathbf{O}} \longrightarrow \mathbb{S}$ adjoint to the identity of S^m induces an isomorphism of naive homotopy groups.*

(ii) *The morphism $\eta : F_m^{\mathbf{\Sigma}} \longrightarrow U(F_m^{\mathbf{O}})$ adjoint to the map $S^0 \longrightarrow O(m)^+ = U(F_m^{\mathbf{O}})_m$ which sends the non-basepoint to the unit element of $O(m)$ is a stable equivalence of symmetric spectra.*

Proof. (i) We use the space $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m+n})$ of linear isometries from $\mathbb{R}^m$ to $\mathbb{R}^{m+n}$. Precomposition with $i : \mathbb{R}^{m-1} \longrightarrow \mathbb{R}^m, i(x) = (x, 0)$ is a locally trivial fiber bundle $i^* : \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \longrightarrow \mathcal{L}(\mathbb{R}^{m-1}, \mathbb{R}^{m+n})$ with fiber an n-sphere, so i^* induces isomorphisms of homotopy groups below dimension n. Since $\mathcal{L}(\mathbb{R}^0, \mathbb{R}^{m+n})$ is a one-point space, we conclude by induction that $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m+n})$ is $(n-1)$-connected.

A homeomorphism

$$(S^m \wedge F_m^{\mathbf{O}})_{m+n} = S^m \wedge \left(O(m+n)^+ \wedge_{1\times O(n)} S^n\right) \;\cong\; S^{m+n} \wedge \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m+n})^+$$

is given by sending $x \wedge [A \wedge y]$ to $A \cdot (x \wedge y) \wedge \rho(A)$ where $\rho(A)$ is the restriction of a linear isometry $A \in O(m+n) = \mathcal{L}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n})$ to $\mathbb{R}^m$. Under this homeomorphisms the $(m+n)$-th level of the morphism $\lambda^{\mathbf{O}}$ corresponds to the map

$$S^{m+n} \wedge \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m+n})^+ \xrightarrow{\ \mathrm{Id}\wedge i^*\ } S^{m+n}$$

which is induced by the unique map $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \longrightarrow *$. The space $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^{m+n})$ is $(n-1)$-connected, so the map $\lambda^{\mathbf{O}}_{m+n}$ induces an isomorphism of homotopy groups below dimension $2n$. As n goes to infinity, we conclude that $\lambda^{\mathbf{O}}$ induces an isomorphism on naive homotopy groups.

(ii) Since suspension preserves and detects stable equivalences (Proposition 4.29), it suffices to show that the morphism $S^m \wedge \eta : S^m \wedge F_m^{\Sigma} \longrightarrow S^m \wedge U(F_m^{\mathbf{O}})$ is a stable equivalence. In the commutative triangle

$$\begin{tikzcd} F_m^{\mathbf{\Sigma}} S^m \arrow[rr, "S^m\wedge\eta"] \arrow[dr, "\lambda"'] & & U(F_m^{\mathbf{O}} S^m) \arrow[dl, "U(\lambda^{\mathbf{O}})"] \\ & \mathbb{S} = U(F_0^{\mathbf{O}}) & \end{tikzcd}$$

the right diagonal morphism is a $\hat{\pi}_*$-isomorphism by part (i), hence a stable equivalence. The left diagonal morphism is a stable equivalence by (by Example 4.26). So the morphism $S^m \wedge \eta$, and thus η itself, are stable equivalences. $\square$

By Corollary 7.47 the forgetful functor $U : \mathcal{S}p^{\mathbf{O}} \longrightarrow \mathcal{S}p_{\mathbf{T}}$ from orthogonal spectra to symmetric spectra of spaces has a left adjoint P. [since UP is lax symmetric monoidal, it makes flat ring/module/algebra spectra semistable, preserving symmetries] [exercise: For every strictly fibrant Ω-spectrum X [not yet defined] the adjunction counit $\eta : U(RX) \longrightarrow X$ is a level equivalence of symmetric spectra, where R is the right adoint.]

Proposition 7.4. *For every flat symmetric spectrum of spaces A the adjunction unit $\eta : X \longrightarrow U(PX)$ is a stable equivalence of symmetric spectra.*

Proof. We start with the special case of a semifree symmetric spectrum G_mL generated in level m by a cofibrant based Σ_m-space L. By representability, the prolonged semifree symmetric spectrum $P(G_kL)$ is trivial below dimension m and is otherwise given by

$$P(G_mL)_{m+n} \;\cong\; O(m+n)^+ \wedge_{\Sigma_m \times O(n)} L \wedge S^n \;;$$

this orthogonal spectrum is naturally isomorphic to $F_m^{\mathbf{O}} \wedge_{\Sigma_m} L$. We choose a based Σ_m-CW-complex $\bar{L}$ with free action (away from the basepoint) and an equivariant based map $\varphi : \bar{L} \longrightarrow L$ which is a weak equivalence of underlying spaces (or simplicial sets); for example, we could take $\bar{L} = E\Sigma_m^+ \wedge L$. Then the morphisms $G_m\varphi : G_m\bar{L} \longrightarrow G_mL$ and $UP(G_m\varphi) : UP(G_m\bar{L}) \longrightarrow UP(G_mL)$ and are level equivalences by inspection. In the commutative square

$$\begin{tikzcd}
G_m\bar{L} \arrow[r] \arrow[d, "G_m\varphi"'] & UP(G_m\bar{L}) \arrow[d, "UP(G_m\varphi)"] \\
G_mL \arrow[r] & UP(G_mL)
\end{tikzcd}$$

the two vertical morphisms are thus level equivalences. The upper vertical morphism is obtained from the stable equivalence (by Proposition 7.3 (ii)) $F_m^{\mathbf{\Sigma}} \longrightarrow U(F_m^{\mathbf{O}})$ by taking smash product with $\bar{L}$ over Σ_m. Since $\bar{L}$ has a free action, the functor $- \wedge_{\Sigma_m} L$ preserves stable equivalences by Proposition 4.31 (xi) [this in not quite what that says...]. So the lower horizontal adjunction unit is also a stable equivalence.

For a general flat symmetric spectrum X we use the skeleton filtration introduced in Construction 5.29. For a flat symmetric spectrum X we show by induction on k that the adjunction unit $F^kX \longrightarrow UP(F^kX)$ is a stable equivalence, where F^kX is the k-skeleton of X. We start with $F^{-1}X = *$, the trivial spectrum, for which the adjunction unit is an isomorphism.

By definition, the $(m-1)$-skeleton and the k-skeleton of X are part of a pushout diagram of symmetric spectra [ref] which implies that the quotient spectrum $F^kX/F^{k-1}X$ of the k-skeleton by the $(k-1)$-skeleton is isomorphic to the semifree spectrum $G_k(X_k/L_kX)$, where L_kX is the k-th latching space. The prolongation functor P and the forgetful functor U both have right adjoints, so they preserve colimits and in the commutative diagram

$$\begin{tikzcd}
F^{k-1}X \arrow[r, "j_k"] \arrow[d] & F^kX \arrow[r] \arrow[d] & G_k(X_k/L_kX) \arrow[d] \\
UP(F^{k-1}X) \arrow[r, "UP(j_k)"'] & UP(F^kX) \arrow[r] & UP(G_k(X_k/L_kX))
\end{tikzcd}$$

both rows are cofiber sequences of symmetric spectra. The map $UP(j_k)$ is levelwise an h-cofibration since it is a cobase change of the h-cofibration $UP(G_k\nu_k)$[justify]. The left and right vertical maps are stable equivalences by induction respectively the special above. So the middle vertical map is a stable equivalence by the five lemma, applies to the long exact sequences of true homotopy groups (compare Proposition 6.11 (iii)).

The symmetric spectrum X is the colimit of the skeleta F^kX along the morphisms $j_k : F^{k-1}X \longrightarrow F^kX$, which are level cofibrations since X is flat (Proposition 5.47 (iii)). The spectrum UPX is thus the colimit of the spectra $UP(F^kX)$. A sequential colimit, over level cofibrations, of stable equivalence is a stable equivalence by [...] So the adjunction unit $\eta : X \longrightarrow UPX$ is also a stable equivalence for the flat symmetric spectrum X. □

As a corollary of the previous two propositions we record that for every flat symmetric spectrum of spaces A the adjunction unit $\eta : A \longrightarrow U(PA)$ is a stable equivalence to a semistable symmetric spectrum.

7.2. Unitary spectra. Unitary spectra are the complex analogues of orthogonal spectra. As the name suggests, symmetric or orthogonal groups are replaced by unitary orthogonal groups. There is another difference, however, coming from the fact that the n-th unitary group $U(n)$ acts naturally on the sphere of dimension $2n$ (and not n). This also shows up in the structure maps, which involve a 2-sphere (and not a circle).

Definition 7.5. A *unitary spectrum* consists of the following data:

- a sequence of pointed spaces X_n for $n \geq 0$
- a base-point preserving continuous left action of the unitary group $U(n)$ on X_n for each $n \geq 0$
- based maps $\sigma_n : X_n \wedge S^2 \longrightarrow X_{n+1}$ for $n \geq 0$.

This data is subject to the following condition: for all $n, m \geq 0$, the iterated structure map

$$\sigma^m \ : \ X_n \wedge S^{2m} \ \longrightarrow \ X_{n+m}$$

(defined as in (1.2)) is $U(n) \times U(m)$-equivariant. Here $U(m)$ acts on S^{2m} via the isomorphism with the one-point compactification of $\mathbb{C}^m$ [make the isomorphism explicit in the $\mathbb{R}$-basis $\{1, i\}$ of $\mathbb{C}$]. The group $U(n) \times U(m)$ acts on the target by restriction, along orthogonal direct sum, of the $U(n+m)$-action.

A morphism $f : X \longrightarrow Y$ of unitary spectra consists of $U(n)$-equivariant based maps $f_n : X_n \longrightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge \mathrm{Id}_{S^1})$ for all $n \geq 0$. We denote the category of unitary spectra by $\mathcal{S}p^{\mathbf{U}}$.

Having seen the definition of symmetric ring spectra and orthogonal ring spectra, it should now be clear what *unitary ring spectra* and their morphisms are. We omit the details.

There are two functors $\Psi : \mathcal{S}p^{\mathbf{O}} \longrightarrow \mathcal{S}p^{\mathbf{U}}$ and $\Phi : \mathcal{S}p^{\mathbf{U}} \longrightarrow \mathcal{S}p^{\mathbf{O}}$ relating unitary spectra to orthogonal spectra that are compatible with multiplicative structures. Given an orthogonal spectrum Y we define a unitary spectrum $\Psi(Y)$ by

$$\Psi(Y)_n \ = \ Y_{2n} \tag{7.6}$$

where the unitary group $U(n)$ acts via the restriction along the monomorphism $U(n) \longrightarrow O(2n)$ that arises by identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ [...]. The structure map $\sigma_n : \Psi(Y)_n \wedge S^2 \longrightarrow \Psi(Y)_{n+1}$ is the iterated structure map

$$Y_{2n} \wedge S^2 \ \xrightarrow{\sigma^2} \ Y_{2n+2} = Y_{2(n+1)} \ .$$

Given a unitary spectrum X we can produce an orthogonal spectrum $\Phi(X)$ as follows. We set

$$\Phi(X)_n \ = \ \mathrm{map}(S^n, X_n) \ . \tag{7.7}$$

The orthogonal group acts on S^n, on X_n via the complexification map $O(n) \longrightarrow U(n)$ and on the mapping space by conjugation. The structure map $\sigma_n : \Phi(X)_n \wedge S^1 \longrightarrow \Phi(X)_{n+1}$ is adjoint to the map

$$\begin{aligned} \mathrm{map}(S^n, X_n) \wedge S^1 \wedge S^{n+1} \ &\cong \ \mathrm{map}(S^n, X_n) \wedge S^n \wedge S^2 \\ &\xrightarrow{\mathrm{eval} \wedge \mathrm{Id}} \ X_n \wedge S^2 \ \xrightarrow{\sigma_n} \ X_{n+1} \end{aligned}$$

where we have used a shuffle isomorphism $S^1 \wedge S^{n+1} \cong S^n \wedge S^2$ [specify]. The composite $\Phi\Psi : \mathcal{S}p^{\mathbf{O}} \longrightarrow \mathcal{S}p^{\mathbf{O}}$ is the analog of the construction $X \mapsto X[1/\iota_1]$ [...]

An example of this is the complex cobordism spectrum MU of Example 1.18 which arises naturally as a unitary ring spectrum, made into an orthogonal spectrum via the functor Φ. More precisely, the sequence of based spaces denoted $\overline{MU}$ in Example 1.18 comes from a unitary spectrum with nth space

$$\overline{MU}_n \ = \ EU(n)^+ \wedge_{U(n)} S^{2n} \ ,$$

the Thom space of the vector bundle over $BU(n)$ with total space $EU(n) \times_{U(n)} \mathbb{C}^n$.

Example 7.8 (Periodic complex cobordism)**.** We define the *periodic complex cobordism* spectrum MUP, a unitary spectrum, as follows. For a complex inner product space V we consider the 'full Grassmannian' $Gr(V \oplus V)$ of $V \oplus V$. A point in $Gr(V \oplus V)$ is any complex sub-vectorspace of $V \oplus V$, and this space is topologized as the disjoint union of the Grassmannians of k-dimensional subspaces of $V \oplus V$ for $k = 0, \dots, 2\dim(V)$. Over the full Grassmannian $Gr(V \oplus V)$ sits a tautological hermitian vector bundle (of non-constant rank!): the total space of this bundle consist of pairs (U, x) where U is a complex sub-vectorspace of $V \oplus V$ and $x \in U$.

We define $(MUP)(V)$ as the Thom space of this tautological vector bundle, i.e., the quotient space of the unit disc bundle by the sphere bundle. The multiplication

$$(MUP)(V) \wedge (MUP)(W) \longrightarrow (MUP)(V \oplus W) \tag{7.9}$$

sends $(U, x) \wedge (U'x')$ to $(U + U', (x, x'))$ where $U + U'$ is the image of $U \oplus U'$ under the isometry $\mathrm{Id} \wedge \tau \wedge \mathrm{Id} : (V \oplus V) \oplus (W \oplus W) \cong (V \oplus W) \oplus (V \oplus W)$. The unit map $S^V \longrightarrow (MUP)(V)$ sends $v \in V$ to $(\Delta(V), (v, v))$ where $\Delta(V)$ is the diagonal copy of V in $V \oplus V$.

As the name suggests, MUP is a periodic version of the Thom spectrum MU. More precisely, we claim that MUP is $\mathbb{Z}$-graded unitary ring spectrum whose piece of degree k is $\hat{\pi}_*$-isomorphic to a $2k$-fold suspension of MU. For every integer k and complex inner product space V we let $Gr^{(k)}(V \oplus V)$ be the subspace of $Gr(V \oplus V)$ consisting of subspaces of dimension $\dim(V) + k$. So the full Grassmannian is the disjoint union of the spaces $Gr^{(k)}(V \oplus V)$ for $k = -\dim(V), \dots, \dim(V)$. We let $(MUP^{(k)})(V)$ be the Thom space of the tautological $(\dim V + k)$-plane bundle over $Gr^{(k)}(V \oplus V)$, so that $(MUP)(V)$ is the one-point union of the Thom spaces $(MUP^{(k)})(V)$ for $k = -\dim(V), \dots, \dim(V)$. We note that the unit map $S^V \longrightarrow (MUP)(V)$ has image in the degree 0 summand $(MUP^{(0)})(V)$. The multiplication map (7.9) is 'graded' in the sense that its retriction to $(MUP^{(k)})(V) \wedge (MUP^{(l)})(W)$ has image in $(MUP^{(k+l)})(V \oplus W)$. Together this implies that $MUP^{(k)}$ is a unitary subspectrum of MUP and altogether the periodic spectrum decomposes as

$$MUP \;=\; \bigvee_{k \in \mathbb{Z}} MUP^{(k)} \; .$$

Now we explain why $MUP^{(k)}$ is, up to $\hat{\pi}_*$-isomorphism, a $2k$-fold suspension of $MUP^{(0)}$. We observe that the Grassmannian $Gr^{(-1)}(\mathbb{C} \oplus \mathbb{C})$ has only one point (the zero subspace of $\mathbb{C} \oplus \mathbb{C}$), and so $MUP^{(-1)}(\mathbb{C})$ is a 0-sphere. So a special case of the multiplication map is

$$MUP^{(1+k)}(V) \cong MUP^{(-1)}(\mathbb{C}) \wedge MUP^{(1+k)}(V) \;\longrightarrow\; MUP^{(k)}(\mathbb{C} \oplus V) \; .$$

If we let V vary, these maps form a morphism of unitary spectra

$$MUP^{(1+k)} \;\longrightarrow\; \mathrm{sh}_{\mathbb{C}} \, MUP^{(k)} \; ,$$

where the right hand side is the *shift* of a unitary spectrum by the inner product space $\mathbb{C}$. Since the map $MUP^{(1+k)}(V) \longrightarrow MUP^{(k)}(\mathbb{C} \oplus V)$ is highly connected [prove] the previous morphism of unitary spectra is in fact a $\hat{\pi}_*$-isomorphism.

It remains to relate the spectrum $MUP^{(0)}$ to MU through morphisms of unitary spectra which are $\hat{\pi}_*$-isomorphisms. [...]

If we replace *complex* inner product spaces by *real* inner product spaces throughout, we obtain an commutative orthogonal ring spectrum MOP which is a periodic version of the unoriented Thom spectrum MO in much the same way.

7.3. Continuous and simplicial functors. We denote by **CW** the category category of based spaces which admit the structure of a finite CW-complex. Every such space is in particular a compact Hausdorff space and the category **CW** contains the spheres S^n and is closed under smash product. By a *continuous functor* we mean a functor $F : \mathbf{CW} \longrightarrow \mathbf{T}$ which is pointed in that it takes one-point spaces to one-point spaces and continuous in the sense that for all pointed spaces K and L the map

$$F \;:\; \mathbf{T}(K, L) \;\longrightarrow\; \mathbf{T}(F(K), F(L))$$

is continuous with respect to the compact open topology on the mapping spaces. The (continuous !) map

$$L \xrightarrow{\; l \,\mapsto (k \mapsto k \wedge l) \;} \mathbf{T}(K, K \wedge L) \xrightarrow{\;F\;} \mathbf{T}(F(K), F(K \wedge L)) \; .$$

then has an adjoint
$$F(K) \wedge L \longrightarrow F(K \wedge L)$$
which we call the *assembly map*. The assembly map is natural in K and L, it is unital in the sense that the composite
$$F(K) \cong F(K) \wedge S^0 \xrightarrow{\text{assembly}} F(K \wedge S^0) \cong F(K)$$
is the identity and it is associative in the sense that the diagram
$$\begin{tikzcd}
(F(K) \wedge L) \wedge M \arrow[r, "\text{ass.}\wedge\mathrm{Id}"] \arrow[d, "\cong"'] & F(K \wedge L) \wedge M \arrow[r, "\text{ass.}"] & F((K \wedge L) \wedge M) \arrow[d, "F(\cong)"] \\
F(K) \wedge (L \wedge M) \arrow[rr, "\text{assembly}"'] & & F(K \wedge (L \wedge M))
\end{tikzcd}$$
commutes for all K, L and M, where the vertical maps are associativity isomorphisms for the smash product.

As usual, there is also a simplicial version. A *simplicial functor* is an enriched, pointed functor $F : \mathbf{sS}^f \longrightarrow \mathbf{sS}$ from the category of finite pointed simplicial sets to itself. So F assigns to each pointed simplicial set K a pointed simplicial set $F(K)$ and to each pair K, L of pointed simplicial sets a morphism of pointed simplicial sets
$$F \ : \ \mathrm{map}(K, L) \longrightarrow \mathrm{map}(F(K), F(L))$$
which is associative and unital and such that $F(*) \cong *$. The restriction of F to vertices is then a functor in the usual sense. The same kind of adjunctions as for continuous functors provides a simplicial functor with an *assembly map* $F(K) \wedge L \longrightarrow F(K \wedge L)$, again unital and associative.

To every continuous (respectively simplicial) functor F we can associate a symmetric spectrum of spaces (respectively of simplicial sets) $F(\mathbb{S})$ by
$$F(\mathbb{S})_n \ = \ F(S^n)$$
where Σ_n permutes the coordinates of S^n. The structure map $\sigma_n : F(S^n) \wedge S^1 \longrightarrow F(S^{n+1})$ is an instance of the assembly map. In the setting of topological spaces, the action of Σ_n on $F(S^n)$ even extends to an action of the orthogonal group $O(n)$, since $O(n)$ acts continuously on S^n. In other words, 'evaluation at spheres' defines forgetful functors
$$\mathrm{ev}_{\mathbb{S}} \ : \ (\text{continuous functors}) \longrightarrow \mathcal{S}p^{\mathbf{O}} \quad \text{and} \quad \mathrm{ev}_{\mathbb{S}} \ : \ (\text{simplicial functors}) \longrightarrow \mathcal{S}p_{\mathbf{sS}} \ .$$
Similarly, evaluating on the sequence of even spheres S^{2n}, viewed as the one-point compactification of $\mathbb{C}^n$, provides a forgetful functor
$$\mathrm{ev}_{\mathbb{S}} \ : \ (\text{continuous functors}) \longrightarrow \mathcal{S}p^{\mathbf{U}}$$
to unitary spectra. The composite of this 'unitary' evaluation functor with the 'realification functor' $\Phi : \mathcal{S}p^{\mathbf{U}} \longrightarrow \mathcal{S}p^{\mathbf{O}}$ (see (7.7)) is *not* equal, nor isomorphic, to the 'orthogonal' evaluation functor. However, there is a natural $\hat{\pi}_*$-isomorphism of orthogonal spectra
$$\mathrm{ev}_{\mathbb{S}}^{\mathbf{U}}(F) \longrightarrow \Phi(\mathrm{ev}_{\mathbb{S}}^{\mathbf{O}}(F))$$
given at level n by the map
$$F(S^n) \longrightarrow \mathrm{map}(S^n, F(S^{2n}))$$
adjoint to the assembly map $F(S^n) \wedge S^n \longrightarrow F(S^{2n})$ [check...].[semistability of $F(\mathbb{S})$ in the simplicial context]

The functor of 'evaluation on spheres' from $\mathbf{\Gamma}$-spaces or continuous functors to symmetric spectra factors through orthogonal spectra, so the last claim follows. Hence every symmetric spectrum of spaces which arises from a $\mathbf{\Gamma}$-space or continuous functor by evaluation on spheres is semistable.

More generally, we can evaluate a continuous (simplicial) functor F on a symmetric spectrum X and get a new symmetric spectrum $F(X)$ by defining $F(X)_n = F(X_n)$ with structure map the composite
$$F(X_n) \wedge S^1 \xrightarrow{\text{assembly}} F(X_n \wedge S^1) \xrightarrow{F(\sigma_n)} F(X_{n+1}) \ .$$

In the context of topological spaces, the analogous construction can be performed within orthogonal or unitary spectra, i.e., $F(X)$ is naturally an orthogonal (respectively unitary) spectrum whenever X is. [this only works if F is defined on all spaces][semistability?]

Some of the symmetric spectra which we described in this chapter are evaluations of continuous (or simplicial) functors on spheres, for example suspension spectra and Eilenberg-Mac Lane spectra. Free symmetric spectra $F_m K$ or semifree symmetric spectra $G_m K$ do not arise this way (they don't even extend to orthogonal spectra unless $m = 0$ or $K = *$) and cobordism spectra like MO and MU don't either.

Example 7.10 (Connective topological K-theory from a continuous functor)**.** The symmetric spectra representing connective and periodic real respectively complex topological K-theory (compare Example 1.20) do not arise from continuous functors, but a small variation of the previous constructions of the connective spectra ko and ku does extend to continuous functors. Let us define two continuous functors by

$$F(X) \;=\; \bigcup_{n\geq 0} \hom(C_0(X); M_n) \qquad \text{respectively} \qquad \hat{F}(X) \;=\; \hom(C_0(X); \mathcal{K}) \; .$$

Here M_n is the algebra of $n \times n$ matrices with real entries, and the union on the left is taken along the (non-unital) algebra maps

$$M_n \;\longrightarrow\; M_{n+1} \; , \quad A \;\longmapsto\; \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \; .$$

On the right hand side, $\mathcal{K}$ is the C^*-algebra of compact operators on the fixed infinite-dimensional, complex separable Hilbert space H with orthonormal basis $\{e_i\}_{i\geq 1}$ [topology]. A natural map $F(X) \longrightarrow \hat{F}(X)$ is induced by the compatible $*$-morphisms $M_n \longrightarrow \mathcal{K}$ which is extension by zero on the orthogonal complement of the basis elements $\{e_1, \dots, e_n\}$.

The symmetric spectrum $F(\mathbb{S})$ is a connective positive Ω-spectrum, the morphism $F(\mathbb{S}) \longrightarrow \hat{F}(\mathbb{S})$ is a level equivalence of symmetric spectra (even of orthogonal spectra) and that $\hat{F}(\mathbb{S})_n = \hat{F}(S^n)$ is homotopy equivalent to ku_n [justify]. The symmetric spectrum ku of Example 1.20 has the advantage over $F(\mathbb{S})$ and $\hat{F}(\mathbb{S})$ that it comes with a commutative multiplication. [is there a morphism of symmetric spectra relating ku and $F(\mathbb{S})$ or $\hat{F}(\mathbb{S})$?]

We call a continuous of simplicial functor F *diagonalizable* if there exists a funtor G in two variables which is reduced and continuous respectively simplicial in each variable seperately, and a natural isomorphism $F(X) \cong G(X, X)$ of functors in one variable.

Lemma 7.11. *Let F be a simplicial or continuous functor which is* diagonalizable*. Then the symmetric spectrum $F(\mathbb{S})$ has trivial naive homotopy groups. Hence $F(\mathbb{S})$ is stably contractible.*

Proof. Suppose $F(X) = G(X, X)$ where G is a functor in two variables which is reduced and simplicial or continuous in each variable seperately. Then the structure map $\sigma_n : F(S^n) \wedge S^1 \longrightarrow F(S^{n+1})$ of the symmetric spectrum $F(\mathbb{S})$ equals the composite

$$G(S^n, S^n) \wedge S^1 \xrightarrow{G(S^n,S^n)\wedge\Delta} G(S^n, S^n) \wedge S^1 \wedge S^1 \xrightarrow{\sigma_{n,n}} G(S^{n+1}, S^{n+1}) \; .$$

Since the diagonal map $\Delta : S^1 \longrightarrow S^1 \wedge S^1$ is null-homotopic, the structure map σ_n is null-homotopy. So the stabilization map $\iota : \pi_{k+n} G(S^n, S^n) \longrightarrow \pi_{k+n+1} G(S^{n+1}, S^{n+1})$ is trivial, and so is the colimit $\pi_k F(\mathbb{S})$. $\square$

A consequence of the formal properties of the assembly map is that the structure of a triple [def or ref] on a continuous or simplicial functor T yields a multiplication on the symmetric spectrum $T(\mathbb{S})$. Indeed, using the assembly map twice and the triple structure map produces multiplication maps

$$T(K) \wedge T(L) \;\longrightarrow\; T(K \wedge T(L)) \;\longrightarrow\; T(T(K \wedge L)) \;\longrightarrow\; T(K \wedge L) \; ;$$

here K and L are pointed spaces. If we apply this to spheres, we get $\Sigma_p \times \Sigma_q$-equivariant maps

$$T(S^p) \wedge T(S^q) \;\longrightarrow\; T(S^{p+q})$$

which provide the multiplication. The unit maps come from the natural transformation $\mathrm{Id} \longrightarrow T$ by evaluating on spheres. In the context of topological spaces, evaluating a triple at spheres as above even gives an *orthogonal* ring spectrum.

Here are some examples.

- The identity triple gives the sphere spectrum as a symmetric ring spectrum.
- Let Gr be the reduced free group triple, i.e., it sends a pointed set K to the free group generated by K modulo the normal subgroup generated by the basepoint. Since $Gr(S^n)$ is weakly equivalent to ΩS^{n+1}, which in the stable range is equivalent to S^n, the unit maps form a $\hat{\pi}_*$-isomorphism $\mathbb{S} \longrightarrow Gr(\mathbb{S})$. The same conclusion would hold with the free reduced monoid functor, also known as the 'James construction' J, since $J(S^n)$ is also weakly equivalence to ΩS^{n+1} as soon as $n \geq 1$.
- Let M be a topological monoid and consider the pointed continuous functor $K \mapsto M^+ \wedge K$. The multiplication and unit of M make this into a triple whose algebras are pointed sets with left M-action. The associated symmetric ring spectrum is the spherical monoid ring $\mathbb{S}M$.
- Let A be a ring and consider the free reduced A-module triple $\tilde{A}[K] = A[K]/A[*]$. Then $\tilde{A}[\mathbb{S}] = HA$, the Eilenberg-Mac Lane ring spectrum. We shall see later [ref] that for every symmetric spectrum of simplicial sets X the symmetric spectrum $\tilde{A}[X]$ is $\hat{\pi}_*$-isomorphic to the smash product $HA \wedge X$.
- Let B be a commutative ring and consider the triple $X \mapsto I(\tilde{B}(X))$, the augmentation ideal of the reduced polynomial algebra over B, generated by the pointed set X. The algebras over this triple are non-unital commutative B-algebras, or augmented commutative B-algebras (which are equivalent categories). The ring spectrum associated to this triple is denoted DB, and it is closely related to topological André-Quillen homology for commutative B-algebras. The ring spectrum DB is rationally equivalent to the Eilenberg-Mac Lane ring spectrum HB, but DB has torsion in higher homotopy groups.

More generally, if we evaluate a triple T on a symmetric ring spectrum R, then the resulting spectrum $T(R)$ is naturally a ring spectrum with multiplication maps

$$T(R_n) \wedge T(R_m) \ \to\ T(R_n \wedge R_m) \ \xrightarrow{T(\mu_{n,m})}\ T(R_{n+m}) \ .$$

In the topological context, this evaluation process takes orthogonal ring spectra to orthogonal ring spectra and has an analog for unitary ring spectra.

7.4. $\mathbf{\Gamma}$-spaces. Many continuous or simplicial functors arise from so called $\mathbf{\Gamma}$-*spaces*, and then the associated symmetric spectra have special properties. The category $\mathbf{\Gamma}$ is a skeletal category of the category of finite pointed sets: there is one object $n^+ = \{0, 1, \dots, n\}$ for every non-negative integer n, and morphisms are the maps of sets which send 0 to 0. A $\mathbf{\Gamma}$-*space* is a covariant functor from $\mathbf{\Gamma}$ to the category of spaces or simplicial sets taking 0^+ to a one point space (simplicial set). A morphism of $\mathbf{\Gamma}$-spaces is a natural transformation of functors. We follow the established terminology to speak of $\mathbf{\Gamma}$-*spaces* even if the values are simplicial sets.

A $\mathbf{\Gamma}$-space X can be extended to a continuous (respectively simplicial, depending on the context) functor by a coend construction. If X is a $\mathbf{\Gamma}$-space and K a pointed space or simplicial set, the value of the extended functor on K is given by

$$X(K) \ = \ \int^{n^+ \in \mathbf{\Gamma}} K^n \times X(n^+) \ = \ \left(\coprod_{n \geq 0} K^n \times X(n^+) \right) / \sim \ , \tag{7.12}$$

where we use that $K^n = \mathrm{map}(n^+, K)$ is contravariantly functorial in n^+. In more detail $X(K)$ is obtained from the disjoint union of the spaces $K^n \times X(n^+)$ by modding out the equivalence relation generated by

$$(k_1, \dots, k_n;\, \alpha_*(x)) \ \sim\ (k_{\alpha(1)}, \dots, k_{\alpha(m)};\, x)$$

for all morphisms $\alpha : m^+ \longrightarrow n^+$ in $\mathbf{\Gamma}$, all $(k_1, \dots, k_n) \in K^n$ and all $x \in X(m^+)$. Here $k_{\alpha(i)}$ is to be interpreted as the basepoinf of K whenever $\alpha(i) = 0$,. We will not distinguish notationally between the original $\mathbf{\Gamma}$-space and its extension. The extended functor is continuous respectively simplicial.

In the simplicial context, the extension of a $\mathbf{\Gamma}$-space admits the following different (but naturally isomorphic) description. First, X can be prolonged, by direct limit, to a functor from the category of pointed sets, not necessarily finite, to pointed simplicial sets. Then if K is a pointed simplicial set we get a bisimplicial set $[k] \mapsto X(K_k)$ by evaluating the (prolonged) $\mathbf{\Gamma}$-space degreewise. The simplicial set $X(K)$ defined by the coend above is naturally isomorphic to the diagonal of this bisimplicial set.

Symmetric spectra which arise from $\mathbf{\Gamma}$-spaces have special properties. First of all, they are always semistable. For $\mathbf{\Gamma}$-spaces X with values in $\mathbf{T}$ this results from Proposition 3.16 (vi) because the symmetric spectrum $X(\mathbb{S})$ is underlying an orthogonal spectrum. For $\mathbf{\Gamma}$-spaces X with values in simplicial sets, essentially the same argument applies, with the following slight extra twist. We let X be a $\mathbf{\Gamma}$-spaces of simplicial sets and K a based simplicial set. If we postcompose the prolonged functor $X(-) : \mathbf{sS} \longrightarrow \mathbf{sS}$ with geometric realization, we obtain a functor $|X(-)| : \mathbf{sS} \longrightarrow \mathbf{T}$ from based simplicial sets to based spaces. However, this composite functor has 'extra functoriality' that is not directly visible from its definition, and we now explain.

Geometric realization of simplicial sets is a left adjoint, so it preserves coends. In particular, the space $|X(K)|$ is a coend of the functor

$$\mathbf{\Gamma}^{\mathrm{op}} \times \mathbf{\Gamma} \longrightarrow \mathbf{T} , \quad (n, m) \longmapsto |K^n \times X(n^+)| .$$

In the category $\mathbf{T}$ of compactly generated weak Hausdorff spaces, geometric realization preserves finite products. So the space $|X(K)|$ is also a coend of the functor

$$\mathbf{\Gamma}^{\mathrm{op}} \times \mathbf{\Gamma} \longrightarrow \mathbf{T} , \quad (n, m) \longmapsto |K|^n \times |X(n^+)| .$$

Such a coend is, by definition, the value of the $\mathbf{\Gamma}$-space $|X|$ on the topological space $|K|$. In other words, we have obtained a natural homeomorphism

$$|X(K)| \longrightarrow |X|(|K|) . \tag{7.13}$$

If we let K run through the simplicial spheres S^n an isomorphism of symmetric spectra of spaces

$$|X(\mathbb{S})| \cong |X|(\mathbb{S}) .$$

In particular, after geometric realization the symmetric spectrum of simplicial sets $X(\mathbb{S})$ admits an extension to an orthogonal spectrum, hence it is semistable.

Moreover, if K is a finite based simplicial set, we can view it as a functor $K : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{\Gamma}$. Then the space $X(|K|)$ is homeomorphic to the geometric realization of the simplicial space $[k] \mapsto X(K_k)$, i.e., the composite functor

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{K} \mathbf{\Gamma} \xrightarrow{X} \mathbf{T} .$$

[prove...] From now on we restrict to $\mathbf{\Gamma}$-spaces of simplicial sets, where the homotopy types of values at spheres are easier to analyze.

Proposition 7.14. *Let X be a $\mathbf{\Gamma}$-space with values in simplicial sets.*

(i) *The prolonged functor $X(-)$ preserves weak equivalences of based simplicial sets and level equivalence of symmetric spectra of simplicial sets.*

(ii) *The symmetric spectrum $X(\mathbb{S})$ is semistable, connective and flat.*

(iii) *For every symmetric spectrum of simplicial sets E the assembly map*

$$X(\mathbb{S}) \wedge E \longrightarrow X(E)$$

is a $\hat{\pi}_*$-isomorphism.

(iv) *Let $f : E \longrightarrow F$ be a morphism of symmetric spectra of simplicial set. If f is a $\hat{\pi}_*$-isomorphism respectively stable equivalence, then so is $X(f) : X(E) \longrightarrow X(F)$.*

Proof. (i) The second claim is a direct consequence of the first. So we show that for every weak equivalence $f : A \longrightarrow B$ of based simplicial sets the map $X(f) : X(A) \longrightarrow X(B)$ is again a weak equivalence. The geometric realization $|f| : |A| \longrightarrow |B|$ is a based homotopy equivalence; we let $g : |B| \longrightarrow |A|$ be a homotopy inverse. There is no reason to assume that g can be chosen as the geometric realization of a morphism from B to A: However, the 'extra functoriality' of the composite functor $|X(-)|$ given by the natural isomorphism (7.13)

$$|X(A)| \;\cong\; |X|(|A|)$$

for provides a continuous map

$$|X|(g) \;:\; |X(B)| \;\longrightarrow\; |X(A)|\;.$$

Since the functor $|X|(-)$ is a continuous functor from based spaces to itself, it preserves the based homotopy relation. Since $g\circ|f|$ and $|f|\circ g$ are homotopic to the respective identity maps, the composites $|X|(g)\circ|X|(|f|)$ and $|X|(|f|)\circ|X|(g)$ are again homotopic to the respective identity maps. So $|X|(|f|)$ is a homotopy equivalence, hence so is the isomorphic map $|X(f)|$. This means that $X(f)$ is a weak equivalence of simplicial sets, as claimed.

(ii) The fact that $X(\mathbb{S})$ is semistable was already shown above, so we only deal with the other properties.

(iii) We first reduce to the special case where the symmetric spectrum E is flat. For this purpose we choose a level equivalence $r : E' \longrightarrow E$ whose source is a flat symmetric spectrum of simplicial sets (for example, the flat resolution of Construction 5.53). In the commutative square

$$\begin{tikzcd}
X(\mathbb{S})\wedge E' \arrow[r, "\text{assembly}"] \arrow[d, "X(\mathbb{S})\wedge r"'] & X(E') \arrow[d, "X(r)"] \\
X(\mathbb{S})\wedge E \arrow[r, "\text{assembly}"'] & X(E)
\end{tikzcd}$$

the left vertical map is a level equivalence by Proposition 5.50 because $X(\mathbb{S})$ is flat by part (i). The right vertical map is a level equivalence by part (i). So the upper assembly map is a $\hat{\pi}_*$-isomorphism if and only if the lower assembly map is a $\hat{\pi}_*$-isomorphism. We can thus assume without loss of generality that the symmetric spectrum E is flat.

Now we first treat a special case, namely where X is the $\mathbf{\Gamma}$-space $\mathbf{\Gamma}^m$ given by

$$\mathbf{\Gamma}^m(A) \;=\; A^m\;,$$

the m-fold cartesian product (this $\mathbf{\Gamma}$-space is in fact representable by the object based set m^+). For every symmetric spectrum E the spectrum $\mathbf{\Gamma}(E)$ is the m-fold cartesian product of copies of E. The assembly map for this $\mathbf{\Gamma}$-space fits into a commutative square:

$$\begin{tikzcd}
(m^+\wedge\mathbb{S})\wedge E \arrow[r, "i_{\mathbb{S}}\wedge E", "\sim"'] \arrow[d, "\cong"'] & \mathbf{\Gamma}(\mathbb{S})\wedge E \arrow[r, "\text{assembly}"] & \mathbf{\Gamma}^m(E) \arrow[d, equal] \\
m^+\wedge E \arrow[rr, "\sim", "i_E"'] & & E^m
\end{tikzcd}$$

The lower morphism i_E is the map from the m-fold wedge to the m-fold product of copies of E. This map is a $\hat{\pi}_*$-isomorphism by Proposition 2.19 (iii). As a special case the morphism $i_{\mathbb{S}} : m^+\wedge\mathbb{S}\longrightarrow\mathbf{\Gamma}^m(\mathbb{S})$ is a $\hat{\pi}_*$-isomorphism. Since E is flat, the morphism $i_{\mathbb{S}}\wedge E$ is a $\hat{\pi}_*$-isomorphism (by Proposition 5.50). So the assembly map $\mathbf{\Gamma}^m(\mathbb{S})\wedge E\longrightarrow\mathbf{\Gamma}^m(E)$ is a $\hat{\pi}_*$-isomorphism.

Now we proceed towards the general case. Every $\mathbf{\Gamma}$-space X has a natural filtration

$$* = X_{[0]} \;\subseteq\; X_{[1]} \;\subseteq\; \dots \;\subseteq\; X_{[m]} \;\subseteq\; \dots$$

by 'cardinality of support': for $m\geq 0$ and a finite based set A the value $X_{[m]}(A)$ is given by

$$X_{[m]}(A) \;=\; \bigcup_{\alpha : m^+\longrightarrow A} \text{image}(\alpha_* : X(m^+)\longrightarrow X(A))\;.$$

Equivalently, $X_{[m]}$ is the smallest sub-$\mathbf{\Gamma}$-space of X that contains $X(m^+)$.

Now we suppose that X has finite filtration, i.e., $X = X_{[m]}$ for some $m \geq 0$. If $X = X_{[0]}$, then X is constant with value a one-point simplicial set, so $X(\mathbb{S})$ and $X(E)$ are both trivial symmetric spectra, and both claims hold. Now we suppose that $m \geq 1$. The value

$$\mathbf{\Gamma}^m_{[m-1]}(A) \ \subseteq \ A^m$$

of the sub-$\mathbf{\Gamma}$-space $\mathbf{\Gamma}^m_{[m-1]}$ at a based set A is the subset of those m-tuples $(a_1, \ldots, a_m)$ where one of the coordinates a_i is the basepoint or $a_i = a_j$ for some pair $i \neq j$. The assembly maps for $\mathbf{\Gamma}^m$, $\mathbf{\Gamma}^m_{[m-1]}$ and the quotient $\mathbf{\Gamma}$-space $\mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]}$ fit into a commutative diagram of symmetric spectra:

$$\begin{tikzcd}
\mathbf{\Gamma}^m_{[m-1]}(\mathbb{S}) \wedge E \arrow[r] \arrow[d] & \mathbf{\Gamma}^m(\mathbb{S}) \wedge E \arrow[r] \arrow[d] & (\mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]})(\mathbb{S}) \wedge E \arrow[d] \\
\mathbf{\Gamma}^m_{[m-1]}(E) \arrow[r] & \mathbf{\Gamma}^m(E) \arrow[r] & (\mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]})(E)
\end{tikzcd}$$

Since E is flat, the upper left morphism is a level cofibration, and so is the lower left morphism. So both morphisms become h-cofibrations after geometric realization, and both rows given rise to a long exact sequence of naive homotopy groups by Corollary 2.13. The assembly map for $\mathbf{\Gamma}^m_{[m-1]}$ is a $\hat{\pi}_*$-isomorphism by induction and the assembly map for $\mathbf{\Gamma}^m$ is a $\hat{\pi}_*$-isomorphism by the first paragraph. So the five-lemma lets us conclude that the assembly map for the quotient $\mathbf{\Gamma}$-space $\mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]}$ a $\hat{\pi}_*$-isomorphism.

We let the symmetric group Σ_m act on A^m by permuting coordinates; this is then an action on $\mathbf{\Gamma}^m$ by automorphisms of $\mathbf{\Gamma}$-spaces. This action is free on the complement of $\mathbf{\Gamma}^m_{[m-1]}(A)$, so the induced action on the quotient $\mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]}(A)$ is free away from the basepoint. Hence also the action of Σ_m on the symmetric spectra $\mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]}(\mathbb{S})$ and on $\mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]}(E)$ is levelwise free (away from the basepoint). So for every based Σ_m-simplicial set L the induced map

$$L\wedge_{\Sigma_m} \ : \ L \wedge_{\Sigma_m} (\mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]}(\mathbb{S}) \wedge E) \ \longrightarrow \ L \wedge_{\Sigma_m} \mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]}(E)$$

is a $\hat{\pi}_*$-isomorphism. This last map is isomorphic to the assembly map for the $\mathbf{\Gamma}$-space $(L \wedge_{\Sigma_m} \mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]}$, so this proves claim (iii) for $\mathbf{\Gamma}$-spaces of the form $L \wedge_{\Sigma_m} \mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]}$.

Smasing with E preserves colimits, so the natural map from $L \wedge_{\Sigma_m} (\mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]}(\mathbb{S}) \wedge E)$ to $(L \wedge_{\Sigma_m} \mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]}(\mathbb{S})) \wedge E$ is an isomorphism. After this identification, the assembly map for the $\mathbf{\Gamma}$-space $L \wedge_{\Sigma_m} \mathbf{\Gamma}^m/(\mathbf{\Gamma}^m)^{[m-1]}$ takes the form

$$L \wedge_{\Sigma_m} (\mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]}(\mathbb{S}) \wedge E) \ \xrightarrow{L\wedge_{\Sigma_m}\text{assembly}} \ L \wedge_{\Sigma_m} (\mathbf{\Gamma}^m/\mathbf{\Gamma}^m)_{[m-1]}(E)) \ .$$

Now we return to a general $\mathbf{\Gamma}$-space X subject to the condition $X = X_{[m]}$. We note that the simplicial subset $X_{[m]}(A)$ is precisely the image of the tautological map

$$X(m^+) \wedge \mathbf{\Gamma}^m(A) \ \longrightarrow \ X(A) \ ;$$

so the condition $X = X_{[m]}$ is equivalent to the property that the tautological morphism of $\mathbf{\Gamma}$-spaces

$$X(m^+) \wedge \mathbf{\Gamma}^m \ \longrightarrow \ X$$

is an epimorphism. The composite with the quotient morphism $X \longrightarrow X/X_{[m-1]}$ factors over an epimorphism

$$X(m^+) \wedge_{\Sigma_m} \mathbf{\Gamma}^m/\mathbf{\Gamma}^m_{[m-1]} \ \longrightarrow \ X/X_{[m-1]} \ .$$

This epimorphism is in fact an isomorphism [...]; so claim (iii) holds for the $\mathbf{\Gamma}$-space $X/X_{[m-1]}$. The assembly maps for X, $X_{[m-1]}$ and the quotient $\mathbf{\Gamma}$-space $X/X_{[m-1]}$ fit into a commutative diagram of symmetric spectra:

$$\begin{tikzcd}
X_{[m-1]}(\mathbb{S}) \wedge E \arrow[r] \arrow[d] & X(\mathbb{S}) \wedge E \arrow[r] \arrow[d] & (X/X_{[m-1]})(\mathbb{S}) \wedge E \arrow[d] \\
X_{[m-1]}(E) \arrow[r] & X(E) \arrow[r] & (X/X_{[m-1]})(E)
\end{tikzcd}$$

Since E is flat the upper left morphism is a level cofibration, and so is the lower left morphism. So both morphisms become h-cofibrations after geometric realization, and both rows given rise to a long exact sequence of naive homotopy groups by Corollary 2.13. The assembly map $X_{[m-1]}(\mathbb{S}) \wedge E \longrightarrow X_{[m-1]}(E)$ is a $\hat{\pi}_*$-isomorphism by induction and the assembly map $(X/X_{[m-1]})(\mathbb{S}) \wedge E \longrightarrow (X/X_{[m-1]})(E)$ is a $\hat{\pi}_*$-isomorphism by the special case above. So the five-lemma lets us conclude that the assembly map $X(\mathbb{S}) \wedge E \longrightarrow X(E)$ is also a $\hat{\pi}_*$-isomorphism.

An arbitrary $\mathbf{\Gamma}$-space X is the union of the nested sequence of sub-$\mathbf{\Gamma}$-spaces $X_{[m]}$. So $X(\mathbb{S})$ and $X(E)$ is the union of the symmetric subspectra $X_{[m]}(\mathbb{S})$ respectively $X_{[m]}(E)$ and $X(\mathbb{S}) \wedge E$ is a colimit of the sequence of symmetric spectra $X_{[m]}(\mathbb{S}) \wedge E$. We consider the commutative square

$$\begin{tikzcd}
\mathrm{colim}_m\, \hat{\pi}_k(X_{[m]}(\mathbb{S}) \wedge E) \arrow[r] \arrow[d] & \hat{\pi}_k(X(\mathbb{S}) \wedge E) \arrow[d] \\
\mathrm{colim}_m\, \hat{\pi}_k(X_{[m]}(E)) \arrow[r] & \hat{\pi}_k X(E)
\end{tikzcd}$$

in which the horizontal maps are the canonical ones and the vertical maps are induced by the assembly maps. Both the horizontal maps are isomorphisms [...], and the left vertical map is an isomorphism by the special cases above. So the assembly map $X(\mathbb{S}) \wedge E \longrightarrow X(E)$ is a $\hat{\pi}_*$-isomorphism. □

Moreover, up to $\hat{\pi}_*$-isomorphisms, $\mathbf{\Gamma}$-spaces model all connective spectra (see Theorem 5.8 of [**13**] [also reference to [**73**]?])

Given two finite based sets A and B, we denote by $p^A : A \vee B \longrightarrow A$ the 'projection' that sends B to the basepoint and is the identity on A. The map $\nabla : 2^+ \longrightarrow 1^+$ is defined by $\nabla(1) = 1 = \nabla(2)$.

Definition 7.15. A $\mathbf{\Gamma}$-space X is *special* if the map

$$(p^A_*, p^B_*) \ : \ X(A \vee B) \ \longrightarrow \ X(A) \times X(B)$$

is a weak equivalence for all pairs of finite based sets A and B. A special $\mathbf{\Gamma}$-space X is *very special* if in addition the map

$$(p^1_*, \nabla_*) \ : \ X(2^+) \ \longrightarrow \ X(1^+) \times X(1^+)$$

is a weak equivalence.

Remark 7.16. 'Special' can be defined equivalently by requiring the [...] map

$$X(n^+) \ \longrightarrow \ X(1^+)^n$$

to be a weak equivalence for all $n \geq 2$. If X is special, then 'very special' can be defined equivalently by requiring the [...] map for every morphism $\Phi : A \vee B \longrightarrow B$ such that $\Phi(b) = b$ for all $b \in B$, the map

$$(p^A_*, \Phi_*) \ : \ X(A \vee B) \ \longrightarrow \ X(A) \times X(B)$$

is a weak equivalence.

If X is special, then the weak map

$$X(1^+) \times X(1^+) \ \xleftarrow{\ \sim\ } \ X(2^+) \ \xrightarrow{X(\nabla)} \ X(1^+)$$

induces an abelian monoid structure on $\pi_0\left(X(1^+)\right)$. A special $\mathbf{\Gamma}$-space X is very special if and only if the monoid $\pi_0\left(X(1^+)\right)$ is a group, i.e., if it has additive inverses.

The following two Theorem 7.17 and 7.21 are due to Segal [**73**], and are probably the most important results about $\mathbf{\Gamma}$-spaces.

Theorem 7.17. *Let X be a cofibrant very special $\mathbf{\Gamma}$-space. Then the symmetric spectrum $X(\mathbb{S})$ is an Ω-spectrum.*

Proof. In a first step we show that the adjoint structure map $\tilde{\sigma}_0 : X(S^0) \longrightarrow \Omega X(S^1)$ of the symmetric spectrum $X(\mathbb{S})$ is a weak equivalence. By [...], the space $X(S^1)$ is homeomorphic to the geometric realization of the simplicial space $BX : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{T}$ defined as the composite

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{S^1} \mathbf{\Gamma} \xrightarrow{X} \mathbf{T} ,$$

where S^1 denotes the based finite simplicial set $\Delta[1]/\partial\Delta[1]$. We define a functor $P : \mathbf{\Delta} \longrightarrow \mathbf{\Delta}$ be $P([n]) = [n+1]$ on objects and on morphisms by $P(\alpha)(0) = 0$ and $P(\alpha)(i) = \alpha(i-1)+1$ for $i \geq 1$. Then we define a new simplicial space $EX : \mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{T}$ as $EX = BX \circ P$. The morphisms $d_0 : [n] \longrightarrow [n+1] = P([n])$ form a natural transformation from the identity of $\mathbf{\Delta}$ to the functor P. So the morphism $d_0^* : (EX)_n = (BX)_{n+1} \longrightarrow (BX)_n$ form a morphism $d_0 : EX \longrightarrow BX$ of simplicial spaces.

We claim that for every morphism $v : [0] \longrightarrow [n]$ in $\mathbf{\Delta}$ the square

$$\begin{tikzcd}
(EX)_n \arrow[r, "v^*"] \arrow[d, "d_0^*"'] & (EX)_0 \arrow[d, "d_0^*"] \\
(BX)_n \arrow[r, "v^*"'] & (BX)_0
\end{tikzcd}$$

is homotopy cartesian. Since $(BX)_0 = X(S^1_0)$ is a point, this means showing that

$$(d_0^*, P(v)^*) \;:\; X(S^1_{n+1}) = (EX)_n \;\longrightarrow\; (BX)_n \times (EX)_0 \;=\; X(S^1_1) \times X(S^1_n) \tag{7.18}$$

is a weak equivalence. If $v(0) = 0$, then $d_0^* : S^1_{n+1} \longrightarrow S^1_n$ and $P(v)^* : S^1_{n+1} \longrightarrow S^1_1$ are complementary projections, so the map (7.18) is a weak equivalence because X is special. If $v(0) \geq 1$, then $P(v)^* : S^1_{n+1} \longrightarrow S^1_1$ is a fold map, possibly iterated.

Now we let $d_i : [n-1] \longrightarrow [n]$ be any face map in $\mathbf{\Delta}$. By the above the right square and the outer rectangle in the commutative diagram

$$\begin{tikzcd}
(EX)_n \arrow[r, "d_i^*"] \arrow[d, "d_0^*"'] & (EX)_{m+1} \arrow[r, "v^*"] \arrow[d, "d_0^*"] & (EX)_0 \arrow[d, "d_0^*"] \\
(BX)_n \arrow[r, "d_i^*"'] & (BX)_m \arrow[r, "v^*"'] & (BX)_0
\end{tikzcd}$$

are homotopy cartesian; so the left square is also homotopy cartesian.

By Proposition 7.19, applied to the morphism $d_0 : EX \longrightarrow BX$, the square

$$\begin{tikzcd}
X(S^1_1) = (EX)_0 \arrow[r] \arrow[d] & ||EX|| \arrow[d, "||d_0||"] \\
* = (BX)_0 \arrow[r] & ||BX||
\end{tikzcd}$$

is homotopy cartesian, where $||-||$ is the 'fat realization' (i.e., where degeneracies are ignored). Since X is a cofibrant $\mathbf{\Gamma}$-space the simplicial spaces EX and BX are proper, and so the natural maps $||EX|| \longrightarrow |EX|$ and $||BX|| \longrightarrow |BX|$ from the fat to the ordinary realization are weak equivalences. So the square

$$\begin{tikzcd}
X(S^1_1) \arrow[r] \arrow[d, "d_0^*"'] & |EX| \arrow[d] \\
* \arrow[r] & |BX| = X(S^1)
\end{tikzcd}$$

is also homotopy cartesian. The space $|EX|$ is contractible by [...], so the map $\tilde{\sigma}_0 : X(S^0) \longrightarrow \Omega X(S^1)$ is a weak equivalence.

To show that also the higher adjoint structure maps of the spectrum $X(\mathbb{S})$ are weak equivalence we consider the 'n-fold shifted' $\mathbf{\Gamma}$-space $\mathrm{sh}^n X = X(S^n \wedge -)$; the with value of this $\mathbf{\Gamma}$-space on a finite based set given by

$$(\mathrm{sh}^n X)(A) \;=\; X(S^n \wedge A)$$

(where the right hand side is the prolonged functor). Then the $\mathbf{\Gamma}$-space $\mathrm{sh}^n X$ is again special [...]; on the other hand, for all $n \geq 1$, the underlying space $(\mathrm{sh}^n X)(S^0) \cong X(S^n)$ is connected, so $\mathrm{sh}^n X$ is even very special. By the first part, the adjoint structure map $|\mathrm{sh}^n X(S^0)| \longrightarrow \Omega|\mathrm{sh}^n X(S^1)|$ is a weak equivalence; but this map is isomorphic to the adjoint structure map

$$\tilde{\sigma}_n \;:\; X(S^n) \;\longrightarrow\; \Omega X(S^{n+1})$$

of the symmetric spectrum $X(\mathbb{S})$. So $X(\mathbb{S})$ is a positive Ω-spectrum. □

('Iff') The effect of the adjoint structure map on path component

$$\pi_0 X(S^0) \;\longrightarrow\; \pi_0(\Omega|X(S^1)|)$$

is a homomorphism of monoids [...] and the target monoid is actually group. If the adjoint structure map is a weak equivalent, this monoid homomorphism is an isomorphism and $\pi_0 X(S^0)$ also has inverses, i.e., X is very special.

Proposition 7.19. *(to appendix?) Let $f : X \longrightarrow Y$ be a morphism of $\mathbf{\Delta}$-spaces such that for every face map $d_i : [n-1] \longrightarrow [n]$ in the category $\mathbf{\Delta}$ the square*

$$\begin{tikzcd} X_n \arrow[r, "d_i^*"] \arrow[d, "f_n"'] & X_{n-1} \arrow[d, "f_m"] \\ Y_n \arrow[r, "d_i^*"'] & Y_{n-1} \end{tikzcd}$$

is homotopy cartesian. Then for every $m \geq 0$ the square

$$\begin{tikzcd} X_m \times \nabla^m \arrow[r] \arrow[d, "f_m \times \nabla^m"'] & ||X|| \arrow[d, "||f||"] \\ Y_m \times \nabla^m \arrow[r] & ||Y|| \end{tikzcd}$$

is homotopy cartesian.

Proof. In a first step we show by induction on m that the squares

$$\begin{tikzcd} X_m \times \nabla^n \arrow[r] \arrow[d, "f_m \times \nabla^m"'] & ||X||^{(m)} \arrow[d, "||f||^{(m)}"] \\ Y_m \times \nabla^m \arrow[r] & ||Y||^{(m)} \end{tikzcd} \tag{7.20}$$

is homotopy cartesian, where $||-||^{(m)}$ is the m-skeleton of the fat realization. For $m = 0$, both horizontal maps in the square (7.20) are identities, so there is nothing to show. For $m \geq 1$ the m-skeleta are obtained as the horizontal pushouts in the commutative diagram

$$\begin{tikzcd} X_m \times \nabla^m \arrow[d] & X_m \times \partial\nabla^m \arrow[l] \arrow[r] \arrow[d] & ||X||^{(m-1)} \arrow[d] \\ Y_m \times \nabla^m & Y_m \times \partial\nabla^m \arrow[l] \arrow[r] & ||Y||^{(m-1)} \end{tikzcd}$$

The left square is homotopy cartesian [clear], and the right square is homotopy cartesian by the inductive hypothesis [show]. So the glued square (7.20) is homotopy cartesian. In the diagram

$$\begin{tikzcd} X_0 = ||X||^{(0)} \arrow[r] \arrow[d, "||f||^{(0)}"'] & ||X||^{(1)} \arrow[r] \arrow[d, "||f||^{(1)}"] & ||X||^{(2)} \arrow[r] \arrow[d, "||f||^{(2)}"] & \cdots \\ Y_0 = ||Y||^{(0)} \arrow[r] & ||Y||^{(1)} \arrow[r] & ||Y||^{(2)} \arrow[r] & \cdots \end{tikzcd}$$

all squares a homotopy cartesian. Hence [...] the composite square [...] is homotopy cartesian. □

Theorem 7.21. *Let X be a cofibrant special $\mathbf{\Gamma}$-space. The adjoint structure map $\tilde{\sigma}_0 : X(S^0) \longrightarrow \Omega X(S^1)$ induces an isomorphism*

$$H_*(X(S^0), \mathbb{Z})[\pi^{-1}] \;\longrightarrow\; H_*(\Omega X(S^1), \mathbb{Z})$$

from the localization of the homology of $X(1^+)$ at the multiplicative subset $\pi = \pi_0 X(S^0)$ to the homology of $\Omega X(S^1)$. Moreover, the symmetric spectrum $X(\mathbb{S})$ is a positive Ω-spectrum.

Proof. Below we define another $\mathbf{\Gamma}$-space $\hat{X}$ and a morphism $f : X \longrightarrow \hat{X}$ of $\mathbf{\Gamma}$-spaces and prove the following properties.

(a) The map $f(1^+) : X(1^+) \longrightarrow \hat{X}(1^+)$ induces an isomorphism

$$H_*(X(1^+), \mathbb{Z})[\pi^{-1}] \;\longrightarrow\; H_*(\hat{X}(1^+), \mathbb{Z})$$

from the localization of the homology of $X(1^+)$ to the homology of $\hat{X}(1^+)$.

(b) The map $X(S^1) \longrightarrow \hat{X}(S^1)$ is a weak equivalence.

(c) The $\mathbf{\Gamma}$-space $\hat{X}$ is very special.

In the commutative square

$$\begin{tikzcd} X(1^+) \arrow[r, "f(1^+)"] \arrow[d, "\sigma_0"'] & \hat{X}(1^+) \arrow[d, "\sigma_0"] \\ \Omega X(S^1) \arrow[r, "\Omega f(S^1)"'] & \Omega \hat{X}(S^1) \end{tikzcd}$$

the right vertical map is a weak equivalence because $\hat{X}$ is very special, and the lower horizontal map is a weak equivalence by (ii). The square

$$\begin{tikzcd} H_*(X(1^+), \mathbb{Z})[\pi^{-1}] \arrow[r, "f(1^+)_*"] \arrow[d, "(\sigma_0)_*"'] & H_*(\hat{X}(1^+), \mathbb{Z}) \arrow[d, "\cong"', "(\sigma_0)_*"] \\ H_*(\Omega X(S^1), \mathbb{Z}) \arrow[r, "\cong", "(\Omega f(S^1))_*"'] & H_*(\Omega \hat{X}(S^1), \mathbb{Z}) \end{tikzcd}$$

of integral homology rings then commute. Since the other three maps are isomorphisms, so is the map $(\sigma_0)_*$.

Now we define the $\mathbf{\Gamma}$-space X. For a based finite set A we consider the simplicial space

$$[k] \;\mapsto\; P_k(A) = \text{homotopy pullback}(X(\Delta[1]_k \wedge A) \;\longrightarrow\; X(S^1_k \wedge A) \;\longleftarrow\; X(\Delta[1]_k \wedge A)\ .$$

Here the left copy of $\Delta[1]$ is pointed by 0, the right copy of $\Delta[1]$ is pointed by 1. As k varies the structure maps of $\Delta[1]$ and S^1 and functoriality of homotopy pullback make this into a simplicial space. We set

$$\hat{X}(A) \;=\; |[k] \mapsto P_k(A)|\ .$$

We let $\mathbb{F}$ be any field and denote by $H = H_*(X(1^+), \mathbb{F})$ the homology of $X(1^+)$ with $\mathbb{F}$-coefficients. The homology has the structure of a graded $\mathbb{F}$-bialgebra: the diagonal $X(1^+) \longrightarrow X(1^+) \times X(1^+)$ and the Kunneth isomorphism define the comultiplication $\Delta : H \longrightarrow H \otimes H$, and the composite

$$H_*(\hat{X}(1^+) \times \hat{X}(1^+), \mathbb{F}) \xrightarrow{H_*(p^1_*, p^2_*)^{-1}} H_*(\hat{X}(2^+), \mathbb{F}) \xrightarrow{H_*(p^1_*, \nabla_*)} H_*(\hat{X}(1^+) \times \hat{X}(1^+), \mathbb{F})$$

and the Kunneth isomorphism define the multiplication $\mu : H \otimes H \longrightarrow H$.

Since $\hat{X}(1^+)$ is the geometric realization of a simplicial space, the skeleton filtration on the realization provides a spectral sequence

$$E^1_{p,q} = H_p(P_q, \mathbb{F}) \implies H_{p+q}(\hat{X}(1^+), \mathbb{F}) \tag{7.22}$$

converging to the homology of $\hat{X}(1^+)$ with coefficients in $\mathbb{F}$. In simplicial degree $[k]$ the structure maps of the $\mathbf{\Gamma}$-space X provide a commutative diagram

$$\begin{tikzcd} X(\Delta[1]_k) \arrow[r] \arrow[d, "\sim"'] & X(S^1_k) \arrow[d, "\sim"'] & X(\Delta[1]_k) \arrow[l] \arrow[d, "\sim"] \\ X(1^+)^{k+1} \arrow[r] & X(1^+)^k & X(1^+)^{k+1} \arrow[l] \end{tikzcd}$$

in which the three vertical maps are weak equivalences. So the induced map on homotopy pullbacks is a weak equivalence. The homotopy pullback of the lower diagram is weakly equivalent to $X(1^+)^{k+2}$. Thus

$$H_*(P_k(1^+), \mathbb{F}) \;\cong\; H_*(X(1^+)^{k+2}, \mathbb{F}) \;\cong\; H^{\otimes(k+2)} ,$$

where the second step is the Kunneth isomorphism. Under this isomorphism, the simplicial structure maps become the maps in the simplicial bar construction $\mathcal{B}(H \otimes H, H, \mathbb{F})$ for the bialgebra H; here $H \otimes H$ is an H-module via the diagonal map. The homology of this bar construction calculates Tor groups, so the E^2-term of the spectral sequence is isomorphic to

$$E^2_{p,*} = \mathrm{Tor}^H_{p,*}(H \otimes H, \mathbb{F}) .$$

The multiplicative subset π acts trivially (so in particular invertibly) on $\mathbb{F}$, so $\mathbb{F}$ is already π-local. So for every H-module N and every $p \geq 0$ the natural map

$$\mathrm{Tor}^H_{p,*}(N, \mathbb{F}) \;\longrightarrow\; \mathrm{Tor}^{H[\pi^{-1}]}_{p,*}(N[\pi^{-1}], \mathbb{F})$$

induced by the localization morphism $N \longrightarrow N[\pi^{-1}]$ is an isomorphism. In the special case where $N = H \otimes H$ with diagonal H-action, we can use an antipode $c : H[\pi^{-1}] \longrightarrow H[\pi^{-1}]$ to 'untwist' the diagonal action on $(H \otimes H)[\pi^{-1}]$ as follows. The homology algebra H is commutive, so the composite

$$H \xrightarrow{\;\iota\;} H[\pi^{-1}] \xrightarrow{s \mapsto s^{-1}} H[\pi^{-1}]$$

is a homomorphism of graded $\mathbb{F}$-algebras. Since the map takes the set π to itself, it extends uniquely to a graded $\mathbb{F}$-algebra homomorphism

$$c \;:\; H[\pi^{-1}] \;\longrightarrow\; H[\pi^{-1}] .$$

The map

$$H \otimes H \;\longrightarrow\; H[\pi^{-1}] \otimes H[\pi^{-1}] , \quad x \otimes y \;\longmapsto\; x \cdot c(y) \otimes y$$

is an $\mathbb{F}$-algebra homomorphism and sends the elements of the form $\Delta(s) = s \otimes s$ to units, for all $s \in \pi$. So the map extends uniquely over an $\mathbb{F}$-algebra homomorphism

$$\Phi \;:\; (H \otimes H)[\pi^{-1}] \;\longrightarrow\; H[\pi^{-1}] \otimes H[\pi^{-1}] .$$

The morphism Φ is an isomorphism [...] Moreover, the composite

$$H \xrightarrow{\;\Delta\;} H \otimes H \xrightarrow{\;\iota \otimes c\;} H[\pi^{-1}] \otimes H[\pi^{-1}] \xrightarrow{\;\mu\;} H[\pi^{-1}]$$

factors over the augmentation $H \longrightarrow \mathbb{F}$ [show]. This means that Φ is H-linear with respect to the diagonal H-action on the source and the action on the target that is via the right factor only. So Φ is an isomorphism of $H[\pi^{-1}]$-modules to the free $H[\pi^{-1}]$-module generated by the underlying graded $\mathbb{F}$-vector space of $H[\pi^{-1}]$. We deduce that with respect to the diagonal $H[\pi^{-1}]$-action, $(H \otimes H)[\pi^{-1}]$ is free, hence flat, as an $H[\pi^{-1}]$-module. So the higher Tor groups vanish, the spectral sequence (7.22) collapses at the E^2-term, and the edge homomorphism

$$(H \otimes H) \otimes_H \mathbb{F} \;\longrightarrow\; H_*(\hat{X}(1^+), \mathbb{F})$$

is an isomorphism. Moreover, the isomorphism Φ induces an isomorphism

$$(H \otimes H) \otimes_H \mathbb{F} \;\cong\; (H \otimes H)[\pi^{-1}] \otimes_{H[\pi^{-1}]} \mathbb{F} \;\cong\; H[\pi^{-1}] \otimes (H[\pi^{-1}] \otimes_{H[\pi^{-1}]} \mathbb{F}) \;\cong\; H[\pi^{-1}] \ .$$

Combining these two proves chaim (a).

Now we show that the $\mathbf{\Gamma}$-space $\hat{X}$ is special. For finite based sets A and B we contemplate the commutative diagram:

$$\begin{tikzcd}
X(\Delta[1]_k \wedge (A \vee B)) \arrow[r] \arrow[d, "\sim"'] & X(S^1_k \wedge (A \vee B)) \arrow[d, "\sim"'] & X(\Delta[1]_k \wedge (A \vee B)) \arrow[l] \arrow[d, "\sim"] \\
X(\Delta[1]_k \wedge A) \times X(\Delta[1]_k \wedge B) \arrow[r] & X(S^1_k \wedge A) \times X(S^1_k \wedge B) & X(\Delta[1]_k \wedge A) \times X(\Delta[1]_k \wedge B) \arrow[l]
\end{tikzcd}$$

The three vertical maps are weak equivalences because X is special. So the induced map

$$P_k(A \vee B) \;\longrightarrow\; \text{ho pullback} \;\cong\; P_k(A) \times P_k(B)$$

on vertical homotopy pullbacks is a weak equivalence. Since geometric realization commutes with product, the map

$$\hat{X}(A \vee B) \;\longrightarrow\; \hat{X}(A) \times \hat{X}(B)$$

is also a weak equivalence, i.e., the $\mathbf{\Gamma}$-space $\hat{X}$ is special.

The proof of claim (b) uses a similar spectral sequence. We exploit that $X(S^1)$ is the geometric realization of the simplicial space $[k] \mapsto X(S^1_k)$, so its homology comes with a spectral sequence

$$E^1_{p,q} = H_p(X(S^1_k), \mathbb{F}) \implies H_{p+q}(X(S^1), \mathbb{F}) \ ,$$

where $\mathbb{F}$ is any field. Since X is special the space $X(S^1_k)$ is weakly equivalent to $X(1^+)^k$. Thus

$$H_*(S^1_k, \mathbb{F}) \;\cong\; H_*(X(1^+)^k, \mathbb{F}) \;\cong\; H^{\otimes k} \ ,$$

using again the Kunneth isomorphism, and where again $H = H_*(X(1^+), \mathbb{Z})$. Under this isomorphism, the simplicial structure maps become the maps in the simplicial bar construction $\mathcal{B}(\mathbb{F}, H, \mathbb{F})$. The homology of this bar construction calculates Tor groups, so the E^2-term of the spectral sequence is isomorphic to

$$E^2_{p,*} = \mathrm{Tor}^H_{p,*}(\mathbb{F}, \mathbb{F}) \ .$$

The same reasoning applies to the special $\mathbf{\Gamma}$-space $\hat{X}$, and the morphism $f : X \longrightarrow \hat{X}$ induces a map of the two spectral sequences

$$\begin{tikzcd}
E^2_{p,*} = \mathrm{Tor}^H_{p,*}(\mathbb{F}, \mathbb{F}) \arrow[r, Rightarrow] \arrow[d] & H_*(X(S^1), \mathbb{F}) \arrow[d, "f(S^1)_*"] \\
\hat{E}^2_{p,*} = \mathrm{Tor}^{H[\pi^{-1}]}_{p,*}(\mathbb{F}, \mathbb{F}) \arrow[r, Rightarrow] & H_*(\hat{X}(S^1), \mathbb{F})
\end{tikzcd}$$

calculating the homology of $X(S^1)$ respectively of $\hat{X}(S^1)$.

Here we have used claim (a) in identifying the homology of $\hat{X}(1^+)$ with the localization of $H = H_*(X(1^+), \mathbb{F})$ at the multiplicative subset π. Since π acts invertibly on $\mathbb{F}$, the H-module $\mathbb{F}$ is already π-local, and the induced map of Tor groups is an isomorphism. The spectral sequences are concentrated in the first quadrant, so we conclude that the induced map $f(S^1)_* : H_*(X(S^1), \mathbb{F}) \longrightarrow H_*(\hat{X}(S^1), \mathbb{F})$ is an isomorphism. Since the $\mathbf{\Gamma}$-spaces X and $\hat{X}$ are both special, $X(S^1)$ and $\hat{X}(S^1)$ are loop spaces (of $X(S^2)$ respectively $\hat{X}(S^2)$, and the map $f(S^1)$ is a loop map. Since $f(S^1)$ is not only a homology isomorphism, but in fact a weak homotopy equivalence. This concludes the proof of claim (b).

It remains to show claim (c), i.e., that the special $\mathbf{\Gamma}$-space $\hat{X}$ is in fact very special. By part (a) the homology algebra structure $\hat{H} = H_*(\hat{X}(1^+), \mathbb{F})$ with coefficients in a field $\mathbb{F}$ is defined so that the diagram

$$\begin{tikzcd}
\hat{H} \otimes \hat{H} \arrow[rr, "x \otimes y \longmapsto x \otimes xy"] \arrow[d, "\cong"'] & & \hat{H} \otimes \hat{H} \arrow[d, "\cong"] \\
H_*(\hat{X}(1^+) \times \hat{X}(1^+), \mathbb{F}) \arrow[r, "\cong", "H_*(p^1_*, p^2_*)^{-1}"'] & H_*(\hat{X}(2^+), \mathbb{F}) \arrow[r, "H_*(p^1_*, \nabla_*)"'] & H_*(\hat{X}(1^+) \times \hat{X}(1^+), \mathbb{F})
\end{tikzcd}$$

commutes, where the vertical maps are the Kunneth isomorphisms. By part (i) the homology $\hat{H}$ is isomorphic, as a bialgebra, to the localization $H_*(X(1^+), \mathbb{F})[S^{-1}]$. In particular, $\hat{H}$ has an antipode $c : \hat{H} \longrightarrow \hat{H}$, so it is in fact a Hopf algebra. In the presence of an antipode, the homomorphism

$$\hat{H} \otimes \hat{H} \longrightarrow \hat{H} \otimes \hat{H} \ , \quad x \otimes y \longmapsto x \otimes xy$$

is invertible, with inverse given by [...] Since the map $(p^1_*, p^2_*) : X(2^+) \longrightarrow X(1^+) \times X(1^+)$ is a weak equivalence we deduce that the map $(p^1_*, \nabla_*) : X(2^+) \longrightarrow X(1^+) \times X(1^+)$ is a homology isomorphism. Since this map is a morphism of H-spaces, it is in fact a weak equivalence. This shows that the $\mathbf{\Gamma}$-space $\hat{X}$ is very special. □

In particular, the component map

$$\pi_0 X(1^+) \longrightarrow \pi_0(\Omega X(S^1))$$

is an algebraic group completion (i.e., Grothendieck group) of the abelian monoid $\pi_0 X(1^+)$.

Now we establish a result that allows us, in many cases, to identity the infinite loop space represented by a Γ-space that is special (but not necessarily very special). We let X be a special $\mathbf{\Gamma}$-space and suppose further that we are given a continuous map

$$\psi \ : \ X(1^+) \longrightarrow X(2^+)$$

such that $p^1_* \circ \psi : X(1^+) \longrightarrow X(1^+)$ is a constant map with value ψ_0 and $p^2_* \circ \tilde{x}$ is based homotopic to the identity of $X(1^+)$. We define λ as the composite

$$X(1^+) \xrightarrow{\psi} X(2^+) \xrightarrow{\nabla_*} X(1^+) \ .$$

Then λ rigidifies the left translation map of ψ_0 (which a priori only makes sense up to homotopy); in particular, the effect of λ on $\pi_0 X(1^+)$ is precisely translation by $[\psi_0]$ in the abelian monoid structure of [...]. We define X_∞ as the mapping telescope of the sequence

$$X(1^+) \xrightarrow{\lambda_x} X(1^+) \xrightarrow{\lambda_x} X(1^+) \xrightarrow{\lambda_x} \dots$$

The map $\sigma : X(1^+) \longrightarrow \Omega X(S^1)$ extends [explain]to a map

$$\bar{\sigma} \ : \ X_\infty \longrightarrow \Omega X(S^1) \ .$$

Proposition 7.23. *Let X be a cofibrant special Γ-space and $\psi : X(1^+) \longrightarrow X(2^+)$ a map as in [...] such that the path component $[\psi_0]$ of $\psi_0 \in X(1^+)$ is cofinal in the abelian monoid $\pi_0 X(1^+)$. Then the morphism*

$$\bar{\sigma} \ : \ X_\infty \longrightarrow \Omega X(S^1)$$

is a weak equivalence.

We note that if X is very special, then the translation map $\lambda_x : X(1^+) \longrightarrow X(1^+)$ is a weak equivalence [explain] and so the canonical map $X(1^+) \longrightarrow X_\infty$ is a weak equivalence. So for very special Γ-spaces the following theorem specializes to [...]

Example 7.24 (Connective topological K-theory as a $\mathbf{\Gamma}$-space)**.** We have already seen two models for topological K-theory spectra, see Examples 1.20 and 7.10. Now we discuss a special $\mathbf{\Gamma}$-space X whose associated spectrum $X(\mathbb{S})$ is level equivalent to the symmetric spectrum ku as defined in Example 7.10.

For a finite based set A we let $F(A)$ be the space of tuples (V_a), indexed by the non-basepoint elements of A, of finite dimensional, pairwise orthogonal subspaces of the hermitian vector space $\mathbb{C}^\infty$. [topology] The basepoint of $F(A)$ is the tuple where each V_a is the zero subspace. For a based map $\alpha : A \longrightarrow B$ the induced map $F(\alpha) : F(A) \longrightarrow F(B)$ sends (V_a) to (W_b) where

$$W_b \;=\; \bigoplus_{\alpha(a)=b} V_a \ .$$

We claim that the $\mathbf{\Gamma}$-space F is special [...].

For a based topological space K the value $F(K)$ of the extended $\mathbf{\Gamma}$-space on K is given by the coend formula (7.12). So an element of $F(K)$ is represented by a tuple $(k_1, \dots, k_n)$ of points of K 'labelled' with pairwise orthogonal vector spaces $(V_1, \dots, V_n)$ of $\mathbb{C}^\infty$ for some n. The topology is such that, informally speaking, the vector spaces V_i and V_j are summed up whenever two points k_i and k_j collide and V_i disappears when k_i approaches the basepoint of K.

Since the $\mathbf{\Gamma}$-space F is special, the symmetric spectrum $F(\mathbb{S})$ is a positive Ω-spectrum by the general theory. In particular, the space $F(\mathbb{S})_1 = F(S^1)$ is an infinite loop space. A nice feature of this $\mathbf{\Gamma}$-space model for ku is that $F(S^1)$ 'is' a familiar space, namely the infinite unitary group U. We get a preferred homeomorphism if we replace S^1 (the one-point compactification of $\mathbb{R}$) by $S(\mathbb{C})$, by the homeomorphic unit sphere in the complex numbers, with basepoint 1. Given a point $(\lambda_1, \dots, \lambda_n) \in (S(\mathbb{C}))^n$ and a tuple $(V_1, \dots, V_n)$ of pairwise orthogonal subspaces of $\mathbb{C}^\infty$ we let $\psi(k_1, \dots, k_n, V_1, \dots, V_n)$ be the unitary transformation of $\mathbb{C}^\infty$ that is multiplication by λ_i on V_i and the identity on the orthogonal complement of $\bigoplus_{i=1}^n V_i$. As n varies, these maps are compatible with the equivalence relation and so they assemble into a continuous map

$$F(S(\mathbb{C})) \;=\; \int^{n^+ \in \mathbf{\Gamma}} S(\mathbb{C})^n \wedge F(n^+) \;\longrightarrow\; U \ .$$

This map is bijective because every unitary transformation is diagonalizable with finitely many eigenvalues in the unit circle and pairwise orthogonal eigenspaces.

A morphism of $\mathbf{\Gamma}$-spaces $\dim : F \longrightarrow H\mathbb{Z}$ is given by the dimension function, i.e., on an object n^+ the map $\dim : F(n^+) \longrightarrow H\mathbb{Z}(n^+)$ is given by

$$\dim(V_1, \dots, V_n) \;=\; \sum_{i=1}^n (\dim V_i) \cdot i \ .$$

Evaluating this morphism on spheres provides a morphism of symmetric spectra $\dim : ku \simeq F(\mathbb{S}) \longrightarrow H\mathbb{Z}$ which induces an isomorphism on π_0.

Now we use Theorem 7.21 about the 'group completion theorem' of a specila $\mathbf{\Gamma}$-spaces to derive Bott periodicity.

Proposition 7.25. *The $\mathbf{\Gamma}$-space F is cofibrant and special*

Since the $\mathbf{\Gamma}$-space F is cofibrant and special, Theorem 7.21 lets us conclude that the map of graded rings

$$(\tilde{\sigma}_0)_* \;:\; H_*(F(S^0), \mathbb{Z}) \;\longrightarrow\; H_*(\Omega F(S^1), \mathbb{Z})$$

induced by the structure map $\tilde{\sigma}_0 : F(S^0) \longrightarrow \Omega F(S^1)$ is a localization at the multiplicative subset $\pi = \pi_0 F(S^0)$ of $H_*(F(S^0), \mathbb{Z})$. The dimension functor identifies the path components of the space

$$F(S^0) \;=\; \coprod_{n \geq 0} \mathrm{Gr}(n, \mathbb{C}^\infty)$$

with the set of natural numbers, and this is an isomorphism of abelian monoids $\dim : \pi_0(F(S^0)) \longrightarrow \mathbb{N}$. We let $W_0 = \mathbb{C} \cdot (1, 0, 0. \dots)$ be the complex line in $\mathbb{C}^\infty$ spanned by the first standard basis vector, and we let $\mathrm{sh} : \mathbb{C}^\infty \longrightarrow \mathbb{C}^\infty$ be the linear isometry given by

$$\mathrm{sh}(x_1, x_2, x_2, \dots) \;=\; (0, x_1, x_2, x_2, \dots) \;.$$

We define a continous map $\psi : F(1^+) \longrightarrow F(2^+)$ by

$$\psi(V) \;=\; (W_0, \mathrm{sh}\, V) \;,$$

The $\mathbf{\Gamma}$-space F and the associated symmetric spectra do not have an obvious multiplication, but a minor variation suffices to upgrade the previous construction to a commutative symmetric ring spectrum. For $n \geq 0$ we let $F^{(n)}$ denote the $\mathbf{\Gamma}$-space defined in the same way as F, but using pairwise orthogonal subspaces of $(\mathbb{C}^\infty)^{\otimes n}$ instead of $\mathbb{C}^\infty$. So we have $F^{(1)} = F$ and for $n \geq 2$, $(\mathbb{C}^\infty)^{\otimes n}$ is non-canonically isomorphic to $\mathbb{C}^\infty$. Any choice of unitary isomorphism between $(\mathbb{C}^\infty)^{\otimes n}$ and $\mathbb{C}^\infty$ induces an isomorphism of $\mathbf{\Gamma}$-spaces between $F^{(n)}$ and F. The unitary Σ_n-action on $(\mathbb{C}^\infty)^{\otimes n}$ by permuting the tensor factors induces an action of Σ_n on $F^{(n)}$ by automorphisms of $\mathbf{\Gamma}$-spaces.

Now we define a symmetric spectrum of topological spaces ku in level n as

$$ku_n \;=\; F^{(n)}(S^n) \;,$$

the values of the prolonged $\mathbf{\Gamma}$-space $F^{(n)}$ at the n-sphere. We let Σ_n acts diagonally, by permuting the sphere coordinates and by the action on $F^{(n)}$ from permutation of the tensor powers. Multiplication maps

$$\mu_{n,m} \;:\; ku_n \wedge ku_m = F^{(n)}(S^n) \wedge F^{(m)}(S^m) \;\longrightarrow\; F^{(n+m)}(S^{n+m}) = ku_{n+m}$$

are obtained by prolonging the $\Sigma_n \times \Sigma_m$-equivariant pairing of $\mathbf{\Gamma}$-spaces

$$F^{(n)}(A) \wedge F^{(m)}(B) \;\longrightarrow\; F^{(n+m)}(A \wedge B) \;, \quad (V_a) \wedge (W_b) \;\longmapsto\; (V_a \otimes W_b)_{a \wedge b} \;.$$

The multiplication maps $\mu_{n,m}$ are associative and commutative, so all that is missing to get a commutative symmetric ring spectrum are the unit maps.

The space $ku_0 = F^{(0)}(S^0)$ consists of all subspaces $(\mathbb{C}^\infty)^{\otimes 0} = \mathbb{C}$, so it has two points, the basepoint 0 and the point $\mathbb{C}$. We let $\iota_0 : S^0 \longrightarrow ku_0$ be the unique based bijection; then the unitality condition for the maps $\mu_{n,0}$ and $\mu_{0,n}$ holds. We let $\iota_1 : S^1 \longrightarrow ku_1 = F(S^1)$ be the map sending $x \in S^1$ to the class of the pair (x, V_0) where V_0 is the subspace of $\mathbb{C}^\infty$ spanned by the first vector $(1, 0, 0, \dots)$ of the canonical basis. Under the isomorphism between $F(S(\mathbb{C}))$ and the unitary group U this corresponds to the embedding $S(\mathbb{C}) = U(1) \longrightarrow U$ sending λ to the matrix

$$\begin{pmatrix} \lambda & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

Since the multiplication maps are commutative, the centrality condition for ι_1 is automatically satisfied. To sum up, we have defined a commutative symmetric ring spectrum ku that is a positive Ω-spectrum [justify] and that in level 1 is homeomorphic to the infinite unitary group. [get periodic KU by inverting an map $S^3 \longrightarrow ku_1$ representing the Bott map]

One should beware that even though $\mathbf{\Gamma}$-spaces were involved heavily in the construction of ku, this symmetric spectrum is not itself the evaluation of any $\mathbf{\Gamma}$-space in spheres.

7.5. Permutative categories.

Definition 7.26. A *permutative category* is a category $\mathcal{C}$ equipped with a functor $\oplus : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ and a natural isomorphism $\tau_{a,b} : a \oplus b \longrightarrow b \oplus a$ such that the following conditions hold:

- the two functors

$$\oplus \circ (\mathrm{Id} \times \oplus) \;,\; \oplus \circ (\oplus \times \mathrm{Id}) \;:\; \mathcal{C} \times \mathcal{C} \times \mathcal{C} \;\longrightarrow\; \mathcal{C}$$

  are equal;

- there is an object 0 such that the two functors
$$0 \oplus - \ , \ - \oplus 0 \ : \ \mathcal{C} \ \longrightarrow \ \mathcal{C}$$
are the identity functors;
- $\tau_{a,b} = \tau_{b,a}^{-1}$
- for every object A the isomorphism
$$\tau_{0,a} \ : \ a = 0 \oplus a \ \longrightarrow \ a \oplus 0 = a$$
is the identity;
- for all objects a, b and c the triangle
$$\begin{tikzcd} & a \oplus b \oplus c \arrow[dl, "\tau_{a,b}\oplus c"'] \arrow[dr, "\tau_{a,b\oplus c}"] & \\ b \oplus a \oplus c \arrow[rr, "b\oplus\tau_{a,c}"'] & & b \oplus c \oplus a \end{tikzcd}$$
commutes.

The above definition can be summarizing by saying that a permutative category is a special kind of symmetric monoidal category, namely one which is 'strictly' associative and unital, i.e., associativity and unit isomorphism are identities. We will give various examples of permutative categories below, after explaining how to associate to a permutative category a special **Γ**-spaces and hence, by evluation on spheres, a symmtric spectrum that is a positive Ω-spectrum).

Remark 7.27. The object 0 that exists by the second property of a permutative category is unique: if 0' is another such object, then $0 = 0 \oplus 0' = 0'$. For later reference we also recall why the endomorphism monoid $\mathcal{C}(0,0)$ of the zero object in a permutative category is automatically commutative. The value of the functor $\oplus$ at the object $(0,0)$ of $\mathcal{C}\times\mathcal{C}$ in particular provides a monoid homomorphism $\oplus : \mathcal{C}(0,0)\times\mathcal{C}(0,0) \longrightarrow \mathcal{C}(0,0)$, which means that
$$(a \cdot b) \oplus (c \cdot d) \ = \ (a \oplus c) \cdot (b \oplus d)$$
for all endomorphisms a, b, c and d of 0. Since $0 \oplus -$ and $- \oplus 0$ are the identity functors, we also have $\mathrm{Id} \oplus a = a = a \oplus \mathrm{Id}$, and that forces
$$\begin{aligned} a \cdot b \ &= \ (a \oplus \mathrm{Id}) \cdot (\mathrm{Id} \oplus b) \ = \ (a \cdot \mathrm{Id}) \oplus (\mathrm{Id} \cdot b) \\ &= \ a \oplus b \ = \ (\mathrm{Id} \cdot a) \oplus (b \cdot \mathrm{Id}) \ = \ (\mathrm{Id} \oplus b) \cdot (a \oplus \mathrm{Id}) \\ &= \ b \cdot a \end{aligned}$$
In other words the composition and direct sum operations in the endomorphism monoid of the zero object coincide and are commutative.

Construction 7.28. We let $\mathcal{C}$ be a small permutative category (i.e., the objects of $\mathcal{C}$ form a set). For every finite based set A we define a category $\underline{\mathcal{C}}(A)$ of 'A-indexed sum diagrams' in $\mathcal{C}$. An object of $\underline{\mathcal{C}}(A)$ is a collection $X = \{X_S, \rho_{S,T}\}$ consisting of

- an object X_S of $\mathcal{C}$ for every subset S of A that does not contain the basepoint,
- an isomorphism $\rho_{S,T} : X_S \oplus X_T \longrightarrow X_{S\cup T}$ for every pair of *disjoint* subsets S, T of A as above.

This dats is subject to the following conditions:

- $X_\emptyset = 0$ and $\rho_{S,\emptyset} = \mathrm{Id}_{X_S} : X_S \oplus X_\emptyset \longrightarrow X_S$ for all S;
- for all mutually disjoint subsets S, T, U of A the following squares commute:
$$\begin{tikzcd} X_S \oplus X_T \arrow[r, "\rho_{S,T}"] \arrow[d, "\tau_{X_S,X_T}"'] & X_{S\cup T} \arrow[d, equal] \\ X_T \oplus X_S \arrow[r, "\rho_{T,S}"'] & X_{T\cup S} \end{tikzcd} \qquad \begin{tikzcd} X_S \oplus X_T \oplus X_U \arrow[r, "\rho_{S,T}\oplus X_U"] \arrow[d, "X_S\oplus\rho_{T,U}"'] & X_{S\cup T} \oplus X_U \arrow[d, "\rho_{S\cup T,U}"] \\ X_S \oplus X_{T\cup U} \arrow[r, "\rho_{S,T\cup U}"'] & X_{S\cup T\cup U} \end{tikzcd}$$

A morphism $f : X \longrightarrow X'$ in the category $\underline{\mathcal{C}}(A)$ consists of morphism $f_S : X_S \longrightarrow X'_S$ for all subsets S of A not containing the basepoint such $f_\emptyset = \mathrm{Id}_0$ and such that the square

$$\begin{tikzcd} X_S \oplus X_T \arrow[r, "\rho_{S,T}"] \arrow[d, "f_S \oplus f_T"'] & X_{S\cup T} \arrow[d, "f_{S\cup T}"] \\ X'_S \oplus X'_T \arrow[r, "\rho'_{S,T}"'] & X'_{S\cup T} \end{tikzcd}$$

commutes for every pair of disjoint subsets S and T.

We observe that the definition of morphism is redundant: because the morphisms $\rho_{S,T}$ are isomorphisms, a morphisms is completely determined by its values on one-element subsets of A, and can be chosen freely there.

Now we let A vary and make the assignment $A \mapsto \underline{\mathcal{C}}(A)$ into a functor from the category of finite based sets to the category of small categories. For a based map $\alpha : B \longrightarrow A$ between finite based sets we define a functor $\alpha_* : \underline{\mathcal{C}}(B) \longrightarrow \underline{\mathcal{C}}(A)$ as follows. An object $X = \{X_S, \rho_{S,T}\}$ of $\underline{\mathcal{C}}(B)$ is sent to the object $\alpha_*(X)$ of $\underline{\mathcal{C}}(A)$ given by

$$(\alpha_*(X))_S \;=\; X_{\alpha^{-1}(S)}$$

and

$$(\alpha_*\rho)_{S,T} \;=\; \rho_{\alpha^{-1}(S),\alpha^{-1}(T)} \;,$$

Since α is based, $\alpha^{-1}(S)$ does not contain the basepoint if S does not, so this definition makes sense. The behaviour of α_* on morphisms is essentially the same: given $f : X \longrightarrow X'$ we define $\alpha_* f : \alpha_* X \longrightarrow \alpha_* X'$ at a subset S of A as

$$(\alpha_* f)_S \;=\; f_{\alpha^{-1}(X)} \;.$$

We omit the straightforward verification that this really defined a functor $\alpha_* : \underline{\mathcal{C}}(B) \longrightarrow \underline{\mathcal{C}}(A)$. Given another based map $\beta : C \longrightarrow B$ of finite based sets we have $(\alpha\beta)^*(S) = \beta^*(\alpha^* S)$ for all subsets S of A, and hence

$$(\alpha\beta)_* \;=\; \alpha_* \circ \beta_* \;:\; \underline{\mathcal{C}}(C) \;\longrightarrow\; \underline{\mathcal{C}}(A) \;.$$

So altogether we constructed a covariant functor $\underline{\mathcal{C}} : \mathbf{\Gamma} \longrightarrow \mathbf{cat}$ from the category of finite based sets to the category of small categories, i.e., a $\mathbf{\Gamma}$-*category.*

The morphism $p^A : A \vee B \longrightarrow A$ sends B to the basepoint and is the identity on A.

Proposition 7.29. *For every small permutative category $\mathcal{C}$ the $\mathbf{\Gamma}$-category $\underline{\mathcal{C}}$ is* special *in the following sense:*

- *if A consists only of the basepoint, then category $\underline{\mathcal{C}}(A)$ is terminal;*
- *for every pair of finite based sets the functor*

$$(p^A_*, p^B_*) \;:\; \underline{\mathcal{C}}(A \vee B) \;\longrightarrow\; \underline{\mathcal{C}}(A) \times \mathcal{C}(B)$$

is an equivalence of categories.

Proof. □

Given a permutative category $\mathcal{C}$, we follow the previous construction of the $\mathbf{\Gamma}$-category $\mathcal{C}$ by the nerve functor to obtain a $\mathbf{\Gamma}$-space (of simplicial sets). We can then evaluate on spheres and obtain a symmetric spectrum of simplicial sets:

Definition 7.30. The *K-theory spectrum* of a permutative category $\mathcal{C}$ is the symmetric spectrum

$$\mathbf{K}(\mathcal{C}) \;=\; (N\underline{\mathcal{C}})(\mathbb{S}) \;.$$

The nerve construction is a covariant functor from the category of small categories to the category of simplicial sets that preserves limits. In particular, it preserves products and sends any terminal category to a constant simplicial set with one vertex. Moreover, the nerve construction sends equivalences of categories to homotopy equivalences of simplicial sets. So for every permutative category $\mathcal{C}$ the composite functor

$$\mathbf{\Gamma} \xrightarrow{\underline{\mathcal{C}}} \mathbf{cat} \xrightarrow{N} \mathbf{sS}$$

is a special $\mathbf{\Gamma}$-space in the sense of Definition 7.15. Moreover, the functor

$$\mathcal{C}(1^+) \longrightarrow \mathcal{C}\ , \quad X \longmapsto X_{\{1\}} \tag{7.31}$$

is an isomorphism of categories, so the underlying space $N\underline{\mathcal{C}}(1^+)$ of the $\mathbf{\Gamma}$-space $N\underline{\mathcal{C}}$ is isomorphic to the nerve of the category $\mathcal{C}$.

For any special $\mathbf{\Gamma}$-space X the underlying space $X(1^+)$ has a 'weak multiplication' [...]. In the case of a permutative category, this weak multiplication for the special $\mathbf{\Gamma}$-space $N \circ \underline{\mathcal{C}}$ is strict in the following sense. Indeed, we can pull back the permutative structure on $\mathcal{C}$ along the isomorphism (7.31) to a permutative structure $\oplus : \mathcal{C}(1^+) \times \mathcal{C}(1^+) \longrightarrow \mathcal{C}(1^+)$. Since the functor $\oplus$ is strictly associative and has a strict unit, the induced map on nerves

$$N\oplus \ :\ N\mathcal{C}(1^+) \times N\mathcal{C}(1^+) \longrightarrow N\mathcal{C}(1^+)$$

makes the $N\mathcal{C}(1^+)$ into a simplicial monoid (where we have used implicitely that the nerve preserves products).

The multiplication of the simplicial monoid is honestly associative and unital, but in general *not* commutative (because $A \oplus B$ and $B \oplus A$ are only isomorphic, but typically not equal).

Then the diagram of categories

$$\begin{tikzcd}
 & \underline{\mathcal{C}}(2^+) \arrow[dl, "(p^1_*{,}p^2_*)"', "\sim"] \arrow[dr, "\nabla_*"] & \\
\underline{\mathcal{C}}(1^+) \times \underline{\mathcal{C}}(1^+) \arrow[rr, "\oplus"'] & & \underline{\mathcal{C}}(1^+)
\end{tikzcd}$$

commutes up to the built-in natural isomorphism (given at an object X of $\mathcal{C}(2^+)$ by $\rho_{\{1\},\{2\}} : X_{\{1\}} \oplus X_{\{2\}} \longrightarrow X_{\{1,2\}}$). Thus the diagram of simplicial sets

$$\begin{tikzcd}
 & N\underline{\mathcal{C}}(2^+) \arrow[dl, "(Np^1_*{,}Np^2_*)"', "\sim"] \arrow[dr, "N\nabla_*"] & \\
N\underline{\mathcal{C}}(1^+) \times N\underline{\mathcal{C}}(1^+) \arrow[rr, "N\oplus"'] & & N\underline{\mathcal{C}}(1^+)
\end{tikzcd}$$

commutes up to preferred homotopy

Corollary 7.32. *For every permutative category $\mathcal{C}$ the symmetric spectrum $\mathbf{K}(\mathcal{C})$ is a connective positive Ω-spectrum. If the isomorphism classes of objects in $\mathcal{C}$ from a group under the operation induced by $\oplus$, then $\mathbf{K}(\mathcal{C})$ is an Ω-spectrum.*

Example 7.33. (Eilenberg-Mac Lane spectrum, revisited) Let A be an abelian monoid. We denote by $\underline{A}$ the 'discrete' permutative category with object set A and only identity morphism. The functor $\oplus : \underline{A} \times \underline{A} \longrightarrow \underline{A}$ is given on object by the addition in A. The symmetry isomorphism $\tau_{a,b} : a + b \longrightarrow b + a$ is (necessarily) the identity of $a + b$. The axioms of a permutative category are clearly satisfied.

Conversely, if $\mathcal{C}$ is any small permutative category that only has identity morphisms, then in particular the symmetry isomorphism have to be identities. Thus the functor $\oplus$ makes the object set of $\mathcal{C}$ into a commutative monoid, and $\mathcal{C}$ is equal to discrete permutative category of this abelian monoid. In other words, abelian monoids 'coincide with' the permutative category that only have identity morphisms.

We will show now that when the abelian monoid A is a group, then the symmetric spectrum $\mathbf{K}(\underline{A})$ associated to the permutative category of A is isomorphic to the Eilenberg-Mac Lane spectrum HA (as defined in Example 1.14).

Since the only morphisms in $\underline{A}$ are identities, for finite based set A, every object X of $\underline{A}(A)$ all the isomorphisms $\rho_{S,T}$ must be identities. In particular, for all subsets $S = \{s_1, \dots, s_n\}$ of A (not containing the basepoint) we must have

$$X_S \ = \ X_{\{s_1\}} + \cdots + X_{\{s_n\}}$$

as elements of A. We observe first that here for all finite based sets A and B the functor

$$(p_*^A, p_*^B) \ : \ \underline{A}(A \vee B) \ \longrightarrow \ \underline{A}(A) \times \underline{A}(B)$$

is bijective on objects, hence an *isomorphism* (and not merely an equivalence) of categories.

The category $\underline{A}(A)$ is again discrete (i.e., the only morphisms are identity morphisms), so its nerve $N\underline{A}(A)$ is a constant simplicial sets (i.e., all structure maps are bijective). Moreover, the map

$$((N\underline{A})(A))_0 \ = \ \text{objects}((N\underline{A})(A)) \ \cong \ A[A]$$

is an isomorphism of simplicial sets, where the right hand side is the reduced A-linearization discussed in Example 1.14. So in summary, the $\mathbf{\Gamma}$-space $N \circ \underline{A}$ is isomorphic to the reduced A-linearization. So evaluation on spheres becomes degreewise reduced A-linearization; this shows that $\mathbf{K}(\underline{A})$ is isomorphic to $HA = A[\mathbb{S}]$, the symmetric Eilenberg-Mac Lane spectrum.

Example 7.34 (K-theory of finite sets)**.** As before we denote by $\mathcal{F}in$ the category of standard finite sets whose objects are the sets $\mathbf{n} = \{1, \dots, n\}$ for $n \geq 0$ and whose morphisms are all set maps. The sum functor $+ : \mathcal{F}in \times \mathcal{F}in \longrightarrow \mathcal{F}in$ is given by by addition on objects and by 'disjoint union' on morphisms. More precisely, for morphisms $f : \mathbf{n} \longrightarrow \mathbf{n}'$ and $g : \mathbf{m} \longrightarrow \mathbf{m}'$ we define $f + g : \mathbf{n} + \mathbf{m} \longrightarrow \mathbf{n}' + \mathbf{m}'$ by

$$(f+g)(i) \ = \ \begin{cases} \quad f(i) & \text{if } 1 \leq i \leq n, \text{ and} \\ g(i-n) + n' & \text{if } n+1 \leq i \leq n+m. \end{cases}$$

The sum functor makes $\mathcal{F}in$ into a permutative category, with respect to the shuffle maps $\chi_{n,m} : \mathbf{n} + \mathbf{m} \longrightarrow \mathbf{m} + \mathbf{n}$ as symmetry isomorphism. We let $i\mathcal{F}in$ denote the (non-full) subcategory of $\mathcal{F}in$ consisting of all isomorphisms (bijections). This is again a permutative category, by restriction of The sum functor restricts to a permutative structure on $i\mathcal{F}in$, so the K-theory construction of Definition 7.30 yield a symmetric spectrum $\mathbf{K}(i\mathcal{F}in)$. The simplicial set $\mathbf{K}(i\mathcal{F}in)_0$ in level 0 is isomorphic to the nerve of the category $i\mathcal{F}in$, which in turn is isomorphic to the disjoint union, over $n \geq 0$, of the classifying spaces $B\Sigma_n$ of all symmetric groups.

The object of $\underline{i\mathcal{F}in}(1^+)$ whose value at the set $\{1\}$ is the object $\mathbf{1}$ of $i\mathcal{F}in$ is a vertex in the simplicial set $\mathbf{K}(i\mathcal{F}in)_0 = N\underline{i\mathcal{F}in}(1^+)$. So it freely generates a morphism of symmetric spectra

$$\mathbb{S} \ \longrightarrow \ \mathbf{K}(i\mathcal{F}in) \ .$$

A consequence of the Kahn-Priddy-Quillen theorem is that this morphism is an $\hat{\pi}_*$-isomorphism. So in this sense the K-theory of finite sets 'is' the sphere spectrum.

Example 7.35 (K-theory of rings)**.** We let R be an associative and unital ring. We denote the $\mathbf{Gl}(R)$ the category whose objects are the natural numbers $n \geq 0$ and with morphism sets given by

$$\mathbf{Gl}(n,m) \ = \ \begin{cases} Gl_n(R) & \text{for } n = m, \text{ and} \\ \quad \emptyset & \text{for } n \neq m. \end{cases}$$

Composition is is given by multiplication in the general linear groups. A sum functor $+ : \mathbf{Gl}(R) \times \mathbf{Gl}(R) \longrightarrow \mathbf{Gl}(R)$ is given by by addition on objects and by 'block sum' of matrices on morphisms, i.e., by

$$+ \ : \ Gl_n(R) \times Gl_m(R) \ \longrightarrow \ Gl_{n+m}(R) \ , \quad (A, B) \ \longmapsto \ A + B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \ .$$

This sum functor makes $\mathbf{Gl}(R)$ into a permutative category, with symmetry isomorphism $\tau_{n,m} \in Gl_{n+m}(R)$ given by the permuation matrices of the shuffle permutations $\chi_{n,m} : \mathbf{n} + \mathbf{m} \longrightarrow \mathbf{m} + \mathbf{n}$. The *free algebraic K-theory spectrum* of R is the associated symmetric spectrum as in Definition 7.30, i.e.,

$$\mathbf{K}^{\text{free}}(R) \ = \ \mathbf{K}(\mathbf{Gl}(R)) \ .$$

The adjective 'free' refers to the fact that the category $\mathbf{Gl}(R)$ is equivalent to the category of finitely generated free R-modules and isomorphisms (as opposed to all projective modules). The simplicial set $(\mathbf{K}^{\text{free}}(R))_0$ in level 0 is isomorphic to the nerve of the category $\mathbf{Gl}(R)$, which in turn is isomorphic to the disjoint union, over $n \geq 0$, of the classifying spaces $BGl_n(R)$ of all general linear groups of R.

A variation of this yields the *projective algebraic K-theory spectrum* of the ring R. We let $\mathbf{P}(R)$ denote the category whose objects are given by pairs (n, P) where $P \in M_n(R)$ is an idempotent $n \times n$ matrix. A morphism from (n, P) to (n', P') in $\mathbf{P}(R)$ is a left R-linear isomorpphism from the image of P to the image of P'. A sum functor $+ : \mathbf{P}(R) \times \mathbf{P}(R) \longrightarrow \mathbf{P}(R)$ is given on objects by $(n, P) + (n', P') = (n + n', P + P')$ where last $+$ is again block sum of matrices. The sum of morphisms in $\mathbf{P}(R)$ is given by direct sum of isomorphisms, using the preferred identification of the image of $P + P'$ with the direct sum of the images of P and P'. This sum functor makes $\mathbf{P}(R)$ into a permutative category, one more time with symmetry isomorphism given by the permuation matrices of the shuffle permutations. The *algebraic K-theory spectrum* of R is the associated symmetric spectrum, i.e.,

$$\mathbf{K}(R) \;=\; \mathbf{K}(\mathbf{P}(R)) \;.$$

The category $\mathbf{Gl}(R)$ embeds fully faithfully into $\mathbf{P}(R)$ by sending n to the pair (n, E_n), where E_n is the $n \times n$ identity matrix. [on morphisms]. This embedding models the inclusions of the R-free modules into the projective R-modules (in both cases finitely generated). The induced morphism

$$\mathbf{K}^{\text{free}}(R) = \mathbf{K}(\mathbf{Gl}(R)) \;\longrightarrow\; \mathbf{K}(\mathbf{P}(R)) = \mathbf{K}(R)$$

is an isomorphism on homotopy groups in positive dimensions (i.e., possibly excluding dimension zero).

Example 7.36 (Picard categories and 1-Postnikov stages)**.** A *strict Picard category* is a small permutative cateogory $(\mathcal{C}, \oplus)$ with the additional properties that

- the underlying category $\mathcal{C}$ is a groupoid, i.e., every morphism is an isomorphism;
- the set of isomorphism classes of objects is a group under the operation induced by $\oplus$.

The K-theory spectrum $\mathbf{K}(\mathcal{C})$ of a strict Picard category is then a connective Ω-spectrum by Corollary 7.32. In particular, the its non-negative naive and true homotopy groups coincide with the homotopy groups of the 0-th space $\mathbf{K}(\mathcal{C})_0$ which is turn isomorphic to the nerve $N\mathcal{C}$ of the category $\mathcal{C}$. The components of $N\mathcal{C}$ are the isomorphism classes of objects; since the category $\mathcal{C}$ is a category, the fundamental group based at the zero object is isomorphic to the automorphism group, which is commutative by Remark 7.27. So we obtain that

$$\hat{\pi}_k \mathbf{K}(\mathcal{C}) \;\cong\; \pi_k \mathbf{K}(\mathcal{C}) \;\cong\; \begin{cases} \text{ob}(\mathcal{C})/\text{isomorphism} & \text{for } k = 0, \\ \quad \text{Aut}_{\mathcal{C}}(0) & \text{for } k = 1, \text{ and} \\ \qquad 0 & \text{else.} \end{cases}$$

We will show later that strict Picard categories model the homotopy category of spectra with homotopy groups concentrated in dimension 0 and 1. More precisely, the K-theory construction induces a fully faithful functor

$$\mathbf{K} \;:\; \text{Ho(Pic)} \;\longrightarrow\; \mathcal{SHC}$$

from the homotopy category of strict Picard categories to the stable homotopy category whose essential image consists of those symmetric spectrum X such that $\pi_k X = 0$ for all $k \neq 0, 1$.

For the K-theory spectra of Picard categories, the multiplication by the class $\eta \in \pi_1 \mathbb{S}$ of Hopf map

$$\cdot\eta \;:\; \pi_0 \mathbf{K}(\mathcal{C}) \;\longrightarrow\; \pi_1 \mathbf{K}(\mathcal{C})$$

(compare Exmaple 1.11) admits the following 'combinatorial' description. For every object b of $\mathcal{C}$ the map

$$- \oplus \text{Id}_b \;:\; \text{Aut}_{\mathcal{C}}(0) \;\longrightarrow\; \text{Aut}_{\mathcal{C}}(b)$$

is a group isomorphism. So for every object a of $\mathcal{C}$ the symmetry automorphism $\tau_{a,a}$ of $a + a$ is on the form $\eta(a) \oplus \text{Id}_{a+a}$ for a unique element $\eta(a) \in \text{Aut}_{\mathcal{C}}(0)$. The element $\eta(a)$ only depends on the isomorphism of a [...] and satisfies $\eta(a \oplus a') = \eta(a) \cdot \eta(a')$. In other word, this construction descends to a homomorphism of abelian group $\eta : \text{ob}(\mathcal{C})/\cong \longrightarrow \text{Aut}_{\mathcal{C}}(0)$.

Proposition 7.37. *For every Picard category $\mathcal{C}$ the square*

$$\begin{tikzcd} \mathrm{ob}(\mathcal{C})/\mathrm{iso} \arrow[r, "\eta"] \arrow[d, "\cong"'] & \mathrm{Aut}_{\mathcal{C}}(0) \arrow[d, "\cong"] \\ \pi_0\mathbf{K}(\mathcal{C}) \arrow[r, "\cdot\eta"'] & \pi_1\mathbf{K}(\mathcal{C}) \end{tikzcd}$$

commutes.

[for 1-Postnikov of $K(R)$ take $\mathcal{C}$ as the category of finitely generated projective R-modules with morphisms $\mathcal{C}(P,Q) = \mathrm{Iso}_R(P,Q)/\sim$ where two isomorphisms are equivalent if the have the same determinant. Baues, Jibladze and Pirashvili [**?**] construct all ring spectra with homotopy concentrated in dimensions 0 and 1 from classes in the third Mac Lane cohomology groups]

The 0-th level of the K-theory spectrum of a permutative category $\mathcal{C}$ is isomorphic to the nerve of the category $\mathcal{C}$. In particular, the monoidal product plays no role here, but it does make $\mathbf{K}(\mathcal{C})_0 = N\mathcal{C}$ into a simplicial monoid. The symmetry isomorphism, however, is not seen at the 0-th level, but it play a crucial role in the 'deloopings', i.e., in the homotopy types of the higher levels of the positive Ω-spectrum $\mathbf{K}(\mathcal{C})$. We illustrate this with a specific example now.

Example 7.38. We let $\mathcal{C}$ be the category with object set $\mathbb{Z}$, with the cyclic group $C_2 = \{1, -1\}$ of order 2 as automorphism group of every object, and with no morphisms between different objects. We define a monoidal product $\oplus : \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}$ by addition $\mathbb{Z}$ on objects and by group multiplication in C_2 on morphisms. The nerve $N\mathcal{C}$ is a disjoint union, indexed by the integers, of copies of the classifying space $BC_2 = C_2[S^1]$. The structure of simplicial monoid induced by $\oplus$ is as the product $\mathbb{Z} \times BC_2$ of the discrete group $\mathbb{Z}$ and the simplicial abelian group BC_2.

Now we endow the strict monoidal category $(\mathbb{C}, \oplus)$ with two different symmetry isomorphism. The 'trivial' symmetry isomorphism $\tau^{\mathrm{tr}}_{a,b} : a + b \longrightarrow b + a$ is the identity morphisms. The 'natural' symmetry isomorphisms $\tau^{\mathrm{nat}}_{a,b} : a + b \longrightarrow b + a$ is defined as $(-1)^{ab} \in C_2 = \mathcal{C}(a + b, a + b)$. We omit the straightforward verification that both symmetry isomorphisms τ^{tr} and τ^{nat} obey the axioms of a permutative category (even of a Picard category).

We denote by $\mathcal{C}^{\mathrm{tr}}$ respectively $\mathcal{C}^{\mathrm{nat}}$ the monoidal category $(\mathcal{C}, \oplus)$ endowed with the trivial respectively natural symmetry isomorphisms. Then the K-theory spectra $\mathbf{K}(\mathcal{C}^{\mathrm{tr}})$ and $\mathbf{K}(\mathcal{C}^{\mathrm{nat}})$ have the same 0-th level, isomorphic to $\mathbb{Z} \times BC_2$. The first level is a delooping of the 0-th level with respect to the monoid structure induced by $\oplus$, so even the 1-levels $\mathbf{K}(\mathcal{C}^{\mathrm{tr}})_1$ and $\mathbf{K}(\mathcal{C}^{\mathrm{nat}})_1$ are weakly equivalent as simplicial sets, and both have the homotopy type of $B\mathbb{Z} \times B^2C_2 \simeq S^1 \times C_2[S^2]$. However, the spectra $\mathbf{K}(\mathcal{C}^{\mathrm{tr}})$ and $\mathbf{K}(\mathcal{C}^{\mathrm{nat}})$ are *not* stably equivalent, as we shall now explain.

There is a unique functor $r : \mathcal{C} \longrightarrow \underline{\mathbb{Z}}$ to the discrete permutative category with object group $\mathbb{Z}$ (compare Example 7.33) that is the identity on objects. This functor is strictly monoidal and symmetric with respect to both symmetry isomorphisms τ^{tr} and τ^{nat}, so it induces morphisms of symmetric spectra

$$r^{\mathrm{nat}}_* \ : \ \mathbf{K}(\mathcal{C}^{\mathrm{nat}}) \ \longrightarrow \ \mathbf{K}(\underline{\mathbb{Z}}) = H\mathbb{Z} \quad \text{and} \quad r^{\mathrm{tr}}_* \ : \ \mathbf{K}(\mathcal{C}^{\mathrm{tr}}) \ \longrightarrow \ H\mathbb{Z} \ .$$

Since r is bijective on isomorphism classes of objects, both morphisms r^{nat}_* and r^{tr}_* induce bijections on π_0.

So far we did not detect the difference that arise from the different symmetries, but this comes now. We can also map $\mathcal{C}$ to the permutative category $\bar{C}_2$ with one object 0 and automorphism group C_2 [and monoidal structure...] Indeed, there is a unique faithful functor $j : \mathcal{C} \longrightarrow \bar{C}_2$ that takes the automorphism group of every object of $\mathcal{C}$ isomorphically onto the automorphism group of 0 in $\bar{C}_2$. This functor is strictly monoidal and symmetric with respect to the trivial symmetry isomorphisms τ^{tr}. So j induces a morphism of symmetric spectra $j_* : \mathbf{K}(\mathcal{C}^{\mathrm{tr}}) \longrightarrow \mathbf{K}(\bar{C}_2)$ that provides an isomorphism on π_1. So the combined morphism

$$(r^{\mathrm{tr}}_*, j_*) \ : \ \mathbf{K}(\mathcal{C}^{\mathrm{tr}}) \ \longrightarrow \ H\mathbb{Z} \times \mathbf{K}(\bar{C}_2)$$

is a stable equivalence. Since source and target of the map (r^{tr}_*, j_*) are Ω-spectra, it is even a level equivalence. [is this an isomorphism? is $\mathcal{C}^{\mathrm{tr}}$ the product of $\underline{\mathbb{Z}}$ and $\bar{C}_2$ as permutative categories?] So the

K-theory spectrum $\mathbf{K}(\mathcal{C}^{\mathrm{tr}})$ associated to the trivial symmetries is a product of $H\mathbb{Z}$ and a shifted copy of HC_2.

However, the strict monoidal functor $j : \mathcal{C} \longrightarrow \bar{C}_2$ is *not* symmetric with respect to the natural symmetry isomorphism τ^{nat}, so it does *not* induce a morphism of symmetric spectra from $\mathbf{K}(\mathcal{C}^{\mathrm{nat}})$ to $\mathbf{K}(\bar{C}_2)$. What is more, the K-theory spectrum $\mathbf{K}(\mathcal{C}^{\mathrm{nat}})$ does not even split off a copy of the Eilenberg-Mac Lane spectrum $H\mathbb{Z}$ 'up to stable equivalence' (i.e., $\mathbf{K}(\mathcal{C}^{\mathrm{nat}})$ is indecomposable as an object of the stable homotopy category). One way to see this is to observe that multiplication by the class $\eta \in \pi_1\mathbb{S}$ of the sphere spectrum is a non-zero (hence surjective) homomorphism

$$\pi_0\mathbf{K}(\mathcal{C}^{\mathrm{nat}}) \ \longrightarrow \ \pi_1\mathbf{K}(\mathcal{C}^{\mathrm{nat}}) \ .$$

[identify the action of η combinatorially in general...] With a little more work one can show that the difference between the K-theory spectra $\mathbf{K}(\mathcal{C}^{\mathrm{tr}})$ and $\mathbf{K}(\mathcal{C}^{\mathrm{nat}})$ first manifests itself in the homotopy type of the second levels. As we saw above, the simplicial set $\mathbf{K}(\mathcal{C}^{\mathrm{tr}})_n$ is a product of two Eilenberg-Mac Lane spaces of type $(\mathbb{Z}, n)$ and $(\mathbb{Z}/2, n+1)$ for every n. With the natural symmetry isomorphisms, however, the the simplicial set $\mathbf{K}(\mathcal{C}^{\mathrm{nat}})_2$ is weakly equivalent to the homotopy fiber of any morphism

$$\mathbb{Z}[S^2] \ \longrightarrow \ \mathbb{Z}/2[S^4]$$

whose effect in mod-2 cohomology is non-zero; since the cohomology group $H^4(\mathbb{Z}[S^2], \mathbb{Z}/2)$ is cyclic of order 2 (generated by the mod-2 reduction of the cup-square of the fundamental class) such a map exists and is unique up to homotopy.

7.6. Spectra as enriched functors. The $\mathbf{\Gamma}$-spaces, continuous and simplicial functors are, by their very definition, categories of functors from some indexing category to based spaces or simplicial sets. In the latter cases, we only consider 'enriched', i.e., continuous respectively simplicial functors. We will now see how symmetric, sequential, orthogonal and unitary spectra can also be viewed as enriched functors on suitable topological or simplicial indexing categories. In this picture, most of the comparison functors in diagram (7.1) are obtained by restriction along functors on the indexing categories.

The original definition of symmetric, sequential, orthogonal and unitary spectra is the most 'down to earth' way to introduce these objects. In a sense, Definitions 1.1, 2.1, 7.2 and 7.5 describes the structure which the spaces X_n have by 'generators' (the actions of the certain groups and the structure maps σ_n) and 'relations' (the equivariance properties for the iterated structure maps). The 'enriched functor' viewpoint encodes all possible natural operations between the spaces in a symmetric, sequential, orthogonal or unitary spectrum.

Definition 7.39. A $\mathbf{T}$-category, or small based topological category, $\mathbf{J}$ consists of

- a set ob $\mathbf{J}$ of objects,
- for every pair i, j of objects a compactly generated based space $\mathbf{J}(i, j)$
- and for every triple of objects i, j, k a based continuous composition map

$$\circ \ : \ \mathbf{J}(i,j) \wedge \mathbf{J}(j,k) \ \longrightarrow \ \mathbf{J}(i,k) \ .$$

Moreover, composition should be associative in the sense that for every quadrupel of objects i, j, k, l the square

$$\begin{tikzcd} \mathbf{J}(i,j) \wedge \mathbf{J}(j,k) \wedge \mathbf{J}(k,l) \arrow[r, "\circ\wedge\mathrm{Id}"] \arrow[d, "\mathrm{Id}\wedge\circ"'] & \mathbf{J}(i,k) \wedge \mathbf{J}(k,l) \arrow[d, "\circ"] \\ \mathbf{J}(i,j) \wedge \mathbf{J}(j,l) \arrow[r, "\circ"'] & \mathbf{J}(i,l) \end{tikzcd}$$

commutes and composition should have two sided units [state unit 'enriched'].

We emphasize that our convention for the order of the factors in composition is different from the usual convention where $f \circ g$ means 'first g, then f'.

A $\mathbf{T}$-category can equivalently be defined as a small category whose hom-sets are endowed with (compactly generated weak Hausdorff) topologies and basepoint and such that [...] However, the was we have stated the definition above is better adapted to generalizations. For example, if we replace the category $\mathbf{T}$ by the category $\mathbf{sS}$ of based simplicial sets throughout Definition 7.39, we arrive at the definition of a $\mathbf{sS}$-category or small based simplicial category. We note that $\mathbf{sS}$-categories are more special objects than simplicial objects of based categories because the object set is constant.

Definition 7.40. Let $\mathbf{J}$ be a $\mathbf{T}$-category. An *enriched functor* $Y : \mathbf{J} \longrightarrow \mathbf{T}$ consists of

- a based space $Y(a)$ for every object a of $\mathbf{J}$,
- a based, continuous action map $\circ : Y(a) \wedge \mathbf{J}(a,b) \longrightarrow Y(b)$ for every pair of objects a, b

such that $\mathrm{Id}_a \in \mathbf{J}(a,a)$ acts as the identity and the square

$$\begin{tikzcd} Y(a) \wedge \mathbf{J}(a,b) \wedge \mathbf{J}(b,c) \arrow[r, "\circ\wedge\mathrm{Id}"] \arrow[d, "\mathrm{Id}\wedge\circ"'] & Y(b) \wedge \mathbf{J}(b,c) \arrow[d, "\circ"] \\ Y(a) \wedge \mathbf{J}(a,c) \arrow[r, "\circ"'] & Y(c) \end{tikzcd}$$

commutes for all objects a, b, c. A *morphism* $f : Y \longrightarrow Z$ of enriched functors consists of based, continuous maps $f(a) : Y(a) \longrightarrow Z(a)$ for every object a such that the square

$$\begin{tikzcd} Y(a) \wedge \mathbf{J}(a,b) \arrow[r, "f(a)\wedge\mathrm{Id}"] \arrow[d, "\circ"'] & Z(a) \wedge \mathbf{J}(a,b) \arrow[d, "\circ"] \\ Y(b) \arrow[r, "f(b)"'] & Z(b) \end{tikzcd}$$

commutes for all objects a and b of $\mathbf{J}$.

The data of an enriched functor $Y : \mathbf{J} \longrightarrow \mathbf{T}$ is the same as specifying a functor, in the usual sense, which is based and continuous [...]. Again, we stated the definition in the way we did because it generalizes easily to the simplicial context (and others): given a $\mathbf{sS}$-category $\mathbf{J}$, an *enriched functor* $Y : \mathbf{J} \longrightarrow \mathbf{sS}$ is defined just as in 7.40, but with $\mathbf{T}$ replaces by the category of based simplicial sets $\mathbf{sS}$ throughout.

Ordinary category theory can be subsumed under this branch of enriched category theory as follows. Every category $\mathcal{C}$ in the ordinary sense gives rise to a $\mathbf{T}$-category $\mathcal{C}^+$ by adding disjoint basepoints to the morphisms sets and endowing them with the discrete topology and extending composition by [...] We can also turn $\mathcal{C}$ into a $\mathbf{\Sigma}$-category by adding disjoint basepoint and viewing the morphisms sets a constant simplicial sets. Then enriched functors from $\mathcal{C}^+$ to $\mathbf{T}$ (respectively to $\mathbf{\Sigma}$) are in bijective correspondence [isomorphism of categories] with functors, in the classical sense, from $\mathcal{C}$ to $\mathbf{T}$ (respectively $\mathbf{\Sigma}$).

Now we introduce the indexing categories $\mathbf{N}$, $\mathbf{\Sigma}$, and $\mathbf{O}$ which parametrize sequential, symmetric respectively orthogonal spectra. There is also a $\mathbf{T}$-category $\mathbf{U}$ which parametrizes unitary spectra in much the same way, but we leave that part of the story to Exercise E.I.49. We focus on the case of topological spaces and define three *topological* categories. Sequential and symmetric spectra also exists with values in simplicial sets, and we will briefly say how this setting has to be dealt with.

The objects of $\mathbf{N}$, $\mathbf{\Sigma}$ and $\mathbf{O}$ are the natural numbers $0, 1, 2, \ldots$. The based spaces of morphisms from n to m are given, respectively, by

$$(7.41)\qquad \begin{aligned} \mathbf{N}(n,m) &= S^{m-n}\,, \\ \mathbf{\Sigma}(n,m) &= \Sigma_m^+ \wedge_{1\times\Sigma_{m-n}} S^{m-n}\,, \\ \mathbf{O}(n,m) &= O(m)^+ \wedge_{1\times O(m-n)} S^{m-n}\,. \end{aligned}$$

Here negative dimensional spheres have to be interpreted as a point, i.e., the respective hom-space has just one point if $n > m$. The unit element in $\mathbf{N}(n,n) = S^0$ is the non-base point; the unit element in $\mathbf{\Sigma}(n,n)$ respectively $\mathbf{O}(n,n)$ is $[1 \wedge z]$, where 1 denotes the neutral element of the group Σ_n respectively $O(n)$ and $z \in S^0$ the non-basepoint.

In the sequential indexing category, the composition map $\circ : \mathbf{N}(n,m) \wedge \mathbf{N}(m,k) \longrightarrow \mathbf{N}(n,k)$ is the preferred identification $S^{m-n} \wedge S^{k-m} \cong S^{k-n}$. In the case of the symmetric indexing category, the composition map is defined by $\circ : \mathbf{\Sigma}(m,k) \wedge \mathbf{\Sigma}(n,m) \longrightarrow \mathbf{\Sigma}(n,k)$ is defined by

$$[\tau \wedge z] \circ [\gamma \wedge y] \ = \ [\tau(\gamma+1) \wedge (y \wedge z)]$$

where $\tau \in \Sigma_k$, $\gamma \in \Sigma_m$, $z \in S^{k-m}$ and $y \in S^{m-n}$. In the case of the orthogonal indexing category $\mathbf{O}$, composition is given by the same formula, just that τ, γ and $\tau(\gamma+1)$ now belong to orthogonal groups.

Now we make precise in which way the various spectra categories 'are' enriched functors; the precise statements are the isomorphisms of categories in the next Proposition 7.42. Given a sequential (respectively symmetric, orthogonal or unitary) spectrum X we define an enriched functor $e(X)$ from $\mathbf{N}$ (respectively $\mathbf{\Sigma}$ or $\mathbf{O}$) to $\mathbf{T}$ as follows. On objects we set $X(n) = X_n$; the action map is defined in the sequential case by

$$\mathbf{N}(n,m) \wedge X_n \ = \ S^{m-n} \wedge X_n \longrightarrow X_m \ , \quad z \wedge x \mapsto \sigma^{m-n}(x \wedge z)$$

and in the symmetric cases by

$$\mathbf{\Sigma}(n,m) \wedge X_n \ = \ (\Sigma_n^+ \wedge_{1 \times \Sigma_{m-n}} S^{m-n}) \wedge X_n \longrightarrow X_m \ , \quad [\tau \wedge z] \wedge x \ \longmapsto \ \tau_*(\sigma^{m-n}(x \wedge z)) \ .$$

In the orthogonal context the same formula works, but then τ is an element of $O(n)$. It is straightforward to check that these assignments indeed define enriched functors, and that it extends to morphisms.

Proposition 7.42. *The assignments*

$$e : \mathbf{T}^{\mathbf{N}} \ \longrightarrow \ \mathcal{S}p^{\mathbf{N}}_{\mathbf{T}} \ , \quad e : \mathbf{T}^{\mathbf{\Sigma}} \ \longrightarrow \ \mathcal{S}p_{\mathbf{T}} \quad \text{and} \quad e : \mathbf{T}^{\mathbf{O}} \ \longrightarrow \ \mathcal{S}p^{\mathbf{O}}$$

and their analogues in the simplicial context

$$e : \mathbf{sS}^{\mathbf{N}} \ \longrightarrow \ \mathcal{S}p^{\mathbf{N}}_{\mathbf{sS}} \ , \quad e : \mathbf{sS}^{\mathbf{\Sigma}} \ \longrightarrow \ \mathcal{S}p_{\mathbf{sS}} \ ,$$

are isomorphisms of categories.

Proof. We treat only one case in detail, namely the one of symmetric spectra of spaces. All other cases are very similar, and we omit the details. To show that $e : \mathbf{T}^{\mathbf{\Sigma}} \longrightarrow \mathcal{S}p_{\mathbf{T}}$ is an isomorphism, we simply define the inverse isomorphism. Given an enriched functor $Y : \mathbf{\Sigma} \longrightarrow \mathbf{T}$ we define a symmetric spectrum uY by $(uY)_n = Y(n)$ with Σ_n-action given by the composite

$$\Sigma_n \times Y(n) \longrightarrow \mathbf{\Sigma}(n,n) \wedge Y(n) \ \xrightarrow{\circ} \ Y(n) \ .$$

[this is not a left! action] The structure map $\sigma_n : Y(n) \wedge S^1 \longrightarrow Y(n+1)$ is the adjoint of the composite

$$Y(n) \wedge S^1 \ \cong \ S^1 \wedge Y(n) \xrightarrow{1 \wedge -} (\Sigma_{n+1}^+ \wedge_{1 \times \Sigma_1} S^1) \wedge Y(n) = \mathbf{\Sigma}(n,n+1) \wedge Y(n) \ \xrightarrow{\circ} \ Y(n+1) \ .$$

[σ_n is Σ_n-equivariant, hence so are all iterated structure maps] With this definition, the iterated structure map $\sigma^m : Y(n) \wedge S^m \longrightarrow Y(n+m)$ comes out as the adjoint of the composite

$$S^m \ \xrightarrow{1 \wedge -} \ \Sigma_{n+m}^+ \wedge_{1 \times \Sigma_m} S^m = \mathbf{\Sigma}(n,n+m) \ \xrightarrow{\circ} \ \mathbf{T}(Y(n), Y(n+m))$$

(this uses the functoriality of Y). The map $1 \wedge - : S^m \longrightarrow \mathbf{\Sigma}(n,n+m)$ is Σ_m-equivariant with respect to the permutation action on S^m and the action on $\mathbf{\Sigma}(n,n+m)$ by $\gamma \cdot [\tau \wedge z] = [(1+\gamma)\tau \wedge z]$. Then map $Y : \mathbf{\Sigma}(n,n+m) \longrightarrow \mathbf{T}(Y(n), Y(n+m))$ is Σ_m-equivariant by functoriality of Y. Hence altogether the iterated structure map σ_m is $\Sigma_n \times \Sigma_m$-equivariant. [on morphisms]

It is now straightforward to check that $u(eX) = X$ and $e(uY) = Y$ on objects and morphisms, so we have an isomorphism of categories. □

More generally we have an enriched category $\mathbf{\Sigma}_R$ for every symmetric ring spectrum R such that enriched functors $\mathbf{\Sigma}_R \longrightarrow \mathbf{T}$ are R-modules. We leave the details to Exercise E.I.50.

7.7. Change of index category. For every object j of a $\mathbf{T}$-category $\mathbf{J}$ there is a enriched representable functor F_j defined by $F_j = \mathbf{J}(j,-)$ which action of $\mathbf{J}$ given by composition. It comes with a special element $\mathrm{Id}_j \in F_j(j)$. The classical Yoneda lemma about natural transformations out of representable functors has an enriched analog: [mapping spaces/objects]

Proposition 7.43 (Enriched Yoneda lemma)**.** *For every $\mathbf{T}$-category $\mathbf{J}$, every object j of $\mathbf{J}$ and every enriched functor $Y : \mathbf{J} \longrightarrow \mathbf{T}$ the assignment*

$$\mathrm{map}_{\mathbf{T}^{\mathbf{J}}}(F_j, Y) \longrightarrow Y(j) , \quad \tau \mapsto \tau_j(\mathrm{Id}_j)$$

is a homeomorphism.

Corollary 7.44. *For every every object $j \in \mathbf{J}$ the functor $\mathbf{T} \longrightarrow \mathbf{T}^{\mathbf{J}}$ which sends a based space K to the enriched functor $K \wedge F_j$ is left adjoint to evaluation at j.*

For every enriched functor Y and every object i of $\mathbf{J}$ there is map $\kappa_i : F_i \wedge Y(i) \longrightarrow Y$ which is adjoint [...]. Thus the value of κ_i at an object j is given by the map $\mathbf{J}(i,j) \wedge F_i \longrightarrow Y_j$ which is adjoint to the structure map $Y : \mathbf{J}(i,j) \longrightarrow \mathbf{T}(Y(i), Y(j))$. As i varies the squares

commute, which means that the maps κ_i assemble into a natural map $F_\bullet \wedge_{\mathbf{J}} Y_\bullet \longrightarrow Y$ from the coend.

Corollary 7.45. *For every enriched functor Y the natural map*

$$\kappa \ : \ F_\bullet \wedge_{\mathbf{J}} Y_\bullet = \text{coequalizer}\left(\bigvee_{j,k} F_k \wedge \mathbf{J}(j,k) \wedge Y(j) \overset{F\wedge \mathrm{Id}}{\underset{\mathrm{Id}\wedge Y}{\rightrightarrows}} \bigvee_i F_i \wedge Y(i) \right) \longrightarrow Y$$

is an isomorphism of continuous functors.

This really means that for every object $j \in \mathbf{J}$ the map

$$\mathbf{J}(-,j) \wedge_{\mathbf{J}} Y \longrightarrow Y(j)$$

is a homeomorphism. Also, for every $G : \mathbf{J}^{op} \longrightarrow \mathbf{T}$ and every object $j \in \mathbf{J}$ the map

$$G \wedge_{\mathbf{J}} F_j \longrightarrow G^j$$

is a homeomorphism [follows from previous by replacing $\mathbf{J}$ by $\mathbf{J}^{op}$]

[Limits, colimits, ends, coends, realization objectwise]

Any enriched functor $\alpha : \mathbf{J} \longrightarrow \mathbf{J}'$ of indexing $\mathbf{T}$-categories induces a restriction functor

$$\alpha^* \ : \ \mathbf{T}^{\mathbf{J}'} \longrightarrow \mathbf{T}^{\mathbf{J}} , \ Y \mapsto Y \circ \alpha$$

by precomposition with α. The forgetful functors in diagram (7.1) arise in this way from the following continuous functors of index categories:

$$\mathbf{N} \longrightarrow \mathbf{\Sigma} \longrightarrow \mathbf{O} \xrightarrow{\mathbb{S}} \mathbf{T} .$$

The first two functors are the identity on objects; on morphisms they are given by changing the groups involved (trivial versus symmetric versus orthogonal) along the injections $\{1\} \longrightarrow \Sigma_n \longrightarrow O(n)$. The third functor is the orthogonal sphere spectrum; it sends the object n of $\mathbf{O}$ to the n-sphere S^n and is given on morphisms by [...]. [how about $\mathbf{U}$?]

We call α *fully faithful* if the map of spaces $\alpha : \mathbf{J}(i,j) \longrightarrow \mathbf{J}'(\alpha(i), \alpha(j))$ is a homeomorphism for all objects i, j of $\mathbf{J}$. Since there are bijective continuous maps which are not homeomorphisms 'fully faithful' as an enriched functor is a stronger condition than demanding that the underlying ordinary functor is fully faithful.

Proposition 7.46. *For every enriched functor $\alpha : \mathbf{J} \longrightarrow \mathbf{J}'$ of $\mathbf{T}$-categories the restriction functor α^* : $\mathbf{T}^{\mathbf{J}'} \longrightarrow \mathbf{T}^{\mathbf{J}}$ has left adjoint α_* and a right adjoint $\alpha_!$ called* left Kan extension *respectively* right Kan extension *along α.*

If α is fully faithful then the unit $X \longrightarrow \alpha^*(\alpha_* X)$ of the adjunction (α_*, α^*) and the counit $\alpha^*(\alpha_! X) \longrightarrow X$ of the adjunction $(\alpha^*, \alpha_!)$ are isomorphisms for every enriched functor $X : \mathbf{J} \longrightarrow \mathbf{T}$.

Proof. We construct the left Kan extension $\alpha_* X$ of an enriched functor $X : \mathbf{J} \longrightarrow \mathbf{T}$ and leave the rest to references. We set

$$\alpha_* X \;=\; F^{\mathbf{J}'}_{\bullet} \wedge_{\mathbf{J}} X \;=\; \text{coequalizer}\left(\bigvee_{j,k} F^{\mathbf{J}'}_k \wedge \mathbf{J}(j,k) \wedge X_j \overset{F\wedge \mathrm{Id}}{\underset{\mathrm{Id}\wedge X}{\rightrightarrows}} \bigvee_i F^{\mathbf{J}'}_i \wedge X_i \right) .$$

□

Since the evaluation functors $\mathrm{ev}_{\mathbb{S}}$: (continuous functors) $\longrightarrow \mathcal{S}p^{\mathbf{O}}$, $\mathrm{ev}_{\mathbb{S}}$: (simplicial functors) $\longrightarrow \mathcal{S}p_{\mathbf{sS}}$ and the forgetful functors $\mathcal{S}p^{O} \longrightarrow \mathcal{S}p_{\mathbf{T}} \longrightarrow \mathcal{S}p^{\mathbf{N}}_{\mathbf{T}}$ and $\mathcal{S}p_{\mathbf{sS}} \longrightarrow \mathcal{S}p^{\mathbf{N}}_{\mathbf{sS}}$ occuring in the diagram (7.1) can be obtained by restriction of enriched functors along morphisms of indexing categories, Proposition 7.46 has the following corollary:

Corollary 7.47. *The evaluation functors*

$$\mathrm{ev}_{\mathbb{S}} : (\text{continuous functors}) \longrightarrow \mathcal{S}p^{\mathbf{O}} \quad \textit{and} \quad \mathrm{ev}_{\mathbb{S}} : (\text{simplicial functors}) \longrightarrow \mathcal{S}p_{\mathbf{sS}}$$

and the forgetful functors

$$\mathcal{S}p^{O} \xrightarrow{U} \mathcal{S}p_{\mathbf{T}} \xrightarrow{U} \mathcal{S}p^{\mathbf{N}}_{\mathbf{T}} \quad \textit{and} \quad \mathcal{S}p_{\mathbf{sS}} \xrightarrow{U} \mathcal{S}p^{\mathbf{N}}_{\mathbf{sS}}$$

have left and right adjoints.

8. Naive versus true homotopy groups

8.1. $\mathcal{M}$-action on homotopy groups. The naive homotopy groups of a symmetric spectrum do not take the action of the symmetric groups into account; this has the effect that there is extra structure on $\hat{\pi}_k$ which we will discuss now. In order to understand the relationship between $\hat{\pi}_*$-isomorphisms and stable equivalences, it is useful to exploit all the algebraic structure available on the naive homotopy groups of a symmetric spectrum. This extra structure is an action of the *injection monoid* $\mathcal{M}$, the monoid of injective self-maps of the set of natural numbers under composition. The $\mathcal{M}$-modules that come up, however, have a special property which we call *tameness*, see Definition 8.1. Tameness has strong algebraic consequences and severely restricts the kinds of $\mathcal{M}$-modules which can arise as homotopy groups of symmetric spectra.

Definition 8.1. The *injection monoid* is the monoid $\mathcal{M}$ of injective self-maps of the set $\omega = \{1, 2, 3, \dots\}$ of natural numbers under composition. An $\mathcal{M}$-*module* is a left module over the monoid ring $\mathbb{Z}\mathcal{M}$. An $\mathcal{M}$-module W *tame* if for every element $x \in W$ there exists a number $n \geq 0$ with the following property: for every element $f \in \mathcal{M}$ which fixes the set $\mathbf{n} = \{1, \dots, n\}$ elementwise we have $fx = x$.

As we shall soon see, the homotopy groups of a symmetric spectrum have a natural tame $\mathcal{M}$-action. An example of an $\mathcal{M}$-module which is not tame is the free module of rank 1. Tameness has many algebraic consequences which we discuss in the next section.

Construction 8.2. We define an action of the injection monoid $\mathcal{M}$ on the naive homotopy groups of a symmetric spectrum X. Let $f \in \mathcal{M}$ be an injective selfmap of ω and suppose that a naive homotopy class $[x] \in \hat{\pi}_k X$ is represented by an unstable homotopy class $x \in \pi_{k+n} X_n$, where we take n large enough so that $k + n \geq 2$. We set $m = \max\{f(1), \dots, f(n)\}$ and choose a permutation $\gamma \in \Sigma_m$ which agrees with f on $\{1, \dots, n\}$. Then we define $f \cdot [x] \in \hat{\pi}_k X$ as the class of

$$\mathrm{sgn}(\gamma) \cdot \gamma_*(\iota^{m-n}(x)) \;\in \pi_{k+m} X_m \;.$$

In other words: we apply the stabilization map (1.7) $(m - n)$ times to the class of x, apply the action of γ induced by its action on X_m and multiply by the sign of γ.

We have to justify that this definition is independent of the choice of the permutation γ and the representative x. Suppose $\gamma' \in \Sigma_m$ is another permutation which agrees with f on $\{1, \dots, n\}$. Then $\gamma^{-1}\gamma'$ is a permutation in Σ_m which fixes the numbers $1, \dots, n$, so it is of the form $\gamma^{-1}\gamma' = 1_n + \tau$ for some $\tau \in \Sigma_{m-n}$, where 1_n is the unit of Σ_n. It suffices to show that for such permutations the induced action on $\pi_{k+m} X_m$ satisfies the relation

$$(1_n + \tau)_*(\iota^{m-n}(x)) \;=\; \mathrm{sgn}(\tau) \cdot (\iota^{m-n}(x)) \tag{8.3}$$

for all $x \in \pi_{k+n}X_n$. To justify this we let $\alpha : S^{k+n} \longrightarrow X_n$ represent x. Since the iterated structure map $\sigma^{m-n} : X_n \wedge S^{m-n} \longrightarrow X_m$ is $\Sigma_n \times \Sigma_{m-n}$-equivariant, we have a commutative diagram:

$$\begin{tikzcd} S^{k+m} \arrow[r, "\alpha\wedge\mathrm{Id}"] \arrow[d, "\mathrm{Id}\wedge\tau"'] & X_n \wedge S^{m-n} \arrow[r, "\sigma^{m-n}"] \arrow[d, "\mathrm{Id}\wedge\tau"] & X_m \arrow[d, "1_n+\tau"] \\ S^{k+m} \arrow[r, "\alpha\wedge\mathrm{Id}"'] & X_n \wedge S^{m-n} \arrow[r, "\sigma^{m-n}"'] & X_m \end{tikzcd}$$

The composite through the upper right corner represents $(1_n + \tau)_*(\iota^{m-n}(x))$. Since the effect on homotopy groups of precomposing with a coordinate permutation of the sphere is multiplication by the sign of the permutation, the composite through the lower left corner represents $\mathrm{sgn}(\tau) \cdot (\iota^{m-n}(x))$. This proves formula (8.3) and shows that the definition of $f \cdot [x]$ is independent of the permutation chosen permutation.

Now we replace the representative x by $\iota(x) \in \pi_{k+n+1}X_{n+1}$ and show that we end up with the same naive homotopy class. If m and $\gamma \in \Sigma_m$ are chosen as above, then $l = \max\{f(1), \dots, f(n), f(n+1)\} \geq m$. We define a permutation in Σ_l as $\bar\gamma = (\gamma + 1_{l-m})(1_n + \tau)$, where $\tau \in \Sigma_{l-n}$ is chosen so that $(1_n + \tau)$ interchanges $n + 1$ and $(\gamma + 1_{l-m})^{-1}(f(n + 1))$ and fixes all other elements. This $\bar\gamma$ then agrees with f on $\{1, \dots, n, n + 1\}$, so we may use it for the representative $\iota(x)$; then we get

$$\begin{aligned} \mathrm{sgn}(\bar\gamma) \cdot \bar\gamma_*(\iota^{l-(n+1)}(\iota(x))) \;&=\; \mathrm{sgn}(\gamma)\,\mathrm{sgn}(\tau) \cdot (\gamma + 1_{l-m})_*(1_n + \tau)_*(\iota^{l-n}(x)) \\ &=\; \mathrm{sgn}(\gamma) \cdot (\gamma + 1_{l-m})_*(\iota^{l-n}(x)) \;=\; \iota^{l-m}\left(\mathrm{sgn}(\gamma) \cdot \gamma_*(\iota^{m-n}(x))\right) \end{aligned}$$

in $\pi_{k+l}X_l$, where the second equation is an instance of (8.3). Thus $\mathrm{sgn}(\bar\gamma) \cdot \bar\gamma_*(\iota^{l-(n+1)}(\iota(x)))$ and $\mathrm{sgn}(\gamma) \cdot \gamma_*(\iota^{m-n}(x))$ represent the same class in $\hat\pi_k X$. This shows that the action of an injection $f \in \mathcal{M}$ on $\hat\pi_k$ is independent of all choices. [action is associative, unital, additive]

A straightforward but important observation is:

Proposition 8.4. *The action of the injection monoid $\mathcal{M}$ on the naive homotopy groups $\hat\pi_k X$ of a symmetric spectrum X is tame.*

Proof. We can argue directly from the definition: every element $[x] \in \hat\pi_k X$ in the colimit is represented by a class $x \in \pi_{k+n}X_n$ for some $n \geq 0$; then for every element $f \in \mathcal{M}$ which fixes the numbers $1, \dots, n$, we have $f \cdot [x] = [x]$. $\square$

In Exercise E.I.59 we show that the injection monoid $\mathcal{M}$ gives essentially all natural operations on the homotopy groups of symmetric spectra. More precisely, we identify the ring of natural operations $\hat\pi_0 X \longrightarrow \hat\pi_0 X$ with a completion of the monoid ring of $\mathcal{M}$, so that an arbitrary operation is a sum, possibly infinite, of operations by elements from $\mathcal{M}$.

Example 8.5. To illustrate the action of the injection monoid $\mathcal{M}$ on the homotopy groups of a symmetric spectrum X we make it explicit for the injection $d : \omega \longrightarrow \omega$ given by $d(i) = 1 + i$, which we refer to as the *shift operator*. For every $n \geq 1$, the map d and the cycle $(1, 2, \dots, n, n + 1)$ of Σ_{n+1} agree on $\{1, \dots, n\}$, so d acts on $\hat\pi_k X$ as the colimit of the system

$$\begin{tikzcd} \pi_k X_0 \arrow[r, "\iota"] \arrow[d, "\iota"'] & \pi_{k+1}X_1 \arrow[r, "\iota"] \arrow[d, "-(1{,}2)\cdot\iota"] & \pi_{k+2}X_2 \arrow[r, "\iota"] \arrow[d, "(1{,}2{,}3)\cdot\iota"] & \cdots \arrow[r, "\iota"] & \pi_{k+n}X_n \arrow[r, "\iota"] \arrow[d, "(-1)^n(1{,}2{,}\dots{,}n{,}n+1)\cdot\iota"] & {} \\ \pi_{k+1}X_1 \arrow[r, "\iota"'] & \pi_{k+2}X_2 \arrow[r, "\iota"'] & \pi_{k+3}X_3 \arrow[r, "\iota"'] & \cdots \arrow[r, "\iota"'] & \pi_{k+n+1}X_{n+1} \arrow[r, "\iota"'] & {} \end{tikzcd}$$

(at least for $k \geq 0$; for negative values of k only a later portion of the system makes sense). The action of the shift operator d on a tame $\mathcal{M}$-module has strong consequences for the whole action of $\mathcal{M}$ (compare Lemma 8.8 (iii)).

The next proposition explains how the shift operator d is realized 'geometrically' by the natural morphisms $\lambda_X : S^1 \wedge X \longrightarrow \mathrm{sh}\, X$ and $\tilde\lambda_X : X \longrightarrow \Omega\, \mathrm{sh}\, X$.

Lemma 8.6. *For every symmetric spectrum X the effect on naive homotopy groups of the morphism $\lambda_X : S^1 \wedge X \longrightarrow \operatorname{sh} X$ coincides with the action of the shift operator $d \in \mathcal{M}$ in the sense that the square*

$$\begin{tikzcd} \hat\pi_k X \arrow[r, "d\cdot"] \arrow[d, "S^1\wedge -"', "\cong"] & \operatorname{sh}(\hat\pi_k X) \arrow[d, equal] \\ \hat\pi_{1+k}(S^1 \wedge X) \arrow[r, "\hat\pi_{1+k}(\lambda_X)"'] & \hat\pi_{1+k}(\operatorname{sh} X) \end{tikzcd}$$

commutes up to the sign $(-1)^k$. In the context of spaces, or if X is levelwise Kan, the squares

$$\begin{tikzcd} \hat\pi_k X \arrow[r, "d\cdot"] \arrow[d, "\hat\pi_k(\tilde\lambda_X)"'] & \operatorname{sh}(\hat\pi_k X) \arrow[d, equal] \\ \hat\pi_k(\Omega \operatorname{sh} X) \arrow[r, "\cong", "\alpha"'] & \hat\pi_{1+k}(\operatorname{sh} X) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \hat\pi_k(\Omega^n \operatorname{sh}^n X) \arrow[r, "\alpha^n", "\cong"'] \arrow[d, "\hat\pi_k(\Omega^n \tilde\lambda_{\operatorname{sh}^n X})"'] & \operatorname{sh}^n(\hat\pi_k X) \arrow[d, "\operatorname{sh}^n d\cdot"] \\ \hat\pi_k(\Omega^{n+1} \operatorname{sh}^{n+1} X) \arrow[r, "\cong", "\alpha^{n+1}"'] & \operatorname{sh}^{n+1}(\hat\pi_k X) \end{tikzcd}$$

also commute up to the sign $(-1)^k$.

Proof. The level n component $\lambda_n : S^1 \wedge X_n \longrightarrow (\operatorname{sh} X)_n = X_{1+n}$ is the composite

$$S^1 \wedge X_n \xrightarrow[\text{twist}]{\cong} X_n \wedge S^1 \xrightarrow{\sigma_n} X_{n+1} \xrightarrow{(1,\dots,n,n+1)} X_{1+n} \ .$$

So the square

$$\begin{tikzcd} \pi_{k+n} X_n \arrow[r, "\iota_*"] \arrow[d, "S^1\wedge -"'] & \pi_{k+n+1} X_{n+1} \arrow[d, "(1,\dots,n,n+1)_*"] \\ \pi_{1+k+n}(S^1 \wedge X_n) \arrow[r, "\pi_{1+k+n}(\lambda_X)"'] & \pi_{1+k+n} X_{1+n} \end{tikzcd}$$

does not in general commute since both ways around the square differ by the coordinate permutation of S^{1+k+n} which moves the first coordinate past the other ones. So the square commutes up to the sign of this permutation, which is $(-1)^{k+n}$. As n increases, the maps $(-1)^n(1,\dots,n,n+1)_* \circ \iota_* : \pi_{k+n} X_n \longrightarrow \pi_{k+1+n} X_{1+n}$ stabilize to the left multiplication of the shift operator d, see Example 8.5, which completes the proof for the first square.

For the second square we exploit the adjunction relations between loop and suspension. More precisely, we have the relation $\lambda_X = \epsilon \circ (S^1 \wedge \tilde\lambda_X)$ where $\epsilon : S^1 \wedge \Omega X \longrightarrow X$ is the adjunction counit. We deduce

$$\begin{aligned} \hat\pi_{1+k}(\lambda_x) \circ (S^1 \wedge -) &= \hat\pi_{1+k}(\epsilon) \circ \hat\pi_{1+k}(S^1 \wedge \tilde\lambda_X) \circ (S^1 \wedge -) \\ &= \hat\pi_{1+k}(\epsilon) \circ (S^1 \wedge -) \circ \hat\pi_k(\tilde\lambda_X) \ = \ \alpha \circ \hat\pi_k(\tilde\lambda_X) \end{aligned}$$

as maps from $\hat\pi_k X$ to $\hat\pi_{1+k}(\operatorname{sh} X)$. The second and third equalities are naturality of the suspension isomorphism respectively Proposition 2.6. Since the first square commutes (up to the specified sign), so does the second.

The last square can be broken up into three parts as follows:

$$\begin{tikzcd} \operatorname{sh}^n(\hat\pi_k X) \arrow[rr, equal] \arrow[d, "\operatorname{sh}^n d\cdot"'] & & \hat\pi_{k+n}(\operatorname{sh}^n X) \arrow[dl, "d\cdot"'] \arrow[dr, "\hat\pi_{k+n}(\tilde\lambda_{\operatorname{sh}^n X})"] & & \hat\pi_k(\Omega^n \operatorname{sh}^n X) \arrow[ll, "\alpha^n"', "\cong"] \arrow[d, "\hat\pi_k(\Omega^n \tilde\lambda_{\operatorname{sh}^n X})"] \\ \operatorname{sh}^{n+1}(\hat\pi_k X) \arrow[r, equal] & \hat\pi_{k+n+1}(\operatorname{sh}^{n+1} X) & & \hat\pi_{k+n}(\Omega \operatorname{sh}^{n+1} X) \arrow[ll, "\cong"', "\alpha"] & \hat\pi_k(\Omega^{n+1} \operatorname{sh}^{n+1} X) \arrow[l, "\cong"', "\alpha^n"] \end{tikzcd}$$

The left and right parts commute by the naturality of the equality $\hat\pi_{k+1}(\operatorname{sh} A) = \operatorname{sh}(\hat\pi_k A)$ respectively the loop isomorphism; the middle triangle commutes up to the sign $(-1)^k$ by the previous paragraph (for X replaced by $\operatorname{sh}^n X$). $\square$

8.2. Algebraic properties of tame $\mathcal{M}$-modules. In this section we discuss some algebraic properties of tame $\mathcal{M}$-modules. It turns out that tameness is a rather restrictive condition. We introduce some useful notation and terminology. For an injective map $f : \omega \longrightarrow \omega$ we write $|f|$ for the smallest number $i \geq 0$ such that $f(i+1) \neq i+1$. So in particular, f restricts to the identity on $\{1, \dots, |f|\}$. An element x of an $\mathcal{M}$-module W has *filtration* n if for every $f \in \mathcal{M}$ with $|f| \geq n$ we have $fx = x$. We denote by $W^{(n)}$ the subgroup of W of elements of filtration n; for example, $W^{(0)}$ is the set of elements fixed by all $f \in \mathcal{M}$. We say that x has *filtration exactly* n if it lies in $W^{(n)}$ but not in $W^{(n-1)}$. By definition, an $\mathcal{M}$-module W is tame if and only if every element has a finite filtration, i.e., if the groups $W^{(n)}$ exhaust W.

The following lemmas collect some elementary observations, first for arbitrary $\mathcal{M}$-modules and then for tame $\mathcal{M}$-modules.

Lemma 8.7. *Let W be any $\mathcal{M}$-module.*

(i) *If two elements f and g of $\mathcal{M}$ coincide on $\mathbf{n} = \{1, \dots, n\}$, then $fx = gx$ for all $x \in W$ of filtration n.*
(ii) *For $n \geq 0$ and $f \in \mathcal{M}$ set $m = \max\{f(\mathbf{n})\}$. Then $f \cdot W^{(n)} \subseteq W^{(m)}$.*
(iii) *If $x \in W$ has filtration exactly n with $n \geq 1$, then dx has filtration exactly $n+1$, where $d \in \mathcal{M}$ is the shift operator.*
(iv) *Every homomorphism from the injection monoid $\mathcal{M}$ to a group is trivial.*
(v) *Let $V \subseteq W$ be an $\mathcal{M}$-submodule such that the action of $\mathcal{M}$ on V and W/V is trivial. Then the action of $\mathcal{M}$ on W is also trivial.*

Proof. (i) We can choose a bijection $\gamma \in \mathcal{M}$ which agrees with f and g on $\mathbf{n}$, and then $\gamma^{-1}f$ and $\gamma^{-1}g$ fix $\mathbf{n}$ elementwise. So for x of filtration n we have $(\gamma^{-1}f)x = x = (\gamma^{-1}g)x$. Multiplying by γ gives $fx = gx$.

(ii) If $g \in \mathcal{M}$ satisfies $|g| \geq m$, then gf and f agree on $\mathbf{n}$. So for all $x \in W^{(n)}$ we have $gfx = fx$ by (i), which proves that $fx \in W^{(m)}$.

(iii) We have $d \cdot W^{(n)} \subseteq W^{(n+1)}$ by part (ii). To prove that d increases the exact filtration we consider $x \in W^{(n)}$ with $n \geq 1$ and show that $dx \in W^{(n)}$ implies $x \in W^{(n-1)}$.

For $f \in \mathcal{M}$ with $|f| = n-1$ we define $g \in \mathcal{M}$ by $g(1) = 1$ and $g(i) = f(i-1)+1$ for $i \geq 2$. Then we have $gd = df$ and $|g| = n$. We let h be the cycle $h = (f(n)+1, f(n), \dots, 2, 1)$ so that we have $|hd| = f(n) = \max\{f(\mathbf{n})\}$. Then $fx \in W^{(f(n))}$ by part (ii) and so

$$fx \;=\; (hd)(fx) \;=\; h(g(dx)) \;=\; (hd)x \;=\; x\ .$$

Altogether this proves that $x \in W^{(n-1)}$.

(iv) Given any injection $f \in \mathcal{M}$ we define another injection $g \in \mathcal{M}$ by

$$g(i) \;=\; \begin{cases} i & \text{if } i \text{ is odd, and} \\ 2 \cdot g(i/2) & \text{if } i \text{ is even.} \end{cases}$$

Let $s, t \in \mathcal{M}$ be given by $t(i) = 2i$ respectively $s(i) = 2i-1$; then the relations $gt = tf$ and $gs = s$ hold in $\mathcal{M}$.

Let $\varphi : \mathcal{M} \longrightarrow G$ a homomorphism of monoids whose target G is a group. Then we have $\varphi(s) = \varphi(gs) = \varphi(g)\varphi(s)$, so $\varphi(g) = 1$. Morover, $\varphi(t) = \varphi(g)\varphi(t) = \varphi(gt) = \varphi(tf) = \varphi(t)\varphi(f)$, so $\varphi(f) = 1$. Hence φ is the trivial homomorphism.

(v) Since the $\mathcal{M}$-action is trivial on V and W/V, every $f \in \mathcal{M}$ determines an additive map $\delta_f : W/V \longrightarrow V$ such that $x - fx = \delta_f(x+V)$ for all $x \in W$. These maps satisfy $\delta_{fg}(x) = \delta_f(x) + \delta_g(x)$ and so $f \mapsto \delta_f$ is a homomorphism from the injection monoid to the abelian group of additive maps from W/V to V. By part (iv) such a homomorphism is trivial, so $\delta_f = 0$ for all $f \in \mathcal{M}$, i.e., $\mathcal{M}$ acts trivially on W. □

Lemma 8.8. *Let W be a tame $\mathcal{M}$-module.*

(i) *Every element of $\mathcal{M}$ acts injectively on W.*
(ii) *If the filtration of elements of W is bounded, then W is a trivial $\mathcal{M}$-module.*
(iii) *If the shift operator d acts surjectively on W, then W is a trivial $\mathcal{M}$-module.*
(iv) *If W is finitely generated as an abelian group, then W is a trivial $\mathcal{M}$-module.*

PROOF. (i) Consider $f \in \mathcal{M}$ and $x \in W^{(n)}$ with $fx = 0$. Since f is injective, we can choose $h \in \mathcal{M}$ with $|hf| \geq n$. Then $x = (hf)x = h(fx) = 0$, so f acts injectively.

(ii) Lemma 8.7 (iii) implies that if $W = W^{(n)}$ for some $n \geq 0$, then $n = 0$, so the $\mathcal{M}$-action is trivial.

(iii) Suppose $\mathcal{M}$ does not act trivially, so that $W^{(0)} \neq W$. Let n be the smallest positive integer such that $W^{(0)} \neq W^{(n)}$. Then by part (iii) of Lemma 8.7, any $x \in W^{(n)} - W^{(0)}$ is not in the image of d, so d does not act surjectively.

(iv) The union of the nested sequence of subgroups $W^{(0)} \subseteq W^{(1)} \subseteq W^{(2)} \subseteq \cdots$ is W. Since finitely generated abelian groups are Noetherian, we have $W^{(n)} = W$ for all large enough n. By part (ii), the monoid $\mathcal{M}$ must act trivially. □

Parts (i), (iii) and (iv) of Lemma 8.8 can fail for non-tame $\mathcal{M}$-modules: we can let $f \in \mathcal{M}$ act on the abelian group $\mathbb{Z}$ as the identity if the image of $f : \omega \longrightarrow \omega$ has finite complement, and we let f acts as 0 if its image has infinite complement.

Example 8.9. We introduce some important tame $\mathcal{M}$-modules $\mathcal{P}_n$ for $n \geq 0$. We denote by $\mathcal{I}_n$ the set of ordered n-tuples of pairwise distinct elements of ω (or equivalently the set of injective maps from $\{1, \dots, n\}$ to ω). The monoid $\mathcal{M}$ acts from the left on this set by componentwise evaluation, i.e., $f(x_1, \dots, x_n) = (f(x_1), \dots, f(x_n))$. We note that $\mathcal{I}_0$ has only one element, the empty tuple; for $n \geq 1$, the set $\mathcal{I}_n$ is countably infinite. The action of $\mathcal{M}$ on the set $\mathcal{I}_n$ is tame: the filtration of a tuple $(x_1, \dots, x_n)$ is the maximum of the components.

The module $\mathcal{P}_n = \mathbb{Z}\mathcal{I}_n$ is the free abelian group with basis the set $\mathcal{I}_n$ of ordered n-tuples of pairwise distinct elements of ω (or equivalently the set of injective maps from $\mathbf{n}$ to ω). The monoid $\mathcal{M}$ acts from the left by additive extension of the action on the basis $\mathcal{I}_n$. Since $\mathcal{I}_0$ has only one element, $\mathcal{P}_0$ is isomorphic to $\mathbb{Z}$ with trivial $\mathcal{M}$-action. For $n \geq 1$, the basis is countably infinite and the $\mathcal{M}$-action is non-trivial. The module $\mathcal{P}_n$ is tame: the filtration of a basis element $(x_1, \dots, x_n)$ is the maximum of the components. So the filtration subgroup $\mathcal{P}_n^{(m)}$ is generated by the n-tuples all of whose components are less than or equal to m. An equivalent way of saying this is that $\mathcal{P}_n^{(m)}$ is the free abelian group generated by all injections from $\mathbf{n}$ to $\mathbf{m}$; in particular, $\mathcal{P}_n^{(m)}$ is trivial for $m < n$.

The module $\mathcal{P}_n$ represents the functor of taking elements of filtration n: for every $\mathcal{M}$-module W, the map

$$\mathrm{Hom}_{\mathcal{M}\text{-mod}}(\mathcal{P}_n, W) \longrightarrow W^{(n)} , \quad \varphi \mapsto \varphi(1, \dots, n)$$

is bijective.

8.3. Examples. We discuss several classes of symmetric spectra with a view towards the $\mathcal{M}$-action on the stable homotopy groups.

Example 8.10 (Eilenberg-Mac Lane spectra)**.** Every tame $\mathcal{M}$-module W can be realized as the homotopy group of a symmetric spectrum. For this purpose we modify the construction of the symmetric Eilenberg-Mac Lane spectrum of an abelian group. We define a symmetric spectrum HW of simplicial sets by

$$(HW)_n \ = \ W^{(n)} \otimes \mathbb{Z}[S^n] \ ,$$

where $W^{(n)}$ is the filtration n subgroup of W and $\mathbb{Z}[S^n]$ refers to the simplicial abelian group freely generated by the simplicial set $S^n = S^1 \wedge \ldots \wedge S^1$, divided by the subgroup generated by the basepoint. The symmetric group Σ_n takes $W^{(n)}$ to itself and we let it act diagonally on $(HW)_n$, i.e., on S^n by permuting the smash factors. If $\mathcal{M}$ acts trivially on W, then this is just the ordinary Eilenberg-Mac Lane spectrum introduced in Example 1.14. Note that HW is an Ω-spectrum if and only if the $\mathcal{M}$-action on W is trivial.

Since $(HW)_n$ is an Eilenberg-Mac Lane space of type $(W^{(n)}, n)$ the homotopy groups of the symmetric spectrum HW are concentrated in dimension zero where we have $\hat{\pi}_0 HW \cong \bigcup_{n \geq 0} W^{(n)} = W$ as $\mathcal{M}$-modules.

Example 8.11 (Loop and suspension)**.** The loop ΩX and suspension $S^1 \wedge X$ of a symmetric spectrum X are defined by applying the functors Ω respectively $S^1 \wedge -$ levelwise, where the structure maps do not interact with the new loop or suspension coordinates, compare Section 2.1. We already saw in Proposition 2.6

that loop and suspension simply shift the homotopy groups, and we shall now prove that the $\mathcal{M}$-action is unchanged in this process.

For every symmetric spectrum X the map $S^1 \wedge - : \pi_{k+n}X_n \longrightarrow \pi_{1+k+n}(S^1 \wedge X_n)$ is Σ_n-equivariant and compatible with the stabilization maps as n increases. So the induced map $S^1 \wedge - : \hat{\pi}_k X \longrightarrow \hat{\pi}_{1+k}(S^1 \wedge X)$ on colimits is $\mathcal{M}$-linear, and hence, by Proposition 2.6 an isomorphism of $\mathcal{M}$-modules. Also by Proposition 2.6 the loop isomorphism $\alpha : \hat{\pi}_k(\Omega X) \longrightarrow \hat{\pi}_{1+k}X$ is the composite of the suspension isomorphism $S^1 \wedge - : \hat{\pi}_k(\Omega X) \longrightarrow \hat{\pi}_{1+k}(S^1 \wedge (\Omega X))$, which is $\mathcal{M}$-linear by the above, and the map induced by the adjunction counit $\epsilon : S^1 \wedge (\Omega X) \longrightarrow X$, which is $\mathcal{M}$-linear. Hence the loop isomorphism α is $\mathcal{M}$-linear.

Example 8.12 (Shift)**.** The shift is another construction for symmetric spectra which reindexes the homotopy groups, but unlike the suspension, this construction changes the $\mathcal{M}$-action in a systematic way. The shift of a symmetric spectrum X was defined in Example 3.9 by $(\mathrm{sh}\, X)_n = X_{1+n}$ with action of Σ_n via the monomorphism $(1 + -) : \Sigma_n \longrightarrow \Sigma_{1+n}$. The structure maps of $\mathrm{sh}\, X$ are the reindexed structure maps for X.

If we view Σ_n as the subgroup of $\mathcal{M}$ of maps which fix all numbers bigger than n, then the homomorphism $(1 + -) : \Sigma_n \longrightarrow \Sigma_{1+n}$ has a natural extension to a *shift homomorphism* $\mathrm{sh} : \mathcal{M} \longrightarrow \mathcal{M}$ given by $(\mathrm{sh}\, f)(1) = 1$ and $(\mathrm{sh}\, f)(1+i) = 1 + f(i)$ for $i \geq 1$. The image of the shift homomorphism is the submonoid of those $g \in \mathcal{M}$ with $g(1) = 1$. If W is an $\mathcal{M}$-module, we denote by $\mathrm{sh}\, W$ the *shift* of W, the $\mathcal{M}$-module with the same underlying abelian group, but with $\mathcal{M}$-action by restriction along the shift homomorphism. In other words, we set

$$f \cdot \mathrm{sh}\, x \;=\; \mathrm{sh}((\mathrm{sh}\, f) \cdot x)\ .$$

Here we denote by $\mathrm{sh}\, x$ an element x of W when we think of it as an element of $\mathrm{sh}\, W$. Since $|\,\mathrm{sh}\, f| = 1 + |f|$, shifting an $\mathcal{M}$-module shifts the filtration subgroups, i.e., we have $(\mathrm{sh}\, W)^{(n)} = W^{(1+n)}$ for all $n \geq 0$. Thus the $\mathcal{M}$-module $\mathrm{sh}\, W$ is tame if and only if W is.

For any symmetric spectrum X, integer k and large enough n we have

$$\pi_{(k+1)+n}(\mathrm{sh}\, X)_n \;=\; \pi_{k+(1+n)}X_{1+n}\ ,$$

and the maps in the colimit system for $\hat{\pi}_{k+1}(\mathrm{sh}\, X)$ are the same as the maps in the colimit system for $\hat{\pi}_k X$. Thus we get $\hat{\pi}_{k+1}(\mathrm{sh}\, X) = \hat{\pi}_k X$ as abelian groups. However, the action of a permutation on $\pi_{k+1+n}(\mathrm{sh}\, X)_n$ is shifted by the homomorphism $1 + -$, so we have

$$\hat{\pi}_{k+1}(\mathrm{sh}\, X) \;=\; \mathrm{sh}(\hat{\pi}_k X) \tag{8.13}$$

as $\mathcal{M}$-modules.

If we iterate the shift construction, there are symmetries around that we want to keep track of. Indeed, on the m-fold shift $\mathrm{sh}^m W$ of an $\mathcal{M}$-module W there is also a Σ_m-action left via the 'inclusion' of Σ_m into $\mathcal{M}$ by extension by the indentity. However, in order to make the iteration of the formula (8.13) equivariant, we twisted this Σ_m-action by the sign action. So in other words, we let $\mathrm{sh}^m W$ denote same underlying abelian group as W, but with $\Sigma_m \times \mathcal{M}$-action by

$$(\sigma, f) \cdot \mathrm{sh}^m x \;=\; \mathrm{sgn}(\sigma) \cdot \mathrm{sh}^m((\sigma + f) \cdot x)\ .$$

Here we denote by $\mathrm{sh}^m x$ an element x of W when we think of it as an element of $\mathrm{sh}^m W$. Another way to rephrase this is to say that $\mathrm{sh}^m W$ equals the restriction of scalars of W along the ring homomorphism

$$\mathbb{Z}[\Sigma_m \times \mathcal{M}] \;\longrightarrow\; \mathbb{Z}\mathcal{M}$$

given on the preferred basis $\Sigma_m \times \mathcal{M}$ by $(\sigma, f) \mapsto \mathrm{sgn}(\sigma) \cdot (\sigma + f)$. We have $1_m + f = \mathrm{sh}^m f$, sos far as the $\mathcal{M}$-action is concerned, $\mathrm{sh}^m W$ is simply the m-fold shift of W.

Now we recall that for any symmetric spectrum X the iterated shift $\mathrm{sh}^m X$ has a left Σ_m-action on the shifted coordinates. We claim that the relation

$$\hat{\pi}_{k+m}(\mathrm{sh}^m X) \;=\; \mathrm{sh}^m(\hat{\pi}_k X) \tag{8.14}$$

holds as $\Sigma_m \times \mathcal{M}$-modules. Indeed, as far as the $\mathcal{M}$ action is concerned, this is simply the m-fold iteration of (8.13). For the Σ_m-action we recall construction 8.2 of the $\mathcal{M}$-action: If $[x] \in \pi_k X$ is represented by $[x] \in \pi_{k+n} X_n$, we can assume that $n \geq m$. Then, by definition,

$$\sigma \cdot [x] \;=\; [\operatorname{sgn}(\gamma) \cdot (\gamma + 1_{m-n})_*(x)] \;.$$

The sign $\operatorname{sgn}(\gamma + 1_{n-m})$ which appear on the right hand side is compensated by the sign that we built into the Σ_m-action on $\operatorname{sh}^m W$; hence (8.14) holds $\Sigma_m \times \mathcal{M}$-equivariantly.

Example 8.15 (Twisted smash products)**.** So we consider a symmetric spectrum X and a based Σ_m-space (or any pointed Σ_m-simplicial set) L, for some $m \geq 0$, and describe the homotopy groups of a twisted smash product $L \triangleright_m X$ (see Example 3.27) as a functor of the homotopy groups of $L \wedge X$, using all available structure. This is essentially a reinterpretation of the additive isomorphism (3.30) paying close attention to the action on the injection monoid. Since $L \triangleright_m X$ is isomorphic to $G_m L \wedge X$ this gives a description of the naive homotopy groups of smash products with semifree spectra. Since free and semifree symmetric spectra are special cases of twisted smash products, this will specialize to formulas for the naive homotopy groups of free and semifree symmetric spectra.

We start from the Σ_m-equivariant morphism $\eta_{L,X} \,:\, L \wedge X \longrightarrow \operatorname{sh}^m(L \triangleright_m X)$ of symmetric spectra which was defined in (3.28) in level n as the map

$$[1 \wedge -] \;:\; L \wedge X_n \;\longrightarrow\; \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge X_n = (L \triangleright_m X)_{m+n} = \operatorname{sh}^m((L \triangleright_m X))_n \;.$$

The induced map on naive homotopy groups

$$(\eta_{L,X})_* \;:\; \hat\pi_{k+m}(L \wedge X) \;\longrightarrow\; \hat\pi_{k+m}(\operatorname{sh}^m(L \triangleright_m X)) \;=\; \operatorname{sh}^m(\hat\pi_k(L \triangleright_m X))$$

is $\Sigma_m \times \mathcal{M}$-linear, where the target is the m-fold algebraic shift, compare (8.12), of the $\mathcal{M}$-module $\hat\pi_k(L \triangleright_m X)$ that we want to calculate. The $\Sigma_m \times \mathcal{M}$-action on the target is obtained from the original $\mathcal{M}$-action on $\hat\pi_k(L \triangleright_m X)$ by restriction of scalars along the ring homomorphism $\mathbb{Z}[\Sigma_m \times \mathcal{M}] \longrightarrow \mathbb{Z}\mathcal{M}$ defined on the preferred basis by

$$(\sigma, f) \;\longmapsto\; \operatorname{sgn}(\sigma) \cdot (\sigma + f) \;.$$

We denote by $\mathbb{Z}\mathcal{M}^{\langle m \rangle}$ the monoid ring of $\mathcal{M}$ with its usual left multiplication action, but with a right action by the monoid $\Sigma_m \times \mathcal{M}$ via restriction along the above ring homomorphism $\mathbb{Z}[\Sigma_m \times \mathcal{M}] \longrightarrow \mathbb{Z}\mathcal{M}$. Then the $\Sigma_m \times \mathcal{M}$-linear map $(\eta_{L,X})_*$ is adjoint to the $\mathcal{M}$-linear map

$$\begin{aligned} \widehat{\eta_*} \;:\; \mathbb{Z}\mathcal{M}^{\langle m \rangle} \otimes_{\Sigma_m \times \mathcal{M}} \hat\pi_{k+m}(L \wedge X) \;&\longrightarrow\; \hat\pi_k(L \triangleright_m X) \\ f \otimes x \;&\longmapsto\; f \cdot (\eta_{L,X})_*(x) \;. \end{aligned} \tag{8.16}$$

Proposition 8.17. *For every $m \geq 0$, every based Σ_m-space (or Σ_m-simplicial set) L and every symmetric spectrum X the map* (8.16)

$$\widehat{\eta_*} \;:\; \mathbb{Z}\mathcal{M}^{\langle m \rangle} \otimes_{\Sigma_m \times \mathcal{M}} \hat\pi_{k+m}(L \wedge X) \;\longrightarrow\; \hat\pi_k(L \triangleright_m X)$$

is an isomorphism of $\mathcal{M}$-modules.

We remark that as a right $\Sigma_m \times \mathcal{M}$-module, $\mathbb{Z}\mathcal{M}^{\langle m \rangle}$ is free of countably infinite rank. One possible basis is given by the '(m,∞)-shuffles', i.e., by those bijections $f \in \mathcal{M}$ which satisfy $f(i) < f(i+1)$ for all $i \neq m$. In other words, all bijective f which keep the sets $\mathbf{m} = \{1, \dots, m\}$ and $\{m+1, m+2, \dots\}$ in their natural order. So Proposition 8.17 in particular implies that the underlying abelian group of $\hat\pi_k(L \triangleright_m X)$ is a countably infinite sum of copies of the underlying abelian group of $\hat\pi_{k+m}(L \wedge X)$, i.e., this proposition refines the additive calculation of (3.30).

Proof. We denote by O_m the set of order preserving injections $f : \mathbf{m} \longrightarrow \omega$ and observe that the composite

$$\bigoplus_{f \in O_m} \hat\pi_{k+m}(L \wedge X) \;\longrightarrow\; \mathbb{Z}\mathcal{M}^{\langle m \rangle} \otimes_{\Sigma_m \times \mathcal{M}} \hat\pi_{k+m}(L \wedge X) \;\xrightarrow{\widehat{\eta_*}}\; \hat\pi_k(L \triangleright_m X)$$

is the bijection (3.30), where the first map sends a class $x \in \hat\pi_{k+m}(L \wedge X)$ in the summand indexed by $f \in O_m$ to $\tilde f \otimes x \in \mathbb{Z}\mathcal{M}^{\langle m \rangle} \otimes_{\Sigma_m \times \mathcal{M}} \hat\pi_{k+m}(L \wedge X)$ where $\tilde f \in \mathcal{M}$ is the unique (m,∞)-shuffle which restricts to f on $\mathbf{m}$. Since the (m,∞)-shuffles form a right $\Sigma_m \times \mathcal{M}$-basis of $\mathbb{Z}\mathcal{M}^{\langle m\rangle}$, the first map, and hence the map $\widehat{\eta_*}$, is bijective. □

Example 8.18 (Free and semifree symmetric spectra)**.** We saw in Example 3.20 that the zeroth stable homotopy group of the free symmetric spectrum F_1S^1 is free abelian of countably infinite rank. We now refine this calculation to an isomorphism of $\mathcal{M}$-modules $\hat\pi_0(F_mS^m) \cong \mathcal{P}_m$, see (8.20) below; here $\mathcal{P}_m$ is the tame $\mathcal{M}$-module which represents taking filtration m elements, see Example 8.9. So while the groups $\hat\pi_0(F_mS^m)$ are all additively isomorphic for different positive m, the $\mathcal{M}$-action distinguishes them. In particular, there cannot be a chain of $\hat\pi_*$-isomorphisms between F_mS^m and F_nS^n for $n \neq m$.

The calculation of the $\mathcal{M}$-action on free and semifree symmetric spectra is a special case of the very general formula (8.16) for the naive homotopy groups of a twisted smash product. Let L be a pointed space (or simplicial set) with a left action by the symmetric group Σ_m, for some $m \geq 0$. Recall that G_mL denotes the semifree symmetric spectrum generated by L in level m, defined in Example 3.23, which is also equal to the twisted smash product $L \triangleright_m \mathbb{S}$ of L with the sphere spectrum. The functor G_m is left adjoint to evaluating a symmetric spectrum at level m, viewed as a functor with values in pointed Σ_m-spaces. The Σ_m-equivariant homomorphism $\eta_{L,\mathbb{S}} : \Sigma^\infty L = L \wedge \mathbb{S} \longrightarrow \mathrm{sh}^m(G_mL)$ induces a $\Sigma_m \times \mathcal{M}$-linear map

$$(\eta_{L,X})_* \ : \ \pi^{\mathrm{s}}_{k+m}L = \hat\pi_{k+m}(\Sigma^\infty L) \ \longrightarrow \ \hat\pi_{k+m}(\mathrm{sh}^m(G_mL)) \ = \ \mathrm{sh}^m(\hat\pi_k(G_mL))$$

on naive homotopy groups, where the target is the m-fold algebraic shift, compare (8.12), of the $\mathcal{M}$-module $\hat\pi_k(G_mL)$. Adjoint to this is the morphism of $\mathcal{M}$-modules

$$\widehat{\eta_*} \ : \ \mathbb{Z}\mathcal{M}^{\langle m\rangle} \otimes_{\Sigma_m\times\mathcal{M}} \hat\pi_{k+m}(L\wedge\mathbb{S}) \ \longrightarrow \ \hat\pi_k(G_mL) \ , \quad f\otimes x \ \longmapsto f\cdot(\eta_{L,X})_*(x) \ .$$

which is an isomorphism by Proposition 8.17 for $X = \mathbb{S}$.

Since the $\mathcal{M}$-action on $\pi^{\mathrm{s}}_{k+m}L = \hat\pi_{k+m}(L\wedge\mathbb{S})$ is trivial [justify] we get

$$\mathbb{Z}\mathcal{M}^{\langle m\rangle} \otimes_{\Sigma_m\times\mathcal{M}} \hat\pi_{k+m}(L\wedge\mathbb{S}) \ \cong \ (\mathbb{Z}\mathcal{M}^{\langle m\rangle} \otimes_{1\times\mathcal{M}} \mathbb{Z}) \otimes_{\Sigma_m} \pi^{\mathrm{s}}_{k+m}L \ .$$

The tame $\mathcal{M}$-module $\mathcal{P}_m$ has a compatible right Σ_m-action which is given on the basis by permuting the components of an m-tuple, i.e., $(x_1,\dots,x_m)\gamma = (x_{\gamma(1)},\dots,x_{\gamma(m)})$. The map

$$\mathbb{Z}\mathcal{M}^{\langle m\rangle} \otimes_{1\times\mathcal{M}} \mathbb{Z} \ \longrightarrow \ \mathcal{P}_m \ , \quad f\otimes 1 \ \longmapsto \ (f(1),\dots,f(m))$$

is an isomorphism of $\mathcal{M}$-Σ_m-bimodules; so combining all these isomorphisms we finally get a natural isomorphism of $\mathcal{M}$-modules

$$(8.19) \qquad \mathcal{P}_m \otimes_{\Sigma_m} (\pi^{\mathrm{s}}_{k+m}L)(\mathrm{sgn}) \ \xrightarrow{\cong} \ \hat\pi_k(G_mL) \ , \quad f\otimes x \ \longmapsto \ f\cdot(\eta_{L,X})_*(x) \ .$$

On the left of the tensor symbol, the group Σ_m acts by what is induced on stable homotopy groups by the action on L, twisted by sign.

Free symmetric spectra are special cases of semifree symmetric spectra. For a pointed space K (without any group action) we have $F_mK \cong G_m(\Sigma_m^+\wedge K)$ and $\pi^{\mathrm{s}}_{k+m}(\Sigma_m^+\wedge K) \cong \mathbb{Z}\Sigma_m \otimes \pi^{\mathrm{s}}_{k+m}K$ as Σ_m-modules. So (8.19) specializes to a natural isomorphism of $\mathcal{M}$-modules

$$(8.20) \qquad \mathcal{P}_m \otimes \pi^{\mathrm{s}}_{k+m}K \ \cong \ \hat\pi_k(F_mK) \ .$$

Here $\pi^{\mathrm{s}}_{k+m}K$ is the $(k+m)$th stable homotopy group of K; the monoid $\mathcal{M}$ acts only on $\mathcal{P}_m$.

Example 8.21 (Induction)**.** The shift functor has a left adjoint induction functor $\triangleright$ given by $(\triangleright X)_0 = *$ and

$$(\triangleright X)_{1+n} \ = \ \Sigma^+_{1+n} \wedge_{1\times\Sigma_n} X_n$$

for $n \geq 0$. Here Σ_n acts from the right on Σ_{1+n} via the monomorphism $(1+-) : \Sigma_n \longrightarrow \Sigma_{1+n}$. The structure map $(\Sigma^+_{1+n}\wedge_{\Sigma_n} X_n)\wedge S^1 \longrightarrow \Sigma^+_{1+n+1}\wedge_{\Sigma_{n+1}} X_{n+1}$ is induced by $(-+1) : \Sigma_{1+n} \longrightarrow \Sigma_{1+n+1}$ (the 'inclusion') and the structure map of X.

The effect of induction on naive homotopy groups is given as a special case of the general formula (8.16) for the homotopy groups of a twisted smash product. Indeed, since $\triangleright X = S^0 \triangleright_1 X$, that formula specializes to a natural isomorphism of $\mathcal{M}$-modules

$$\mathbb{Z}\mathcal{M}^{\langle 1\rangle} \otimes_{\mathcal{M}} \hat{\pi}_{k+1} X \;\cong\; \hat{\pi}_k(\triangleright X) \ . \tag{8.22}$$

Here $\mathbb{Z}\mathcal{M}^{\langle 1\rangle}$ denotes the monoid ring of $\mathcal{M}$ with its usual left action, but with right action through the shift homomorphism $\mathrm{sh} : \mathcal{M} \longrightarrow \mathcal{M}$ given by $(\mathrm{sh}\, f)(1) = 1$ and $(\mathrm{sh}\, f)(i) = f(i-1)+1$ for $i \geq 2$. As a right $\mathcal{M}$-module, $\mathbb{Z}\mathcal{M}^{\langle 1\rangle}$ is free of countably infinite rank (one possible basis is given by the transpositions $(1,n)$ for $n \geq 1$). So the isomorphism (8.22) in particular implies that the underlying abelian group of $\hat{\pi}_k(\triangleright X)$ is a countably infinite sum of copies of the underlying abelian group of $\hat{\pi}_{k+1}X$, a fact which we already observed in Example 3.17.

The functor $\mathbb{Z}\mathcal{M}^{\langle 1\rangle} \otimes_{\mathcal{M}} -$ is left adjoint to $\mathrm{Hom}_{\mathbb{Z}\mathcal{M}}(\mathbb{Z}\mathcal{M}^{\langle 1\rangle}, -)$, which is a fancy way of writing the algebraic shift functor $W \mapsto \mathrm{sh}\, W$. Under the isomorphism (8.22) and the identification (8.13), the adjunction between shift and induction as functors of symmetric spectra corresponds exactly to the adjunction between $W \mapsto \mathrm{sh}\, W$ and $\mathbb{Z}\mathcal{M}^{\langle 1\rangle} \otimes_{\mathbb{Z}\mathcal{M}} -$ as functors of tame $\mathcal{M}$-modules.

Example 8.23 (Infinite products)**.** Finite products of symmetric spectra are $\hat{\pi}_*$-isomorphic to finite wedges, so stable homotopy groups commute with finite products. But homotopy groups do not in general commute with infinite products. This should not be surprising because stable homotopy groups involves a sequential colimit, and these generally do not preserve infinite products.

There are even two different ways in which commutation with products can fail. First we note that an infinite product of a family $\{W_i\}_{i\in I}$ of tame $\mathcal{M}$-modules is only tame if almost all the modules W_i have trivial $\mathcal{M}$-action. Indeed, if there are infinitely many W_i with non-trivial $\mathcal{M}$-action, then by Lemma 8.8 (ii) the product $\prod_{i\in I} W_i$ contains tuples of elements whose filtrations are not bounded. We define the *tame product* of the family $\{W_i\}_{i\in I}$ by

$$\prod_{i\in I}^{\text{tame}} W_i \;=\; \bigcup_{n\geq 0} \left(\prod_{i\in I} W_i^{(n)} \right) \ ,$$

which is the largest tame submodule of the product and thus the categorical product in the category of tame $\mathcal{M}$-modules.

Now we consider a family $\{X_i\}_{i\in I}$ of symmetric spectra. Since the monoid $\mathcal{M}$ acts tamely on the homotopy groups of any symmetric spectrum, the natural map from the homotopy groups of the product spectrum to the product of the homotopy groups always lands in the tame product. But in general, this natural map

$$\hat{\pi}_k \left(\prod_{i\in I} X_i \right) \;\longrightarrow\; \prod_{i\in I}^{\text{tame}} \hat{\pi}_k X_i \tag{8.24}$$

need not be an isomorphism.

In Remark 2.20 we exhibited a countable family of symmetric spectra with trivial naive homotopy groups whose product has non-trivial (even infinitely generated) naive homotopy groups. Now we modify that example slightly and consider the symmetric spectra $(F_1S^1)^{\leq i}$ obtained by truncating the free symmetric spectrum F_1S^1 above level i, i.e.,

$$((F_1S^1)^{\leq i})_n \;=\; \begin{cases} (F_1S^1)_n & \text{for } n \leq i, \\ \quad * & \text{for } n \geq i+1 \end{cases}$$

with structure maps as a quotient spectrum of F_1S^1. Then $(F_1S^1)^{\leq i}$ has trivial naive homotopy groups for all i. The 0-th homotopy group of the product $\prod_{i\geq 1}(F_1S^1)^{\leq i}$ is the colimit of the sequence of maps

$$\prod_{i\geq n} \mathcal{P}_1^{(n)} \;\longrightarrow\; \prod_{i\geq n+1} \mathcal{P}_1^{(n+1)}$$

which first projects away from the factor indexed by $i = n$ and then takes a product of inclusions $\mathcal{P}_1^{(n)} \longrightarrow \mathcal{P}_1^{(n+1)}$. The colimit is the quotient of the tame product $\prod_{i\geq 1}^{\text{tame}} \mathcal{P}_1$ by the sum $\bigoplus_{i\geq 1} \mathcal{P}_1$; so $\hat{\pi}_0$ of the product is non-zero and even has a non-trivial $\mathcal{M}$-action. [check whether this is stably contractible; is the product of semistable spectra again semistable; is the product of $\hat{\pi}_*$-isos between semistable spectra]

We now return to the important class of semistable symmetric spectra and collect various equivalent characterizations of this class of symmetric spectra. Many symmetric spectra which arise naturally are semistable, compare Example 8.27. We recall from Definition 3.14 that a symmetric spectrum X is semistable if the morphism $\lambda_X : S^1 \wedge X \longrightarrow \mathrm{sh}\, X$ is a $\hat{\pi}_*$-isomorphism. The morphism $\tilde{\lambda}_X : X \longrightarrow \Omega(\mathrm{sh}\, X)$ is adjoint to λ_X. In Section 4.2 we used a spectrum $\Omega^\infty \mathrm{sh}^\infty X$ as a tool for showing that $\hat{\pi}_*$-isomorphisms are stable equivalences. The spectrum $\Omega^\infty \mathrm{sh}^\infty X$ was defined as the mapping telescope of a sequence of spectra $\Omega^n \mathrm{sh}^n X$, see (4.18). This construction comes with a canonical natural morphism $\lambda_X^\infty : X \longrightarrow \Omega^\infty \mathrm{sh}^\infty X$. For a description of the naive homotopy groups of $\Omega^\infty \mathrm{sh}^\infty X$ as a functor of $\hat{\pi}_* X$ see Exercise E.I.54.

Theorem 8.25. *For every symmetric spectrum X the following conditions are equivalent.*

(i) *The symmetric spectrum X is semistable.*
(ii) *The shift operator d acts bijectively on all naive homotopy groups of X.*
(iii) *The injection monoid $\mathcal{M}$ acts trivially on all naive homotopy groups of X.*
(iv) *The map $c : \hat{\pi}_k X \longrightarrow \pi_k X$ from naive to true homotopy groups is an isomorphism for all integers k.*
(v) *There exists a $\hat{\pi}_*$-isomorphism from X to an Ω-spectrum.*

In the context of spaces, or if X is levelwise Kan, then conditions (i)–(v) *are furthermore equivalent to the following conditions:*

(vi) *The morphism $\tilde{\lambda}_X : X \longrightarrow \Omega(\mathrm{sh}\, X)$ is a $\hat{\pi}_*$-isomorphism.*
(vii) *The morphism $\lambda_X^\infty : X \longrightarrow \Omega^\infty \mathrm{sh}^\infty X$ is a $\hat{\pi}_*$-isomorphism.*
(viii) *The symmetric spectrum $\Omega^\infty \mathrm{sh}^\infty X$ is an Ω-spectrum.*
(ix) *The symmetric spectrum $\Omega^\infty \mathrm{sh}^\infty X$ is semistable.*

In the context of simplicial sets, then conditions (i)–(v) *are furthermore equivalent to the condition:*

(x) *There exists a $\hat{\pi}_*$-isomorphism from X to an injective Ω-spectrum.*

Proof. The equivalence of conditions (i) and (ii) follows from the first commutative square of Lemma 8.6. The equivalence of conditions (ii) and (iii) is a general algebraic property of tame $\mathcal{M}$-modules, see Lemma 8.8 (iii). Condition (i) implies condition (iv) by Proposition 6.3. If condition (iv) holds, then the morphism $\eta_X : X \longrightarrow QX$ is a $\hat{\pi}_*$-isomorphism to an Ω-spectrum, so condition (v) holds. Every Ω-spectrum is semistable, so if condition (v) holds, then X is $\hat{\pi}_*$-isomorphic to a semistable symmetric spectrum, hence itsself semistable (compare Proposition 3.15 (ii)). This shows that conditions (i) through (v) are equivalent.

Now we assume that we are in the context of spaces, or that X is levelwise Kan. The equivalence of conditions (i) and (vi) is then the content of Proposition 3.15 (iii). If X is semistable, then the morphism $\lambda_X^\infty : X \longrightarrow \Omega^\infty \mathrm{sh}^\infty X$ is a $\hat{\pi}_*$-isomorphism with semistably target, by Proposition 4.24. So conditions (vii) and (viii) hold. Every Ω-spectrum is semistable, so condition (viii) implies condition (ix).

(ix) $\Rightarrow$ (i): If $\Omega^\infty \mathrm{sh}^\infty X$ is semistable, then by definition the injection monoid $\mathcal{M}$ acts trivially on all naive homotopy groups of $\Omega^\infty \mathrm{sh}^\infty X$. The map $\lambda_X^\infty : X \longrightarrow \Omega^\infty \mathrm{sh}^\infty X$ induces an $\mathcal{M}$-linear monomorphism on naive homotopy groups. So $\mathcal{M}$ also acts trivially on $\hat{\pi}_* X$, and thus X is semistable. □

Proposition 8.26. *A symmetric spectrum X is semistable if it satisfies one of the following conditions.*

(i) *For every $k \in \mathbb{Z}$ there is an $n \geq 0$ such that the canonical map $\pi_{k+n} X_n \longrightarrow \hat{\pi}_k X$ is surjective.*
(ii) *The naive homotopy groups of X are dimensionwise finitely generated as abelian groups.*

Proof. We show that in both cases the injection monoids acts trivially on the naive homotopy groups of X; so X is semistable by Theorem 8.25.

(i) Under the assumption every element of $\hat{\pi}_k X$ has filtration n. But tame $\mathcal{M}$-modules with bounded filtration necessarily have trivial $\mathcal{M}$-action by Lemma 8.8 (ii).

(ii) If $\hat{\pi}_k X$ is finitely generated as an abelian group, then tameness forces the $\mathcal{M}$-action to be trivial on $\hat{\pi}_k X$ (Lemma 8.8 (iv)). □

Example 8.27. An important special case where condition (i) in Proposition 8.26 above holds is when the homotopy groups of a symmetric spectrum X stabilize, i.e., for each $k \in \mathbb{Z}$ there exists an $n \geq 0$ such that from the group $\pi_{k+n}X_n$ on, all maps in the sequence (1.7) defining $\hat{\pi}_k X$ are isomorphisms.

Examples of symmetric spectra with stabilizing homotopy groups include all suspension spectra, Ω-spectra, or Ω-spectra from some point X_n on. So it includes Eilenberg-Mac Lane spectra HA associated to an abelian group (see Example 1.14) as well as spectra of topological K-theory (Example 1.20) and algebraic K-theory (Example 3.50). So all these kinds of symmetric spectra are semistable.

Remark 8.28. By definition, a symmetric spectrum X is semistable if $\lambda_X : S^1 \wedge X \longrightarrow \operatorname{sh} X$ is a $\hat{\pi}_*$-isomorphism. So for semistable X the morphism λ_X is also a stable equivalence. The converse is not true: in Exercise E.I.57 we discuss a symmetric spectrum X which is not semistable, but such that $S^1 \wedge X$ and $\operatorname{sh} X$ are stably contractible and hence λ_X is a stable equivalence.

Lemma 8.29. *For every symmetric spectrum A the morphism $c : \hat{\pi}_k A \longrightarrow \pi_k A$ from the naive to the true homotopy groups coequalizes the action of the injection monoid $\mathcal{M}$, i.e., we have $c(fx) = c(x)$ for all $f \in \mathcal{M}$ and $x \in \hat{\pi}_k A$.*

Proof. The spectrum QA is an Ω-spectrum, hence semistable, so the injection monoid $\mathcal{M}$ acts trivially on its naive homotopy groups (compare Theorem 8.25). So we have

$$c(fx) \ = f \cdot (cx) \ = \ c(x) \ .$$

in $\pi_k X = \hat{\pi}_k(QX)$ by the naturality of the $\mathcal{M}$-action. □

Example 8.30. We collect some examples of symmetric spectra which are *not* semistable. Example 8.18 identifies the homotopy groups of free and semifree symmetric spectra as

$$\hat{\pi}_k(F_m K) \ \cong \ \mathcal{P}_m \otimes \pi^{\mathrm{s}}_{k+m} K \qquad \text{respectively} \qquad \hat{\pi}_k(G_m L) \ \cong \ \mathcal{P}_m \otimes_{\Sigma_m} (\pi^{\mathrm{s}}_{k+m} L)(\mathrm{sgn}) \ .$$

Since $\mathcal{P}_m$ is free of countably infinite rank as a right Σ_m-module, the free or semifree symmetric spectra generated in positive level m are never semistable unless K respectively L has trivial stable homotopy groups. For a semifree symmetric spectrum generated in positive level m we have $\operatorname{sh}(G_m) \cong G_{m-1}(\operatorname{sh} L) \vee (S^1 \wedge G_m L)$ by (3.26), and the morphism $\lambda_{G_m L} : S^1 \wedge G_m L \longrightarrow \operatorname{sh}(G_m L)$ is the inclusion of a wedge summand.

If W is a tame $\mathcal{M}$-module with non-trivial $\mathcal{M}$-action, then $\hat{\pi}_0 HW \cong W$ as $\mathcal{M}$-modules and so the generalized Eilenberg-Mac Lane spectrum HW as defined in Example 8.10 is not semistable.

Example 8.23 shows that an infinite product of symmetric spectra with trivial homotopy groups can have homotopy groups with non-trivial $\mathcal{M}$-action. In particular, infinite products of semistable symmetric spectra need not be semistable. As we already discussed in Example 3.17, if X has at least one non-trivial homotopy group, then the induced spectrum $\triangleright X$ is not semistable. We can now also deduce this from the formula (8.22) for the naive homotopy groups of $\triangleright X$ in terms of the naive homotopy groups of X. This formula shows that Mc acts non-trivially on $\hat{\pi}_k(\triangleright X)$ whenever $\hat{\pi}_{k+1} X$ is nonzero.

The 'trivial $\mathcal{M}$-action' criterion is often handy for showing that semistability is preserved by certain constructions. Here are some examples of this.

Example 8.31. If $f : X \longrightarrow Y$ is any morphism of symmetric spectra, then the homotopy groups of the spectra X, Y and the mapping cone $C(f) = [0,1]^+ \wedge X \cup_f Y$ are related by a long exact sequence of naive homotopy groups as in Proposition I.2.12. This is even an exact sequence of $\mathcal{M}$-modules; indeed, $\mathcal{M}$-lineariy is clear for the two maps induced by homomorphisms of symmetric spectra. The connecting homomorphism is $\mathcal{M}$-linear because the $\mathcal{M}$-action does not change under loop and suspension isomorphisms. Trivial tame $\mathcal{M}$-modules are closed under taking submodules, quotient modules and extensions (Lemma 8.7 (iv)); so if two out of three graded $\mathcal{M}$-modules $\hat{\pi}_* X$, $\hat{\pi}_* Y$ and $\hat{\pi}_* C(f)$ have trivial $\mathcal{M}$-action, then so does the third. Thus the mapping cone of any morphism between semistable symmetric spectra is semistable.

Example 8.32. Let $F : J \longrightarrow \mathcal{S}p$ be a functor from a small category J to the category of symmetric spectra. If $F(j)$ is semistable for each object j of J, then the homotopy colimit of F over J is semistable.

Indeed, the homotopy colimit is the geometric realization of the *simplicial replacement* $\amalg_* F$ in the sense of Bousfield and Kan [**12**, Ch. XII, 5.1], a simplicial object of symmetric spectra. The spectrum of n-simplices of $\amalg_* F$ is a wedge, indexed over the n-simplices of the nerve of J, of spectra which occur as values of F. The geometric realization $|\amalg_* F|$ is the sequential colimit, over h-cofibrations, of the realizations of the skeleta $\mathrm{sk}_n \amalg_* F$ in the simplicial direction, so it suffices to show that each of these is semistable. The skeleton inclusion realizes to an h-cofibration $|\,\mathrm{sk}_{n-1} \amalg_* F| \longrightarrow |\,\mathrm{sk}_n \amalg_* F|$ whose quotient symmetric spectrum is a wedge, indexed over the *non-degenerate* n-simplices of the nerve of J, of n-fold suspensions of spectra which occur as values of F. So the quotient spectra are semistable, and so by induction the symmetric spectra $|\,\mathrm{sk}_n \amalg_* F|$ are semistable.

We revisit the diagonal construction first discussed in Example 2.21. More previsely, we'll investigate a generalization.

Construction 8.33 (Diagonal of an **I**-spectrum)**.** We recall the category **I**: it has an object $\mathbf{n} = \{1, \dots, n\}$ for every non-negative integer n, including $\mathbf{0} = \emptyset$. Morphisms in **I** are all injective maps.

Suppose we are given a functor $H : \mathbf{I} \longrightarrow \mathcal{S}p$ from the category **I** to the category of symmetric spectra. We define a new symmetric spectrum $\mathrm{diag}\, H$, the *diagonal* of H. The levels of the diagonal are given by

$$(\mathrm{diag}\, H)_n \;=\; H(\mathbf{n})_n \ ,$$

i.e., we take the n-th level of the symmetric spectrum $H(\mathbf{n})$ with the diagonal Σ_n-action. The structure map $(\mathrm{diag}\, H)_n \wedge S^1 \longrightarrow (\mathrm{diag}\, H)_{n+1}$ is the composite around either way in the commutative square

$$\begin{tikzcd}
H(\mathbf{n})_n \wedge S^1 \arrow[r, "\sigma_n^n"] \arrow[d, "H(\iota)_n \wedge \mathrm{Id}"'] & H(\mathbf{n})_{n+1} \arrow[d, "H(\iota)_{n+1}"] \\
H(\mathbf{n+1})_n \wedge S^1 \arrow[r, "\sigma_n^{n+1}"'] & H(\mathbf{n+1})_{n+1}
\end{tikzcd}$$

where $\iota : \mathbf{n} \longrightarrow \mathbf{n+1}$ is the inclusion. This constructions generalizes the diagonal of a sequence of symmetric spectra as discussed in Example 2.21. Indeed, given a sequence (X^i, f^i) we can define a **I**-spectrum by $H(\mathbf{n}) = X^n$ and $\alpha : \mathbf{n} \longrightarrow \mathbf{m}$ induces $f^{m-1} \cdots f^n : X^n \longrightarrow X^m$. Then the diagonal $\mathrm{diag}_i X^i$ as defined in Example 2.21 equals the diagonal $\mathrm{diag}\, H$ as defined here.

[From **I**-functors to tame $\mathcal{M}$-modules.] Let $F : \mathbf{I} \longrightarrow \mathcal{C}$ be a functor from the category **I** to a category $\mathcal{C}$ which has filtered colimits. We construct a natural tame left action by the injection monoid $\mathcal{M}$ on the colimit of F, formed over the subcategory $\mathbb{N}$ of inclusions. [... spell out]

Example 8.34 (From symmetric spectra to **I**-functors.)**.** One can break the construction of the $\mathcal{M}$-action on the naive homotopy groups of a symmetric spectrum up into two steps and pass through the intermediate category of **I**-*functors*.

Given a symmetric spectrum X and an integer k we assign an **I**-functor $\underline{\pi}_k X$ to the symmetric spectrum X. On objects, this **I**-functor is given by

$$(\underline{\pi}_k X)(\mathbf{n}) \;=\; \pi_{k+n} X_n$$

if $k + n \geq 2$ and $(\underline{\pi}_k X)(\mathbf{n}) = 0$ for $k + n < 2$. If $\alpha : \mathbf{n} \longrightarrow \mathbf{m}$ is an injective map and $k + n \geq 2$, then $\alpha_* : (\underline{\pi}_k X)(\mathbf{n}) \longrightarrow (\underline{\pi}_k X)(\mathbf{m})$ is given as follows. We choose a permutation $\gamma \in \Sigma_m$ such that $\gamma(i) = \alpha(i)$ for all $i = 1, \dots, n$ and set

$$\alpha_*(x) \;=\; \mathrm{sgn}(\gamma) \cdot \gamma(\iota^{m-n}(x))$$

where $\iota : \pi_{k+n} X_n \longrightarrow \pi_{k+n+1} X_{n+1}$ is the stabilization map (1.7). Justify that this definition is independent of the choice of permutation γ and really defines a functor on the category **I**.

We will now identify the naive homotopy groups of the diagonal with the sequential colimit of the naive homotopy groups of the symmetric spectra $H(\mathbf{n})$. The action of the injection monoid comes with an extra twist as follows. For every integer k we can consider the **I**-functor $\mathbf{n} \mapsto \hat{\pi}_k H(\mathbf{n})$. As n varies over the subcategory $\mathbb{N}$ of inclusions, we can take the colimit $\mathrm{colim}_{n \in \mathbb{N}}\, \hat{\pi}_k H(\mathbf{n})$ in the category of tame

$\mathcal{M}$-modules. Such colimits are created on underlying abelian groups (and even on underlying sets), so may as well take the colimit of the sequence of underlying abelian groups and endow it with the tame $\mathcal{M}$-action inherited from the tame $\mathcal{M}$-actions of $\hat{\pi}_k H(\mathbf{n})$ for varying n. We refer to this as the *internal* $\mathcal{M}$-action on the sequential colimit.

However, as explained above, this colimit comes with another *external* $\mathcal{M}$-action, because it came from a functor defined on the category $\mathbf{I}$. The external action is by homomorphisms of $\mathcal{M}$-modules; so altogether the group $\operatorname{colim}_{n\in\mathbb{N}} \hat{\pi}_k H(\mathbf{n})$ comes with a natural $\mathcal{M}\times\mathcal{M}$-action. In the following proposition we consider the diagonal $\mathcal{M}$-action, i.e., the restriction along the diagonal homomorphism $\mathcal{M} \longrightarrow \mathcal{M}\times\mathcal{M}$, and denote it by 'diag'.

By the very definition of the structure maps of $\operatorname{diag} H$, the square

$$\begin{tikzcd} \pi_{k+n} H(\mathbf{n})_n \arrow[r] \arrow[d, "\sigma_n"'] & \hat{\pi}_k H(\mathbf{n}) \arrow[d, "\hat{\pi}_k H(\iota)"] \\ \pi_{k+n+1} H(\mathbf{n+1})_{n+1} \arrow[r] & \hat{\pi}_k H(\mathbf{n+1}) \end{tikzcd}$$

commutes, where the horizontal maps are the canonical maps to the colimit. So in the colimit over $n\in\mathbb{N}$, ther horizontal maps conspire into a natural morphism of abelian groups

$$\hat{\pi}_k(\operatorname{diag} H) \ = \ \operatorname{colim}_n \pi_{k+n} H(\mathbf{n})_n \ \longrightarrow \ \operatorname{colim}_{n\in\mathbb{N}} \hat{\pi}_k H(\mathbf{n}) \ . \tag{8.35}$$

Proposition 8.36. *For any* $\mathbf{I}$*-symmetric spectrum* $H : \mathbf{I} \longrightarrow \mathcal{S}p$ *the map* (8.35)

$$\hat{\pi}_k(\operatorname{diag} H) \ \longrightarrow \ \operatorname{diag}\left(\operatorname{colim}_{n\in\mathbb{N}} \hat{\pi}_k H(\mathbf{n})\right)$$

is an isomorphism of $\mathcal{M}$*-modules.*

Proof. The group $\operatorname{colim}_{n\in\mathbb{N}} \hat{\pi}_k H(\mathbf{n})$ is obtained from the $\mathbb{N}\times\mathbb{N}$-functor $(n,m) \mapsto \pi_{k+m} H(\mathbf{n})_m$ by first taking colimit over m and then over n. Since the diagonal $\mathbb{N}$ is cofinal in $\mathbb{N}\times\mathbb{N}$, the colimit over the diagonal of any such functor maps isomorphically to the colimit over $\mathbb{N}\times\mathbb{N}$. In the case at hand, this show that the map (8.35) is an isomorphism of abelian groups. The fact that the map is also $\mathcal{M}$-linear with respect to the diagonal $\mathcal{M}$-action on the target is straightforward from the definitions. □

In Example 4.26 we considered the stable equivalence $\lambda : F_1 S^1 \longrightarrow \mathbb{S}$ which is adjoint to the identity in level 1. This morphism is not a $\hat{\pi}_*$-isomorphism, but a consequence of Theorem 6.2 is that the stable equivalence λ also induces isomorphisms of true homotopy groups. Since the sphere spectrum $\mathbb{S}$ is semistable, its naive and true homotopy groups 'coincide' and so the homotopy groups of the spectrum $F_1 S^1$ are isomorphic to the stable stems. However, we can also calculate the true homotopy groups $\pi_*(F_m S^n)$ directly with the tools developed so far.

Remark 8.37. In addition to the properties of Proposition **??**, we also know (by Proposition 2.6) that the left Σ_n-action on the sphere coordinates of $F_m S^n$ induces the sign action on naive homotopy groups, so in particular on the naive fundamental class. By naturality, the induced action on true fundamental class is then also by sign.

By Example 6.13 the right action by a permutation $\sigma \in \Sigma_m$ on the free coordinates of $F_m S^n$ takes ι^n_m to $\operatorname{sgn}(\sigma)\cdot \iota^n_m$. We emphasize that the right action by a permutation $\sigma\in\Sigma_m$ on the free coordinates of $F_m S^n$ does *not* generally multiply the naive fundametal class ι^n_m by a sign. In fact, we have an isomorphism of $\mathcal{M}$-modules $\hat{\pi}_{n-m}(F_m S^n) \cong \mathcal{P}_m$ which takes the fundamental class ι^n_m to the basis element $(1,\dots,m)$. This isomorphism is not only left $\mathcal{M}$-linear, but also equivariant for the right Σ_m-action. Hence the naive homotopy group $\hat{\pi}_{n-m}(F_m S^n)$ is actually free as a right $\mathbb{Z}\Sigma_m$-module.

8.4. A spectral sequence for true homotopy groups. For semistable spectra the naive and true homotopy groups agree via the natural homomorphism $c : \hat{\pi}_* X \longrightarrow \pi_* X$. If X is not semistable, then there must be at least one dimension k for which the injection monoid $\mathcal{M}$ acts non-trivially on $\hat{\pi}_k X$; since c factors over the $\mathcal{M}$-coinvariants (compare Lemma 8.29), the map $c : \hat{\pi}_k X \longrightarrow \pi_k X$ is then not injective.

For spectra which are not semistable it would thus be interesting to describe the true homotopy groups in terms of the naive homotopy groups, which are often more readily computable from an explicit presentation of the symmetric spectrum. The bad news is that the true homotopy groups are *not* a functor of the classical homotopy groups, not even if one takes the $\mathcal{M}$-action into account [give an example as exercise]. But the next best thing is true: the naive and true homotopy groups are only a spectral sequence apart from each other.

In this section we construct a spectral sequence (see Theorem 8.41)

$$E^2_{p,q} = H_p(\mathcal{M}, \hat{\pi}_q X) \quad \Longrightarrow \quad \pi_{p+q} X$$

which converges strongly to the true homotopy groups of a symmetric spectrum X and whose E^2-term is given by the homology of the naive homotopy groups, viewed as modules over the injection monoid $\mathcal{M}$. The homology groups above are defined as Tor groups over the monoid ring of $\mathcal{M}$, i.e., $H_p(\mathcal{M}, W) = \mathrm{Tor}_p^{\mathbb{Z}\mathcal{M}}(\mathbb{Z}, W)$. We refer to this spectral sequence as the *naive-to-true spectral sequence.*

We will see below that the naive-to-true spectral sequence collapses in many cases, for example for semistable symmetric spectra (Example 8.45) and for free symmetric spectra (Example 8.46), and it always collapses rationally (Example 8.49). The naive-to-true spectral sequence typically does not collapse for semifree symmetric spectra, see Example 8.47.

We need to develop more homological algebra of tame $\mathcal{M}$-modules. We recall from Example 8.9 that $\mathcal{I}_n$ denotes the set of ordered n-tuples of pairwise distinct elements of ω (or equivalently the set of injective maps from $\{1, \dots, n\}$ to ω). This is a tame $\mathcal{M}$-set with action by componentwise evaluation, i.e., $f(x_1, \dots, x_n) = (f(x_1), \dots, f(x_n))$.

Lemma 8.38. *The classifying space $B\mathcal{M}$ of the injection monoid $\mathcal{M}$ is contractible. More generally, the simplicial set $E\mathcal{M} \times_{\mathcal{M}} \mathcal{I}_n$ is weakly contractible.*

Proof. The classifying space $B\mathcal{M}$ is the geometric realization of the nerve of the category $\underline{B}\mathcal{M}$ with one object whose monoid of endomorphisms is $\mathcal{M}$. Let $t \in \mathcal{M}$ be given by $t(i) = 2i$. We define an injective endomorphism $c_t : \mathcal{M} \longrightarrow \mathcal{M}$ as follows. For $f \in \mathcal{M}$ and $i \in \omega$ we set

$$c_t(f)(i) \;=\; \begin{cases} i & \text{if } i \text{ is odd, and} \\ 2 \cdot f(i/2) & \text{if } i \text{ is even.} \end{cases}$$

Even though t is not bijective, the endomorphism c_t behaves like conjugation by t in the sense that the formula $c_t(f) \cdot t = t \cdot f$ holds. Thus t provides a natural transformation from the identity functor of $\underline{B}\mathcal{M}$ to $\underline{B}(c_t)$. On the other hand, if $s \in \mathcal{M}$ is given by $s(i) = 2i - 1$, then $c_t(f) \cdot s = s$ for all $f \in \mathcal{M}$, so s provides a natural transformation from the constant functor of $\underline{B}\mathcal{M}$ with values $1 \in \mathcal{M}$ to $\underline{B}(c_t)$. Thus via the homotopies induced by t and s, the identity of $B\mathcal{M}$ is homotopic to a constant map, so $B\mathcal{M}$ is contractible.

The simplicial set $E\mathcal{M} \times_{\mathcal{M}} \mathcal{I}_n$is isomorphic to the nerve of the translation category $T(\mathcal{M}, \mathcal{I}_n)$ whose objects are the elements of $\mathcal{I}_n$ and whose morphism from x to y are those monoid elements f which satisfy $fx = y$. We consider the functor $\mathbf{n} + - : \underline{B}\mathcal{M} \longrightarrow T(\mathcal{M}, \mathcal{I}_n)$ which sends the unique object of $\underline{B}\mathcal{M}$ to the element $(1, 2, \dots, n)$ of $\mathcal{I}_n$ and whose behavior on morphisms is given by $f \mapsto \mathbf{n} + f$. This functor sends the monoid $\mathcal{M}$ isomorphically onto the endomorphism monoid of $(1, 2, \dots, n)$ in $T(\mathcal{M}, \mathcal{I}_n)$, which means that $\mathbf{n} + -$ is fully faithful as a functor. For every element $x \in \mathcal{I}_n$ there exists a bijection $\sigma \in \mathcal{M}$ such that $\sigma \cdot (1, \dots, n) = x$, so every object of $T(\mathcal{M}, \mathcal{I}_n)$ is isomorphic to the object $(1, \dots, n)$. Thus the functor $\mathbf{n} + - : \underline{B}\mathcal{M} \longrightarrow T(\mathcal{M}, \mathcal{I}_n)$ is an equivalence of categories, so it induces a weak equivalence of nerves. Altogether, $E\mathcal{M} \times_{\mathcal{M}} \mathcal{I}_n$ is weakly equivalent to the classifying space $B\mathcal{M}$, which we showed is weakly contractible. □

In the following proposition, we let $\mathbb{Z}\mathcal{M}^{\langle 1 \rangle}$ denote the monoid ring of $\mathcal{M}$ with its usual left action, but with right action through the shift homomorphism $\mathrm{sh} : \mathcal{M} \longrightarrow \mathcal{M}$ given by $(\mathrm{sh}\, f)(1) = 1$ and $(\mathrm{sh}\, f)(1+i) =$

$1 + f(1)$ for $i \geq 1$. The relation $(\mathrm{sh}\, f)d = df$ implies that for every $\mathcal{M}$-module W the map

$$\mathbb{Z}\mathcal{M}^{\langle 1\rangle} \otimes_{\mathcal{M}} W \longrightarrow W \ , \quad f \otimes x \longmapsto fdx$$

[name it?] is well-defined.

Proposition 8.39. (i) *Then for every $n \geq 0$ the map*

$$\kappa \ : \ \mathcal{P}_{1+n} \longrightarrow \mathbb{Z}\mathcal{M}^{\langle 1\rangle} \otimes_{\mathcal{M}} \mathcal{P}_n$$

which sends the generator $(1,\dots,1+n)$ to the element $1\otimes(1,\dots,n)$ of filtration $1+n$ in $\mathbb{Z}\mathcal{M}^{\langle 1\rangle}\otimes_{\mathcal{M}}\mathcal{P}_n$ is an isomorphism of $\mathcal{M}$-modules.

(ii) *For every $\mathcal{M}$-module W, the homomorphism*

$$\mathbb{Z}\mathcal{M}^{\langle 1\rangle} \otimes_{\mathcal{M}} W \longrightarrow W \ , \quad f \otimes x \longmapsto fdx$$

[name it?] induces isomorphisms

$$H_*(\mathcal{M}, \mathbb{Z}\mathcal{M}^{\langle 1\rangle} \otimes_{\mathcal{M}} W) \ \cong \ H_*(\mathcal{M}, W)$$

on all homology groups.

(iii) *For every $n \geq 0$ and every abelian group A, the homology groups $H_p(\mathcal{M}, \mathcal{P}_n \otimes A)$ vanish in positive dimensions.*

(iv) *For every $n \geq 0$ and every Σ_n-module B we have a natural isomorphism*

$$H_*(\mathcal{M}, \mathcal{P}_n \otimes_{\Sigma_n} B) \ \cong \ H_*(\Sigma_n; B) \ .$$

Proof. (i) For any n-tuple $(x_1,\dots,x_n)$ of pairwise distinct natural numbers we can choose $g \in \mathcal{M}$ with $g(i) = x_i$ for $1 \leq i \leq n$. Because of

$$f \otimes (x_1,\dots,x_n) \ = \ f \otimes g(1,\dots,n) \ = \ f(1+g)\cdot(1\otimes(1,\dots,n))$$

the element $1\otimes(1,\dots,n)$ generates $\mathbb{Z}\mathcal{M}^{\langle 1\rangle}\otimes_{\mathcal{M}}\mathcal{P}_n$, so the map κ is surjective. The map $\mathbb{Z}\mathcal{M}^{\langle 1\rangle}\otimes_{\mathcal{M}}\mathcal{P}_n \longrightarrow \mathcal{P}_{1+n}$ which sends $f\otimes(x_1,\dots,x_n)$ to $(f(1), f(x_1+1),\dots,f(x_n+1))$ is right inverse to κ since the composite sends the generator $(1,\dots,1+n)$ to itself. So κ is also injective.

(ii) The $\mathcal{M}$-bimodule $\mathbb{Z}\mathcal{M}^{\langle 1\rangle}$ is free as a left and right module separately. So if P_* is a resolution of W by projective left $\mathcal{M}$-modules, then $\mathbb{Z}\mathcal{M}^{\langle 1\rangle}\otimes_{\mathcal{M}} P_*$ is a a resolution of $\mathbb{Z}\mathcal{M}^{\langle 1\rangle}\otimes_{\mathcal{M}} W$ by projective left $\mathcal{M}$-modules. Thus we get isomorphisms

$$\begin{aligned}\mathrm{Tor}_*^{\mathbb{Z}\mathcal{M}}(\mathbb{Z}, \mathbb{Z}\mathcal{M}^{\langle 1\rangle}\otimes_{\mathcal{M}} W) \ &= \ H_*(\mathbb{Z}\otimes_{\mathcal{M}}(\mathbb{Z}\mathcal{M}^{\langle 1\rangle}\otimes_{\mathcal{M}} W)) \\ &\cong H_*((\mathbb{Z}\otimes_{\mathcal{M}}\mathbb{Z}\mathcal{M}^{\langle 1\rangle})\otimes_{\mathcal{M}} W) \ \cong \ \mathrm{Tor}_*^{\mathbb{Z}\mathcal{M}}(\mathbb{Z}, W)\end{aligned}$$

since $\mathbb{Z}\otimes_{\mathcal{M}}\mathbb{Z}\mathcal{M}^{\langle 1\rangle}$ is again the trivial right $\mathcal{M}$-module $\mathbb{Z}$.

(iii) The groups $\mathrm{Tor}_p^{\mathbb{Z}\mathcal{M}}(\mathbb{Z}, A)$ are isomorphic to the singular homology groups with coefficients in A of the classifying space $B\mathcal{M}$ of the monoid $\mathcal{M}$. This classifying space is contractible by Lemma 8.38, so the groups $\mathrm{Tor}_p^{\mathbb{Z}\mathcal{M}}(\mathbb{Z}, A)$ vanish for $p \geq 1$, which proves the case $n = 0$. For $n \geq 1$ we use induction and parts (i) and (ii).

(iv) Since $\mathcal{P}_n$ is free as a right Σ_n-module, the functor $\mathcal{P}_n\otimes_{\Sigma_n} -$ is exact. The functor takes the free Σ_n-module of rank 1 to $\mathcal{P}_n$, so by part (ii) it takes projective Σ_n-modules to tame $\mathcal{M}$-modules which are acyclic for the functor $\mathbb{Z}\otimes_{\mathcal{M}} -$.

Thus if $P_\bullet \longrightarrow B$ is a projective resolution of B by Σ_n-modules, then $\mathcal{P}_n\otimes_{\Sigma_n} P_\bullet$ is a resolution of $\mathcal{P}_n\otimes_{\Sigma_n} B$ which can be used to calculate the desired Tor groups. Thus we have isomorphisms

$$\mathrm{Tor}_*^{\mathbb{Z}\mathcal{M}}(\mathbb{Z}, \mathcal{P}_n\otimes_{\Sigma_n} B) \ = \ H_*(\mathbb{Z}\otimes_{\mathcal{M}}\mathcal{P}_n\otimes_{\Sigma_n} P_\bullet) \ \cong \ H_*(\mathbb{Z}\otimes_{\Sigma_n} P_\bullet) \ = \ H_*(\Sigma_n; B) \ . \qquad \square$$

To construct the spectral sequence from naive to true homotopy groups we use the cotripleobtained from the adjunction between free symmetric spectra and evaluation. We denote this cotriple by

$$PX \ = \ \bigvee_{n\geq 0} F_n X_n \ ;$$

it comes with an adjunction counit

$$\epsilon \;:\; PX \;\longrightarrow\; X$$

which takes the n-th summand to X by the adjoint of the identity of X_n.

Proposition 8.40. *For every symmetric spectrum X the following properties hold.*

(i) *The map $\hat{\pi}_*\epsilon : \hat{\pi}_*(PX) \longrightarrow \hat{\pi}_*X$ induced by the adjunction counit on naive homotopy groups is surjective.*
(ii) *For every $p \geq 1$ and all integers q the homology group $H_p(\mathcal{M}, \hat{\pi}_q(PX))$ is trivial.*
(iii) *The map $\bar{c} : \mathbb{Z} \otimes_{\mathcal{M}} (\hat{\pi}_*(PX)) \longrightarrow \pi_*(PX)$ induced by $c : \hat{\pi}_*(PX) \longrightarrow \pi_*(PX)$ is bijective.*

Proof. (i) In every level $n \geq 0$ the map $\epsilon_n : (PX)_n \longrightarrow X_n$ is a split surjection as a map of based spaces (or simplicial sets), so it induces a split epimorphism on $\pi_{k+n}(-)$. So the induced map on naive homotopy groups is also surjective.

(ii) We have

$$\hat{\pi}_k(PX) \;\cong\; \bigoplus_n \hat{\pi}_k(F_n X_n) \;\cong\; \bigoplus_n \mathcal{P}_n \otimes \pi^{\mathrm{s}}_{k+n} X_n \;.$$

as $\mathcal{M}$-modules, compare (8.20). The $\mathcal{M}$-homology of the right hand side vanishes in positive dimensions by Proposition 8.39 (iii).

Part (iii) follows from the stable equivalence $F_n K \longrightarrow \Omega^n(\Sigma^\infty K)$ with semistable target and the isomorphism (8.20). □

Now we are ready to construct the spectral sequence that attempts to calculate the true homotopy groups of a symmetric spectrum from its naive homotopy groups as an $\mathcal{M}$-module.

Theorem 8.41 (Naive-to-true spectral sequence)**.** *There is a strongly convergent, natural half-plane spectral sequence*

$$E^2_{p,q} = H_p(\mathcal{M}, \hat{\pi}_q X) \;\Longrightarrow\; \pi_{p+q} X$$

with d^r-differential of bidegree $(-r, r-1)$. The E^2-term is given by the homology of the monoid $\mathcal{M}$ with coefficients in the $\mathcal{M}$-module $\hat{\pi}_* X$. The edge homomorphism

$$E^2_{0,q} = H_0(\mathcal{M}, \hat{\pi}_q X) = \mathbb{Z} \otimes_{\mathcal{M}} \hat{\pi}_q X \;\longrightarrow\; \pi_q X$$

is induced by the natural transformation $c : \hat{\pi}_q X \longrightarrow \pi_q X$.

Proof. It suffices to work in the context of symmetric spectra of spaces. We define symmetric spectra X_n and P_n inductively, starting with $X_0 = X$. In each step we set $P_n = P(X_n)$ and we let $f_n = \epsilon : P_n = P(X_n) \longrightarrow X_n$ be the adjunction counit. Then we define $X_{n+1} = F(f_n)$ as the homotopy fibre of f_n, denote by $i_n : X_{n+1} \longrightarrow P_n$ the inclusion, and iterate the construction. Then by parts (i) and (ii) of Proposition 8.40 the homotopy groups of the sequence of symmetric spectra

$$\cdots \;\longrightarrow\; P_{n+1} \;\xrightarrow{i_n f_{n+1}}\; P_n \;\longrightarrow\; \cdots \;\longrightarrow\; P_0 \;\xrightarrow{f_0}\; X_0 = X$$

form a resolution

$$\cdots \;\longrightarrow\; \hat{\pi}_k P_{n+1} \;\xrightarrow{(i_n f_{n+1})_*}\; \hat{\pi}_k P_n \;\longrightarrow\; \cdots \;\longrightarrow\; \hat{\pi}_k P_0 \;\xrightarrow{(f_0)_*}\; \hat{\pi}_k X \;\longrightarrow\; 0 \tag{8.42}$$

of the k-th naive homotopy group $\hat{\pi}_k X$ by tame $\mathcal{M}$-modules which are acyclic for the functor $H_0(\mathcal{M}, -) = \mathbb{Z} \otimes_{\mathcal{M}} -$.

By Proposition 6.11 (ii) the homotopy fiber sequence

$$X_{n+1} \;\xrightarrow{i_n}\; P_n \;\xrightarrow{f_n}\; X_n$$

gives rise to a long exact sequence of true homotopy groups. These homotopy groups thus assemble into an exact couple with

$$E^1_{p,q} = \pi_q P_p \quad \text{and} \quad D^1_{p,q} = \pi_q X_p$$

and morphisms

$$\begin{aligned} j \ &: \ D^1_{p+1,q} \longrightarrow E^1_{p,q} \qquad \text{induced by } i_p : X_{p+1} \longrightarrow P_p, \\ k \ &: \ E^1_{p,q} \quad \longrightarrow D^1_{p,q} \qquad \text{induced by } f_p : P_p \longrightarrow X_p, \\ i \ &: \ D^1_{p,q} \quad \longrightarrow D^1_{p+1,q-1} \end{aligned}$$

given by the connecting homomorphism (6.10) $\delta : \pi_q X_p \longrightarrow \pi_{q-1} X_{p+1}$ of the homotopy fiber sequence.

Part (iii) of Proposition 8.40 the natural map $c : \hat{\pi}_q P_p \longrightarrow \pi_q P_p$ from naive to true homotopy groups factors over an isomorphism

$$\mathbb{Z} \otimes_{\mathcal{M}} (\hat{\pi}_q P_p) \ \cong \ \pi_q P_p = E^1_{p,q} \ ;$$

under this isomorphism, the differential $d^1 = jk : E^1_{p,q} \longrightarrow E^1_{p-1,q}$ becomes the map obtained by applying $\mathbb{Z} \otimes_{\mathcal{M}} -$ to the resolution (8.42) of $\hat{\pi}_* X$. Since the naive homotopy groups of the spectra P_p are acyclic for the functor $\mathbb{Z} \otimes_{\mathcal{M}} -$, the E^2-term calculates the homology groups $H_p(\mathcal{M}, \hat{\pi}_q X)$.

It remains to discuss convergence of the spectral sequence. The p-th filtration subgroup F^p of the abutment $\pi_* X$ is the kernel of the map

$$i^p \ : \ \pi_* X = D^1_{0,*} \ \longrightarrow \ D^1_{p,*-p} = \pi_{*-p}(X_p) \ .$$

To prove that the spectral sequence converges to the true homotopy groups of X we show that the filtration is exhaustive, i.e., $\pi_q X = \bigcup_p F^p_q$.

The map $i : \pi_q X_p \longrightarrow \pi_{q-1} X_{p+1}$ is induced (up to the suspension isomorphism) by the morphism of symmetric spectra $X_p \longrightarrow S^1 \wedge X_{p+1}$ [which]. By construction, [...] induces the trivial map on naive homotopy groups, so the mapping telescope of the sequence

$$X = X_0 \ \longrightarrow \ S^1 \wedge X_1 \ \longrightarrow \ S^2 \wedge X_2 \ \longrightarrow \ \cdots \tag{8.43}$$

has trivial naive homotopy groups and is thus stably contractible. Thus the true homotopy groups of the mapping telescope of the sequence (8.43) which are isomorphic to the colimit of homotopy groups, are trivial. Since each instance of the map $i : D^1_{p,q} \longrightarrow D^1_{p+1,q-1}$ is induced by a connecting homomorphism $X_k \longrightarrow S^1 \wedge X_{k+1}$ this shows that the kernels of the maps i^p exhaust all of $\pi_* X$. The spectral sequence is concentrated in a half-plane and has exiting differentials (in the sense of Boardman [**7**, II.6]), so it is strongly convergent.

All constructions that go into the naive-to-true spectral sequence are functorial and all maps involved are natural. So the naive-to-true spectral sequence is natural in X. □

The abutment $\pi_{p+q} X$ of the naive-to-true spectral sequence comes with an exhaustive natural filtration

$$F^0_q \ \subseteq \ F^1_q \ \subseteq \ \cdots \ \subseteq \ F^p_q \ \subseteq \ \cdots$$

such that $E^\infty_{p,q}$ is isomorphic to F^p_q / F^{p-1}_q. We make this filtration more explicit in Exercise E.I.62.

Remark 8.44. An important point about of naive homotopy groups of symmetric spectra is that the action of the injection monoid $\mathcal{M}$ is tame. So one could expect that the E^2-term of the naive-to-true spectral sequence should be given by homological algebra in the abelian category of tame $\mathcal{M}$-modules. In other words, it may be a little surprising that the E^2-term is given by Tor groups over the monoid ring $\mathbb{Z}\mathcal{M}$, which are the *absolute* derived functors of $W \mapsto \mathbb{Z} \otimes_{\mathcal{M}} W$ (as opposed to some 'tamely derived' or *relative* derived functors). The explanation is the following: while the abelian category of tame $\mathcal{M}$-modules has no nonzero projective objects, the modules $\mathcal{P}_n$ play a role analogous to projective generators. Proposition 8.39 (which ultimately is a consequence of the contractibility of the classifying space $B\mathcal{M}$) says that the particular Tor groups $\mathrm{Tor}^{\mathbb{Z}\mathcal{M}}_*(\mathbb{Z}, \mathcal{P}_n)$ vanish in positive dimension (even though $\mathcal{P}_n$ is *not* flat as a left $\mathcal{M}$-module). So to calculate $\mathrm{Tor}^{\mathbb{Z}\mathcal{M}}_*(\mathbb{Z}, W)$ for tame $\mathcal{M}$-modules we can use resolutions by sum of modules of the form $\mathcal{P}_n$, as opposed to projective resolutions.

Example 8.45 (Semistable spectra)**.** When X is a semistable symmetric spectrum, the injection monoid acts trivially on the naive homotopy groups of X, so $H_0(\mathcal{M}, \hat{\pi}_k X)$ is isomorphic to $\hat{\pi}_k X$. and the higher homology groups vanish by part (ii) of Proposition 8.39. Thus $E^2_{p,q} = 0$ for $p \neq 0$ in the naive-to-true spectral sequence and the edge homomorphism

$$\hat{\pi}_* X \longrightarrow \pi_* X$$

is an isomorphism. We recover the fact that the natural map $c : \hat{\pi}_k X \longrightarrow \pi_k X$ is an isomorphism for semistable X.

Example 8.46 (Free spectra)**.** For the free symmetric spectrum generated by a pointed space (or simplicial set) K in level m, (8.20) provides an isomorphism of $\mathcal{M}$-modules

$$\hat{\pi}_k(F_m K) \cong \mathcal{P}_m \otimes \pi^{\mathrm{s}}_{k+m} K \ .$$

The higher Tor groups for such $\mathcal{M}$-modules vanish by part (ii) of Proposition 8.39. Thus $E^2_{p,q} = 0$ for $p \neq 0$ in the naive-to-true spectral sequence and the edge homomorphism

$$\mathbb{Z} \otimes_{\mathcal{M}} \hat{\pi}_*(F_m K) \longrightarrow \pi_*(F_m K)$$

is an isomorphism. The left hand side is isomorphic to $\mathbb{Z} \otimes_{\mathcal{M}} \mathcal{P}_m \otimes \pi^{\mathrm{s}}_{k+m} K \cong \pi^{\mathrm{s}}_{k+m} K$, the $(k+m)$th stable homotopy group of K.

We can also use Example 4.35 instead of the spectral sequence of Theorem 8.41 to calculate the true homotopy groups of the free spectrum $F_m K$. Indeed, there we introduced a stable equivalence from $F_m K$ to $\Omega^m(\Sigma^\infty K)$. The spectrum $\Omega^m(\Sigma^\infty K)$ is semistable, so its naive and true homotopy groups coincide.

Example 8.47 (Semifree spectra)**.** For semifree symmetric spectra (see Example 3.23) the naive-to-true spectral sequence typically does not degenerate. If L is a cofibrant based Σ_m-space (or simplicial set), then we know from Example 4.36 that the semifree spectrum $G_m L$ is stably equivalent to the semistable homotopy orbit spectrum $(\Omega^m(\Sigma^\infty L))_{h\Sigma_m}$. So the true homotopy groups of $G_m L$, which are the abutment of the naive-to-true spectral sequence, are isomorphic to the naive homotopy groups of the homotopy orbit spectrum $(\Omega^m(\Sigma^\infty L))_{h\Sigma_m}$.

The isomorphism (8.19) and Proposition 8.39 (iii) allows us to rewrite the E^2-term of the spectral sequence as

$$H_q(\mathcal{M}, \hat{\pi}_q(G_m L)) \cong H_q(\mathcal{M}, \mathcal{P}_m \otimes_{\Sigma_m} (\pi^{\mathrm{s}}_{q+m} L)(\mathrm{sgn})) \cong H_q(\Sigma_m, (\pi^{\mathrm{s}}_{q+m} L)(\mathrm{sgn})) \ .$$

The homotopy orbit spectral sequence [ref]

$$E^2_{p,q} = H_p(\Sigma_m, \hat{\pi}_q(\Omega^m(\Sigma^\infty L))) \implies \hat{\pi}_{p+q}(\Omega^m(\Sigma^\infty L))_{h\Sigma_m}$$

has isomorphic E^2-term and isomorphic abutment, which makes it very likely that it is in fact isomorphic to the naive-to-true spectral sequence for $G_m L$.

As a specific example we consider the semifree symmetric spectrum $G_2 S^2$, where S^2 is a Σ_2-space by coordinate permutations. We first identify the stable equivalence type of $G_2 S^2$. The spectrum $G_2 S^2$ is isomorphic to the quotient spectrum of Σ_2 permuting the smash factors of $(F_1 S^1)^{\wedge 2}$. Since the Σ_2-action on $(F_1 S^1)^{\wedge 2}$ is free [not yet shown], the map

$$E\Sigma_2^+ \wedge_{\Sigma_2} (F_1 S^1)^{\wedge 2} \longrightarrow (F_1 S^1)^{\wedge 2}/\Sigma_2 = G_2 S^2$$

which collapses $E\Sigma_2$ to a point is a level equivalence. On the other hand, the stable equivalence $\lambda^{(2)} : (F_1 S^1)^{\wedge 2} \longrightarrow \mathbb{S}$ is Σ_2-equivariant, so it induces a stable equivalence

$$E\Sigma_2^+ \wedge_{\Sigma_2} (F_1 S^1)^{(2)} \longrightarrow E\Sigma_2^+ \wedge_{\Sigma_2} \mathbb{S} = \Sigma^\infty B\Sigma_2^+$$

on homotopy orbit spectra. Altogether we conclude that $G_2 S^2$ is stably equivalent to $\Sigma^\infty B\Sigma_2^+$.

The naive-to-true spectral sequence for $G_2 S^2$ has as E^2-term the Tor groups of $\hat{\pi}_*(G_2 S^2)$. According to (8.19) these homotopy groups are isomorphic to $\mathcal{P}_2 \otimes_{\Sigma_2} (\pi^{\mathrm{s}}_{*+2} S^2)(\mathrm{sgn})$. The sign representation cancels the sign action induced by the coordinate flip of S^2, so we have an isomorphism of $\mathcal{M}$-modules $\hat{\pi}_q(G_2 S^2) \cong \mathcal{P}_2 \otimes_{\Sigma_2} \pi^{\mathrm{s}}_q S^0$, this time with trivial action on the stable homotopy groups of spheres. Using part (iii) of Proposition 8.39, the naive-to-true spectral sequence for $G_2 S^2$ takes the form

$$E^2_{p,q} \cong H_p(\Sigma_2; \pi^{\mathrm{s}}_q S^0) \implies \pi^{\mathrm{s}}_{p+q}(B\Sigma_2^+) \ .$$

This spectral sequence has non-trivial differentials and it seems likely that it coincides with the Atiyah-Hirzebruch spectral sequence for the stable homotopy of the space $B\Sigma_2^+$.

Example 8.48 (Eilenberg-Mac Lane spectra)**.** In Example 8.10 we associated an Eilenberg-Mac Lane spectrum HW to every tame $\mathcal{M}$-module W. The homotopy groups of HW are concentrated in dimension 0, where we get the $\mathcal{M}$-module W back. So the naive-to-true spectral sequence for HW collapses onto the axis $q = 0$ to isomorphisms

$$\pi_p(HW) \;\cong\; H_p(\mathcal{M}, W) \ .$$

In particular, the true homotopy groups of HW need not be concentrated in dimension 0. One can show that HW is in fact stably equivalent to the product of the Eilenberg-Mac Lane spectra associated to the groups $H_p(\mathcal{M}, W)$, shifted up p dimensions. [make exercise]

Here is an example which shows that for non-trivial W the Eilenberg-Mac Lane spectrum HW can be stably contractible: we let W be the kernel of a surjection $\mathcal{P}_n \longrightarrow \mathbb{Z}$. Proposition 8.39 and the long exact sequence of Tor groups show that the groups $H_p(\mathcal{M}, W)$ vanish for all $p \geq 0$. Thus the true homotopy groups of HW are trivial, i.e., HW is stably contractible.

Example 8.49 (Rational collapse)**.** We claim that for every tame $\mathcal{M}$-module W and all $p \geq 1$, we have $\mathbb{Q} \otimes H_p(\mathcal{M}, W) = 0$. So the spectral sequence of Theorem 8.41 always collapses rationally and the edge homomorphism is a rational isomorphism

$$\mathbb{Q} \otimes_{\mathcal{M}} \hat{\pi}_* X \;\longrightarrow\; \mathbb{Q} \otimes \pi_* X \ .$$

In particular, for every symmetric spectrum X the tautological map $c : \hat{\pi}_* X \longrightarrow \pi_* X$ is rationally surjective. The rational vanishing of higher Tor groups is special for *tame* $\mathcal{M}$-modules.

To prove the claim we consider a monomorphism $i : V \longrightarrow W$ of tame $\mathcal{M}$-modules and show that the kernel of the map $\mathbb{Z} \otimes_{\mathcal{M}} i : \mathbb{Z} \otimes_{\mathcal{M}} V \longrightarrow \mathbb{Z} \otimes_{\mathcal{M}} W$ is a torsion group. The inclusions $W^{(n)} \longrightarrow W$ induce an isomorphism

$$\operatorname{colim}_n \ \mathbb{Z} \otimes_{\Sigma_n} W^{(n)} \ \xrightarrow{\cong}\ \mathbb{Z} \otimes_{\mathcal{M}} W \ .$$

For every $n \geq 0$, the kernel of $\mathbb{Z} \otimes_{\Sigma_n} i^{(n)} : \mathbb{Z} \otimes_{\Sigma_n} V^{(n)} \longrightarrow \mathbb{Z} \otimes_{\Sigma_n} W^{(n)}$ is annihilated by the order of the group Σ_n. Since the kernel of $\mathbb{Z} \otimes_{\mathcal{M}} i$ is the colimit of the kernels of the maps $\mathbb{Z} \otimes_{\Sigma_n} i^{(n)}$, it is torsion. Thus the functor $\mathbb{Q} \otimes_{\mathcal{M}} -$ is exact on short exact sequences of tame $\mathcal{M}$-modules and the higher Tor groups vanish as claimed.

Example 8.50 (Connective spectra)**.** Let A be a symmetric spectrum which is 'naively $(k-1)$-connected' for some integer k in the sense that the naive homotopy groups below dimension k are trivial. Then the homology group $H_p(\mathcal{M}, \hat{\pi}_q A)$ is trivial whenever $p + q < k$, and so the true homotopy groups also vanish below dimension k, by the naive-to-true spectral sequence. Moreover, the edge homomorphism of the spectral sequence is an isomorphism

$$\mathbb{Z} \otimes_{\mathcal{M}} \hat{\pi}_k A \;\cong\; \pi_k A$$

for the k-th true homotopy group of A.

Somewhat more generally, we can deduce that, roughly speaking, the true homotopy groups up to certain dimension only depend on the naive homotopy groups up to that dimension (as long as induced by a morphism). More precisely: let $f : A \longrightarrow B$ be a morphism of symmetric spectrum which induces an isomorphism on $\hat{\pi}_k$ for $k < n$ and an epimorphism on $\hat{\pi}_n$. Then f also induces an isomorphism on π_k for $k < n$ and an epimorphism on π_n. Indeed, under this hypothesis the mapping cone $C(f)$ is naively n-connected (by the long exact sequence of naive homotopy groups, Proposition 2.12) and thus the true homotopy groups of $C(f)$ vanish in dimensions n and below (by the above). So the long exact sequence of true homotopy groups (Proposition 6.11) shows the claim about f.

8.5. Detection functors. By a *detection functor* we mean an endofunctor $D : \mathcal{S}p \longrightarrow \mathcal{S}p$ on the category of symmetric spectra with values in semistable spectra and such that D is related by a chain of natural stable equivalences to the identity functor. For any such detection functor and symmetric spectrum X, the naive homotopy groups of DX are then naturally isomorphic to the true homotopy groups of X,

$$\pi_k X \;\cong\; \hat{\pi}_k(DX) \ .$$

Thus a morphism $f : X \longrightarrow Y$ is a stable equivalence if and only if the morphism $Df : DX \longrightarrow DY$ is a $\hat{\pi}_*$-isomorphism. In this sense the naive homotopy groups of DX 'detect' stable equivalences, hence the name. We have already seen one detection functor, namely the functor Q of Proposition 4.39 which takes values in Ω-spectra and comes with a natural stable equivalence $\eta_X : A \longrightarrow QA$.

Now we can define the *orthogonal detection functor* $\mathbb{P}$. This detection functor is essentially the composite of the forgetful functor $U : \mathcal{S}p^{\mathbf{O}} \longrightarrow \mathcal{S}p_{\mathbf{T}}$ and its left adjoint prolongation functor P; however, we can only expect a good homotopical behaviour of P on flat spectra, so we have to combine this with some flat resolution. For definiteness, we use the flat resolution $X^\flat$ of Construction 5.53 and define $\mathbb{P}$ as the composite functor

$$\mathbb{P}X \;=\; UP(X^\flat) \ .$$

Since $\mathbb{P}X$ is underlying an orthogonal spectrum, it is semistable by Proposition 3.16 (vi). The two natural morphisms

$$X \xleftarrow{\;r_X\;} X^\flat \xrightarrow{\;\eta_{X^\flat}\;} UP(X^\flat) = \mathbb{P}X$$

are stable equivalences: the resolution morphism $r_X : X^\flat \longrightarrow X$ is a level equivalence by construction and the morphism $\eta_{X^\flat} : X^\flat \longrightarrow UP(X^\flat)$ is a stable equivalence by Proposition 7.4, since $X^\flat$ is flat. So the functor $\mathbb{P}$ is a detection functor. In particular, a morphism $f : X \longrightarrow Y$ of symmetric spectra of spaces is a stable equivalence if and only if the morphism $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a $\hat{\pi}_*$-isomorphism.

The orthogonal detection functor $\mathbb{P}$ above works on symmetric spectra of spaces. A slight modification yields a detection functor for symmetric spectra of simplicial set: we set

$$\mathbb{P}'Y \;=\; \mathcal{S}(UP(|Y^\flat|))$$

for a symmetric spectrum Y of simplicial sets. Since the flat resolution for symmetric spectra of spaces was defined by $X^\flat = |\mathcal{S}(X)^\flat|$, the only difference between $\mathbb{P}$ and $\mathbb{P}'$ is in the place where singular complex and geometric realization are taken. So for every symmetric spectrum of spaces X, we have $\mathbb{P}'(\mathcal{S}X) = \mathcal{S}(\mathbb{P}X)$, by definition. The symmetric spectrum $\mathbb{P}'Y$ is also semistable and the chain of natural stable equivalences

$$Y \xleftarrow{\;r_Y\;} Y^\flat \xrightarrow{\;\eta_Y\;} \mathcal{S}(|Y^\flat|) \xrightarrow{\;\mathcal{S}(\eta_{|Y^\flat|})\;} \mathcal{S}(UP(|Y^\flat|) = \mathbb{P}'Y$$

relates Y to $\mathbb{P}'Y$.

Now we discuss a third detection functor $\mathbb{D}$, due to Shipley. As before, we denote by $\mathbf{I}$ the category with objects the sets $\mathbf{n} = \{1, \ldots, n\}$ for $n \geq 0$ (where $\mathbf{0} = \emptyset$) and with morphisms all injective maps. For a symmetric spectrum of simplicial sets X we define a functor $\underline{\mathbb{D}}X : \mathbf{I} \longrightarrow \mathcal{S}p_{\mathbf{T}}$ from the category $\mathbf{I}$ to symmetric spectra of spaces. On objects, the functor is given by

$$(\underline{\mathbb{D}}X)(\mathbf{n}) \;=\; \Omega^n(\Sigma^\infty|X_n|) \ .$$

A permutation $\gamma \in \Sigma_n = \mathbf{I}(\mathbf{n}, \mathbf{n})$ acts by conjugation, using the given action on X_n and permutation of the loop coordinates. The inclusion $\iota : \mathbf{n} \longrightarrow \mathbf{n} + \mathbf{m}$ induces the morphism

$$\iota_* \;:\; (\underline{\mathbb{D}}X)(\mathbf{n}) \;=\; \Omega^n(\Sigma^\infty|X_n|) \xrightarrow{\;\Omega^n(\widetilde{\sigma^m})\;} \Omega^{n+m}(\Sigma^\infty|X_{n+m}|) \;=\; (\underline{\mathbb{D}}X)(\mathbf{n} + \mathbf{m}) \ .$$

In more detail: the adjoint $\tilde{\sigma}^m : |X_n| \longrightarrow \Omega^m|X_{n+m}|$ of the iterated structure map of the spectrum $|X|$ freely generates a morphism of symmetric spectra $\widetilde{\sigma^m} : \Sigma^\infty|X_n| \longrightarrow \Omega^m(\Sigma^\infty|X_{n+m}|)$, and ι_* is given by the n-fold loop of this morphism.

If $\alpha : \mathbf{n} \longrightarrow \mathbf{n} + \mathbf{m}$ is an arbitrary injective map, then we choose a permutation $\gamma \in \Sigma_{n+m}$ such that $\gamma(i) = \alpha(i)$ for all $i = 1, \ldots, n$ and define $\alpha_* : (\underline{\mathbb{D}}X)(\mathbf{n}) \longrightarrow (\underline{\mathbb{D}}X)(\mathbf{n} + \mathbf{m})$ as the composite

$$(\underline{\mathbb{D}}X)(\mathbf{n}) \xrightarrow{\;\iota_*\;} (\underline{\mathbb{D}}X)(\mathbf{n} + \mathbf{m}) \xrightarrow{\;\gamma_*\;} (\underline{\mathbb{D}}X)(\mathbf{n} + \mathbf{m}) \ .$$

If $\bar{\gamma} \in \Sigma_{n+m}$ is another permutation that agrees with α on $\mathbf{n}$, then $\gamma^{-1}\bar{\gamma}$ fixes the set $\mathbf{n}$ elementwise, so $\gamma^{-1}\bar{\gamma} = 1_{\mathbf{n}}+\tau$ for a unique permutation $\tau \in \Sigma_m$. Since $(1_{\mathbf{n}}+\tau)_* \circ \iota_* = \iota_*$, the definition of α_* is independent of the choice of permutation. Functoriality in α is then straightforward.

We observe that in spectrum level 0, the functor $\underline{\mathbb{D}}X$ is given by

$$(\underline{\mathbb{D}}X)(\mathbf{n})_0 \ = \ \Omega^n|X_n| \ \cong \ \mathrm{map}(S^n, |X_n|) \ = \ (\Omega^\bullet|X|)(\mathbf{n}) \ ,$$

where the $\mathbf{I}$-space $\Omega^\bullet|X|$ was defined in Example 3.52. This last homeomorphism is natural in $\mathbf{n}$, so the $\mathbf{I}$-spectrum $\underline{\mathbb{D}}X$ is a spectrum level extensions of the $\mathbf{I}$-space $\Omega^\bullet|X|$.

Definition 8.51. Given a symmetric spectrum of simplicial sets X, the symmetric spectrum $\mathbb{D}X$ is given by

$$\mathbb{D}X \ = \ \mathrm{hocolim}_{\mathbf{I}}\, \underline{\mathbb{D}}X \ ,$$

the homotopy colimit [reference] of the functor $\underline{\mathbb{D}}X$.

For a symmetric spectrum of spaces Y we define the detection functor by first taking singular complex, i.e., we set $\mathbb{D}Y = \mathbb{D}(\mathcal{S}Y)$. We hope that using the same symbol for two different (but closely related) detection functors causes no trouble.

The symmetric spectrum $\mathbb{D}X$ is semistable. Indeed, for every $n \geq 0$ the symmetric spectrum $\Omega^n(\Sigma^\infty|X_n|)$ is underlying an orthogonal spectrum, and for every injection α, the morphism $\alpha_* : \mathbf{n} \longrightarrow \mathbf{n}+\mathbf{m}$ preserves the orthogonal group actions. Hence the homotopy colimit $\mathbb{D}X$ is underlying an orthogonal spectrum, and so it is semistable by Proposition 3.16 (vi).

The underlying sequence of the $\mathbf{I}$-symmetric spectrum $\underline{\mathbb{D}}X$, i.e., the restriction to the subcategory $\mathbb{N} \subset \mathbf{I}$ of inclusions, was already considered in [...], where we compared the mapping telescope of this sequence to the symmetric spectrum $\Omega^\infty\mathrm{sh}^\infty X$. A natural map

$$\mathrm{tel}_m\, \Omega^m(\Sigma^\infty|X_m|) = \mathrm{tel}_m\, (\underline{\mathbb{D}}X)(\mathbf{m}) \ \to \ \mathrm{hocolim}_{\mathbf{I}}\, \underline{\mathbb{D}}X$$

from the mapping telescope of the underlying sequence to the homotopy colimit over the entire category $\mathbf{I}$ is given by [...]

Proposition 8.52. *For every semistable symmetric spectrum X the morphism*

$$\mathrm{tel}_m\, \Omega^m(\Sigma^\infty|X_m|) = \mathrm{tel}_m(\underline{\mathbb{D}}X)(\mathbf{m}) \ \to \ \mathrm{hocolim}_{\mathbf{I}}\, \underline{\mathbb{D}}X$$

is a level equivalence. So the symmetric spectrum $\mathbb{D}X$ is an Ω-spectrum.

We combine Proposition 4.24, 5.59 and 8.52 and obtain:

Corollary 8.53. *For every semistable symmetric spectrum X the two morphisms*

$$X \ \xrightarrow{\lambda_X^\infty} \ \Omega^\infty\mathrm{sh}^\infty X \ \xleftarrow{\Phi_X} \ \mathrm{tel}_m\, \Omega^m(\Sigma^\infty|X_m|) \ \to \ \mathbb{D}X$$

are $\hat{\pi}_*$-isomorphisms, hence stable equivalences. Moreover, the symmetric spectra $\Omega^\infty\mathrm{sh}^\infty X$, $\mathrm{tel}_m\, \Omega^m(\Sigma^\infty|X_m|)$ and $\mathbb{D}X$ are Ω-spectra.

If X is not semistable, then we cannot control the stable homotopy type of the symmetric spectra $\Omega^\infty\mathrm{sh}^\infty X$ and $\mathrm{tel}_m\, \Omega^m(\Sigma^\infty|X_m|)$. The symmetric spectrum $\mathbb{D}X$, however, is always stably equivalent to X. We obtain a chain of four natural stable equivalences

$$X \ \to \ \Omega^\infty\mathrm{sh}^\infty(QX) \ \leftarrow \ \mathrm{tel}_m\, \Omega^m(\Sigma^\infty|(QX)_m|) \ \xrightarrow{\Phi_{QX}} \ \mathbb{D}(QX) \ \xleftarrow{\mathbb{D}\eta_X} \ \mathbb{D}X$$

as follows. The left morphism is the composite of the stable equivalence $\eta : X \longrightarrow QX$ whose target is an Ω-spectrum (compare Proposition 4.39) with $\lambda_{QX}^\infty : QX \longrightarrow \Omega^\infty\mathrm{sh}^\infty(QX)$. By using a different intermediate functor, this chain can actually be reduced to a chain of just two natural stable equivalences between the identity functor and the detection functor $\mathbb{D}$ [ref].

[we need that $\mathbb{D}$ preserves stable equivalence]

Proposition 8.54. *Let X be a symmetric spectrum of simplicial sets. If $f : A \longrightarrow B$ is a level equivalence, then so is $\mathbb{D}f : \mathbb{D}A \longrightarrow \mathbb{D}B$.*

PROOF. Since f is a level equivalence, the natural transformation $\underline{\mathbb{D}}f : \underline{\mathbb{D}}A \longrightarrow \underline{\mathbb{D}}B$ of functors $\mathbf{I} \longrightarrow \mathcal{S}p$ is a level equivalence at every object $\mathbf{n}$ of $\mathbf{I}$. Since homotopy colimits of symmetric spectra a formed levelwise, the induced map on homotopy colimits $\mathbb{D}f : \mathbb{D}A \longrightarrow \mathbb{D}B$ is also a level equivalence [need that $\underline{\mathbb{D}}X$ is levelwise and objectwise cofibrant space...] □

Theorem 8.55. *The following are equivalent for a morphism $f : A \longrightarrow B$ of symmetric spectra.*

(i) *$f : A \longrightarrow B$ is a stable equivalence.*
(ii) *The morphism $\mathbb{D}f : \mathbb{D}A \longrightarrow \mathbb{D}B$ is a $\hat{\pi}_*$-isomorphism.*
(iii) *The morphism $\mathbb{D}^2 f : \mathbb{D}^2 A \longrightarrow \mathbb{D}^2 B$ is a level equivalence.*

Exercises

Exercise E.I.1. The definition of a symmetric spectrum contains some redundancy. Show that the equivariance condition for the iterated structure map is already satisfied if for every $n \geq 0$ the following two conditions hold:

(i) the structure map $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ is Σ_n-equivariant where Σ_n acts on the target by restriction from Σ_{n+1} to the subgroup Σ_n.
(ii) the composite

$$X_n \wedge S^2 \xrightarrow{\sigma_n \wedge \mathrm{Id}} X_{n+1} \wedge S^1 \xrightarrow{\sigma_{n+1}} X_{n+2}$$

is Σ_2-equivariant.

Exercise E.I.2. Let X be a symmetric spectrum such that for infinitely many n the action of Σ_n on X_n is trivial. Show that all naive homotopy groups of X are trivial. (Hint: identify the quotient space of the Σ_2-action on S^2.) What can be said if infinitely many of the alternating groups act trivially?

Exercise E.I.3. Find a family $\{X^i\}_{i\in I}$ of symmetric spectra for which the natural map

$$\pi_0\left(\prod_{i\in I} X^i\right) \longrightarrow \prod_{i\in I} \pi_0(X^i)$$

is not surjective.

Exercise E.I.4. We recall that the m-th stable homotopy group $\pi^{\mathrm{s}}_m K$ of a based space (or simplicial set) is defined as the m-th maive homotopy groups $\hat{\pi}_m(\Sigma^\infty K)$, i.e., as the colimit of the sequence of abelian groups

$$\pi_m K \xrightarrow{-\wedge S^1} \pi_{m+1}(K \wedge S^1) \xrightarrow{-\wedge S^1} \pi_{m+2}(K \wedge S^2) \xrightarrow{-\wedge S^1} \cdots .$$

Show that the naive homotopy group $\hat{\pi}_k X$ of a symmetric spectrum X can also be calculated from the system of *stable* as opposed to *unstable* homotopy groups of the individual spaces X_n. where we stabilize from the left. Smashing with the identity of S^1 from the left provides a map $S^1 \wedge - : \pi^{\mathrm{s}}_m K \longrightarrow \pi^{\mathrm{s}}_{1+m}(S^1 \wedge K)$ which is a special case of the suspension isomorphism [ref] for the suspension spectrum of K.

Exercise E.I.5 (Coordinate free symmetric spectra)**.** There is an equivalent definition of symmetric spectra which is, in a certain sense, 'coordinate free'. If A is a finite set we denote by $\mathbb{R}^A$ the set of functions from A to $\mathbb{R}$ with pointwise structure as a $\mathbb{R}$-vector space. We let S^A denote the one-point compactification of $\mathbb{R}^A$, a sphere of dimension equal to the cardinality of A. A *coordinate free symmetric spectrum* consists of the following data:

- a based space $X(A)$ for every finite set A
- a based continuous map $\alpha_* : X(A) \wedge S^{B\backslash\alpha(A)} \longrightarrow X(B)$ for every injective map $\alpha : A \longrightarrow B$ of finite sets, where $B\backslash\alpha(A)$ is the complement of the image of α.

This data is subject to the following conditions:

- (Unitality) For every finite set A, the composite
$$X(A) \cong X(A) \wedge S^{\emptyset} \xrightarrow{(\mathrm{Id}_A)_*} X(A)$$
is the identity.
- (Associativity) For every pair of composable injections $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ the diagram
$$\begin{tikzcd}
X(A) \wedge S^{B\backslash\alpha(A)} \wedge S^{C\backslash\beta(B)} \arrow[r, "\mathrm{Id}\wedge\beta^!"] \arrow[d, "\alpha_*\wedge\mathrm{Id}"'] & X(A) \wedge S^{C\backslash\beta(\alpha(A))} \arrow[d, "(\beta\alpha)_*"] \\
X(B) \wedge S^{C\backslash\beta(B)} \arrow[r, "\beta_*"'] & X(C)
\end{tikzcd}$$
commutes. In the top vertical map we use the homeomorphism $\beta^! : S^{B\backslash\alpha(A)} \wedge S^{C\backslash\beta(B)} \cong S^{C\backslash\beta(\alpha(A))}$ which is one-point compactified from the linear isomorphism $\mathbb{R}^{B\backslash\alpha(A)} \times \mathbb{R}^{C\backslash\beta(B)} \cong \mathbb{R}^{C\backslash\beta(\alpha(A))}$ which uses β on the basis elements indexed by $B\backslash\alpha(A)$ and the identity on the basis elements indexed by $C\backslash\beta(B)$.

A coordinate free symmetric spectrum X gives rise to a symmetric spectrum by remembering only the values on the 'standard' finite sets $\mathbf{n} = \{1, \dots, n\}$. In other words, we take $X_n = X(\mathbf{n})$ as the nth level of the underlying symmetric spectrum. A permutation $\gamma \in \Sigma_n$ acts on X_n as the composite
$$X(\mathbf{n}) \cong X(\mathbf{n}) \wedge S^{\emptyset} \xrightarrow{\gamma_*} X(\mathbf{n}) \ .$$
We define the structure map $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ as the map
$$X(\mathbf{n}) \wedge S^1 \cong X(\mathbf{n}) \wedge S^{\{n+1\}} \xrightarrow{\ \iota_*\ } X(\mathbf{n+1})$$
where $\iota : \mathbf{n} \longrightarrow \mathbf{n+1}$ is the inclusion, and the homeomorphism $S^{\{n+1\}} \cong S^1$ arises from the linear isomorphism $\mathbb{R}^{\{n+1\}} \cong \mathbb{R}^1$ respecting the preferred bases.

(i) Show that the above construction indeed defines a symmetric spectrum in the sense of Definition 1.1.
(ii) Show that this 'forgetful' functor from coordinate free symmetric spectra to symmetric spectra is an equivalence of categories.
(iii) Work out how symmetric ring spectra are formulated in this language.

Exercise E.I.6. For an abelian group A we define a symmetric spectrum KA of simplicial sets as follows. We set $KA_n = A[\Delta[n]/\partial\Delta[n]]$, the A-linearization of the simplicial n-simplex modulo its boundary. The symmetric group acts on KA_n by multiplication by sign. The structure map
$$\sigma_n \ : \ A[\Delta[n]/\partial\Delta[n]] \wedge S^1 \ \longrightarrow \ A[\Delta[n+1]/\partial\Delta[n+1]]$$
is the 'A-linear extension' of the map $(\Delta[n]/\partial\Delta[n]) \wedge S^1 \longrightarrow A[\Delta[n+1]/\partial\Delta[n+1]]$ that sends the ith generating $(n+1)$-simplex of the source to $(-1)^i$ times the generating $(n+1)$-simplex of the target.

Show that the above really defines a symmetric spectrum and calculate the naive homotopy groups of KA in terms of the group A. For which A is KA an Ω-spectrum?

Exercise E.I.7. For any $m \geq 0$ and any symmetric spectrum Z the m-fold shift $\mathrm{sh}^m Z$ has a natural Σ_m-action as explained in Example 3.9. Show that the functor
$$\mathrm{sh}^m \ : \mathcal{S}p \ \longrightarrow \ \Sigma_m\text{-}\mathcal{S}p$$
has a left adjoint $\triangle_m : \mathcal{S}p \longrightarrow \Sigma_m\text{-}\mathcal{S}p$. Construct natural isomorphisms
$$\triangle_m(\Sigma_m^+ \wedge X) \ \cong \ \underbrace{\triangleright(\cdots(\triangleright}_{m} X)\cdots) \quad \text{and} \quad \triangle_m(L \wedge X) \ \cong \ L \triangleright_m X$$
where X is a (non-equivariant) symmetric spectrum and L any based Σ_m-space (or Σ_m-simplicial set).

Exercise E.I.8. As we discussed in Example 3.17, the shift functor for symmetric spectra has a left adjoint induction functor $\triangleright$. Show that the shift functor also has a *right* adjoint. If we denote the right adjoint by 'hs', construct a natural splitting

$$\triangleright^{1+m}(L, \mathrm{hs}Z) \;\cong\; \triangleright^{m}(\mathrm{sh}\, L, Z) \times \mathrm{hs}(\triangleright^{1+m}(L, Z))$$

where L is any based Σ_{1+m}-space (or simplicial set) and Z any symmetric spectrum.

Exercise E.I.9. In this exercise we show that the shift of a twisted smash product decomposes into two pieces which are themselves twisted smash products. For a symmetric spectrum X we define two natural morphisms

$$\zeta_{L,X} \;:\; (\mathrm{sh}\, L) \triangleright_m X \;\longrightarrow\; \mathrm{sh}(L \triangleright_{1+m} X) \qquad \text{and} \qquad \xi_{L',X} \;:\; L' \triangleright_m \mathrm{sh}\, X \;\longrightarrow\; \mathrm{sh}(L' \triangleright_m X)$$

as follows, where L is a based Σ_{1+m}-space and L' is a based Σ_m-space (or simplicial set). In level $m+n$ the morphism $\zeta_{L,X}$ is given by

$$((\mathrm{sh}\, L) \triangleright_m X)_{m+n} = \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} (\mathrm{sh}\, L) \wedge X_n \;\longrightarrow\; \Sigma^+_{1+m+n} \wedge_{\Sigma_{1+m} \times \Sigma_n} L \wedge X_n = \mathrm{sh}(L \triangleright_{1+m} X)_{m+n}$$

obtained by smashing the monomorphism $1 + - : \Sigma_{m+n} \longrightarrow \Sigma_{1+m+n}$ with the identity of L and X_n. We define the morphism $\xi_{L',X}$ by specifying a Σ_m-equivariant morphism $\bar{\xi}_{L',X} : L' \wedge \mathrm{sh}\, X \longrightarrow \mathrm{sh}^m(\mathrm{sh}(L' \triangleright_m X))$ as follows. In level n we take the composite is given by

$$\begin{aligned}(L' \wedge \mathrm{sh}\, X)_n = L' \wedge X_{1+n} \;&\xrightarrow{[1\wedge -]}\; \Sigma^+_{m+1+n} \wedge_{\Sigma_m \times \Sigma_{1+n}} L' \wedge X_{1+n} = (L' \triangleright_m X)_{m+1+n} \\ &\xrightarrow{\chi_{m,1}+1_n}\; (L' \triangleright_m X)_{1+m+n} = (\mathrm{sh}^m(\mathrm{sh}(L' \triangleright_m X)))_n \;\;.\end{aligned}$$

By the adjunction type isomorphism (3.29), there is thus a unique morphism $\xi_{L',X} \;:\; L' \triangleright_m \mathrm{sh}\, X \longrightarrow \mathrm{sh}(L' \triangleright_m X)$ such that $(\mathrm{sh}^m \xi_{L',X}) \circ \eta_{L',\mathrm{sh}\, X} = \bar{\xi}_{L',X}$.

Show that for every pointed Σ_{1+m}-space (or simplicial set) L and symmetric spectrum X the morphism

$$\zeta_{L,X} \vee \xi_{L,X} \;:\; (\mathrm{sh}\, L) \triangleright_m X \;\vee\; L \triangleright_{1+m} (\mathrm{sh}\, X) \;\longrightarrow\; \mathrm{sh}(L \triangleright_{1+m} X)$$

is an isomorphism.

Show that the morphism $\lambda_{L \triangleright_{1+m} X} : S^1 \wedge (L \triangleright_{1+m} X) \longrightarrow \mathrm{sh}(L \triangleright_{1+m} X)$ equals the composite

$$S^1 \wedge (L \triangleright_{1+m} X) \;\cong\; L \triangleright_{1+m} (S^1 \wedge X) \;\xrightarrow{L \triangleright_{1+m} \lambda_X}\; L \triangleright_{1+m} (\mathrm{sh}\, X) \;\xrightarrow{\xi_{L,X}}\; \mathrm{sh}(L \triangleright_{1+m} X) \;.$$

Show that the map

$$\lambda_{G_{1+m}L} \vee \;\xi :\; (S^1 \wedge G_{1+m}L) \;\vee\; G_m(\mathrm{sh}\, L) \;\longrightarrow\; \mathrm{sh}(G_{1+m}L)$$

is an isomorphism

Exercise E.I.10. Let R be a symmetric ring spectrum. Define mapping spaces (simplicial sets) and function symmetric spectra of homomorphisms between two given R-modules. Check that for all $k \geq 0$ the endomorphism ring spectrum $\mathrm{Hom}_R(k^+ \wedge R, k^+ \wedge R)$ of the R-module $k^+ \wedge R$ is isomorphic, as a symmetric ring spectrum, to the matrix ring spectrum $M_k(R)$ (see Example 3.44).

Exercise E.I.11. We let L be a based Σ_m-space or a based Σ_m-simplicial set and we let X and Z be two symmetric spectra in the appropriate context of spaces or simplicial sets. We discussed various adjunction bijections involving the semifree symmetric spectrum G_mL, the twisted smash product $L \triangleright_m X$ and the equivariant function spectrum $\triangleright^m(L, Z)$. The purpose of this exercise is to promote the bijections of morphism sets to isomorphisms of mapping spaces or even symmetric function spectra.

(i) Construct natural homeomorphisms of spaces respectively isomorphisms of simplicial sets

$$\begin{tikzcd}
 & \mathrm{map}(G_m L, \mathrm{Hom}(X,Z)) \arrow[d] \\
\mathrm{map}^{\Sigma_m}(L \wedge X, \mathrm{sh}^m Z) \arrow[r] & \Sigma_m\text{-}\mathbf{T}(L, \mathrm{map}(X, \mathrm{sh}^m Z)) \\
\mathrm{map}(L \rhd_m X, Z) \arrow[r] \arrow[u] & \mathrm{map}(X, \rhd^m(L,Z)) \arrow[u]
\end{tikzcd}$$

(ii) Construct natural isomorphisms of symmetric spectra

$$\begin{tikzcd}
 & \mathrm{Hom}(G_m L, \mathrm{Hom}(X,Z)) \arrow[d] \\
\mathrm{Hom}^{\Sigma_m}(L \wedge X, \mathrm{sh}^m Z) \arrow[r] & \rhd^m(L, \mathrm{Hom}(X,Z)) \\
\mathrm{Hom}(L \rhd_m X, Z) \arrow[r] \arrow[u] & \mathrm{Hom}(X, \rhd^m(L,Z)) \arrow[u]
\end{tikzcd}$$

Exercise E.I.12. In Example 3.23 we saw that every symmetric spectrum is naturally a coequalizer of semifree symmetric spectra. This fact has a dual version: every symmetric spectrum X is naturally an equalizer made up of co-semifree symmetric spectra $P_m L$ (see Example 4.2). Show that the diagram

$$X \longrightarrow \prod_{m\geq 0} P_m X_m \underset{I}{\overset{\tilde{\sigma}}{\rightrightarrows}} \prod_{n \geq 0} P_n(\Omega X_{n+1})$$

is an equalizer diagram. Here the upper morphism is the product of the morphisms $P_m\tilde{\sigma}_m : P_m X_m \longrightarrow P_m(\Omega X_{m+1})$; the lower map is the product of the morphisms $P_{m+1}X_{m+1} \longrightarrow P_m(\Omega X_{m+1})$ which are adjoint to the identity of $(P_{m+1}X_{m+1})_m = \Omega X_{m+1}$. [Morphisms to X are the same as morphisms to the equalizer]

Exercise E.I.13. In (4.18) we defined a functor $\Omega^\infty\mathrm{sh}^\infty : \mathcal{S}p \longrightarrow \mathcal{S}p$; for a symmetric spectrum X, $\Omega^\infty\mathrm{sh}^\infty X$ is a certain mapping telescope of the spectra $\Omega^m \mathrm{sh}^m X$. Let $f : A \longrightarrow B$ be a $\hat{\pi}_*$-isomorphism of symmetric spectra of spaces. Show that then the morphism $\Omega^\infty\mathrm{sh}^\infty f : \Omega^\infty\mathrm{sh}^\infty A \longrightarrow \Omega^\infty\mathrm{sh}^\infty B$ is a level equivalence.

Exercise E.I.14. We recall that a commutative square of space or simplicial sets

$$\begin{tikzcd}
V \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & W \arrow[d, "\psi"] \\
X \arrow[r, "\beta"'] & Y
\end{tikzcd}$$

is called *homotopy cartesian* if for some (hence any) factorization of the morphism ψ as the composite of a weak equivalence $w : W \longrightarrow Z$ followed by a Serre respectively Kan fibration $Z \longrightarrow Y$ the induced morphism

$$V \xrightarrow{(\varphi, w\alpha)} X \times_Y Z$$

is a weak equivalence. The definition is in fact symmetric in the sense that the square is homotopy cartesian if and only if the square obtained by interchanging X and W (and the morphisms) is homotopy cartesian. So if the square is homotopy cartesian and ψ (respectively β) is a weak equivalence, then so is φ (respectively α).

Consider a pullback square of symmetric spectra of simplicial sets

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "j"'] & D \end{tikzcd}$$

in which the morphism g is levelwise a Kan fibration. Show that for every injective Ω-spectrum X the commutative square of simplicial sets

$$\begin{tikzcd} \mathrm{map}(D,X) \arrow[r, "\mathrm{map}(g,X)"] \arrow[d, "\mathrm{map}(j,X)"'] & \mathrm{map}(B,X) \arrow[d, "\mathrm{map}(i,X)"] \\ \mathrm{map}(C,X) \arrow[r, "\mathrm{map}(f,X)"'] & \mathrm{map}(A,X) \end{tikzcd}$$

is homotopy cartesian.

Exercise E.I.15. This exercise elaborates on the term 'bimorphism' which we used in the first way to introduce the smash product of symmetric spectra. Let X, Y and Z be symmetric spectra.

(i) Let $b_{p,q} : X_p \wedge Y_q \longrightarrow Z_{p+q}$ be a collection of $\Sigma_p \times \Sigma_q$-equivariant maps. Show that the commutativity of the left part of (5.1) is equivalent to the condition that for every $p \geq 0$ the maps $b_{p,q} : X_p \wedge Y_q \longrightarrow Z_{p+q}$ form a morphism $b_{p,\bullet} : X_p \wedge Y \longrightarrow \mathrm{sh}^p Z$ of symmetric spectra as q varies. Show that the commutativity of the right part of (5.1) is equivalent to the condition that for every $q \geq 0$ the composite maps

$$Y_q \wedge X_p \xrightarrow{\text{twist}} X_p \wedge Y_q \xrightarrow{b_{p,q}} Z_{p+q} \xrightarrow{\chi_{p,q}} Z_{q+p}$$

form a morphism $Y_q \wedge X \longrightarrow \mathrm{sh}^q Z$ of symmetric spectra as p varies.

(ii) Let $b = \{b_{p,q} : X_p \wedge Y_q \longrightarrow Z_{p+q}\}$ be a bimorphism. Define $\bar{b}_p : X_p \longrightarrow \mathrm{map}(Y, \mathrm{sh}^p Z)$ as the adjoint of the morphism of symmetric spectra $b_{p,\bullet} : X_p \wedge Y \longrightarrow \mathrm{sh}^p Z$ (compare part (i)). Show that as p varies, the maps $\bar{b}_p$ form a morphism of symmetric spectra $\bar{b} : X \longrightarrow \mathrm{Hom}(Y, Z)$. Show then that the assignment

$$\mathrm{Bimor}((X,Y),Z) \longrightarrow \mathcal{S}p(X, \mathrm{Hom}(Y,Z)) , \quad b \mapsto \bar{b}$$

is bijective and natural in all three variables.

Exercise E.I.16. The way Hovey, Shipley and Smith introduce the smash product in their original paper [**36**] is quite different from our exposition, and this exercise makes the link. Thus the paper [**36**] has the solutions to this exercise. A *symmetric sequence* consists of pointed spaces (or simplicial set) X_n, for $n \geq 0$, with based, continuous (respectively simplicial) Σ_n-action on X_n. Morphisms $f : X \longrightarrow Y$ are sequences of equivariant based maps $f_n : X_n \longrightarrow Y_n$. The *tensor product* $X \otimes Y$ of two symmetric sequences X and Y is the symmetric sequence with nth term

$$(X \otimes Y)_n \;=\; \bigvee_{p+q=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \ .$$

(i) Make the tensor product into a closed symmetric monoidal product on the category of symmetric sequences.

(ii) Show that the sequence of spheres $\mathbb{S} = \{S^n\}_{n \geq 0}$ forms a commutative monoid in the category of symmetric sequences. Show that the category of symmetric spectra is isomorphic to the category of right $\mathbb{S}$-modules in the monoidal category of symmetric sequences.

(iii) Given a commutative monoid R in the monoidal category of symmetric sequences and two right R-modules M and N, show that the coequalizer $M \wedge_R N$ of the two morphisms

$$\alpha_M \otimes \mathrm{Id},\ \mathrm{Id} \otimes (\alpha_N \circ \tau_{R,N}) \ : \ M \otimes R \otimes N \ \to \ M \otimes N$$

is naturally a right R-module. Show that the smash product over R is a closed symmetric monoidal product on the category of right R-modules.

(iv) Show that the smash product over $\mathbb{S}$ corresponds to the smash product of symmetric spectra under the isomorphism of categories of part (ii).

Exercise E.I.17. Let X be a flat symmetric spectrum and Y a symmetric spectrum such that Y_0 is cofibrant and all structure morphisms $\sigma_n : Y_n \wedge S^1 \longrightarrow Y_{n+1}$ are cofibrations. Show that then the morphism

$$\xi^{1,0}_{X,Y} \ : \ (\mathrm{sh}\, X) \wedge Y \ \longrightarrow \ \mathrm{sh}(X \wedge Y)$$

defined in (5.15) is a level cofibration.

Hint: the special case $Y = \bar{\mathbb{S}}$ featured in the proof of Proposition **??**.

Exercise E.I.18. Show that the morphism of symmetric spectra

$$\wedge \ : \ \mathrm{Hom}(A, X) \wedge \mathrm{Hom}(B, Y) \ \longrightarrow \ \mathrm{Hom}(A \wedge B, X \wedge Y)$$

defined in (5.21) is adjoint to the composite

$$\mathrm{Hom}(A, X) \wedge \mathrm{Hom}(B, Y) \wedge A \wedge B \xrightarrow{\mathrm{Id} \wedge \tau_{\mathrm{Hom}(B,Y) \wedge A}}$$
$$\mathrm{Hom}(A, X) \wedge A \wedge \mathrm{Hom}(B, Y) \wedge B \xrightarrow{\mathrm{ev} \wedge \mathrm{ev}} X \wedge Y$$

where the morphisms $\mathrm{Hom}(A, X) \wedge A \longrightarrow X$ and $\mathrm{Hom}(B, Y) \wedge B \longrightarrow Y$ are the evaluation maps, adjoint to the identity morphism.

Exercise E.I.19. Let $\mathcal{C}$ be a symmetric monoidal category with monoidal product $\Box$, associativity isomorphism α and symmetric isomorphism τ. For every object X of $\mathcal{C}$ and every natural number $n \geq 1$ we define $X^{(n)}$ inductively by $X^{(1)} = X$ and $X^{(n)} = X^{(n-1)} \Box X$. For $n \geq 2$ and $1 \leq i \leq n-1$ we define an automorphism t_i of $X^{(n)}$ as follows. For $n = 2$ we take $t_1 = \tau_{X,X}$, the symmetry automorphism of $X^{(2)} = X \Box X$. For $n \geq 3$ and $i < n-1$ we take t_i as the automorphism $t_i \Box X$ of $X^{(n)} = X^{(n-1)} \Box X$. Finally, for $n \geq 3$ we take t_{n-1} as the composite automorphism of $X^{(n)} = (X^{(n-2)} \Box X) \Box X$

$$(X^{(n-2)} \Box X) \Box X \xrightarrow{\alpha_{X^{(n-2)},X,X}} X^{(n-2)} \Box (X \Box X)$$
$$\xrightarrow{X^{(n-2)} \Box \tau_{X,X}} X^{(n-2)} \Box (X \Box X) \xrightarrow{\alpha^{-1}_{X^{(n-2)},X,X}} (X^{(n-2)} \Box X) \Box X \ .$$

Show that we obtain an action of the symmetric group Σ_n on $X^{(n)}$ if we let the transposition $(i, i+1)$ act as t_i.

Exercise E.I.20. Define a notion of 'commuting homomorphisms' between symmetric ring spectra such that homomorphism of symmetric ring spectra $R \wedge S \longrightarrow T$ are in natural bijection with pairs of commuting homomorphisms $(R \longrightarrow T, S \longrightarrow T)$. Deduce that the smash product is the categorical coproduct for *commutative* symmetric ring spectra.

Exercise E.I.21. In this exercise we discuss an operator on symmetric spectra which has formal properties very similar to differentiation of functions. We define a functor

$$\partial \ : \ \mathcal{S}p \ \longrightarrow \ \mathcal{S}p$$

by

$$\partial X \ = \ \mathrm{cokernel}(\lambda_X : S^1 \wedge X \longrightarrow \mathrm{sh}\, X)$$

where the map λ_X was defined in (3.12). Show that the functor ∂ has the following properties.

(i) The 'derivative' is additive in the sense that ∂ commutes with colimits and satisfies $\partial(K \wedge X) \cong K \wedge \partial X$ for a pointed space (or simplicial set) K.

(ii) For the smash product of two symmetric spectra X and Y we have the 'Leibniz rule' in the form of a natural isomorphism

$$\partial(X \wedge Y) \ \cong \ (\partial X) \wedge Y \ \vee \ X \wedge (\partial X) \ .$$

(iii) For the semifree symmetric spectrum generated by a pointed Σ_m-space (or simplicial set) L we have $\partial(G_m L) = G_{m-1}(\mathrm{sh}\, L)$ where $\mathrm{sh}\, L$ is the restriction of L along the homomorphism $1 + - : \Sigma_{m-1} \longrightarrow \Sigma_m$.

(iv) For every pointed Σ_m-space (or simplicial set) L and every symmetric spectrum X, the twisted smash product 'differentiates' according to the rule

$$\partial(L \triangleright_m X) \;\cong\; (\mathrm{sh}\, L) \triangleright_{m-1} X \;\vee\; L \triangleright_m (\partial X) \ .$$

In particular, induction 'differentiates' as

$$\partial(\triangleright X) \;\cong\; X \;\vee\; \triangleright(\partial X) \ .$$

Show that $\partial(\triangleright X)$ is stably equivalent to $\triangleright(\mathrm{sh}\, X)$. Show that $\mathrm{sh}(\triangleright X)$ is stably equivalent to $X \vee X \vee \triangleright(\partial X)$.

(v) The free symmetric spectrum $F_n S^0$ 'differentiates' formally like the function x^n: there is an isomorphism

$$\partial(F_n) \;\cong\; F_{n-1} \wedge_{\Sigma_{n-1}} \Sigma_n^+$$

which is equivariant for the right Σ_m-action on the free coordinates. So non-equivariantly, $\partial(F_n)$ is a wedge of n copies of F_{n-1}.

(vi) The n-fold iterated derivative $\partial^{(n)} X$ has a natural action of the symmetric group Σ_n in such a way that its 0th level admits a natural isomorphism of Σ_n-spaces

$$(\partial^{(n)} X)_0 \;\cong\; \mathrm{cokernel}(\nu_n : L_n X \longrightarrow X_n) \ .$$

(vii) There is an analogue of the Taylor expansion $f(x) = \sum_{n\geq 0} f^{(n)}(0)/n! \cdot x^n$, but only 'up to extensions' in the following sense: the graded symmetric spectrum associated to the filtration of X by the spectra $F^n X$ (see Construction 5.29) is isomorphic to

$$\bigvee_{n\geq 0} (\partial^{(n)} X)_0 \wedge_{\Sigma_n} (F_1 S^0)^{\wedge n} \ .$$

(viii) Suppose that X is flat. Then ∂X is flat and X is semistable if and only if all naive homotopy groups of ∂X are trivial.

Exercise E.I.22. Show that the k-skeleton $F^k X$ of a symmetric spectrum X is a coequalizer of the two morphisms

$$\text{(E.I.23)} \qquad \bigvee_{0\leq n\leq k-1} G_{n+1}(\Sigma_{n+1}^+ \wedge_{\Sigma_n \times 1} X_n \wedge S^1) \;\underset{I}{\overset{\sigma}{\rightrightarrows}}\; \bigvee_{0\leq n\leq k} G_n X_n \ .$$

Here the upper map σ takes the nth wedge summand to the $(n+1)$st wedge summand by the adjoint of

$$\sigma_n \;:\; X_n \wedge S^1 \;\longrightarrow\; X_{n+1} \;=\; (G_{n+1} X_{n+1})_{n+1} \ .$$

The other map I takes the nth wedge summand to the nth wedge summand by the adjoint of the wedge summand inclusion

$$X_n \wedge S^1 \;\longrightarrow\; \Sigma_{n+1}^+ \wedge_{\Sigma_n \times \Sigma_1} (X_n \wedge S^1) \;=\; (G_n X_n)_{n+1}$$

indexed by the identity of Σ_{n+1}.

(Hint: note that the coequalizer (E.I.23) differs from the coequalizer (3.25) for X only by upper bounds on the wedge indices.)

Exercise E.I.24. In this exercise we characterize the latching space by the functor that it represents. Since $(F^{k-1}X)_{k-1} = X_{k-1}$, the structure map of the $(k-1)$-skeleton is a morphism

$$\sigma_{n-1} \;:\; X_{k-1} \wedge S^1 = (F^{k-1}X)_{k-1} \wedge S^1 \;\longrightarrow\; (F^{k-1}X)_k = L_k X$$

whose composite with the latching morphism $\nu_k : L_k X \longrightarrow X_k$ equals the structure map $\sigma_{k-1} : X_{k-1} \wedge S^1 \longrightarrow X_k$ of the spectrum X. Moreover, for every $n = 0, \ldots, k-1$ the composite of $\sigma_{k-1} : X_{k-1} \wedge S^1 \longrightarrow L_k X$ with $\sigma^{k-n-1} \wedge \mathrm{Id} : X_k \wedge S^{k-n} \longrightarrow X_{k-1} \wedge S^1$ is $\Sigma_n \times \Sigma_{k-n}$-equivariant.

(i) Show that $\tilde{\sigma}_{k-1}$ is the universal example of a map with the above properties: for every symmetric spectrum X and based Σ_k-space (or simplicial set) Z the map

$$\mathrm{map}^{\Sigma_k}(L_k X, Z) \longrightarrow \mathrm{map}(X_{k-1} \wedge S^1, Z)$$

given by precomposition with $\sigma_{k-1} : X_{k-1} \wedge S^1 \longrightarrow L_k X$ is a bijection from the space of based Σ_k-maps $L_k X \longrightarrow Z$ onto the subspace of those maps $f : X_{k-1} \wedge S^1 \longrightarrow Z$ such that the composite

$$f \circ (\sigma^{k-n-1} \wedge \mathrm{Id}) : X_n \wedge S^{k-n} \longrightarrow Z$$

is $\Sigma_n \times \Sigma_{k-n}$-equivariant for all $n = 0, \dots, k-1$. For example, the latching map $\nu_k : L_k X \longrightarrow X_k$ corresponds, under this bijection, to the structure map $\sigma_{k-1} : X_{k-1} \wedge S^1 \longrightarrow X_k$ of the spectrum X.

(ii) Describe the functor which the Σ_m-space $(F^i X)_m$ represents in a way which generalizes (i).

Exercise E.I.25. Let $\mathbb{S}^{[k]}$ denote the symmetric subspectrum of the sphere spectrum obtained by truncating below level k, i.e.,

$$(\mathbb{S}^{[k]})_n = \begin{cases} * & \text{for } n < k \\ S^n & \text{for } n \geq k. \end{cases}$$

For example we have $\mathbb{S}^{[0]} = \mathbb{S}$ and $\mathbb{S}^{[1]} = \bar{\mathbb{S}}$.

(i) Show that for every symmetric spectrum X and all $k \geq 0$ the inclusion $\mathbb{S}^{[k]} \longrightarrow \mathbb{S}$ induces a $\hat{\pi}_*$-isomorphism $\mathbb{S}^{[k]} \wedge X \longrightarrow \mathbb{S} \wedge X \cong X$.

(ii) In Proposition 5.39 we identified the latching space $L_m X = (F^{m-1}X)_m$ of a symmetric spectrum X with the m-th level of the spectrum $X \wedge \bar{\mathbb{S}}$. Generalize this by constructing a natural Σ_m-equivariant isomorphism between $(F^i X)_m$ and $(X \wedge \mathbb{S}^{[m-i]})_m$ for $i \leq m$.

Exercise E.I.26. A *k-truncated symmetric spectrum* is a sequence of pointed spaces (or simplicial sets) X_n for $0 \leq n \leq k$, a basepoint preserving continuous left action of the symmetric group Σ_n on X_n for each $0 \leq n \leq k$ and based structure maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ for $0 \leq n \leq k-1$. This data is subject to the same equivariance condition as for symmetric spectrum on the iterated structure maps $\sigma^m : X_n \wedge S^m \longrightarrow X_{n+m}$, but only where it makes sense, i.e., as long as $n + m \leq k$. A *morphism* $f : X \longrightarrow Y$ of k-truncated symmetric spectra consists of Σ_n-equivariant based maps $f_n : X_n \longrightarrow Y_n$ for $0 \leq n \leq k$ which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge \mathrm{Id}_{S^1})$ for all $0 \leq n \leq k-1$. The category of k-truncated symmetric spectra is denoted by $\mathcal{S}p_{\leq k}$. There is a forgetful functor $\tau_{\leq k} : \mathcal{S}p \longrightarrow \mathcal{S}p_{\leq k}$ which forgets all information above level k, and which we call *truncation at level k*.

(i) Show that the truncation functor $\tau_{\leq k}$ has both a left and right adjoint.

(ii) Let $l : \mathcal{S}p_{\leq k} \longrightarrow \mathcal{S}p$ be a left adjoint to the truncation functor $\tau_{\leq k}$. Show that the k-th skeleton $F^k X$ of a symmetric spectrum X is isomorphic to $l(\tau_{\leq k} X)$ in such a way that the morphism $i_k : F^k X \longrightarrow X$ corresponds to the adjunction counit.

(iii) We define (the object function of) a functor $r_k : \mathcal{S}p_{\leq k} \longrightarrow \mathcal{S}p$ as follows. For a k-truncated symmetric spectrum X we set

$$(r_k X)_n = \begin{cases} * & \text{for } n > k, \\ X_n & \text{for } n \leq k. \end{cases}$$

The symmetric groups actions and structure maps of $r_k X$ are those of X wherever this makes sense. Show that r_k is a right adjoint of the truncation functor $\tau_{\leq k}$.

(iv) Show that the commutative square

$$\begin{tikzcd} r_k(\tau_{\leq k} X) \arrow[r] \arrow[d] & r_{k-1}(\tau_{\leq k-1} X) \arrow[d] \\ P_k X_k \arrow[r] & r_{k-1}(\tau_{\leq k-1}(P_k X_k)) \end{tikzcd}$$

is a pullback for every symmetric spectrum X.

Exercise E.I.27. Let X and Y be symmetric spectra; define the *homotopy smash product* $X \wedge^h Y$ as the homotopy colimit (as opposed to colimit...). Show that this is homotopy invariant in the context of simplicial sets, or if both factors are levelwise cofibrant when in the context of spaces. Show that whenever one factor is flat and the other one level cofibrant, then $X \wedge^h Y \longrightarrow X \wedge Y$ is a stable equivalence [even more?]

Exercise E.I.28. Let X be a flat symmetric spectrum and Y level cofibrant. Show that $X \wedge Y$ is stably equivalent to

$$\mathrm{hocolim}_{(\mathbf{n},\mathbf{m})\in \mathbf{I}\times\mathbf{I}} \ \Omega^{n+m}(\Sigma^\infty(X_n \wedge Y_m))$$

When is $X \wedge Y$ $\hat{\pi}_*$-isomorphic to this homotopy colimit? How about more than two smash factors?

Exercise E.I.29. Symmetric spectra can be smashed together, and they can be smashed with a based space or simplicial set. We have also defined two isomorphisms

$$(K \wedge X) \wedge Y \xrightarrow{a_1} K \wedge (X \wedge Y) \xleftarrow{a_2} X \wedge (K \wedge Y)$$

that intertwine these different types of smash product constructions, where K is a based space (or simplicial set) and X and Y are symmetric spectra. Show that these isomorphisms satisfy the 'mixed' coherence condition, i.e., the following diagrams commute:

$$\begin{tikzcd}
 & ((K\wedge X)\wedge Y)\wedge Z \arrow[dl, "a_1^{K,X,Y}\wedge Z"'] \arrow[dr, "\alpha_{K\wedge X,Y,Z}"] & \\
(K\wedge(X\wedge Y))\wedge Z \arrow[d, "a_1^{K,X\wedge Y,Z}"'] & & (K\wedge X)\wedge(Y\wedge Z) \arrow[d, "a_1^{K,X,Y\wedge Z}"] \\
K\wedge((X\wedge Y)\wedge Z) \arrow[rr, "K\wedge\alpha_{X,Y,Z}"'] & & K\wedge(X\wedge(Y\wedge Z)))
\end{tikzcd}$$

$$\begin{tikzcd}
(K\wedge X)\wedge Y \arrow[r, "a_1"] \arrow[d, "\tau_{K\wedge X,Y}"'] & K\wedge(X\wedge Y) \arrow[d, "K\wedge\tau_{X,Y}"] \\
Y\wedge(K\wedge X) \arrow[r, "a_2"'] & K\wedge(Y\wedge X)
\end{tikzcd}$$

Exercise E.I.30. Recall that for symmetric spectra of simplicial set A and B, the spectra of spaces $|A\wedge B|$ and $|A| \wedge |B|$ are naturally isomorphic. The singular complex, however, does not commute with smash product. There is a natural and lax monoidal map $\mathcal{S}(X) \wedge \mathcal{S}(Y) \longrightarrow \mathcal{S}(X \wedge Y)$, compare (5.23).

Show that for all symmetric spectra of spaces X and Y such that X is flat and Y is level cofibrant, the composite

$$\mathcal{S}(X)^\flat \wedge \mathcal{S}(Y) \xrightarrow{r\wedge \mathrm{Id}} \mathcal{S}(X) \wedge \mathcal{S}(Y) \longrightarrow \mathcal{S}(X \wedge Y)$$

is a level equivalence, where $r : \mathcal{S}(X)^\flat \longrightarrow \mathcal{S}(X)$ is the flat resolution.

Exercise E.I.31. We have shown in Proposition 4.28 that for every symmetric spectrum X the morphism $\hat{\lambda}_X : S^1 \wedge \triangleright X \longrightarrow X$ is a stable equivalence. So we can define a natural *induction isomorphism*

$$\triangleright \ : \ \pi_{1+k} X \longrightarrow \pi_k(\triangleright X) \qquad \text{by} \qquad \triangleright x \ = \ S^{-1} \wedge (\hat{\lambda}_X)_*^{-1}(x) \ .$$

In other words, $\triangleright$ is defined so that the triangle

$$\begin{tikzcd}
\pi_{1+k}X \arrow[rr, "\triangleright"] & & \pi_k(\triangleright X) \arrow[dl, "S^1\wedge -"] \\
 & \pi_{1+k}(S^1\wedge \triangleright X) \arrow[ul, "(\hat{\lambda}_X)_*"] &
\end{tikzcd}$$

of isomorphisms commutes, starting at any of the vertices.

(i) Show that for every symmetric spectrum X of spaces and integer k the composite

$$\pi_k(\rhd X) \xrightarrow{\pi_k(\bar\lambda_X)} \pi_k(\Omega X) \xrightarrow{\alpha} \pi_{1+k} X$$

is the *negative* of the inverse of the induction isomorphism $\rhd : \pi_{k+1} X \longrightarrow \pi_k(\rhd X)$. Here the morphism $\bar\lambda_A : \rhd A \longrightarrow \Omega A$ was defined in (4.27) and the loop isomorphism α was discussed in Proposition 6.7.

(ii) Show that the diagram

$$\begin{tikzcd} \pi_{1+k}X \arrow[rr, "S^1\wedge -"] & & \pi_k(S^1\wedge X) \arrow[dl, "\rhd"] \\ & \pi_{1+k}(\rhd(S^1\wedge X)) \arrow[ul, "(\hat\lambda'_X)_*"] & \end{tikzcd}$$

of isomorphisms commutes up to the sign -1, starting at any of the vertices, where $\hat\lambda'_X : \rhd(S^1 \wedge X) \longrightarrow X$ is the stable equivalence adjoint to $\lambda_X : S^1 \wedge X \longrightarrow \operatorname{sh} X$. (The induction isomorphism $\rhd : \pi_{1+k} X \longrightarrow \pi_k(\rhd X)$ and the suspension isomorphism satisfy the following relations:

(E.I.32) $$\hat\lambda_X(S^1 \wedge (\rhd x)) \ = \ x \ = \ -\hat\lambda'_X(\rhd(S^1 \wedge x)) \ .$$

Indeed, the first relation holds by definition of the induction isomorphism $\rhd$, the second was just shown. The sign is to be expected somewhere because the suspension and induction operators have degree 1; so when they change places, a sign is supposed to come up. More formally, the proof of this exercise shows that when comparing $\hat\lambda_X(\cong (\rhd(S^1 \wedge x)))$ and $\hat\lambda'_X(\rhd(S^1 \wedge x))$, a twist isomorphism of $S^1 \wedge S^1$ comes up, so these two expressions differ by a sign. At which of the two sides the sign occures is a matter of convention.)

(iii) The effect of the isomorphism $b_2 : S^n \wedge \rhd X \longrightarrow \rhd(S^n \wedge X)$ on true homotopy groups is given by

$$(b_2)_*(S^n \wedge \rhd x) \ = \ (-1)^n \cdot \rhd(S^n \wedge x) \ .$$

[definition of b_2]

(iv) The maps induced by the isomorphisms

$$(\rhd X) \wedge Y \xrightarrow{b_1} \rhd(X \wedge Y) \xleftarrow{b_2} X \wedge (\rhd Y)$$

on true homotopy groups satisfy

(E.I.33) $$(b_1)_*((\rhd x) \cdot y) \ = \ \rhd(x \cdot y) \ = \ (-1)^k \cdot (b_2)_*(x \cdot (\rhd y))$$

for all integers k, l and true homotopy classes $x \in \pi_k X$ and $y \in \pi_l Y$.

Exercise E.I.34. The shift construction for symmetric spectra preserves naive homotopy groups, but it does *not* in general preserve stable equivalences, see Example 4.34. Hence we cannot expect the true homotopy groups of $\operatorname{sh} X$ to be a functor in the true homotopy groups of X. However, some things can be saif about the stable homotopy type of $\operatorname{sh} X$.

We define a natural *shift morphism*

(E.I.35) $$\operatorname{sh}_! \ : \ \pi_{k+1}(\operatorname{sh} X) \ \longrightarrow \ \pi_k X \quad \text{by} \quad \operatorname{sh}_!(x) \ = \ (-1)^{k+1} \cdot \epsilon_*(\rhd x) \ ,$$

where the induction isomorphism $\rhd$ is as in Exercise E.I.31 and $\epsilon : \rhd(\operatorname{sh} X) \longrightarrow X$ is the adjunction counit, see Example 3.17.

(i) Show that the shift homomorphism is an isomorphisms whenever X is semistable.

(ii) Show that the composite

$$\pi_k X \xrightarrow{S^1\wedge -} \pi_{1+k}(S^1 \wedge X) \xrightarrow{(-1)^k(\lambda_X)_*} \pi_{k+1}(\operatorname{sh} X)$$

is a section to the shift homomorphism $\operatorname{sh}_! : \pi_{k+1}(\operatorname{sh} X) \longrightarrow \pi_k X$, which is thus a split epimorphism..

(iii) Show that the map
$$\mathrm{sh}_! \oplus i_* \ : \ \pi_{k+1}(\mathrm{sh}\, X) \ \longrightarrow \ \pi_k X \oplus \pi_{k+1} C(\lambda_X)$$
is an isomorphism where $i : \mathrm{sh}\, X \longrightarrow C(\lambda_X)$ is the mapping cone inclusion of the morphism $\lambda_X : S^1 \wedge X \longrightarrow \mathrm{sh}\, X$.

(iv) Show that the map
$$\mathrm{sh}\, X \ \xrightarrow{(\mathrm{sh}(\eta_X), i)} \ \mathrm{sh}(QX) \times C(\lambda_X)$$
is a stable equivalence.

Exercise E.I.36. The naive and true fundamental classes $\hat{\iota}^n_m \in \hat{\pi}_{n-m}(F_m S^n)$ respectively $\iota^n_m \in \pi_{n-m}(F_m S^n)$ were defined in (6.4).

(i) Show that the canonical isomorphism
$$a \ : \ S^1 \wedge F_m S^n \ \cong \ F_m S^{1+n}$$
satisfies
$$a_*(S^1 \wedge \hat{\iota}^n_m) \ = \ \hat{\iota}^{1+n}_m \quad \text{and} \quad a_*(S^1 \wedge \iota^n_m) \ = \ \iota^{1+n}_m \ .$$

(ii) Let $\lambda : F_{m+1} S^1 \longrightarrow F_m$ be the morphism adjoint to the map
$$1 \wedge - \ : \ S^1 \ \longrightarrow \ \Sigma_{m+1} \wedge S^1 = (F_m)_{m+1} \ .$$
Show that $\lambda_*(\hat{\iota}^1_{m+1}) = \hat{\iota}_m$ and $\lambda_*(\iota^1_{m+1}) = \iota_m$.

(iii) Let $b : \triangleright F_m \longrightarrow F_{1+m}$ be the isomorphism [...] Show that $b_*(\triangleright \hat{\iota}_m) = \hat{\iota}_{1+m}$ and $b_*(\triangleright \iota_m) = \iota_{1+m}$.

(iv) Let $b : F_p S^n \wedge F_q S^m \longrightarrow F_{p+q} S^{n+m}$ be the morphism (in fact an isomorphism) corresponding to the unique bimorphism whose (p,q)-component is
$$(F_m S^n)_m \wedge (F_p S^q)_p \ = \ (\Sigma^+_m \wedge S^n) \wedge (\Sigma^+_p \wedge S^q)$$
$$\xrightarrow{(\sigma \wedge x)\wedge(\tau \wedge y) \ \mapsto \ (\sigma + \tau)\wedge x \wedge y} \ \Sigma^+_{m+p} \wedge S^{n+q} = (F_{m+p} S^{n+q})_{m+p} \ .$$
Show that $b_*(\hat{\iota}^n_m \cdot \hat{\iota}^q_p) = \hat{\iota}^{n+q}_{m+p}$ and $b_*(\iota^n_m \cdot \iota^q_p) = \iota^{n+q}_{m+p}$. **[check the sign...]**

Exercise E.I.37. Let X be a symmetric spectrum and $x \in \pi_0 X$ a true homotopy class. Then there exists a flat symmetric spectrum Z, a stable equivalence $f : Z \longrightarrow \mathbb{S}$ and a morphism $g : Z \longrightarrow X$ such that $x = g_*(f_*^{-1}(1))$.

Exercise E.I.38 (Variations on true homotopy groups)**.** Let $\mathcal{C}$ be a class of symmetric spectra and define a 'homotopy group relative to $\mathcal{C}$' by
$$\pi^{\mathcal{C}}_k A \ = \ \mathrm{Nat}_{\mathcal{C} \to \mathbf{set}}(\mathcal{S}p(A, -)|_{\mathcal{C}}, \hat{\pi}_k) \ .$$
(In this generality, the 'collection' of natural transformations need not form a set, but that will be the case of interest below.) For $\mathcal{B} \subset \mathcal{C}$ we get a homomorphism $\pi^{\mathcal{C}}_k \longrightarrow \pi^{\mathcal{B}'}_k$ by restricting a natural transformation to the smaller category.

(i) Let $\mathcal{B} \subset \mathcal{C}$ be two classes of symmetric spectra such that there is a functor $F : \mathcal{C} \longrightarrow \mathcal{C}$ with values in $\mathcal{B}'$ and a natural $\hat{\pi}_*$-isomorphism $X \longrightarrow FX$. Show that the restriction homomorphism $\pi^{\mathcal{C}}_k \longrightarrow \pi^{\mathcal{B}}_k$ is an isomorphism.

(ii) Show that the true homotopy group $\pi_k A$ is naturally isomorphic to the homotopy group relative any of the following classes of symmetric spectra: (a) Ω-spectra; (b) flat fibrant Ω-spectra; (c) injective Ω-spectra; (d) positive Ω-spectra; (e) flat fibrant positive Ω-spectra; (f) injective positive Ω-spectra; (g) flat fibrant semistable spectra; (h) injective semistable spectra. Can you think of other classes of symmetric spectra that can be used here?

(iii) Suppose that the class $\mathcal{C}$ contains a symmetric spectrum which is not semistable. Show that then $\pi^{\mathcal{C}}_k A$ is not naturally isomorphic to the true homotopy group functor.

(iv) Let $\mathcal{C}$ be the class of symmetric spectra X such that for all $n \geq 0$ the alternating group A_n acts trivially on X_n. Show that every spectrum in $\mathcal{C}$ is semistable and that the restriction map $\pi_k A \longrightarrow \pi^{\mathcal{C}}_k A$ extends to an isomorphism $\mathbb{Q} \otimes \pi_k A \cong \pi^{\mathcal{C}}_k A$.

Exercise E.I.39. The true homotopy groups of a symmetric spectrum A of simplicial sets were defined in 6.1 via the geometric realization. This exercise shows that instead we could have defined $\pi_k A$ as a group of natural transformations, analogous to Definition 6.1, but within the category of semistable symmetric spectra of simplicial sets.

We define a group $\pi_k^{\mathbf{sS}} A$ by

$$\pi_k^{\mathbf{sS}} A \;=\; \mathrm{Nat}_{\mathcal{S}p^{\mathrm{ss}}_{\mathbf{sS}} \to \mathbf{set}}(\mathcal{S}p(A,-), \hat{\pi}_k) \ ,$$

the set of natural transformations, of functors from semistable symmetric spectra *of simplicial sets* to sets, from the restriction of the representable functor to the restriction of k-th naive homotopy group functor. Construct a natural isomorphism of abelian groups between $\pi_k^{\mathbf{sS}} A$ and $\pi_k|A|$.

Exercise E.I.40. We let $b : (K \wedge S^n) \wedge F_{m+n} \longrightarrow K \wedge F_m$ be the morphism adjoint to the map

$$[- \wedge 1 \wedge -] \ : \ K \wedge S^n \ \longrightarrow \ K \wedge \Sigma^+_{m+n} \wedge_{1 \times \Sigma_n} S^n \ = \ (K \wedge F_m)_{m+n} \ .$$

(i) For which permutation $\sigma \in \Sigma_{m+1}$ is the composite

$$S^1 \wedge F_{1+m} \ \xrightarrow{\cdot\sigma} \ S^1 \wedge F_{m+1} \ \xrightarrow{b} F_m$$

equal to the morphism $\lambda : S^1 \wedge F_{1+m} \longrightarrow F_m$?

(ii) Show that the map b is a stable equivalence.

(iii) As a special case for $K = S^p$ we obtain a stable equivalence $b : S^{p+n} \wedge F_{m+n} \longrightarrow S^p \wedge F_m$. The true homotopy group $\pi_{p-m}(S^{p+n} \wedge F_{m+n})$ is freely generated by the fundamental class ι^{p+n}_{m+n}, and $\pi_{p-m}(S^p \wedge F_m)$ is freely generated ι^p_m, so we must have $b_*(\iota^{p+n}_{m+n}) = \pm \iota^p_m$. Determine the correct sign as a function of m, n and p.

Exercise E.I.41. For $n \geq 1$, let $F : \mathcal{S}p^n \longrightarrow$ (sets) be a functor of n variables to the category of sets which takes stable equivalences between flat symmetric spectra in each variable to bijections. Then evaluation at $(1, \dots, 1) \in (\pi_0 \mathbb{S})^n$ is a bijection from the set of natural transformations

$$\pi_0 X^1 \times \cdots \times \pi_0 X^n \ \longrightarrow \ F(X^1, \dots, X^n)$$

to $F(\mathbb{S}, \dots, \mathbb{S})$.

Exercise E.I.42. In Examples 1.20. and 6.58 we introduced the commutative symmetric ring spectra KU and KO of periodic complex and real topological K-theory.

(i) Show that every graded $\pi_* KU$-module is realizable as the homotopy of a KU-module spectrum.

(ii) Show that the only cyclic graded $\pi_* KO$-modules which are realizable as the homotopy of a KO-module spectrum are the free module, the trivial module and $\mathbb{Z}/n \otimes \pi_* KO$ for n an odd integer. (Hint: produce enough nonzero Toda brackets and use Proposition 2.13)

(iii) Recall from Example 6.39 how to 'mod out' a homotopy class x of a symmetric ring spectrum R on an R-module M, producing an R-module M/x. Calculate the homotopy groups of $KO/2$ as a $\pi_* KO$-module and show that the double suspension of KO/ξ is stably equivalent, as a KO-module spectrum, to $KO/2$. [can we show that $\eta\beta^n$ and $\eta^2\beta^n$ are in the Hurewicz image in ko_*, using bracket manipulations?]

Exercise E.I.43. The construction $f^{-1}X$ that inverts a 'graded selfmap' $f : S^k \wedge X \longrightarrow \mathrm{sh}^n X$ of a symmetric spectrum X (see Construction 6.46) has some naturality and functoriality properties that we work out in this exercise.

(i) Consider another symmetric spectrum Y and a graded selfmap $g : S^l \wedge Y \longrightarrow \mathrm{sh}^m Y$. Construct a pairing of symmetric spectra

$$M(f, \mathbf{p}) \wedge N(g, \mathbf{p}) \ \longrightarrow \ (M \wedge N)(f \cdot g, \mathbf{p}) \ ,$$

where $f \cdot g$ is the graded endomorphism of $M \wedge N$ defined as the composite

$$S^{k+l} \wedge (M \wedge N) \cong (S^k \wedge M) \wedge (S^l \wedge N) \ \xrightarrow{f \wedge g} \ \mathrm{sh}^n M \wedge \mathrm{sh}^m N \ \xrightarrow{\xi} \mathrm{sh}^{n+m}(M \wedge N)$$

which is the identity for $p = 0$. Show that this pairing is suitably associative, commutative and unital.

(ii) Consider a second graded selfmap $f' : S^l \wedge X \longrightarrow \mathrm{sh}^m X$ of X. Define the *composite* $f \circ f' : S^{k+l} \wedge X \longrightarrow \mathrm{sh}^{n+m} X$ of f and f' the composite

$$S^{k+l} \wedge X \xrightarrow{S^k \wedge f'} S^k \wedge \mathrm{sh}^m X \;=\; \mathrm{sh}^m(S^k \wedge X) \xrightarrow{\mathrm{sh}^m f} \mathrm{sh}^m(\mathrm{sh}^n X) \;=\; \mathrm{sh}^{n+m} X \; .$$

Construct a morphism of $\mathbf{I}$-functors $f_* : M(f', -) \longrightarrow M(f \circ f', -)$ which is the identity on the object $\mathbf{0}$ of $\mathbf{I}$ and which is associative in the sense that such that for a third graded selfmap f'' the composite

$$M(f'', -) \xrightarrow{f'_*} M(f' \circ f'', -) \xrightarrow{f_*} M(f \circ f' \circ f'', -)$$

equals $(f \circ f')_*$.

Exercise E.I.44. [right to left...] Let R be a symmetric ring spectrum and $x : S^{k+n} \longrightarrow R_n$ and $y : S^{l+m} \longrightarrow R_m$ based continuous maps. As in [...] we define the product $x \cdot y$ as the composite

$$S^{k+l+n+m} \xrightarrow{k \times \chi_{l,n} \times m} S^{k+n+l+m} \xrightarrow{x \wedge y} R_n \wedge R_m \xrightarrow{\mu_{n,m}} R_{n+m} \; .$$

Given any right R-module M, in (6.36) we defined a homomorphism

$$\rho_x \; : \; M \wedge F_n S^{k+n} \;\longrightarrow\; M \; ,$$

which should be thought of a 'right multiplication by x'. This exercise shows that right multiplication by the product $x \cdot y$ is essentially the composite of ρ_x followed by ρ_y.

More precisely, show that square

$$\begin{tikzcd}
(M \wedge F_n S^{k+n}) \wedge F_m S^{l+m} \arrow[r, "\rho_x \wedge F_m S^{l+m}"] \arrow[d, "\cong"'] & M \wedge F_m S^{l+m} \arrow[d, "\rho_y"] \\
M \wedge F_{n+m} S^{k+l+n+m} \arrow[r, "\rho_{x \cdot y}"'] & M
\end{tikzcd}$$

commutes. [specify the isomorphism]

Exercise E.I.45 (Coordinate free orthogonal spectra)**.** There is a more natural notion where we use vector spaces with inner product to index the spaces in an orthogonal spectrum. In the following, an *inner product space* is a finite dimensional real vector space with a euclidian scalar product. Given two real inner product spaces V and W we define a based space $\mathbf{O}(V, W)$ as follows. First we consider the vector bundle $\xi(V, W)$ over the space $\mathcal{L}(V, W)$ of linear isometric embedding from V to W whose fiber over $\varphi : V \longrightarrow W$ is $W - \varphi(V)$, the orthogonal complement of the image of φ. Then $\mathbf{O}(V, W)$ is the Thom space of the bundle $\xi(V, W)$. If we are given another inner product space U, there is a bundle map [make explicit]

$$\xi(U, V) \times \xi(V, W) \;\longrightarrow\; \xi(U, W)$$

which covers the composition map $\mathcal{L}(U, V) \times \mathcal{L}(V, W) \longrightarrow \mathcal{L}(U, W)$.

A *coordinate free orthogonal spectrum* X consists of the following data:

- a pointed space $X(V)$ for each inner product space V, and
- a based continuous map
$$X(V) \wedge \mathbf{O}(V, W) \;\longrightarrow\; X(W)$$
for each pair V, W of inner product spaces.

This data should satisfy two conditions:

- the element $(\mathrm{Id}_V, 0)$ in $\mathbf{O}(V, V)$ acts as the identity on $X(V)$ for every inner product space V;
- the square

$$\begin{tikzcd}
X(U) \wedge \mathbf{O}(U, V) \wedge \mathbf{O}(V, W) \arrow[r, "\circ \wedge \mathrm{Id}"] \arrow[d, "\mathrm{Id} \wedge \circ"'] & X(V) \wedge \mathbf{O}(V, W) \arrow[d, "\circ"] \\
X(U) \wedge \mathbf{O}(U, W) \arrow[r, "\circ"'] & X(W)
\end{tikzcd}$$

commutes for every triple of inner product spaces U, V and W.

A *morphism* $f : X \longrightarrow Y$ of coordinate free orthogonal spectra consists of based maps $f(V) : X(V) \longrightarrow Y(V)$ for all V which are compatible with the structure maps in the sense that for every pair of inner product spaces V and W the square

$$\begin{tikzcd} X(V) \wedge \mathbf{O}(V,W) \arrow[r] \arrow[d] & X(W) \arrow[d] \\ Y(V) \wedge \mathbf{O}(V,W) \arrow[r] & Y(W) \end{tikzcd}$$

commutes.

(i) prove an equivalence of categories.

A coordinate free orthogonal spectrum X gives rise to a coordinate free symmetric spectrum UX (see Exercise E.I.5) by forgetting symmetry. For a finite set A the space $(UX)_A$ is $X(\mathbb{R}^A)$, the value of X at the inner product space $\mathbb{R}^A$ which has A as orthonormal basis. [define the structure maps $\alpha_* : (UX)_A \wedge S^{B-\alpha(A)} \longrightarrow (UX)_B$]

Exercise E.I.46 (Coordinate free unitary spectra)**.** Give a coordinate free description of unitary spectra, analogous to the coordinate free description of orthogonal spectra in Exercise E.I.45. Find coordinate free descriptions of the functors $\Psi : \mathcal{S}p^{\mathbf{O}} \longrightarrow \mathcal{S}p^{\mathbf{U}}$ and $\Phi : \mathcal{S}p^{\mathbf{U}} \longrightarrow \mathcal{S}p^{\mathbf{O}}$ originally defined in (7.6) respectively (7.7).

Exercise E.I.47. Recall from (7.41) the $\mathbf{T}$-category (or $\mathbf{sS}$-category) $\mathbf{\Sigma}$ that parameterizes symmetric spectra.

(i) We define a functor $+ : \mathbf{\Sigma} \times \mathbf{\Sigma} \longrightarrow \mathbf{\Sigma}$ on objects by addition of natural numbers and on morphisms by
$$[\tau \wedge z] \ + \ [\gamma \wedge y] \ = \ [(\tau + \gamma) \wedge (1 + \chi_{n,\bar{m}} + 1)_*(z \wedge y)] \ \in \ \mathbf{\Sigma}(n + \bar{n}, m + \bar{m})$$
for $[\tau \wedge z] \in \mathbf{\Sigma}(n, m)$ and $[\gamma \wedge y] \in \mathbf{\Sigma}(\bar{n}, \bar{m})$. Show that '+' is strictly associative and unital, i.e., a strict monoidal product on the category $\mathbf{\Sigma}$. Define a symmetry isomorphism to make this into a *symmetric* monoidal product. Show that under the correspondence between symmetric spectra and based continuous functors $\mathbf{\Sigma} \longrightarrow \mathbf{T}$, shifting of spectra corresponds to precomposition with the functor $1 + - : \mathbf{\Sigma} \longrightarrow \mathbf{\Sigma}$.

(ii) Show that the isomorphism of categories of Proposition 7.42 between symmetric spectra and enriched functors from $\mathbf{\Sigma}$ can be extended to an isomorphism between the categories of symmetric ring spectra and strong monoidal functors from $\mathbf{\Sigma}$ to $\mathbf{T}$ such that it takes commutative symmetric ring spectra isomorphically onto the full subcategory of symmetric monoidal functors.

(iii) Show that the smash product of symmetric spectra corresponds, in the picture of enriched functors, to the enriched Kan extension along $+ : \mathbf{\Sigma} \times \mathbf{\Sigma} \longrightarrow \mathbf{\Sigma}$ of an 'external' smash product.

(iv) Present an enriched functor $X : \mathbf{\Sigma} \longrightarrow \mathbf{T}$ as an enriched coend,
$$X \ \cong \ \int_{n \in \mathbf{\Sigma}} X_n \wedge F_n \ .$$
This leads to a presentation of a symmetric spectrum X as a coequalizer
$$\bigvee_{m \geq n \geq 0} X_n \wedge \mathbf{\Sigma}(n, m) \wedge F_m \ \underset{\mathrm{Id}\,\wedge \mathrm{coact}}{\overset{\mathrm{act} \wedge \mathrm{Id}}{\rightrightarrows}} \ \bigvee_{n \geq 0} X_n \wedge F_n \longrightarrow X$$

Exercise E.I.48. Recall from (7.41) the $\mathbf{T}$-category $\mathbf{O}$ that parameterizes orthogonal spectra.

(i) We define a functor $+ : \mathbf{O} \times \mathbf{O} \longrightarrow \mathbf{O}$ on objects by addition of natural numbers and on morphisms by
$$[\tau \wedge z] \ + \ [\gamma \wedge y] \ = \ [(\tau + \gamma) \wedge (1 + \chi_{n,\bar{m}} + 1)_*(z \wedge y)] \ \in \ \mathbf{O}(n + \bar{n}, m + \bar{m})$$
for $[\tau \wedge z] \in \mathbf{O}(n, m)$ and $[\gamma \wedge y] \in \mathbf{O}(\bar{n}, \bar{m})$. Show that '+' is strictly associative and unital, i.e., a strict monoidal product on the category $\mathbf{O}$. Define a symmetry isomorphism to make this into a *symmetric* monoidal product.

(ii) Show that the isomorphism of categories of Proposition 7.42 between orthogonal spectra and enriched functors from **O** can be extended to an isomorphism between the categories of orthogonal ring spectra and strong monoidal functors from **O** to **T** such that it takes commutative orthogonal ring spectra isomorphically onto the full subcategory of symmetric monoidal functors.

Exercise E.I.49. Formulate unitary spectra as continuous functors from a topological category **U** and state and solve the analogue of Exercise E.I.48 in the unitary context.

Exercise E.I.50. Let R be a symmetric ring spectrum. Define a **T**-category (or **sS**-category) $\mathbf{\Sigma}_R$, with objects the natural number, such that enriched functors from $\mathbf{\Sigma}_R$ are 'the same' (in the sense of an isomorphism of categories) as R-modules. In the case where R is commutative, define a symmetric monoidal product on $\mathbf{\Sigma}_R$ which is given by addition on objects. Do the same for orthogonal and unitary ring spectra.

Exercise E.I.51. Let $f : B \longrightarrow A$ be a homomorphism of abelian groups. We let $\mathcal{P}f$ to be the category whose objects are the elements of A, and with morphism sets

$$\mathcal{P}f(a, a') \ = \ \{b \in B \mid a + f(b) = a'\} \ .$$

Composition is addition in B. We define a functor $\oplus : Pf \times Pf \longrightarrow Pf$ on objects by addition in A, and on morphisms by addition in B. Show that $\mathcal{P}f$ is a strict Picard category with respect to the identity symmetry isomorphisms. Describe the homotopy groups of the K-theory spectrum $\mathbf{K}(\mathcal{P}f)$ in terms of the morphism f. Construct a natural $\hat{\pi}_*$-isomorphism between $\mathbf{K}(\mathcal{P}f)$ and the mapping cone of the morphism $Hf : HB \longrightarrow HA$ between Eilenberg-Mac Lane spectra.

Exercise E.I.52. One can break the construction of the $\mathcal{M}$-action on the naive homotopy groups of a symmetric spectrum up into two steps and pass through the intermediate category of **I**-*functors*. As before, the category **I** has an object $\mathbf{n} = \{1, \ldots, n\}$ for every non-negative integer n, including $\mathbf{0} = \emptyset$. Morphisms in **I** all injective maps. An **I**-*functor* is a covariant functor from the category **I** to the category of abelian groups.

(i) (From symmetric spectra to **I**-functors.) Given a symmetric spectrum X and an integer k we assign an **I**-functor $\underline{\pi}_k X$ to the symmetric spectrum X. On objects, this **I**-functor is given by

$$(\underline{\pi}_k X)(\mathbf{n}) \ = \ \pi_{k+n} X_n$$

if $k + n \geq 2$ and $(\underline{\pi}_k X)(\mathbf{n}) = 0$ for $k + n < 2$. If $\alpha : \mathbf{n} \longrightarrow \mathbf{m}$ is an injective map and $k + n \geq 2$, then $\alpha_* : (\underline{\pi}_k X)(\mathbf{n}) \longrightarrow (\underline{\pi}_k X)(\mathbf{m})$ is given as follows. We choose a permutation $\gamma \in \Sigma_m$ such that $\gamma(i) = \alpha(i)$ for all $i = 1, \ldots, n$ and set

$$\alpha_*(x) \ = \ \mathrm{sgn}(\gamma) \cdot \gamma(\iota^{m-n}(x))$$

where $\iota : \pi_{k+n} X_n \longrightarrow \pi_{k+n+1} X_{n+1}$ is the stabilization map (1.7). Justify that this definition is independent of the choice of permutation γ and really defines a functor on the category **I**.

(ii) (From **I**-functors to tame $\mathcal{M}$-modules.) Let F be any **I**-functor F; construct a natural tame left action by the injection monoid $\mathcal{M}$ on the colimit of F, formed over the subcategory $\mathbb{N}$ of inclusions, in such a way that for the **I**-functor $\underline{\pi}_k X$ coming from a symmetric spectrum X as in (i) , this yields the $\mathcal{M}$-action on the stable homotopy group $\hat{\pi}_k X$.

(iii) In Example 3.52 we associated an **I**-space $\Omega^\bullet X$ to every symmetric spectrum X. Construct an isomorphism of **I**-functors $\hat{\pi}_k(\Omega^\bullet X) \cong (\underline{\pi}_k X)$. [$k + n \geq 2$?]

(iv) Show that every **I**-functor arises as the **I**-functor $\underline{\pi}_0$ of a symmetric spectrum. (Hint: revisit and generalize the construction of the Eilenberg-Mac Lane spectrum a second time; Examples 1.14, 4.33 and 8.10 should become special cases.)

(v) Show that the tame $\mathcal{M}$-set $\mathcal{I}_n$ which represents the functor of taking filtration n (see Example 8.9) is isomorphic, as an $\mathcal{M}$-set, to the colimit of the representable, set-valued, **I**-functor $\mathbf{I}(\mathbf{n}, -)$.

Exercise E.I.53. Let W be an $\mathcal{M}$-module. Show that the assignment $\mathbf{n} \mapsto W^{(n)}$ extends to an **I**-functor $W^{(\bullet)}$ in such a way that $W \mapsto W^{(\bullet)}$ is right adjoint to the functor which assigns to an **I**-functor F the $\mathcal{M}$-module $F(\omega)$. The counit of the adjunction $(W^{(\bullet)})(\omega) \longrightarrow W$ is injective with image the subgroup of elements of finite filtration, which is also the largest tame submodule of W. The assignment $W \mapsto (W^{(\bullet)})(\omega) = \bigcup_n W^{(n)}$ is right adjoint to the inclusion of tame $\mathcal{M}$-modules into all $\mathcal{M}$-modules. The restriction of $W \mapsto W^{(\bullet)}$ to tame $\mathcal{M}$-modules is fully faithful.

Exercise E.I.54. Let W be an $\mathcal{M}$-module.

(i) Show that the map $d\cdot : W \longrightarrow \operatorname{sh} W$ given by the left multiplication by the shift operator d is $\mathcal{M}$-linear with respect to the shifted $\mathcal{M}$-action on the target.

(ii) Denote by $\operatorname{sh}^\infty W$ the colimit of the sequence

$$W \xrightarrow{d\cdot} \operatorname{sh} W \xrightarrow{\operatorname{sh} d\cdot} \operatorname{sh}^2 W \xrightarrow{\operatorname{sh}^d \cdot} \cdots$$

of $\mathcal{M}$-modules and homomorphisms. We denote the colimit of this sequence by $\operatorname{sh}^\infty W$. For a symmetric spectrum X, construct a natural isomorphism of $\mathcal{M}$-modules

$$\hat{\pi}_k(\Omega^\infty \operatorname{sh}^\infty X) \;\cong\; \operatorname{sh}^\infty(\hat{\pi}_k X) \; .$$

(iii) Show that the Σ_m-actions on $\operatorname{sh}^m W$ and the morphisms $\operatorname{sh}^m d$ extend to a functor $\operatorname{sh}^\bullet$ from the category T to the category of tame $\mathcal{M}$-modules. Conclude that the colimit $\operatorname{sh}^\infty W$ over the subcategory $\mathbb{N}$ has another 'external' tame $\mathcal{M}$-action which commutes with the 'internal' $\mathcal{M}$-action.

Exercise E.I.55. We now give a construction which associates to an $\mathbf{I}$-functor with Σ_m-action F (i.e., a covariant functor $F : \mathbf{I} \longrightarrow \mathbb{Z}\Sigma_m\text{-mod}$) a new $\mathbf{I}$-functor $\triangleright_m F$ and give a formula for the $\mathcal{M}$-module $(\triangleright_m F)(\omega)$.

Given $F : \mathbf{I} \longrightarrow \mathbb{Z}\Sigma_m$-mod we define a new $\mathbf{I}$-functor $\triangleright_m F$ by $(\triangleright_m F)(\mathbf{k}) = 0$ for $k < m$ and

$$(\triangleright_m F)(\mathbf{m}+\mathbf{n}) \;=\; \mathbb{Z}\Sigma_{m+n} \otimes_{\Sigma_m \times \Sigma_n} F(\mathbf{n}) \; . \tag{E.I.56}$$

We define $\triangleright_m F$ on morphisms $\alpha : \mathbf{m}+\mathbf{n} \longrightarrow \mathbf{m}+\mathbf{k}$ in $\mathbf{I}$ as follows. We choose a permutation $\gamma \in \Sigma_{m+k}$ which agrees with α on $\mathbf{m}+\mathbf{n}$ and define

$$\alpha_* \; : \; (\triangleright_m F)(\mathbf{m}+\mathbf{n}) = \mathbb{Z}\Sigma_{m+n} \otimes_{\Sigma_m \times \Sigma_n} F(\mathbf{n}) \;\longrightarrow\; \mathbb{Z}\Sigma_{m+k} \otimes_{\Sigma_m \times \Sigma_k} F(\mathbf{k}) = (\triangleright_m F)(\mathbf{m}+\mathbf{k})$$

by $\alpha_*(\tau \otimes x) = \gamma(\tau + 1_{k-n}) \otimes \iota_*(x)$ where $\iota : \mathbf{n} \longrightarrow \mathbf{k}$ is the inclusion.

(i) Check that this defines a functor.

(ii) We recall that a homomorphism of monoids $+ : \Sigma_m \times \mathcal{M} \longrightarrow \mathcal{M}$ is given by

$$(\gamma + f)(i) \;=\; \begin{cases} \gamma(i) & \text{for } 1 \le i \le m, \text{ and} \\ f(i-m)+m & \text{for } m+1 \le i. \end{cases}$$

As before we denote by $\mathbb{Z}\mathcal{M}^{\langle m \rangle}$ the monoid ring of $\mathcal{M}$ with its usual left multiplication action, but with action by the monoid $\Sigma_m \times \mathcal{M}$ via restriction along the homomorphism $\times : \Sigma_m \times \mathcal{M} \longrightarrow \mathcal{M}$. Since F takes values in Σ_m-modules, the colimit $F(\omega)$ not only has an action of $\mathcal{M}$, but also a compatible left action by the group Σ_m. So we can form

$$\mathbb{Z}\mathcal{M}^{\langle m \rangle} \otimes_{\Sigma_m \times \mathcal{M}} F(\omega)$$

which is a left $\mathcal{M}$-module via the left multiplication action of $\mathbb{Z}\mathcal{M}$ on itself. Show that for every $\mathbf{I}$-functor with Σ_m-action F the natural map

$$\begin{array}{ccc} \mathbb{Z}\mathcal{M}^{\langle m \rangle} \otimes_{\Sigma_m \times \mathcal{M}} F(\omega) & \longrightarrow & (\triangleright_m F)(\omega) \\ f \otimes [x] & \longmapsto & f \cdot [1 \otimes x] \end{array}$$

is an isomorphism of $\mathcal{M}$-modules.

Relate this to the $\mathbf{I}$-functor of stable homotopy groups of a twisted smash product. [where is the sign?]

Exercise E.I.57. Let W be a tame $\mathcal{M}$-module and let HW denote the associated Eilenberg-Mac Lane spectrum as in Example 8.10. Construct natural $\hat{\pi}_*$-isomorhisms

$$\begin{aligned} S^1 \wedge H(\operatorname{sh} W) &\longrightarrow \operatorname{sh}(HW) \quad \text{and} \\ S^1 \wedge \triangleright(HW) &\longrightarrow H(\mathbb{Z}\mathcal{M}^{\langle 1 \rangle} \otimes_{\mathcal{M}} W) . \end{aligned}$$

Exercise E.I.58. We dconsider the tame $\mathcal{M}$-module

$$W \;=\; \mathcal{P}_3 \otimes_{\Sigma_3} \mathbb{Q}^{\pm}$$

where $\mathbb{Q}^{\pm}$ is the sign representation of the symmetric group Σ_3. Since the $\mathcal{M}$-action on W is non-trivial, the associated Eilenberg-Mac Lane spectrum HW (see Example 8.10) is not semistable and so the morphism $\lambda_{HW} : S^1 \wedge HW \longrightarrow \operatorname{sh}(HW)$ is not a $\hat{\pi}_*$-isomorphism. Show that nevertheless λ_{HW} is a stable equivalence because both $S^1 \wedge HW$ and $\operatorname{sh}(HW)$ are stably contractible.

Exercise E.I.59. We show that the injection monoid $\mathcal{M}$ gives essentially all natural operations on the homotopy groups of symmetric spectra. More precisely, we now identify the ring of natural operations $\hat{\pi}_0 X \longrightarrow \hat{\pi}_0 X$ with a completion of the monoid ring $\mathbb{Z}\mathcal{M}$. Any functor $G : \mathbf{A} \longrightarrow \mathcal{C}$ from a small category $\mathbf{A}$ has an endomorphism monoid $\operatorname{End}(G)$; the elements of $\operatorname{End}(G)$ are the natural self-transformation from the functor G to itself and composition of transformations gives the product.

(i) An **I**-*set* is a functor from the category **I** of standard finite sets and injections to the category of sets. Show that the endomorphism monoid of the 'colimit over inclusions' functor

$$(\mathbf{I}\text{-sets}) \;\longrightarrow\; (\text{sets}) \;, \quad F \mapsto \operatorname{colim}_{n \in \mathbb{N}} F(n)$$

is isomorphic to the injection monoid $\mathcal{M}$.

(ii) We denote by I_n the left ideal of the monoid ring $\mathbb{Z}\mathcal{M}$ which is additively generated by all differences of the form $f - g$ for all $f, g \in \mathcal{M}$ such that f and g agree on $\mathbf{n}$. Define a multiplication on the abelian group

$$\mathbb{Z}[[\mathcal{M}]] \;=\; \lim_n \; \mathbb{Z}\mathcal{M}/I_n$$

such that the natural map $\mathbb{Z}\mathcal{M} \longrightarrow \mathbb{Z}[[\mathcal{M}]]$ is a ring homomorphism. (Warning: I_n is *not* a right ideal for $n \geq 1$, so the individual terms $\mathbb{Z}\mathcal{M}/I_n$ do *not* inherit multiplications.)

(iii) We consider the endomorphism ring $\operatorname{End}(\operatorname{colim}_{\mathbb{N}})$ of the 'colimit over inclusions' functor

$$(\mathbf{I}\text{-functors}) \;\longrightarrow\; (\text{abelian groups}) \;, \quad F(\omega) = F \mapsto \operatorname{colim}_{n \in \mathbb{N}} F(n) \;.$$

Addition in this ring is given by pointwise addition. Show that $\operatorname{End}(\operatorname{colim}_{\mathbb{N}})$ is isomorphic to the completed monoid ring $\mathbb{Z}[[\mathcal{M}]]$.

(iv) Let W be a tame $\mathcal{M}$-module. Show that the action of the monoid ring $\mathbb{Z}\mathcal{M}$ on W extends to an action of the completed monoid ring $\mathbb{Z}[[\mathcal{M}]]$ which is *discrete* in the sense that the action map

$$\mathbb{Z}[[\mathcal{M}]] \times W \;\longrightarrow\; W$$

is continuous with respect to the discrete topology on W and the filtration topology on $\mathbb{Z}[[\mathcal{M}]]$. Show that conversely, if W is discrete module over $\mathbb{Z}[[\mathcal{M}]]$, then its underlying $\mathcal{M}$-module is tame. Show that this establishes an isomorphism between the category of tame $\mathcal{M}$-modules and the category of discrete $\mathbb{Z}[[\mathcal{M}]]$-modules.

(v) Since $\hat{\pi}_0 X$ is a tame $\mathcal{M}$-module for every symmetric spectrum X, part (iv) provides a natural action of the completed monoid ring $\mathbb{Z}[[\mathcal{M}]]$ on $\hat{\pi}_0 X$. As X varies, this is a homomorphism of rings

$$\mathbb{Z}[[\mathcal{M}]] \;\longrightarrow\; \operatorname{End}(\hat{\pi}_0)$$

to the endomorphism ring of the naive homotopy group functor $\hat{\pi}_0 : \mathcal{S}p \longrightarrow (\text{abelian groups})$. Show that this map is an isomorphism.

Exercise E.I.60. Show that for every **I**-functor F there are natural isomorphisms of abelian groups

$$\mathrm{colim}_p^{\mathbf{I}} F \ \cong\ H_p(\mathcal{M}, F(\omega)) \ .$$

for all $p \geq 0$, where '$\mathrm{colim}_p^{\mathbf{I}}$' is the pth left derived functor of the colimit functor from **I**-functors to abelian groups.

Exercise E.I.61. Let $H : \mathbf{I} \longrightarrow \mathcal{S}p$ be an **I**-symmetric spectrum.

(i) The underlying sequential spectra of the mapping telescope $\mathrm{tel}_n\, H(\mathbf{n})$ of the sequence of symmetric spectra $H(\mathbf{n})$ taken over the inclusions in $\mathbb{N}$ and the diagonal $\mathrm{diag}\, H$ are naturally $\hat{\pi}_*$-isomorphic.

(ii) If the external $\mathcal{M}$-action on the sequential colimit $\mathrm{colim}_{n\in\mathbb{N}}\ \hat{\pi}_k H(\mathbf{n})$ is trivial, then mapping telescope $\mathrm{tel}_n\, H(\mathbf{n})$ and the diagonal $\mathrm{diag}\, H$ are naturally stably equivalent as symmetric spectra.

Exercise E.I.62. The true homotopy group $\pi_k X$ of a symmetric spectrum X is the abutment of the naive-to-true spectral sequence (see Theorem 8.41); as such it comes with an exhaustive natural filtration

$$\Pi_0 \subseteq \Pi_1 \subseteq \cdots \subseteq \Pi_i \subseteq \Pi_{i+1}$$

such that the subquotient Π_p/Π_{p-1} is isomorphic to the $E^{\infty}_{p,k-p}$-term of the naive-to-true spectral sequence.

(i) Show that the fltration group Π_0 equals the image of the tautological map $c : \hat{\pi}_q X \longrightarrow \pi_q X$ from naive to true homotopy groups.

(ii) Prove the following characterization of the filtration groups. A true homotopy class $x \in \pi_n X$ belongs to Π_i if and only if the following holds: for every chain

$$X = Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} \cdots Y_i \xrightarrow{f_i} Y_i$$

of $i+1$ composable morphisms of symmetric spectra each of which is trivial on naive homotopy groups, we have $\pi_n(f_i \circ \cdots \circ f_0)(x) = 0$.

Exercise E.I.63. In (E.I.35) we have defined the shift homomorphism $\mathrm{sh}_! : \pi_{k+1}(\mathrm{sh}\, X) \longrightarrow \pi_k X$ which is compatible with the tautological map from naive to true homotopy groups and has the map $(-1)^k(\lambda_X)_*(S^1\wedge -) : \pi_k X \longrightarrow \pi_{k+1}(\mathrm{sh}\, X)$ as a section, by Proposition **??**.

(i) Show that the diagram

$$\begin{tikzcd}
\hat{\pi}_k X \arrow[r, "\mathrm{sh}(d\cdot -)"] \arrow[d, "c"'] & \mathrm{sh}(\hat{\pi}_k X) \arrow[r, equal] & \hat{\pi}_{k+1}(\mathrm{sh}\, X) \arrow[d, "c"] \\
\pi_k X \arrow[rr, "(-1)^k(\lambda_X)_*(S^1\wedge -)"'] & & \pi_{k+1}(\mathrm{sh}\, X)
\end{tikzcd}$$

commutes. There is a certain danger to misinterpretation: The tautological map $c : \hat{\pi}_k Y \longrightarrow \pi_k Y$ coequalizes the action of the injection monoid, so one could be led to think that for a naive homotopy class $y \in \hat{\pi}_k X$ the true homotopy class $c(\mathrm{sh}\, y)$ equals $c(\mathrm{sh}(dy))$ and hence cy. However, the element $\mathrm{sh}(dy)$ is in general *not* the same as $d(\mathrm{sh}\, y)$ and not even of the form $f(\mathrm{sh}\, y)$ for any monoid element $f \in \mathcal{M}$.

(ii) Show that there is no natural transformation $\tau : \pi_k X \longrightarrow \pi_{k+1}(\mathrm{sh}\, X)$ such that the square

$$\begin{tikzcd}
\hat{\pi}_k X \arrow[r, equal] \arrow[d, "c"'] & \hat{\pi}_{k+1}(\mathrm{sh}\, X) \arrow[d, "c"] \\
\pi_k X \arrow[r, "\tau"'] & \pi_{k+1}(\mathrm{sh}\, X)
\end{tikzcd}$$

commutes.

The relations enjoyed by this construction are part of an action of the *injection operad*, see Exercise E.I.69 for details.

Exercise E.I.64 (Products on naive homotopy groups)**.** In this exercise we discuss pairings of the naive homotopy groups of two symmetric spectra. Before we start, we repeat an earlier warning, namely that there is no preferred pairing of naive homotopy groups from $\hat{\pi}_k X \times \hat{\pi}_l Y$ to $\hat{\pi}_{k+l}(X \wedge Y)$ for general symmetric spectra X and Y!

Given two symmetric spectra X and Y and two homotopy classes $x \in \pi_{k+n} X_n$ and $y \in \pi_{l+m} Y_m$ we denote by $x \cdot y$ the homotopy class in $\pi_{k+l+n+m}(X \wedge Y)_{n+m}$ given by the composite

$$S^{k+n+l+m} \xrightarrow{x \wedge y} X_n \wedge Y_m \xrightarrow{i_{n,m}} (X \wedge Y)_{n+m} \tag{E.I.65}$$

multiplied by an appropriate sign [...], where $i_{n,m}$ is a component of the universal bimorphism.

(i) Show that the stable class of $x \cdot y$ in $\hat{\pi}_{k+l}(X \wedge Y)$ only depends on the stable class of y in $\hat{\pi}_l Y$. Give an example showing that replacing x by $\iota(x) \in \pi_{k+n+1} X_{n+1}$ can change the product, i.e., $x \cdot y$ and $\iota(x) \cdot y$ may represent *different* classes in $\hat{\pi}_{k+l}(X \wedge Y)$.

(ii) Let $\varphi : \mathbf{2} \times \omega \longrightarrow \omega$ be an injective map. We consider naive homotopy classes in $\hat{\pi}_k X$ and $\hat{\pi}_l Y$ represented by unstable homotopy classes $x \in \pi_{k+n} X_n$ respectively $y \in \pi_{l+m} Y_m$. We choose an injection $f \in \mathcal{M}$ such that $\varphi(1,i) = f(i)$ for $i = 1, \dots, n$ and $\varphi(2,j) = f(n+j)$ for $j = 1, \dots, m$. Then we define $\varphi_*([x],[y])$ in $\hat{\pi}_{k+l}(X \wedge Y)$ as

$$(-1)^{ln} \cdot f \cdot [x \cdot y] \ ; \tag{E.I.66}$$

in other words, we take the stable class represented by the composite

$$S^{k+n+l+m} \xrightarrow{x \wedge y} X_n \wedge Y_m \xrightarrow{i_{n,m}} (X \wedge Y)_{n+m}$$

where $i_{n.m}$ is a component of the universal bimorphism, multiply by the sign $(-1)^{ln}$ and act by the monoid element f. Show that the definition (E.I.66) of the pairing

$$\varphi_* \ : \ \hat{\pi}_k X \times \hat{\pi}_l Y \ \longrightarrow \ \hat{\pi}_{k+l}(X \wedge Y)$$

is independent of all choices and biadditive.

(iii) Let g, h_1, h_2 be elements of the injection monoid $\mathcal{M}$. Show the relations

$$g \cdot \varphi_*(x,y) \ = \ (g\varphi)_*(x,y) \quad \text{and} \quad \varphi_*(h_1 \cdot x, h_2 \cdot y) \ = \ (\varphi(h_1 + h_2))_*(x,y)$$

hold in $\hat{\pi}_{k+l}(X \wedge Y)$ for all naive homotopy classes $x \in \hat{\pi}_k X$ and $y \in \hat{\pi}_l Y$. Here $h_1 + h_2$ is the selfmap of $\mathbf{2} \times \omega$ given by $(h_1 + h_2)(\alpha, i) = (\alpha, h_\alpha(i))$.

(iv) Show that for $Y = \mathbb{S}$ the composite

$$\hat{\pi}_k X \times \pi_l^{\mathrm{s}} \xrightarrow{\varphi_*} \hat{\pi}_{k+l}(X \wedge \mathbb{S}) \ = \ \hat{\pi}_{k+l} X$$

agrees with the action of the stable homotopy groups of spheres as defined in Example 1.11.

(v) Show that the diagram

$$\begin{tikzcd} \hat{\pi}_k X \times \hat{\pi}_l Y \arrow[r, "\varphi_*"] \arrow[d, "\mathrm{twist}"'] & \hat{\pi}_{k+l}(X \wedge Y) \arrow[d, "(-1)^{kl} \cdot \hat{\pi}_{k+l}(\tau_{X,Y})"] \\ \hat{\pi}_l Y \times \hat{\pi}_k X \arrow[r, "(\varphi\tau)_*"'] & \hat{\pi}_{l+k}(Y \wedge X) \end{tikzcd}$$

commutes, where τ is the involution of $\mathbf{2} \times \omega$ which interchanges the two copies of ω, i.e., $\tau(\alpha, i) = (3 - \alpha, i)$.

(vi) Show that the square

$$\begin{tikzcd} \hat{\pi}_k X \times \hat{\pi}_l Y \arrow[r, "\varphi_*"] \arrow[d, "c \times c"'] & \hat{\pi}_{k+l}(X \wedge Y) \arrow[d, "c"] \\ \pi_k X \times \pi_l Y \arrow[r, "\cdot"'] & \hat{\pi}_{k+l}(X \wedge Y) \end{tikzcd}$$

commutes.

Exercise E.I.67. Let R be a symmetric ring spectrum. Show that there is a natural structure of a graded ring on the graded subgroup $(\hat{\pi}_* R)^{(0)}$ of $\mathcal{M}$-fixed elements in the homotopy groups of R. If R is commutative, then the product on $(\hat{\pi}_* R)^{(0)}$ is graded-commutative. The homotopy groups of every right R-module naturally form a graded right module over the graded ring $(\hat{\pi}_* R)^{(0)}$.

If R is semistable, then $(\hat{\pi}_* R)^{(0)} = \hat{\pi}_* R$ and this should generalize the product on true homotopy groups as define in Proposition 6.25.

Exercise E.I.68. In this exercise we introduce and study the *injection operad.* Readers familiar with the linear isometries operad and the theory of S-modules in the sense of Elmendorf, Kriz, Mandell and May [**26**] will notice similarity between the injection operad and the linear isometries operad. Indeed, the operad $\underline{\mathcal{M}}$ can be viewed as a 'discrete analog' of the linear isometries operad.

The injection operad $\underline{\mathcal{M}}$ is defined by letting $\underline{\mathcal{M}}(n)$ be the set of injections from the set $\omega \times \mathbf{n}$ into ω, for $n \geq 0$. Note that for $n = 0$ the source is the empty set, so $\underline{\mathcal{M}}(0)$ has exactly one element, and $\underline{\mathcal{M}}(1)$ is the monoid $\mathcal{M}$. The symmetric groups permute the second coordinates in $\omega \times \mathbf{n}$. The operad structure is via disjoint union and composition, i.e., $\mathcal{M}$ is a suboperad of the endomorphism operad of the set ω in the symmetric monoidal category of sets under disjoint union. More precisely, the operad structure morphism

$$\gamma \ : \ \underline{\mathcal{M}}(n) \times \underline{\mathcal{M}}(i_1) \times \cdots \times \underline{\mathcal{M}}(i_n) \ \longrightarrow \ \underline{\mathcal{M}}(i_1 + \cdots + i_n)$$

sends $(\varphi, f_1, \ldots, f_n)$ to $\varphi \circ (f_1 + \cdots + f_n)$.

(i) Show that the collection $\{\underline{\mathcal{M}}(n)\}_{n \geq 0}$ is an operad with respect to the structure maps γ defined above.

(ii) For all $n, m \geq 1$ the map

$$a \ : \ \underline{\mathcal{M}}(2) \times_{\mathcal{M}^2} (\underline{\mathcal{M}}(n) \times \underline{\mathcal{M}}(m)) \ \longrightarrow \ \underline{\mathcal{M}}(n+m)$$

given by $\varphi(\psi, \lambda) \mapsto \varphi \circ (\psi + \lambda)$ is an isomorphism of $\mathcal{M}$-$\mathcal{M}^{n+m}$-bisets.

(iii) The injection operad of sets has similar formal properties as the linear isometries operad. Show that the operadic composition map

$$\gamma \ : \ \underline{\mathcal{M}}(1) \times \underline{\mathcal{M}}(n) \ \longrightarrow \ \underline{\mathcal{M}}(n)$$

is a free and transitive action of the monoid $\mathcal{M}$ on the set $\underline{\mathcal{M}}(n)$ for $n \geq 1$. Show that the injection operad has 'Hopkins' property' that the map

$$\underline{\mathcal{M}}(2) \times \underline{\mathcal{M}}(1) \times \underline{\mathcal{M}}(1) \times \underline{\mathcal{M}}(i) \times \underline{\mathcal{M}}(j) \underset{\mathrm{Id} \times \gamma^2}{\overset{\gamma \times \mathrm{Id}}{\rightrightarrows}} \underline{\mathcal{M}}(2) \times \underline{\mathcal{M}}(i) \times \underline{\mathcal{M}}(j) \xrightarrow{\ \gamma\ } \underline{\mathcal{M}}(i+j)$$

is a split coequalizer for all $i, j \geq 1$.

(iv) Show that although the monoid $\mathcal{M} \times \mathcal{M}$ does not act transitively on the set $\underline{\mathcal{M}}(2)$, the orbit set $\underline{\mathcal{M}}(2)/\underline{\mathcal{M}}(1) \times \underline{\mathcal{M}}(1) = \underline{\mathcal{M}}(2)/\underline{\mathcal{M}}(1)^2$ has only one element.

Exercise E.I.69. As we have seen in Proposition 6.25 the *true* homotopy groups of any symmetric ring spectrum form a graded ring. However, the *naive* homotopy groups of a symmetric ring spectrum R do in general *not* form a graded ring in any natural way (unless R is semistable, when the naive and true homotopy groups coincide). As we mentioned in [...] the problem is that the smash product pairing

$$\pi_{k+n} R_n \ \times \ \pi_{l+m} R_m \ \longrightarrow \ \pi_{k+n+l+m} R_{n+m}$$

induced by the multiplication $R_n \wedge R_m \longrightarrow R_{n+m}$ is generally *not* compatible with the stabilization in the left factor and so does not usually induce a map $\hat{\pi}_k R \times \hat{\pi}_l R \longrightarrow \hat{\pi}_{k+l} R$ on colimits. In this exercise we reveal the natural structure on the naive homotopy groups of a symmetric ring spectrum, namely a graded algebra structure over the injection operad. For semistable symmetric spectra, the naive and true homotopy groups are isomorphic. By part (iii) of this exercise, in that case the action of the injection operad reduces to the multiplication on the homotopy groups.

(i) Let $X^1, \ldots, X^n$ be symmetric spectra and $\varphi \in \underline{\mathcal{M}}(n)$ an injection from $\mathbf{n} \times \omega$ to ω. Define a natural map

$$\varphi_* \ : \ \hat{\pi}_{k_1} X^1 \times \cdots \times \hat{\pi}_{k_n} X^n \ \longrightarrow \ \pi_{k_1 + \cdots + k_n}(X^1 \wedge \ldots X^n)$$

so that for $n = 1$ we recover the action of the injection monoid $\mathcal{M} = \underline{\mathcal{M}}(1)$.

(ii) Find out what relations the operations φ_* satisfy as n and φ vary.

(iii) Let R be a symmetric ring spectrum. For every $n \geq 1$, every $\varphi \in \underline{\mathcal{M}}(n)$ and all integers k_i, we obtain a multilinear internal pairing of the naive homotopy groups of R as the composite

$$\varphi_* \ : \ \hat{\pi}_{k_1} R \times \cdots \times \hat{\pi}_{k_n} R \ \xrightarrow{\varphi_*} \ \hat{\pi}_{k_1+\cdots+k_n}(R \wedge \ldots \wedge R) \ \xrightarrow{\mu^{(n)}} \ \hat{\pi}_{k_1+\cdots+k_n} R \ ,$$

where $\mu^{(n)} : R^{\wedge n} \longrightarrow R$ is the iterated multiplication. Show that these maps make the naive homotopy groups $\hat{\pi}_* R$ into a graded (non-symmetric) algebra over the (non-symmetric) operad $\mathcal{M}$.

(iv) Show that if the multiplication of R is commutative, then the algebra structure of part (i) makes $\hat{\pi}_* R$ into a graded algebra over the operad $\mathcal{M}$, i.e., the symmetry property also holds.

(v) Suppose A is an algebra over the injection operad in the monoidal category of abelian groups under tensor product. Show that if the action of the injection monoid $\mathcal{M} = \underline{\mathcal{M}}(1)$ on A is trivial, then all injections $\varphi \in \underline{\mathcal{M}}(n)$ induced the same map $A^{\otimes n} \longrightarrow A$. Conclude that the full subcategory of those $\underline{\mathcal{M}}$-algebras for which $\mathcal{M} = \underline{\mathcal{M}}(1)$ acts trivially is equivalent to the category of rings.

Exercise E.I.70. We define a binary product $\square$ for $\mathcal{M}$-modules V, W by

$$V \square W \ = \ \mathbb{Z}\underline{\mathcal{M}}(2) \otimes_{\mathcal{M} \times \mathcal{M}} V \otimes W$$

[explain...] As in Exercise E.I.68, there are strong similarities between the $\square$-product of $\mathcal{M}$-spectra and the smash product of $\mathbb{L}$-spectra defined in [**26**, I.5].

(i) Show that the $\square$-product is coherently associative and symmetric on the category of $\mathcal{M}$-modules. Show that when restricted to *tame* $\mathcal{M}$-modules, then the trivial $\mathcal{M}$-module $\mathbb{Z}$ is a coherent unit object for the $\square$-product.

(ii) Given two symmetric spectra X and Y, construct a natural map of $\mathcal{M}$-modules

$$\cdot \ : \ (\hat{\pi}_k X) \square (\hat{\pi}_l Y) \ \longrightarrow \ \hat{\pi}_{k+l}(X \wedge Y)$$

which constitutes a lax symmetric monodal transformation.

(iii) Suppose that at least on of the symmetric spectra X and Y is flat, and the other one level cofibrant. Suppse that X is naively $(k-1)$-connected and Y is naively $(l-1)$-connected. Show that then the smash product $X \wedge Y$ is naively $(k+l-1)$-connected and the map

$$\hat{\pi}_{k+l}(X \wedge Y) \ \cong \ (\hat{\pi}_k X) \square (\hat{\pi}_l Y)$$

constructed in part (ii) is an isomorphism of $\mathcal{M}$-modules.

(iv) In Example 8.10 we associated to every tame $\mathcal{M}$-module W an Eilenberg-Mac Lane spectrum HW and an $\mathcal{M}$-linear isomorphism $j_W : W \cong \hat{\pi}_0(HW)$. Show that this isomorphism is multiplicative, i.e., that for every pair of tame $\mathcal{M}$-modules W and V the composite map

$$W \square V \ \xrightarrow{j_W \square j_V} \ \hat{\pi}_0 HW \square \hat{\pi}_0 HV \ \xrightarrow{\cdot} \ \hat{\pi}_0(HW \wedge HV) \ \xrightarrow{\hat{\pi}_0(m_{W,V})} \ H(W \square V)$$

equals $j_{W \otimes V}$. (Note that if $\mathcal{M}$ acts trivially on W and V, then this specializes to Example 6.27.)

Exercise E.I.71. Let X be a symmetric spectrum such that all structure maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ are cofibrations. Show that X is semistable if and only if the adjunction counit $GUX \longrightarrow X$ is a stable equivalence. Here U is the forgetful functor to sequential spectra and G its left adjoint.

Exercise E.I.72. Let X and Y be semistable symmetric spectra and let $\varphi : X \longrightarrow Y$ be a $\hat{\pi}_*$-isomorphism of the underlying sequential spectra (so φ need not respect the symmetric group actions). Show that X and Y are stably equivalent as symmetric spectra (by a chain of stable equivalences).

History, credits and further reading

I now summarize the history of symmetric spectra and symmetric ring spectra, and the genesis of the examples which were discussed above, to the best of my knowledge. My point with respect to the examples is not when certain spectra first appeared as homotopy types or ring spectra 'up to homotopy', but rather when a 'highly structured' multiplication was first noticed in one form or another. Additions, corrections and further references are welcome.

Symmetric spectra and symmetric ring spectra were first introduced under this name in the article [**36**] by Hovey, Shipley and Smith. However, these mathematical concepts had been used before, in particular in several papers related to topological Hochschild homology and algebraic K-theory. For example, symmetric ring spectra appeared as *strictly associative ring spectra* in [**31**, Def. 6.1] and as *FSPs defined on spheres* in [**33**, 2.7].

There is a key observation, however, which is due to Jeff Smith and which was essential for the development of symmetric spectra and related spectra categories. Smith noticed that symmetric ring spectra are the monoids in a category of symmetric spectra which has a smash product and a compatible stable model structure. Smith gave the first talks on this subject in 1993. In the fall of 1995, Hovey, Shipley and Smith started a collaboration in which many remaining issues and in particular the model structures were worked out. The results first appeared in a joint preprint on the Hopf algebraic topology server (at `hopf.math.purdue.edu`), the K-theory preprint server (at `www.math.uiuc.edu/K-theory/`) and the ArXiv (under `math.AT/9801077`) in January 1998. This preprint version has a section about symmetric spectra based on topological spaces which did not make it into the published version [**36**] because the referee requested that the paper be shortened.

Several of the examples which we gave in Section 1.1 had been around with enough symmetries before symmetric spectra were formally introduced. For example, Bökstedt and Waldhausen introduced *functors with smash product*, or FSPs for short, in [**8**], from which symmetric ring spectra are obtained by restricting to spheres. Eilenberg-Mac Lane spectra (Example 1.14) and monoid ring spectra (Example 3.42) arise in this way from FSPs and seem to have first appeared in [**8**] Matrix ring spectra (Example 3.44) were also treated as FSP in [**8**] [first published reference ?].

Cobordism spectra first appeared as highly structured ring spectra in the form of as '$\mathscr{I}_*$-prefunctors' in [**56**]. $\mathscr{I}_*$-prefunctors are the same as [commutative ?] orthogonal ring spectra, and the underlying symmetric ring spectra are what we present in Example 1.16.

The model for the connective complex topological K-theory spectrum ku in Example 1.20 is essentially taken from [???]

Free and semifree symmetric spectra, suspensions, loop and shifts of symmetric spectra were first discussed in the original paper [**36**] of Hovey, Shipley and Smith.

The particular method for inverting a homotopy elements in a symmetric ring spectrum described in Examples 3.47, 3.48 and 6.53 seems to be new. The construction of Example 3.49 for adjoining roots of unity to a symmetric ring spectrum is due to Schwänzl, Vogt and Waldhausen [**68**]. They originally wrote up the construction in the context of S-modules, but their argument only needs that one can form monoid rings and invert homotopy elements within the given framework of commutative ring spectra. So as soon as these constructions are available, their argument carries over to symmetric ring spectra.

Waldhausen notes on p. 330 of [**88**] that the iterated $S.$-construction defines a (sequential) spectrum which is an Ω-spectrum from level 1 upwards. Waldhausen's construction predates symmetric spectra, and it was later noticed by Hesselholt [**28**, Appendix] that iterating the $S.$-construction in fact has all the symmetries needed to form a symmetric spectrum. Moreover, bi-exact pairings of input data yields multiplications of associated K-theory spectra. Our treatment of the algebraic K-theory spectrum on Example 3.50 follows very closely the Appendix of [**28**].

Our treatment of stable equivalences is the same as in the original paper [**36**] of Hovey, Shipley and Smith, expanded by a few more details and examples. The arguments of Propositions 4.8 and 4.9 to reduce the lifting property to a set of morphisms with bounded cardinality is taken from [**36**, Lemma 5.1.4 (6)] and ultimately goes back to Bousfield, who used it in [**10**] to establish a 'local' model structure for simplicial sets with respect to a homology theory.

The smash product of symmetric spectra was defined by Hovey, Shipley and Smith in their original paper [**36**]. However, their exposition of the smash product differs substantially from ours. Hovey, Shipley and Smith use the category of *symmetric sequences* (sequences of pointed spaces X_n, for $n \geq 0$, with Σ_n-action on X_n) as an intermediate step towards symmetric spectra and in the construction of the smash product, compare Exercise E.I.16. I chose to present the smash product of symmetric spectra in a different way because I want to highlight its property as the universal target for bimorphisms.

The $\mathcal{M}$-action on the homotopy groups of symmetric spectra was first studied systematically by the author in [**70**]. Some results related to the $\mathcal{M}$-action on homotopy groups are already contained, mostly implicitly, in the papers [**36**] and [**75**]. The definition of semistable symmetric spectra and the characterizations [???] of Theorem 8.25 appear in Section of [**36**]; the criterion of trivial $\mathcal{M}$-action on homotopy groups (Theorem 8.25 (i)) first appears in [**70**].

The definition of true homotopy groups that we use is new; because of its abstractness I am afraid that it may not be to everybody's taste. The more traditional approach, adopted for example in [**36**], has been to choose a stably fibrant replacement functor, i.e., a functor $X \mapsto X^{\mathrm{f}}$ with values in Ω-spectra and a natural stable equivalence $X \longrightarrow X^{\mathrm{f}}$; the 'true' (or 'derived') homotopy groups were then defined as the naive (or 'classical') homotopy groups of the fibrant replacement X^{f}. I decided to work with the present definition of true homotopy groups because I wanted an intrinsic approach independent of any choices.

The idea to construct various spectra from the Thom spectrum MU by killing a regular sequence and possibly inverting an element (see Example 6.63) is taken from Chapter V of [**26**] where this process is carried out in the world of S-modules. This strategy had previously been adapted to symmetric spectra in Weiner's *Diplomarbeit* [**89**].

The original construction of the Brown-Peterson spectrum in the paper [**14**] by Brown and Peterson was quite different. They constructed a spectrum whose mod-p homology realizes a certain polynomial subalgebra of the dual Steenrod algebra. Later Quillen gave a construction of the spectrum BP using the theory of formal groups, and Quillen's approach is still at the heart of most current applications of BP. Quillen used p-typical formal groups to produce an idempotent endomorphism $e : MU_{(p)} \longrightarrow MU_{(p)}$ of the p-localization of MU in the stable homotopy category (see Section II) which is even a homomorphism of homotopy ring spectra (see Section II.3 below). The 'image' of this idempotent is isomorphic, in the stable homotopy category, to the spectrum BP, and Quillen's construction produces it as a homotopy ring spectrum. Part II of Adams' notes [**2**] are a good exposition of Quillen's results in this area. [original paper?]

I learned the model of the periodic complex cobordism spectrum MUP given in Example 7.8 from Morten Brun, who adapted a construction of Strickland [**82**, Appendix] from 'complex S-modules' to unitary spectra.

The category of $\mathbf{\Gamma}$-spaces was introduced by Segal [**73**], who showed that it has a homotopy category equivalent to the usual homotopy category of connective spectra. The category we denote $\mathbf{\Gamma}$ is really equivalent to the opposite of Segal's category Γ, so that covariant functors from $\mathbf{\Gamma}$ are 'the same' as contravariant from Γ. Bousfield and Friedlander [**13**] considered a bigger category of $\mathbf{\Gamma}$-spaces in which the ones introduced by Segal appeared as the *special* $\mathbf{\Gamma}$-spaces. Their category admits a closed simplicial model category structure with a notion of stable weak equivalences giving rise again to the homotopy category of connective spectra. The proof that a prolonged $\mathbf{\Gamma}$-space of simplicial sets preserves weak equivalences of simplicial sets first appears (with a different proof) as Proposition 4.9 in [**13**]. Lydakis [**47**] showed that $\mathbf{\Gamma}$-spaces admit internal function objects and a symmetric monoidal smash product with good homotopical properties. The spectra that arise from $\mathbf{\Gamma}$-spaces have more special properties than the ones we have mentioned above. For example, the colimit systems for the stable homotopy groups stabilize in a uniform way. More specifically, for every $\mathbf{\Gamma}$-space X with values in simplicial sets, the simplicial set $X(S^n)$ is always $(n-1)$-connected [**13**] and the structure map $X(S^n) \wedge S^1 \longrightarrow X(S^{n+1})$ is $2n$-connected [**47**, Prop. 5.21].

After the discovery of smash products and compatible model structures for $\mathbf{\Gamma}$-spaces and symmetric spectra it became obvious that variations of this theme are possible. Simplicial functors were first used for the purposes of describing stable homotopy types by Bökstedt and Waldhausen when they introduced 'FSPs' in [**8**]. Various model structures and the smash product of simplicial functors were systematically studied by Lydakis in [**48**]. The paper [**53**] contains a systematic study of 'diagram spectra', their model structures and smash products, which includes symmetric spectra, $\mathbf{\Gamma}$-spaces and simplicial functors. Here orthogonal spectra and continuous functors (defined on finite CW-complexes) make their first explicit appearance. The category of S-modules is very different in flavor from the categories diagram spectra, and it is defined and studied in the monograph [**26**].

CHAPTER II

The stable homotopy category

[intro to the chapter]

1. The stable homotopy category

Now we introduce the stable homotopy category. For this purpose we choose for each symmetric spectrum of simplicial sets Y a stable equivalence $p_Y : Y \longrightarrow \omega Y$ with target an injective Ω-spectrum, which is possible by combining Propositions I.4.39 and I.4.10. We insist that if Y is already an injective Ω-spectrum, then $\omega Y = Y$ and p_Y is the identity. This is not really necessary, but will simplify some arguments. The following definition depends on these choices, but only very slighty, as we explain in Remark 1.2 below.

Definition 1.1. The *stable homotopy category* $\mathcal{SHC}$ has as objects all symmetric spectra of simplicial sets. For two such spectra, the morphisms from X to Y in $\mathcal{SHC}$ are given by $[X, \omega Y]$, the set of homotopy classes of spectrum morphisms from X to the chosen injective Ω-spectrum ωY. If $f : X \longrightarrow \omega Y$ is a homomorphism of symmetric spectra we denote by $[f] : X \longrightarrow Y$ its homotopy class, considered as a morphism in $\mathcal{SHC}$.

Composition in the stable homotopy category is defined as follows. Let $f : X \longrightarrow \omega Y$ and $g : Y \longrightarrow \omega Z$ be morphism of symmetric spectra which represent morphism from X to Y respectively from Y to Z in $\mathcal{SHC}$. Then there is a morphism $\bar{g} : \omega Y \longrightarrow \omega Z$, of symmetric spectra, unique up to homotopy, such that $\bar{g} \circ p_Y$ is homotopic to g. The composite of $[f] \in \mathcal{SHC}(X, Y)$ and $[g] \in \mathcal{SHC}(Y, Z)$ is then defined by

$$[g] \circ [f] \ = \ [\bar{g} \circ f] \ \in \ \mathcal{SHC}(X, Z) \ .$$

There are a few things to check so that Definition 1.1 makes sense. To see that composition in the stable homotopy category is associative we consider four symmetric spectra X, Y, Z and W and three homomorphisms $f : X \longrightarrow \omega Y$, $g : Y \longrightarrow \omega Z$ and $h : Z \longrightarrow \omega W$ of symmetric spectra. We also pick homomorphisms $\bar{g} : \omega Y \longrightarrow \omega Z$ and $\bar{h} : \omega Z \longrightarrow \omega W$ such that $\bar{g} \circ p_Y \simeq g$ and $\bar{h} \circ p_Z \simeq h$. Then we have

$$([h][g])[f] \ = \ [\bar{h} \circ g][f] \ = \ [(\bar{h} \circ \bar{g}) \circ f] \ = \ [\bar{h} \circ (\bar{g} \circ f)] \ = \ [h][\bar{g} \circ f] \ = \ [h]([g][f])$$

where the second equality uses that $(\bar{h} \circ \bar{g}) \circ p_Y$ is homotopic to $\bar{h} \circ g$. It is straightforward to check that $[p_X]$, the homotopy class of the chosen stable equivalence $p_X : X \longrightarrow \omega X$, is a two-sided unit for composition, so p_X represents the identity of X in $\mathcal{SHC}$.

Remark 1.2. The definition of the stable homotopy category depends on the unspecified choices of stable equivalences $p_X : X \longrightarrow \omega X$ with targets injective Ω-spectra. However, if $p'_X : X \longrightarrow \omega' X$ is another such choice, then there is a unique homotopy class of morphisms of symmetric spectra $\kappa : \omega X \longrightarrow \omega' X$ such that $\kappa \circ p_X$ is homotopic to p'_X. By symmetry and uniqueness, κ is a homotopy equivalence. So if we use p'_X instead of p_X in the definition of the stable homotopy category, the resulting morphism sets are in canonical bijection. Altogether, the stable homotopy category is independent of the choices up to preferred isomorphism of categories which is the identity on objects. This strong uniqueness property is also reflected in the universal property of the stable homotopy category, see Theorem 1.6 below.

Remark 1.3. The choice $p_Y : Y \longrightarrow \omega Y$ of stable equivalence to an injective Ω-spectrum can in fact be made functorially at the pointset level (and not just up to homotopy), since the level equivalent injective replacement $Y \mapsto Y^{\mathrm{inj}}$ of Proposition I.4.10 and the stably equivalent Ω-spectrum replacement Q of Proposition I.4.39 are both functorial. However, if we want this extra functoriality, we cannot simultaneously arrange things so that $\omega Y = Y$ if Y is already an injective Ω-spectrum. Since pointset level functoriality of ω is irrelevant for the current discussion, and so we continue without it.

In Chapter III we will show that the stable equivalences can be complemented by various useful choices of cofibrations and fibrations, thus arriving at different stable model category structures for symmetric spectra. For one particular choice (the *injective stable model structure*), every symmetric spectrum is cofibrant and the fibrant objects are precisely the injective Ω-spectra. Moreover, the 'concrete' homotopy relation using homotopies defined on $\Delta[1]^+ \wedge A$ coincides with the model category theoretic homotopy relation using abstract cylinder objects. Thus the stable homotopy category as introduced above turns out to be the homotopy category, in the sense of model category theory, with respect to the injective stable model structure.

The stable homotopy category comes with a functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ from the category of symmetric spectra of simplicial sets which is the identity on objects. For a morphism $\varphi : X \longrightarrow Y$ of symmetric spectra we set

$$\gamma(\varphi) \ = \ [p_Y \circ \varphi] \quad \text{in} \quad \mathcal{SHC}(X, Y) \ ,$$

where $p_Y : Y \longrightarrow \omega Y$ is the chosen stable equivalence. Note that we have $\gamma(p_Y) = [p_Y]$ since $p_{\omega Y} = \mathrm{Id}$ by convention. Thus for every morphism $f : X \longrightarrow \omega Y$ of symmetric spectra we have the relation $\gamma(p_Y) \circ [f] = \gamma(f)$ as morphisms from X to ωY in the stable homotopy category. Since $\gamma(p_Y) = [p_Y]$ is an isomorphism with inverse $[\mathrm{Id}_{\omega Y}]$, this can also be rewritten as

$$[f] \ = \ \gamma(p_Y)^{-1} \circ \gamma(f) \ \in \mathcal{SHC}(X, Y) \ . \tag{1.4}$$

In other words, every morphism in the stable homotopy category can be written as a 'fraction', i.e., the composite of a morphism of symmetric spectra with the inverse of a stable equivalence.

We also note that for morphisms $\alpha : W \longrightarrow X$ and $f : X \longrightarrow \omega Y$ we have the relation

$$[f] \circ \gamma(\alpha) \ = \ [f\alpha] \ \in \mathcal{SHC}(W, \omega Y) \ . \tag{1.5}$$

Indeed, if $\bar f : \omega X \longrightarrow \omega Y$ is such that $\bar f p_X$ is homotopic to f, then $\bar f p_X \alpha$ is homotopic to $f\alpha$ and so

$$[f] \circ \gamma(\alpha) \ = \ [f] \circ [p_X \circ \alpha] \ = \ [\bar f \circ p_X \circ \alpha] \ = \ [f\alpha] \ .$$

We now show that γ is indeed a functor; even better: γ is the universal functor that takes stable equivalences to isomorphisms.

Theorem 1.6. *The functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ is a localization of the category of symmetric spectra at the class of stable equivalences. More precisely, we have*

(i) *The assignment $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ is a functor which takes stable equivalences to isomorphisms. Moreover, a morphism φ of symmetric spectra is a stable equivalence if and only if $\gamma(\varphi)$ is an isomorphism in the stable homotopy category.*

(ii) *For every functor $F : \mathcal{S}p \longrightarrow \mathcal{C}$ which takes stable equivalences to isomorphisms, there exists a unique functor $\bar F : \mathcal{SHC} \longrightarrow \mathcal{C}$ such that $\bar F \gamma = \varepsilon$.*

(iii) *Let $\tau : F \longrightarrow G$ be a natural transformation between functors $F, G : \mathcal{S}p \longrightarrow \mathcal{C}$ which takes stable equivalences to isomorphisms. Then there exists a unique natural transformation $\bar\tau : \bar F \longrightarrow \bar G$ between the induced functors $\bar F, \bar G : \mathcal{SHC} \longrightarrow \mathcal{C}$ such that $\bar\tau\gamma = \tau$. If τ is a natural isomorphism, so is $\bar\tau$.*

Proof. (i) First we check functoriality. Since the homotopy class of p_Y is the identity of Y in $\mathcal{SHC}$, γ preserves identities. For composable morphism of symmetric spectra $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ we have

$$\gamma(\psi)\gamma(\varphi) \ = \ [p_Z \circ \psi]\gamma(\varphi) \ = \ [p_Z \circ (\psi\varphi)] \ = \ \gamma(\psi\varphi)$$

by (1.5). So γ is indeed functorial.

Now we show that a morphism $\varphi : X \longrightarrow Y$ of symmetric spectra is a stable equivalence if and only if $\gamma(\varphi)$ is an isomorphism in the stable homotopy category. By definition, the morphism φ is a stable equivalence if and only if the map $[\varphi, W]$ is bijective for every injective Ω-spectrum W. If Z runs through the class of all symmetric spectra, then ωZ runs through the class of all injective Ω-spectra. So $\varphi : X \longrightarrow Y$

is a stable equivalence if and only if the map $[\varphi, \omega Z]$ is bijective for every symmetric spectrum Z. The relation (1.5) shows that the square

$$\begin{tikzcd} \mathcal{SHC}(Y,Z) \arrow[r, "\mathcal{SHC}(\gamma(\varphi),Z)"] \arrow[d, equal] & \mathcal{SHC}(X,Z) \arrow[d, equal] \\ {[Y,\omega Z]} \arrow[r, "{[\varphi,\omega Z]}"'] & {[X,\omega Z]} \end{tikzcd}$$

commutes. So φ is a stable equivalence if and only if $\mathcal{SHC}(\gamma(\varphi), Z)$ is bijective for every symmetric spectrum Z. By the Yoneda lemma, this happens if and only if $\gamma(\varphi)$ is an isomorphism.

(ii) We consider a functor $\bar{F} : \mathcal{SHC} \longrightarrow \mathcal{C}$ and prove the uniqueness property by showing that $\bar{F}$ is completely determined by the composite functor $\bar{F} \circ \gamma : \mathcal{S}p \longrightarrow \mathcal{C}$. This is clear on objects since γ is the identity on objects. If $f : X \longrightarrow \omega Y$ is a morphism of symmetric spectra which represents a morphism $[f] : X \longrightarrow Y$ in $\mathcal{SHC}$, then we can apply $\bar{F}$ to the equation (1.4) and obtain

$$\bar{F}([f]) \;=\; \bar{F}(\gamma(p_Y)^{-1} \circ \gamma(f)) \;=\; (\bar{F} \circ \gamma)(p_Y)^{-1} \circ (\bar{F} \circ \gamma)(f) \;.$$

Thus also the behavior of $\bar{F}$ on morphisms is determined by the composite $\bar{F} \circ \gamma$.

Now we tackle the existence property. Given a functor $F : \mathcal{S}p \longrightarrow \mathcal{C}$ which takes stable equivalences to isomorphisms we set $\bar{F}(X) = F(X)$ on objects. Given a homomorphism $f : X \longrightarrow \omega Y$, the uniqueness argument tells us that we have to define the value of $\bar{F}$ on $[f]$ by

$$\bar{F}([f]) \;=\; F(p_Y)^{-1} \circ F(f) \;. \tag{1.7}$$

We have to check that this is well-defined and functorial.

To see that the assignment (1.7) is well-defined we have to show that the $\mathcal{C}$-morphism $F(f)$ only depends on the homotopy class of $f : X \longrightarrow \omega Y$. Indeed, the morphism $c : \Delta[1]^+ \wedge X \longrightarrow X$ that maps all of $\Delta[1]$ to a point is a level equivalence, hence a stable equivalence. So by hypothesis, $F(c) : F(\Delta[1]^+ \wedge X) \longrightarrow F(X)$ is an isomorphism in $\mathcal{C}$. The composite with the two end point inclusions $i_0, i_1 : X \longrightarrow \Delta[1]^+ \wedge X$ satisfy $c \circ i_0 = \mathrm{Id}_X = c \circ i_1$, so we have

$$F(c) \circ F(i_0) \;=\; \mathrm{Id}_{F(X)} \;=\; F(c) \circ F(i_1) \;.$$

Since $F(c)$ is an isomorphism, we deduce $F(i_0) = F(i_1)$.

Now suppose that $f, f' : X \longrightarrow \omega Y$ are homotopic morphisms via some homotopy $H : \Delta[1]^+ \wedge X \longrightarrow \omega Y$. Then we have

$$F(f) \;=\; F(H) \circ F(i_0) \;=\; F(H) \circ F(i_1) \;=\; F(g) \;,$$

which proves that the formula (1.7) is well-defined.

By the various definitions we have

$$\bar{F}(\mathrm{Id}_X) \;=\; \bar{F}([p_X]) \;=\; F(p_X)^{-1} \circ F(p_X) \;=\; \mathrm{Id}_{F(X)}$$

so $\bar{F}$ is unital. For associativity we consider two homomorphisms $f : X \longrightarrow \omega Y$ and $g : Y \longrightarrow \omega Z$ as well as a homomorphism $\bar{g} : \omega Y \longrightarrow \omega Z$ such that $\bar{g} \circ p_Y$ is homotopic to g. Then we have

$$\begin{aligned} \bar{F}([g] \circ [f]) \;&=\; \bar{F}([\bar{g} \circ f]) \;=\; F(p_Z)^{-1} \circ F(\bar{g} \circ f) \\ &=\; F(p_Z)^{-1} \circ F(\bar{g} \circ p_Y) \circ F(p_Y)^{-1} \circ F(f) \\ &=\; (F(p_Z)^{-1} \circ F(g)) \circ (F(p_Y)^{-1} \circ F(f)) \;=\; \bar{F}(g) \circ \bar{F}(f) \end{aligned}$$

where we used functoriality of F and homotopy invariance of F. Thus $\bar{F}$ is a functor.

Finally, we have to check the relation $\bar{F} \circ \gamma = F$. On objects this holds by definition. For a homomorphism $\varphi : X \longrightarrow Y$ of symmetric spectra we have

$$\bar{F}(\gamma(\varphi)) \;=\; \bar{F}([p_Y \circ \varphi]) \;=\; F(p_Y)^{-1} \circ F(p_Y \circ \varphi) \;=\; F(\varphi) \;,$$

which finishes the proof.

(iii) □

A fancier way to rephrase Theorem 1.6 is as follows: for every category $\mathcal{C}$, precomposition with the functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ is an isomorphism of categories

$$\mathrm{Hom}(\gamma, \mathcal{C}) \ : \ \mathrm{Hom}(\mathcal{SHC}, \mathcal{C}) \ \longrightarrow \ \mathrm{Hom}^{\mathrm{st.eq.}}(\mathcal{S}p, \mathcal{C}) \ ,$$

where 'Hom' denotes the (big) category of functors and natural transformations, and the superscript 'st.eq.' denotes the full subcategory consisting of those functors which take stable equivalences to isomorphisms.

Example 1.8 (Homotopy groups)**.** By [...] there are stable equivalences of symmetric spectra that do not induced isomorphisms on all naive homotopy groups. So the k-th naive homotopy group functor $\hat{\pi}_k : \mathcal{S}p \longrightarrow \mathcal{A}b$ does not descend to the stable homotopy category. However, the k-th true homotopy group functor $\pi_k : \mathcal{S}p \longrightarrow \mathcal{A}b$ does take stable equivalences to isomorphisms, so it factors uniquely through the localization functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$, by the universal property. We allow ourselves an abuse of notation and also write $\pi_k : \mathcal{SHC} \longrightarrow \mathcal{A}b$ for the induced functor which is defined on the stable homotopy category.

The true homotopy groups then detect isomorphisms in the stable homotopy category:

Proposition 1.9. *Let $f : X \longrightarrow Y$ be a morphism in the stable homotopy category. Then f is an isomorphism in $\mathcal{SHC}$ if and only the induced map $\pi_k f : \pi_k X \longrightarrow \pi_k Y$ on true homotopy groups is an isomorphism for every integer k.*

Proof. Suppose first that $f = \gamma(\alpha)$ for some morphism α of symmetric spectra. Then f is an isomorphism in $\mathcal{SHC}$ if and only if α is a stable equivalence (by Theorem 1.6 (i)) and that is the case if and only if the maps $\pi_k\alpha$ are all isomorphisms (by Theorem I.6.2). Since the composite of the true homotopy group functor $\pi_k : \mathcal{SHC} \longrightarrow \mathcal{A}b$ with the localization functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ is the true homotopy group functor on the category of symmetric spectra, this proves the claim whenever f is in the image of γ. An arbitrary morphism in $\mathcal{SHC}$ is of the form $f = \gamma(p)^{-1} \circ \gamma(\alpha)$ for a morphism of symmetric spectra $\alpha : X \longrightarrow Z$ and a stable equivalence $p : Y \longrightarrow Z$. Since $\gamma(p)$ is an isomorphism, f is an isomorphism if and only if $\gamma(\alpha)$ is, and we are reduced to the special case considered first. □

The stable homotopy category does *not* have general limits and colimits, but it has coproducts and products of arbitrary size:

Proposition 1.10. (i) *The stable homotopy category has coproducts and the functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ preserves coproducts.*
(ii) *The stable homotopy category has products. The functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ preserves* finite *products.*
(iii) *For every* finite *family of symmetric spectra the canonical morphism in the stable homotopy category from the coproduct to the product is an isomorphism.*

Proof. (i) We show that γ preserves arbitrary coproducts. Since every object of $\mathcal{SHC}$ is in the image of γ, this in particular shows that coproducts exist in the stable homotopy category.

Let $\{X^i\}_{i\in I}$ be a family of symmetric spectra. Let $\alpha^j : X^j \longrightarrow \bigvee_{i\in I} X^i$ denote the inclusion of the jth wedge summand. We then have to show that for every symmetric spectrum Y the map

$$\mathcal{SHC}(\bigvee_{i\in I} X^i, Y) \ \longrightarrow \ \prod_{i\in I} \mathcal{SHC}(X^i, Y) \ , \qquad [f] \quad \longmapsto \quad \big([f]\circ\gamma(\alpha^i)\big)_{i\in I}$$

is a bijection. By definition of morphisms in $\mathcal{SHC}$ this amounts to verifying that the map

$$[\bigvee_{i\in I} X^i, \omega Y] \ \longrightarrow \ \prod_{i\in I} [X^i, \omega Y] \ , \qquad [f] \quad \longmapsto \quad \big([f\alpha^i]\big)_{i\in I}$$

between sets of homotopy classes of morphisms is a bijection, where we used (1.5). This last claim is clear since a morphism out of a wedge corresponds bijectively to a familiy of morphisms from each summand, and similarly for homotopies.

(ii) We start by constructing products in the stable homotopy category, which are in general *not* given by the product as symmetra spectra. Let $\{Y^i\}_{i\in I}$ be a family of symmetric spectra with chosen stable equivalences $p_{Y^i} : Y^i \longrightarrow \omega(Y^i)$. We form the product of the injective Ω-spectra $\omega(Y^i)$; for each index $j \in I$ this symmetric spectrum comes with a projection

$$\pi_j \ : \ \prod_{i\in I} \omega(Y^i) \ \longrightarrow \ \omega(Y_j)$$

which represent a morphism $[\pi_j]$ in $\mathcal{SHC}$ from $\prod_{i\in I} \omega(Y^i)$ to Y_j. We claim that these morphisms make $\prod_{i\in I} \omega(Y^i)$ into a product, in the stable homotopy category, of the family $\{Y^i\}_{i\in I}$.

To see this we have to show that for every symmetric spectrum X the map

$$\mathcal{SHC}(X, \prod_{i\in I} \omega(Y^i)) \ \longrightarrow \ \prod_{i\in I} \mathcal{SHC}(X, Y^i) \ , \qquad [f] \quad \longmapsto \quad \big([\pi_i]\circ[f]\big)_{i\in I}$$

is a bijection. Since $\omega(Y^i)$ is an injective Ω-spectrum for all $i \in I$, so is the product. So by our convention, $\omega\big(\prod_{i\in I} \omega(Y^i)\big) = \prod_{i\in I} \omega(Y^i)$ and the morphism $p_{\prod_{i\in I}\omega(Y^i)}$ is the identity. By definition of morphisms in $\mathcal{SHC}$ we thus have to verifying that the map

$$[X, \prod_{i\in I} \omega(Y^i)] \ \longrightarrow \ \prod_{i\in I} [X, \omega(Y^i)] \ , \qquad [f] \quad \longmapsto \quad \big([\pi_i \circ f]\big)_{i\in I}$$

between sets of homotopy classes of morphisms is a bijection, where we used (1.5). This is again clear since a morphism to a product corresponds bijectively to a familiy of morphisms to each factor, and similarly for homotopies.

It remains to show that γ preserves finite products. For ease of notation we treat the case of two factors Y and Y'; the general case then follows by induction on the number of factors. Since ωY and $\omega Y'$ are injective Ω-spectra, so is their product. There is thus a morphism $\psi : \omega\,(Y\times Y') \longrightarrow (\omega Y)\times(\omega Y')$, unique up to homotopy, such that $\psi\circ p_{Y\times Y'} : Y\times Y' \longrightarrow (\omega Y)\times(\omega Y')$ is homotopic to $p_Y \times p_{Y'}$. The morphisms $p_Y\times p_{Y'} : Y\times Y' \longrightarrow (\omega Y)\times(\omega Y')$ (by Proposition I.4.31 (ii)) and $p_{Y\times Y'}$ are stable equivalences, hence so is ψ. As a stable equivalence between injective Ω-spectra, ψ is thus a homotopy equivalence. The map

$$\mathcal{SHC}(X, Y\times Y') \ \longrightarrow \ \mathcal{SHC}(X,Y)\times \mathcal{SHC}(X,Y') \ , \qquad [f] \quad \longmapsto \quad \big([\pi_Y]\circ[f],\, [\pi_{Y'}]\circ[f]\big)$$

equals the composite

$$[X, \omega(Y\times Y')] \ \xrightarrow{[X,\psi]} \ [X, (\omega Y)\times(\omega Y')] \ \xrightarrow{([X,\pi_{\omega Y}],[X,\pi_{\omega Y'}])} \ [X,\omega Y]\times[X,\omega Y']$$

The first map is a bijection since ψ is a homotopy equivalence. The second map is a bijection since a morphism to a product corresponds bijectively to a pair of morphisms to each factor, and similarly for homotopies.

(iii) It suffices to consider two factors. The canonical morphism $\kappa : X\vee Y \longrightarrow X\times Y$ of symmetric spectra is a stable equivalence by Corollary I.4.25, and so the morphism $\gamma(\kappa) : \gamma(X\vee Y) \longrightarrow \gamma(X\times Y)$ is an isomorphism is the stable homotopy category. Since γ preserves coproducts and finite product, $\gamma(\kappa)$ is the canonical morphism, in the stable homotopy category, from a coproduct of $\gamma(X)$ and $\gamma(Y)$ to a product. This proves the claim since every object in $\mathcal{SHC}$ is in the image of γ. □

The next thing we show is that the stable homotopy category is additive, i.e., there is a natural commutative groups structure on the homomorphism sets such that composition is biadditive. This definition makes it sound as if 'additive category' is extra structure on a category (namely the addition on morphism sets), but in fact, 'additive category' is really a property of a category (namely having finite sums which are isomorphic to products). So we present the construction of the addition on hom-sets in this generality; in Remark 1.14 below we make the addition on the stable homotopy category a bit more explicit.

Construction 1.11. (change product and coproducts for later consistency) Let $\mathcal{C}$ be a category which has finite products and a zero object. Suppose further that 'finite products are coproducts'; more precisely, assume that for every pair of objects A and B the morphisms $i_1 = (\mathrm{Id}, 0) : A \longrightarrow A\times B$ and $i_2 = (0, \mathrm{Id}) : B \longrightarrow A\times B$ make $A\times B$ into a *co*-product of A and B, where '0' is the unique morphism which factors through a zero object. In other words, we demand that for every object X the map

$$\mathcal{C}(A\times B, X) \ \longrightarrow \ \mathcal{C}(A,X)\times\mathcal{C}(B,X) \ , \quad f \mapsto (fi_1, fi_2)$$

is a bijection.

In this situation we can define a binary operation on the morphism set $\mathcal{C}(A, X)$ for every pair of objects A and X. Given morphisms $a, b : A \longrightarrow X$ we let $a \perp b : A \times A \longrightarrow X$ be the unique morphism such that $(a \perp b)i_1 = a$ and $(a \perp b)i_2 = b$. Then we define $a + b : A \longrightarrow X$ as $(a \perp b)\Delta$ where $\Delta = (\mathrm{Id}, \mathrm{Id}) : A \longrightarrow A \times A$ is the diagonal morphism.

Proposition 1.12. *Let $\mathcal{C}$ be a category which has finite products and a zero object, and in which 'finite products are coproducts' in the sense of Construction 1.11.*

(i) *For every pair of objects A and X of $\mathcal{C}$ the binary operation $+$ makes the set $\mathcal{C}(A, X)$ of morphisms into an abelian monoid with the zero morphism as neutral element. Moreover, the monoid structure is natural for all morphisms in both variables, or, equivalently, composition is biadditive.*

(ii) *If, moreover, the shearing morphism $i_1 \perp \Delta : A \times A \longrightarrow A \times A$ is an isomorphism, then the abelian monoid $\mathcal{C}(A, X)$ has additive inverse, i.e., is an abelian group, for every object X.*

(iii) *Let $F : \mathcal{C} \longrightarrow$ (abelian monoids) be a functor that preserves zero objects and finite products. Then for all objects A and X of $\mathcal{C}$ and every element $a \in F(A)$ the evaluation map*

$$\mathcal{C}(A, X) \longrightarrow F(X) , \quad f \longmapsto F(f)(a)$$

is a monoid homomorphism.

Proof. (i) The proof is lengthy, but quite formal. For the associativity of '+' we consider three morphisms $a, b, c : A \longrightarrow X$. Then $a + (b + c)$ respectively $(a + b) + c$ are the two outer composites around the diagram

$$\begin{tikzcd}[column sep=small]
 & & A \arrow[dll, "\Delta"'] \arrow[drr, "\Delta"] & & \\
A \times A \arrow[d, "\mathrm{Id} \times \Delta"'] & & & & A \times A \arrow[d, "\Delta \times \mathrm{Id}"] \\
A \times (A \times A) \arrow[drr, "a \perp (b \perp c)"'] & & & & (A \times A) \times A \arrow[dll, "(a \perp b) \perp c"] \\
 & & X & &
\end{tikzcd}$$

If we fill in the canonical associativity isomorphism $A \times (A \times A) \cong (A \times A) \times A$ then the upper part of the diagram commutes because the diagonal morphism is coassociative. The lower triangle then commutes since the two morphisms $a \perp (b \perp c), (a \perp b) \perp c : A \times (A \times A) \longrightarrow X$ have the same 'restrictions', namely a, b respectively c.

The commutativity is a consequence of two elementary facts: first, $b \perp a = (a \perp b)\tau$ where $\tau : A \times A \longrightarrow A \times A$ is the automorphism which interchanges the two factors; this follows from $\tau i_1 = i_2$ and $\tau i_2 = i_1$. Second, the diagonal morphism is cocommutative, i.e., $\tau \Delta = \Delta : A \longrightarrow A \times A$. Altogether we get

$$a + b \ = \ (a \perp b)\Delta \ = \ (a \perp b)\tau\Delta \ = \ (b \perp a)\Delta \ = \ b + a \ .$$

As before we denote by $0 \in \mathcal{C}(A, X)$ the unique morphism which factors through a zero object. Then we have $a \perp 0 = ap_1$ in $\mathcal{C}(A \times A, X)$ where $p_1 : A \times A \longrightarrow A$ is the projection onto the first factor. Hence $a + 0 = (a \perp 0)\Delta = ap_1\Delta = a$; by commutativity we also have $0 + a = a$.

Now we verify naturality of the addition on $\mathcal{C}(A, X)$ in A and X. To check $(a + b)c = ac + bc$ for $a, b : A \longrightarrow X$ and $c : A' \longrightarrow A$ we consider the commutative diagram

$$\begin{tikzcd}
A' \arrow[r, "c"] \arrow[d, "\Delta"'] & A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A \times A \arrow[d, "a \perp b"] \\
A' \times A' \arrow[r, "c \times c"] \arrow[rr, bend right, "ac \perp bc"'] & A \times A \arrow[r, "a \perp b"] & X
\end{tikzcd}$$

in which the composite through the upper right corner is $(a+b)c$. We have $(a\perp b)(c\times c)i_1 = (a\perp b)(c,0) = ac = (ac\perp bc)i_1$ and similarly for i_2 instead i_1. So $(a\perp b)(c\times c) = ac\perp bc$ since both sides have the same 'restrictions' to the two factors of $A'\times A'$. Since the composite through the lower left corner is $ac+bc$, we have shown $(a+b)c = ac+bc$. Naturality in X is even easier. For a morphism $d : X \longrightarrow Y$ we have $d(a\perp b) = da\perp db : A\times A \longrightarrow Y$ since both sides have the same 'restrictions' da respectively db to the two factors of $A\times A$. Thus $d(a+b) = da+db$ by the definition of '+'.

(ii) An arbitrary abelian monoid M has additive inverses if and only if the map

$$M^2 \;\longrightarrow\; M^2\ , \quad (a,b)\longmapsto (a,a+b)$$

is bijective. Indeed, the inverse of $a\in A$ is the second component of the preimage of $(a,0)$. For the abelian monoid $\mathcal{C}(A,X)$ the square

$$\begin{tikzcd}
\mathcal{C}(A\times A,X) \arrow[r, "(i_1\perp\Delta)^*"] \arrow[d, "(i_1^*,i_2^*)"', "\cong"] & \mathcal{C}(A\times A,X) \arrow[d, "\cong"', "(i_1^*,i_2^*)"] \\
\mathcal{C}(A,X)^2 \arrow[r, "(a,b)\mapsto(a,a+b)"'] & \mathcal{C}(A,X)^2
\end{tikzcd}$$

commutes by definition and both vertical maps are bijective. Since $i_1\perp\Delta$ is an isomorphism, the upper map is bijective, hence so is the lower map, and so the monoid $\mathcal{C}(A,X)$ has inverses.

(iii) We start by showing that the images of the three morphisms $i_1, i_2, \Delta : A \longrightarrow A\times A$ under the functor F satisfy

$$F(\Delta) \;=\; F(i_1) \;+\; F(i_2)$$

where the sum is pointwise addition of functions $F(A)\longrightarrow F(A\times A)$. Indeed, this relation holds after precomposition with the two maps $F(p_1), F(p_2) : F(A\times A)\longrightarrow F(A)$, and that suffices because $(F(p_1),F(p_2)) : F(A\times A)\longrightarrow F(A)\times F(A)$ is bijective by hypothesis on F. So we get

$$\begin{aligned} F(f+g) \;&=\; F(f\perp g)\circ F(\Delta) \;=\; F(f\perp g)\circ(F(i_1)+F(i_2)) \\ &=\; F(f\perp g)\circ F(i_1) \;+\; F(f\perp g)\circ F(i_2) \;=\; F(f) \;+\; F(g)\ . \end{aligned}$$

□

Corollary 1.13. *The stable homotopy category $\mathcal{SHC}$ is an additive category.*

Proof. We apply Proposition 1.12 to the stable homotopy category. Finite coproducts and products exist and are isomorphic by Proposition 1.10. So we have to verify that for every symmetric spectrum A the morphism $i_1\perp\Delta : A\times A\longrightarrow A\times A$ is an isomorphism in the stable homotopy category. For every integer k the composite map

$$\pi_k A\oplus\pi_k A \;\longrightarrow\; \pi_k(A\times A) \;\xrightarrow{\pi_k(i_1\perp\Delta)}\; \pi_k(A\times A) \;\longrightarrow\; \pi_k A\times\pi_k A$$

sends (a,b) to $(a+b,b)$ where the first and last maps are the canonical ones. The composite map is an isomorphism since $\pi_k A$ is a group, i.e., has additive inverses. Since canonical maps are isomorphisms, so is the middle map; thus $i_1\perp\Delta$ induces isomorphisms of true homotopy groups. and is thus a stable equivalence. Since homotopy groups detect isomorphisms in the stable homotopy category, (see Proposition 1.9) $i_1\perp\Delta$ is an isomorphism. □

Remark 1.14. As we have just seen, the stable homotopy category is additive. However, the binary operation '+' on the morphism sets in $\mathcal{SHC}$ was defined in a rather abstract fashion, and we want to make it more explicit.

We let X be an injective Ω-spectrum. The wedge inclusion $\iota : X\vee X\longrightarrow X\times X$ (which is the canonical morphism in $\mathcal{S}p$ from coproduct to product) is a stable equivalence (see Proposition I.4.31 (ii)). Thus the induced map

$$[\iota,X] \;:\; [X\times X,X] \;\longrightarrow\; [X\vee X,X]$$

on homotopy classes of morphisms is a bijection. So there exists a morphism $m : X\times X\longrightarrow X$, unique up to homotopy, such that $m\iota : X\vee X\longrightarrow X$ equals the fold map (which is the identity on each wedge summand). Given any other symmetric spectrum A and two morphisms $f, g : A \longrightarrow X$ we define a new morphism $f+g : A \longrightarrow X$ as $f+g = m(f,g)$. This construction is well-defined on homotopy classes and we claim that induced construction on the set $[A, X]$ of homotopy classes of morphisms from A to X coincides with the operation '+' defined in Construction 1.11 in the case of the stable homotopy category.

Indeed, if A and B are symmetric spectra, then morphisms from A to B in the stable homotopy category are defined as $[A, \omega B]$, the set of homotopy classes of morphisms from A to the chosen stably equivalent replacement. Given morphisms of symmetric spectra $f, g : A \longrightarrow \omega B$, then the composite $m(f \times g) : A \times A \longrightarrow \omega B$ satisfies

$$(m(f \times g)) \circ i_1 = m(f, 0) \simeq f$$

and $(m(f \times g)) \circ i_2$ is similarly homotopic to g. So the homotopy class $[m(f \times g)] \in [A \times A, \omega B]$ equals $[f] \perp [g]$, and thus

$$[f] + [g] \;=\; ([f]\perp[g])\Delta \;=\; [m(f \times g)\Delta] \;=\; [m(f,g)] \; .$$

The operation which sends a pair of morphisms $f, g : A \longrightarrow X$ to $m(f,g)$ is not associative nor commutative on the pointset level, but by Proposition 1.12 the induced operation '+' on the set $[A, X]$ of homotopy classes of morphisms is an abelian group structure. An equivalent way of saying this is that the morphism m is associative, commutative and unital up to homotopy, and has a homotopy inverse.

The additivity of the stable homotopy category is a fundamental result which deserves two different proofs.

Second proof of Corollary 1.13. If X is an injective Ω-spectrum then the morphism $\tilde{\lambda}_X : X \longrightarrow \Omega(\mathrm{sh}\, X)$ (see (4.16) in Chapter I) is a level equivalence between injective Ω-spectra. By Proposition I.4.6, λ_X^* is the a homotopy equivalence, so it induces a homotopy equivalence

$$\mathrm{map}(A, \tilde{\lambda}_X) \;:\; \mathrm{map}(A, X) \;\longrightarrow\; \mathrm{map}(A, \Omega(\mathrm{sh}\, X)) \cong \Omega\, \mathrm{map}(A, \mathrm{sh}\, X)$$

on mapping spaces. Since the target is the simplicial loop space, the loop addition defines a group structure on the set of components $\pi_0\, \mathrm{map}(A, \Omega(\mathrm{sh}\, X))$ which we pull back along the bijection induced by $\mathrm{map}(A, \tilde{\lambda}_X)$ to a natural group structure on $\pi_0\, \mathrm{map}(A, X)$. Now we show that the natural bijection

$$[A, X] \;\cong\; \pi_0\, \mathrm{map}(A, X)$$

takes the operation '+' to the loop product in the components of the mapping space $\mathrm{map}(A, X)$ and we show simultaneously that the product on the right hand side is abelian. For this we consider the commutative diagram

$$\begin{tikzcd}
{[A,X]^2} \arrow[r, "\cong"] \arrow[dd, bend right=60, "+"'] & (\pi_0\, \mathrm{map}(A,X))^2 \arrow[dd, bend left=60, "\text{loop product}"] \\
{[A \times A, X]} \arrow[u, "\cong", "(i_0^*, i_1^*)"'] \arrow[r, "\cong"] \arrow[d, "{[\Delta, X]}"] & \pi_0\, \mathrm{map}(A \times A, X) \arrow[u, "(i_0^*, i_1^*)", "\cong"'] \arrow[d, "\pi_0\, \mathrm{map}(\Delta, X)"'] \\
{[A,X]} \arrow[r, "\cong"'] & \pi_0\, \mathrm{map}(A,X)
\end{tikzcd}$$

of sets in which all horizontal and the left upper vertical map are bijections. The left vertical composite defines '+'. The right vertical composite coincides with the loop product in $\pi_0\, \mathrm{map}(A, X)$ since it is a homomorphism which sends $(f, *)$ and $(*, f)$ to f. Since the group multiplication $(\pi_0\, \mathrm{map}(A, X))^2 \longrightarrow \pi_0\, \mathrm{map}(A, X)$ is a homomorphism of groups, the group $\pi_0\, \mathrm{map}(A, X)$, and thus $[A, X]$, is abelian. $\square$

Now we discuss some examples in which we can identify morphisms in the stable homotopy category with other possibly more familiar expressions. We start by identifying morphisms in $\mathcal{SHC}$ out of the free spectra $F_m S^n$.

Example 1.15. For every symmetric spectrum X and all $m, n \geq 0$ the evaluation map

$$\text{ev}_X \ : \ \mathcal{SHC}(F_m S^n, X) \ \longrightarrow \ \pi_{n-m} X \ , \quad \text{ev}_X(\alpha : F_m S^n \longrightarrow X) \ = \ \alpha_*(\iota_m^n) \tag{1.16}$$

is an isomorphism of abelian groups. Here $\iota_m^n \in \pi_{n-m}(F_m S^n)$ is the true fundamental class, defined in (6.4) of Chapter I.

Indeed, since true homotopy groups commute with products, the evaluation map is additive by Proposition 1.12 (iii). By Proposition I.6.5 there is a unique natural transformation $\tau : \pi_{n-m} \longrightarrow \mathcal{SHC}(F_m S^n, -)$ such that $\tau_{F_m S^n}(\iota_m^n) = \text{Id}_{F_m S^n}$. Then the composite $\text{ev} \circ \tau$ is a natural self-transformation of the functor π_{n-m} that satisfies $\text{ev}_{F_m S^n}(\tau_{F_m S^n}(\iota_m^n)) = \text{ev}_{F_m S^n}(\text{Id}_{F_m S^n}) = \iota_m^n$; so $\text{ev} \circ \tau$ is the identity transformation by the uniqueness clause of Proposition I.6.5. The other composite satisfies

$$\tau_X(\text{ev}_X(\alpha)) \ = \ \tau_X(\alpha_*(\iota_m^n)) \ = \ \mathcal{SHC}(F_m S^n, \alpha)(\tau_{F_m S^n}(\iota_m^n)) \ = \ \mathcal{SHC}(F_m S^n, \alpha)(\text{Id}_{F_m S^n}) \ = \ \alpha \ .$$

So the transformation τ is inverse to the evaluation map.

Example 1.17. Let K be a based simplicial set and X an Ω-spectrum. We claim that then the map

$$[K, X_0]_{\mathbf{sS}} \ \cong \ [\Sigma^\infty K, X]_{\mathcal{S}p} \ \xrightarrow{\ \gamma\ } \ \mathcal{SHC}(\Sigma^\infty K, X)$$

given by the localization functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ is bijective. Every stable equivalence between Ω-spectra is a level equivalence (see Proposition I.4.13), so both sides of the map take stable equivalences in X to bijections. We can thus assume that X is an injective Ω-spectrum, in which case the map $\gamma : [\Sigma^\infty K, X] \longrightarrow \mathcal{SHC}(\Sigma^\infty, X)$ is the identity by definition of morphisms in $\mathcal{SHC}$.

There are also plenty of examples which are not quite Ω-spectra, but as least positive Ω-spectra. For positive Ω-spectra X and any based simplicial set K the map

$$[K, \Omega X_1]_{\mathbf{sS}} \ \cong \ [\Sigma^\infty K, \Omega(\text{sh}\, X)]_{\mathcal{S}p} \ \xrightarrow[\cong]{\ \gamma\ } \ \mathcal{SHC}(\Sigma^\infty K, \Omega(\text{sh}\, X)) \ \xleftarrow[\cong]{\mathcal{SHC}(\Sigma^\infty K, \tilde\lambda_X)} \ \mathcal{SHC}(\Sigma^\infty K, X)$$

is bijective because the spectrum $\Omega(\text{sh}\, X)$ is then an Ω-spectrum (from level 0 on) and $\tilde\lambda_X : X \longrightarrow \Omega(\text{sh}\, X)$ is a stable equivalence. By essentially the same argument the set $\mathcal{SHC}(\Sigma^\infty K, X)$ is in bijection with the homotopy set $[K, \Omega^n X_n]_{\mathbf{sS}}$ whenever X is an 'Ω-spectrum from level n on'.

Example 1.18. We specialize the previous Example 1.17 Eilenberg-Mac Lane spectra and pin down an isomorphism between the k-th reduced cohomology group of a based simplicial set K and the group $\mathcal{SHC}(\Sigma^\infty K, \text{sh}^k(HA))$.

For every abelian group A, the Eilenberg-Mac Lane spectrum HA was introduced in Example I.1.14. The symmetric spectrum HA is an Ω-spectrum, hence so is its k-th shift $\text{sh}^k(HA)$. So for every based simplicial set K, Example 1.17 specializes to a bijection

$$[K, A[S^k]]_{\mathbf{sS}} \ = \ [K, (\text{sh}^k(HA))_0]_{\mathbf{sS}} \ \cong \ \mathcal{SHC}(\Sigma^\infty K, \text{sh}^k(HA)) \ , \quad [f] \ \longmapsto \ \gamma(\hat f) \ ,$$

where $\hat f : \Sigma^\infty K \longrightarrow \text{sh}^k(HA)$ is the morphism of symmetric spctra freely generated by a map of based simplicial sets $f : K \longrightarrow A[S^k]$.

The pointed simplicial set $A[S^k]$ is an Eilenberg-Mac Lane space of type (A, k), so it represents cohomology of simplicial sets. More precisely, $A[S^k]$ has a *fundamental class* $\iota \in H^k(A[S^k], A)$ in the k-th cohomology group with coefficients in A. This fundamental class is uniquely characterized by the property that the cap product map

$$H_k(A[S^k], \mathbb{Z}) \ \longrightarrow \ A \ , \quad x \ \longmapsto \ x \cap \iota$$

is inverse to the composite

$$A \ \xrightarrow{\ l\ } \ \pi_k(A[S^k], 0) \ \xrightarrow{\text{Hurewicz}} \ H_k(A[S^k], \mathbb{Z})$$

of the isomorphism l and the Hurewicz homomorphism. [here $l(a)$ is the 'left multiplication' map, i.e., the homotopy class of the map $S^k \longrightarrow A[S^k]$ sending x to ax. The composite sends a to $a \cdot \iota_{S^k}$]

The representability property then says that for every simplicial set K the evaluation map

$$[K, A[S^k]]_{\mathbf{sS}}^{\text{unbased}} \ \longrightarrow \ H^k(K, A) \ , \quad [f] \mapsto f^*(\iota)$$

at the fundamental class is bijective. If K is based, then this bijection restricts to a bijection

$$[K, A[S^k]]_{\mathbf{sS}} \longrightarrow \tilde{H}^k(K, A) , \quad [f] \mapsto f^*(\iota) .$$

We can thus get a composite bijection

$$\mathcal{SHC}(\Sigma^\infty K, \mathrm{sh}^k(HA)) \xleftarrow[\gamma(\hat{f}) \leftarrow\!\shortmid [f]]{\cong} [K, A[S^k]]_{\mathbf{sS}} \xrightarrow[{[f] \mapsto f^*(\iota)}]{\cong} \tilde{H}^k(K, A) .$$

Example 1.19. As another example we consider the periodic complex K-theory spectrum KU, compare Examples 1.20 and 6.58 of Chapter I. This is a positive Ω-spectrum, and as such, the map

$$[K_+, \Omega(KU)_1] \longrightarrow \mathcal{SHC}(\Sigma^\infty K_+, KU)$$

discussed in Example 1.17 is bijective. As we discussed in I.1.20, the space $(KU)_1$ is weakly equivalent to the infinite unitary group U. The loop space $\Omega(KU_1)$ is thus weakly equivalent to ΩU, which by Bott periodicity is weakly equivalent to $\mathbb{Z} \times BU$. So if K is finite, and hence the realization $|K|$ a compact space, then the set of unbased homotopy classes of maps from K to $\Omega(KU)_1$ is in bijection with the Grothendieck group of complex vector bundles over the realization of K. Altogether this produces a natural bijection

$$\mathcal{SHC}(\Sigma^\infty K_+, KU) \cong K^0(|K|) .$$

[unfortunate notation...] The analogous arguments apply to real topological K-theory; the conclusion is a natural isomorphism

$$\mathcal{SHC}(\Sigma^\infty K_+, KO) \cong KO^0(|K|)$$

to the Grothendieck group of complex vector bundles over the realization of K.

Example 1.20. If K is a based simplicial set and X any symmetric spectrum, we define a natural map to $\mathcal{SHC}(\Sigma^\infty K, X)$ from the set

$$\mathrm{colim}_m \, [K \wedge S^m, X_m]_{\mathbf{sS}}$$

where the colimit is formed over the maps

$$[K \wedge S^m, X_m]_{\mathbf{sS}} \xrightarrow{- \wedge S^1} [K \wedge S^{m+1}, X_m \wedge S^1]_{\mathbf{sS}} \xrightarrow{(\sigma_m)_*} [K \wedge S^{m+1}, X_{m+1}]_{\mathbf{sS}} .$$

The map will turn out to be bijective whenever K is finite and X is semistable and levelwise Kan.

The map from the colimit to $\mathcal{SHC}(\Sigma^\infty K, X)$ is defined as follows. Every based map of simplicial sets $f : K \wedge S^m \longrightarrow X_m$ freely generates a morphism of symmetric spectra $\hat{f} : F_m(K \wedge S^m) \longrightarrow X$. The morphism $\lambda_m : F_m(K \wedge S^m) \longrightarrow \Sigma^\infty K$ freely generated by the identity of $K \wedge S^m = (\Sigma^\infty K)_m$ is a stable equivalence, so it becomes an isomorphism in the stable homotopy category. So we can define

$$[K \wedge S^m, X_m]_{\mathbf{sS}} \longrightarrow \mathcal{SHC}(\Sigma^\infty K, X) \qquad \text{by} \qquad [f] \longmapsto \gamma(\hat{f}) \circ \gamma(\lambda_m)^{-1} .$$

The diagram of morphisms of symmetric spectra

$$\begin{tikzcd} & F_{m+1}(K \wedge S^{m+1}) \arrow[dl, "\lambda_{m+1}"', "\sim"] \arrow[d, "\sim"] \arrow[dr, "\sigma_m \circ \widehat{(f \wedge S^1)}"] & \\ \Sigma^\infty K & & X \\ & F_m(K \wedge S^m) \arrow[ul, "\lambda_m", "\sim"'] \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

commutes, where the middle vertical map is freely generated by the identity of $K \wedge S^{m+1} = (F_m(K \wedge S^m))_{m+1}$. So as m increases the maps to $\mathcal{SHC}(\Sigma^\infty K, X)$ are compatible and assemble into a well-defined natural map

$$\mathrm{colim}_m \, [K \wedge S^m, X_m]_{\mathbf{sS}} \longrightarrow \mathcal{SHC}(\Sigma^\infty K, X) . \tag{1.21}$$

Now assume that the simplicial set K is finite and the symmetric spectrum X is semistable and levelwise Kan. We show that then the map (1.21) is bijective. Since X is levelwise Kan, the simplicial mapping space $\mathrm{map}(K, X_m)$ is Kan for all $m \geq 0$, and the source of the map (1.21) is naturally isomorphic to

$$\mathrm{colim}_m \, [S^m, \mathrm{map}(K, X_m)]_{\mathbf{sS}} \cong \mathrm{colim}_m \, \pi_m \, \mathrm{map}(|K|, |X_m|)| = \hat{\pi}_0 \, \mathrm{map}(|K|, |X|) .$$

Since K is finite, the functor $\mathrm{map}(|K|, -)$ preserves $\hat{\pi}_*$-isomorphisms (see Proposition I.2.19 (v)), so the source of the map (1.21) takes $\hat{\pi}_*$-isomorphisms between symmetric spectra that are levelwise Kan to bijections. The same is true for the target of (1.21) (since $\hat{\pi}_*$-isomorphisms are stable equivalences, hence isomorphisms in $\mathcal{SHC}$). Since X is semistable, it is $\hat{\pi}_*$-isomorphic to an Ω-spectrum (see Theorem I.8.25), so we can assume that X is an Ω-spectrum. For Ω-spectra, however, colimit that is the source of (1.21) is formed over bijections, so we are reduced to Example 1.17.

Example 1.22. An important example of a semistable symmetric spectrum is the sphere spectrum, and we can apply the previous example to $\mathbb{S}$ and its shifts. Unfortunately, the simplicial spheres are not Kan, but we can remedy this by replacing $\mathbb{S}$ be the level equivalent spectrum $\mathcal{S}(\mathbb{S})$, the singular complex of the topological sphere spectrum. This spectrum and its shifts are levelwise Kan and semistable, so for every finite based simplicial set K, the bijection (1.21) specializes to a bijection

$$\mathrm{colim}_m\, [K \wedge S^m, \mathcal{S}(S^{k+m})]_{\mathbf{sS}} \;\cong\; \mathcal{SHC}(\Sigma^\infty K, \mathrm{sh}^k \mathcal{S}(\mathbb{S}))\ .$$

The adjunction between singular complex and geometric realization identifies the previous colimit with the colimit

$$\mathrm{colim}_m\, [|K| \wedge S^m, S^{k+m}]_{\mathbf{T}} \;=\; \pi^k_{\mathrm{s}}(|K|)\ ,$$

the k-th stable cohomotopy group of the geometric realization of K. Moreover, the morphism $\Sigma^\infty S^k \longrightarrow \mathrm{sh}^k \mathcal{S}(\mathbb{S}))$ whose m-th level is the weak equivalence $S^{k+m} \longrightarrow \mathcal{S}(S^{k+m})$ that is adjoint to the homeomorphism (3.3) of Chapter I between $|S^{k+m}|$ and S^{k+m}. As a level equivalence, this morphism induces a bijection on $\mathcal{SHC}(\Sigma^\infty K, -)$, so altogether we constructed a bijection

$$\mathcal{SHC}(\Sigma^\infty K, \Sigma^\infty S^k) \;\cong\; \pi^k_{\mathrm{s}}(|K|)$$

that is natural in K.

2. Triangulated structure

We have seen that the stable homotopy category is an additive category with products and coproducts. In this section we make the stable homotopy category into a triangulated category. First we recall the definition.

Let $\mathcal{T}$ be a category equipped with an endofunctor $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$. A *triangle* in $\mathcal{T}$ (with respect to the functor Σ) is a triple (f, g, h) of composable morphisms in $\mathcal{T}$ such that the target of h is equal to Σ applied to the source of f. We will often display a triangle in the form

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \xrightarrow{\ h\ } \Sigma A\ .$$

A *morphism* from a triangle (f, g, h) to a triangle (f', g', h') is a triple of morphisms $a : A \longrightarrow A'$, $b : B \longrightarrow B'$ and $c : C \longrightarrow C'$ in $\mathcal{T}$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r, "h"] \arrow[d, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A'
\end{tikzcd}$$

commutes. A morphism of triangles is an isomorphism (i.e., has an inverse morphism) if and only all three components are isomorphisms in $\mathcal{T}$.

Definition 2.1. A *triangulated category* is an additive category $\mathcal{T}$ equipped with a self-equivalence $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$ and a collection of triangles, called *distinguished triangles*, which satisfy the following axioms (T0) – (T4).

We refer to the equivalence Σ of a triangulated category as the *suspension*, since that is what it will be in our main example. In algebraic contexts, this equivalence is often denoted $X \mapsto X[1]$ and called the 'shift'.

(T0) The class of distinguished triangles is closed under isomorphism.

(T1) For every object X the triangle $0 \longrightarrow X \xrightarrow{\mathrm{Id}} X \longrightarrow 0$ is distinguished.

(T2) [Rotation] If a triangle (f, g, h) is distinguished, then so is the triangle $(g, h, -\Sigma f)$.

(T3) [Completion of triangles] Given distinguished triangles (f, g, h) and (f', g', h') morphisms (a, b) satisfying $bf = f'a$, there exists a morphism c making the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r, "h"] \arrow[d, dotted, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A'
\end{tikzcd}$$

(T4) [Octahedral axiom] For every pair of composable morphisms $f : A \longrightarrow B$ and $f' : B \longrightarrow D$ there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "f'"'] & C \arrow[r, "h"] \arrow[d, "x"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f'f"'] & D \arrow[r, "g''"'] \arrow[d, "g'"'] & E \arrow[r, "h''"'] \arrow[d, "y"] & \Sigma A \arrow[d, "\Sigma f"] \\
 & F \arrow[r, equal] \arrow[d, "h'"'] & F \arrow[r, "h'"'] \arrow[d, "(\Sigma g)\circ h'"] & \Sigma B \\
 & \Sigma B \arrow[r, "\Sigma g"'] & \Sigma C &
\end{tikzcd}$$

such that the triangles (f, g, h), (f', g', h'), $(f'f, g'', h'')$ and $(x, y, (\Sigma g) \circ h')$ are distinguished.

The above formulation of the axioms appears to be weaker, at first sight, than the original axioms of Verdier [**85**, II.1]; however, we show in Proposition 2.10 below that the weaker axioms imply the stronger properties: part (iii) establishes an 'if and only if' in the rotation axiom (T2), and part (iv) is the octahedral axiom in its original form.

Now we define the triangulated structure for the stable homotopy category. The suspension functor in the stable homotopy category is essentially given by suspension of symmetric spectra. In more detail, we recall from Proposition I.4.29 that the functor $S^1 \wedge -$ preserves stable equivalences of symmetric spectra, and so the composite functor $\gamma \circ (S^1 \wedge -) : \mathcal{S}p \longrightarrow \mathcal{SHC}$ takes stable equivalences to isomorphisms. By the universal property of the functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ (see Theorem 1.6 (ii)), there is a unique functor

$$\Sigma \ : \ \mathcal{SHC} \ \longrightarrow \ \mathcal{SHC}$$

that satisfies $\Sigma \circ \gamma = \gamma \circ (S^1 \wedge -)$. Then Σ is given on objects by $\Sigma X = S^1 \wedge X$, and we claim that the behavior on morphisms is as follows. If $f : X \longrightarrow \omega Y$ represents a morphism $[f]$ from X to Y in $\mathcal{SHC}$, then

$$\Sigma[f] \ = \ [\kappa \circ (S^1 \wedge f)] \ \in \ \mathcal{SHC}(\Sigma X, \Sigma Y),$$

where $\kappa : S^1 \wedge \omega Y \longrightarrow \omega(S^1 \wedge Y)$ is any morphism (uniquely determined up to homotopy) whose composite with the stable equivalence $S^1 \wedge p_Y : S^1 \wedge Y \longrightarrow S^1 \wedge \omega Y$ is homotopic to the chosen stable equivalence $p_{S^1 \wedge Y} : S^1 \wedge Y \longrightarrow \omega(S^1 \wedge Y)$. Indeed, we have

$$\begin{aligned}
[\kappa \circ (S^1 \wedge f)] \ &= \ \gamma(p_{S^1 \wedge Y})^{-1} \circ \gamma(\kappa) \circ \gamma(S^1 \wedge f) \ = \ \gamma(S^1 \wedge p_Y)^{-1} \circ \gamma(S^1 \wedge f) \\
&= \ \Sigma(\gamma(p_Y))^{-1} \circ \Sigma(\gamma(f)) \ = \ \Sigma(\gamma(p_Y)^{-1}\gamma(f)) \ = \ \Sigma[f]
\end{aligned}$$

where we used the relation (1.4) twice, as well as $\gamma(p_{S^1 \wedge Y}) = \gamma(\kappa \circ (S^1 \wedge p_Y))$.

Proposition 2.2. *The suspension functor $\Sigma : \mathcal{SHC} \longrightarrow \mathcal{SHC}$ is a self-equivalence of the stable homotopy category.*

PROOF. The induction functor $\triangleright$ (see Example I.3.17) preserves stable equivalences of symmetric spectra (Proposition I.4.14). The universal property of the localization functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ thus provides a unique functor

$$\Sigma^{-1} \; : \; \mathcal{SHC} \; \longrightarrow \; \mathcal{SHC}$$

that satisfies $\Sigma^{-1} \circ \gamma = \gamma \circ \triangleright$. The two composite constructions $\triangleright(S^1 \wedge A)$ and $S^1 \wedge \triangleright A$ are isomorphic to each other, and they come with a natural stable equivalence $\hat{\lambda}_A : S^1 \wedge \triangleright A \longrightarrow A$ (compare Proposition I.4.28). The isomorphism $\triangleright(S^1 \wedge A) \cong S^1 \wedge \triangleright A$ and the stable equivalence $\hat{\lambda}_A$ descend to natural isomorphisms

$$\Sigma^{-1}(\Sigma A) \; \cong \; \Sigma(\Sigma^{-1} A) \; \cong A$$

of endofunctors on the stable homotopy category (by part (iii) of the universal property of Proposition 1.6). So Σ^{-1} is a quasi-inverse to the suspension functor Σ, which is thus an equivalence of categories. $\square$

Now we define the distinguished triangles in the stable homotopy category. Given any monomorphism $j : A \longrightarrow B$ of symmetric spectra of simplicial sets, we define the *connecting morphism* $\delta(j) : B/A \longrightarrow \Sigma A$ in $\mathcal{SHC}$ as

$$\delta(j) \; = \; \gamma(p) \circ \gamma(0 \cup q)^{-1} \; : B/A \; \longrightarrow \; \Sigma A \; . \tag{2.3}$$

Here $C(j)$ is the mapping cone of the morphism j (compare (2.8) of Chapter I), and $p : C(j) \longrightarrow S^1 \wedge A$ the projection that sends the image of B to the basepoint and is the projection $\Delta[1] \longrightarrow S^1$, smashed with the identity of A, on the cone. Moreover, $q : B \longrightarrow B/A$ is the quotient morphism and $0 \cup q : C(j) \longrightarrow B/A$ is the level equivalence that collapses $\Delta[1] \wedge A$. Since the mapping cone is functorial and the morphisms $p : C(j) \longrightarrow S^1 \wedge A$ and $0 \cup q : C(j) \longrightarrow B/A$ are natural as morphisms of symmetric spectra, the connecting morphism (2.3) is natural in the stable homotopy category, i.e., for every commutative square of symmetric spectra on the left

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A' \arrow[r, "j'"'] & B' \end{tikzcd} \qquad\qquad \begin{tikzcd} B/A \arrow[r, "\delta(j)"] \arrow[d, "\gamma(\beta/\alpha)"'] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\ B'/A' \arrow[r, "\delta(j')"'] & \Sigma A' \end{tikzcd}$$

such that j and j' are monomorphisms, the square on the right commutes in $\mathcal{SHC}$.

The *elementary distinguished triangle* associated to the monomorphism $j : A \longrightarrow B$ is the sequence

$$A \xrightarrow{\gamma(j)} B \xrightarrow{\gamma(q)} B/A \xrightarrow{\delta(j)} \Sigma A \; .$$

A *distinguished triangle* is any triangle in the stable homotopy category which is isomorphic to an elementary distinguished triangle, i.e., such that there is a monomorphism $j : A \longrightarrow B$ of symmetric spectra and isomorphisms $a : X \longrightarrow A$, $b : Y \longrightarrow b$ and $c : Z \longrightarrow B/A$ in $\mathcal{SHC}$ that make the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "g"] \arrow[d, "b"'] & Z \arrow[r, "h"] \arrow[d, "c"] & \Sigma X \arrow[d, "\Sigma a"] \\ A \arrow[r, "\gamma(j)"'] & B \arrow[r, "\gamma(q)"'] & B/A \arrow[r, "\delta(j)"'] & \Sigma X \end{tikzcd}$$

commutes.

Example 2.4. For every symmetric spectrum A the 'cone inclusion' $i_1 : A \longrightarrow \Delta[1] \wedge A = CA$ is a monomorphism with quotient $S^1 \wedge A = \Sigma A$, via the quotient morphism $p : \Delta[1] \wedge A \longrightarrow (\Delta[1]/\partial\Delta[1]) \wedge A = \Sigma A$. We claim that the associated connecting morphism $\delta(i_1) : \Sigma A \longrightarrow \Sigma A$ is the *negative* of the identity of ΣA in the stable homotopy category.

To see this, we use the 'collapse morphism' $\kappa : \Sigma A \longrightarrow \Sigma A \vee \Sigma A$ in $\mathcal{SHC}$ defined as the composite

$$\Sigma A \xrightarrow{\gamma(0 \cup p)^{-1}} CA \cup_A CA \xrightarrow{\gamma(p \cup p)} \Sigma A \vee \Sigma A \; .$$

The collapse morphism κ satisfies the relations

$$(2.5)\qquad (0+\mathrm{Id})\circ\kappa \;=\; \mathrm{Id}\;, \quad (\mathrm{Id}+0)\circ\kappa \;=\; \delta(i_1) \quad\text{and}\quad (\mathrm{Id}+\mathrm{Id})\circ\kappa \;=\; 0$$

as endomorphisms of ΣA in $\mathcal{SHC}$. Indeed, we have $(0+\mathrm{Id})\circ(p\cup p) = 0\cup p$ and $(\mathrm{Id}+0)\circ(p\cup p) = p\cup 0$ as morphisms of symmetric spectra $CA\cup_A CA \longrightarrow \Sigma A$, so

$$(0+\mathrm{Id})\circ\kappa \;=\; (0+\mathrm{Id})\circ\gamma(p\cup p)\circ\gamma(0\cup p)^{-1} \;=\; \gamma(0\cup p)\circ\gamma(0\cup p)^{-1} \;=\; \mathrm{Id}$$

and similarly

$$(\mathrm{Id}+0)\circ\kappa \;=\; =\; \gamma(p\cup 0)\circ\gamma(0\cup p)^{-1} \;=\; \delta(i_1)\;.$$

The square

$$\begin{tikzcd} CA\cup_A CA \arrow[r, "p\cup p"] \arrow[d, "\mathrm{Id}\cup\mathrm{Id}"'] & \Sigma A\vee\Sigma A \arrow[d, "\mathrm{Id}+\mathrm{Id}"] \\ CA \arrow[r, "p"'] & \Sigma A \end{tikzcd}$$

commutes in $\mathcal{S}p$, so the morphism $(\mathrm{Id}+\mathrm{Id})\circ\kappa = (\mathrm{Id}+\mathrm{Id})\circ\gamma(p\cup p)\circ\gamma(0\cup p)^{-1}$ factors through the cone CA, which is a zero object in the stable homotopy category. Thus $(\mathrm{Id}+\mathrm{Id})\circ\kappa = 0$.

The first two relations of (2.5) mean that the collapse morphism $\kappa : \Sigma A \longrightarrow \Sigma A\vee\Sigma A$ equals the morphism $\delta(i_0)\perp\mathrm{Id}$, where the notation is as in the definition of addition in the morphisms sets [...]. So we have

$$\delta(i_1)+\mathrm{Id} \;=\; \nabla\circ(\delta(i_0)\perp\mathrm{Id}) \;=\; (\mathrm{Id}+\mathrm{Id})\circ\kappa \;=\; 0\;.$$

In other words, $\delta(i_1) = -\mathrm{Id}_{\Sigma A}$, i.e., the connecting morphism of the cone inclusions is the negative of the identity.

Example 2.6. Now we let $\varphi : X \longrightarrow Y$ be an arbitrary morphism of symmetric spectra of simplicial sets. The associated mapping cone inclusion $i : Y \longrightarrow C(\varphi)$ is a monomorphism and the projection $p : C(\varphi) \longrightarrow S^1\wedge X$ can serve as the associated quotient morphism. We claim that the relation

$$(2.7)\qquad \delta(i) \;=\; -\Sigma\gamma(\varphi)$$

holds as morphisms $\Sigma X \longrightarrow \Sigma Y$ in $\mathcal{SHC}$. Indeed, in the commutative diagram of pushout squares

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "i_1"'] & Y \arrow[r] \arrow[d, "i"'] & * \arrow[d] \\ \Delta[1]\wedge X \arrow[r] & C(j) \arrow[r, "p"'] & S^1\wedge X \end{tikzcd}$$

the vertical morphisms i_1 and i are monomorphisms, and the morphism $p : C(\varphi) \longrightarrow S^1\wedge X$ can serve as the quotient map associated to the mapping cone inclusions $i : Y \longrightarrow C(\varphi)$. So by naturality of the connecting morphisms we get

$$\delta(i) \;=\; \delta(i)\circ\mathrm{Id}_{\Sigma X} \;=\; \Sigma\gamma(\varphi)\circ\delta(i_1) \;=\; -\Sigma\gamma(\varphi)$$

using that the connecting morphism $\delta(i_1)$ is the negative of the identity.

Proposition 2.8. *Every morphism $\alpha : X \longrightarrow Y$ in $\mathcal{SHC}$ can be written as $\alpha = \gamma(s)^{-1}\gamma(i)$ with $i : X \longrightarrow Z$ and $s : Y \longrightarrow Z$ morphisms of symmetric spectra such that i and s are monomorphisms and s is a stable equivalence. If $f,g : A \longrightarrow B$ are two morphisms of symmetric spectra such that $\gamma(f) = \gamma(g)$ in $\mathcal{SHC}$, then there is an acyclic cofibration $s : B \longrightarrow \bar{B}$ such that sf is homotopic to sg.*

Proof. Let $f : X \longrightarrow \omega Y$ represent α and let $Z = Z(f+p_Y) = \Delta[1]_+\wedge(X\vee Y)\cup_{f+p_Y}\omega Y$ be the mapping cylinder of the morphism $f+p_Y : X\vee Y \longrightarrow \omega Y$. The mapping cylinder comes with a 'front inclusion' $u : X\vee Y \longrightarrow Z$ and a 'projection' $v : Z \longrightarrow \omega Y$ such that $f+p_Y = vu : X\vee Y \longrightarrow \omega Y$. the projection is a homotopy equivalence, hence a stable equivalence. Then u is a monomorphism and $u = i+s$

for monomorphisms $i : X \longrightarrow Z$ and $s : Y \longrightarrow Z$. Since $vs = p_Y$ and v and p_Y are stable equivalences, s is also a stable equivalence. Finally, we have

$$\alpha \ = \ \gamma(p_Y)^{-1}\gamma(f) \ = \ \big(\gamma(p_Y)^{-1}\gamma(v)\big)\big(\gamma(v)^{-1}\gamma(f)\big) \ = \ \gamma(s)^{-1}\gamma(i) \ ,$$

where the last equation uses $f = vi$.

For the second part we start from the relation $[p_B f] = \gamma(f) = \gamma(g) = [p_B g]$ which provides a homotopy $H : \Delta[1]^+ \wedge A \longrightarrow \omega B$ from $Hi_0 = f$ to $Hi_1 = g$. We form the pushout

$$\begin{tikzcd}
A \vee A \arrow[r, "f+g"] \arrow[d, "i_0+i_1"'] & B \arrow[d, "b"] \\
\Delta[1]^+ \wedge A \arrow[r, "h_1"'] & P
\end{tikzcd}$$

and factor the morphism $H \cup p_B : P \longrightarrow \omega B$ as $H \cup p_B = qh_2$ for a monomorphism $h_2 : P \longrightarrow \bar{B}$ followed by a stable equivalence $q : \bar{B} \longrightarrow \omega B$. Then $s = h_2 b : B \longrightarrow \bar{B}$ is a cofibration, and a stable equivalence because q and $qs = qh_2 b = (H \cup p_B)b = p_B$ are stable equivalences. Moreover, $h_2 h_1 : \Delta[1]^+ \wedge A \longrightarrow \bar{B}$ is a homotopy from $h_2 h_1 i_0 = h_2 bf = sf$ to $h_2 h_1 i_1 = h_2 bg = sg$. $\square$

Now we can state and prove the main result of this section.

Theorem 2.9. *The suspension functor and the class of distinguished triangles make the stable homotopy category into a triangulated category.*

Proof. We have see in Corollary 1.13 that the stable homotopy category is additive and in Proposition 2.2 that the suspension functor is an autoequivalence. So it remains to prove the axioms (T0) – (T4). By definition, the class of distinguished triangles is closed under isomorphism, so (T0) holds.

(T1) The unique morphism $* \longrightarrow X$ from the trivial spectrum to any other symmetric spectrum X is a monomorphism with quotient morphism the identity of X. The triangle $(0, \mathrm{Id}_X, 0)$ is the associated elementary distinguished triangle.

(T2 – Rotation) We start with a distinguished triangle (f, g, h) and want to show that the triangle $(g, h, -\Sigma f)$ is also distinguished. It suffices to consider the elementary distinguished triangle $(\gamma(j), \gamma(q), \delta(j))$ associated to a monomorphism $j : A \longrightarrow B$. The diagram of triangles in $\mathcal{SHC}$

$$\begin{tikzcd}
B \arrow[r, "\gamma(i)"] \arrow[d, equal] & C(j) \arrow[r, "\gamma(p)"] \arrow[d, "\gamma(0\cup q)"', "\cong"] & \Sigma A \arrow[r, "\delta(i)"] \arrow[d, equal] & \Sigma B \arrow[d, equal] \\
B \arrow[r, "\gamma(q)"'] & B/A \arrow[r, "\delta(j)"'] & \Sigma A \arrow[r, "-\Sigma\gamma(j)"'] & \Sigma B
\end{tikzcd}$$

commutes by definition of the connecting morphism and the relation (2.7). The upper row is the elementary distinguished triangle of the cone inclusion $i : B \longrightarrow C(j)$, and all vertical maps are isomorphisms, so the lower triangle is distinguished, as claimed.

(T3 – Completion of triangles) We are given two distinguished triangles (f, g, h) and (f', g', h') and two morphisms a and b in $\mathcal{SHC}$ satisfying $bf = f'a$ as in the diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, dashed, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A'
\end{tikzcd}$$

We have to extend this data to a morphism of triangles, i.e., find a morphism c making the entire diagram commute. If we can solve the problem for isomorphic triangles, then we can also solve it for the original triangles. We can thus assume that the triangle (f, g, h) and (f', g', h') are the elementary distinguished triangle arising from two monomorphisms $j : A \longrightarrow B$ and $j' : A' \longrightarrow B'$.

We start with the special case where $a = \gamma(\alpha)$ and $b = \gamma(\beta)$ for morphisms $\alpha : A \longrightarrow A'$ and $\beta : B \longrightarrow B'$ of symmetric spectra. Then $\gamma(j'\alpha) = \gamma(\beta j)$, so Proposition 2.8 provides an acyclic cofibration $s : B' \longrightarrow \bar{B}$ and a homotopy $H : \Delta[1]^+ \wedge A \longrightarrow \bar{B}$ from $Hi_0 = sj'\alpha$ to $Hi_1 = s\beta j$. The following diagram on the left commutes in $\mathcal{S}p$, and all four horizontal morphisms are monomorphisms. So the diagram of elementary distinguished triangles on the right commutes in $\mathcal{SHC}$ by the naturality of the connecting morphisms:

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, equal] & B \\
A \arrow[r, "i_0"] \arrow[d, "\alpha"'] & \Delta[1]^+ \wedge A \cup_{i_1} B \arrow[u, "jp \cup B", "\sim"'] \arrow[d, "H \cup s\beta"] \\
A' \arrow[r, "sj'"] \arrow[d, equal] & \bar{B} \\
A' \arrow[r, "j'"'] & B' \arrow[u, "s"', "\sim"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, equal] & B \arrow[r, "\gamma(q)"] & B/A \arrow[r, "\delta(j)"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(i_0)"] \arrow[d, "\gamma(\alpha)"'] & \Delta[1]^+ \wedge A \cup_{i_1} B \arrow[u, "\gamma(jp \cup B)", "\cong"'] \arrow[r, "\gamma(q)"] \arrow[d, "\gamma(H \cup s\beta)"'] & (\Delta[1]^+ \wedge A \cup_{i_1} B)/A \arrow[u, "\cong", "\gamma((jp \cup B)/A)"'] \arrow[r, "\delta(i_0)"] \arrow[d, "\gamma((H \cup s\beta)/\alpha)"] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\
A' \arrow[r, "\gamma(sj')"] \arrow[d, equal] & \bar{B} \arrow[r, "\gamma(\bar{q})"] & \bar{B}/A' \arrow[r, "\delta(sj')"] & \Sigma A' \arrow[d, equal] \\
A' \arrow[r, "\gamma(j')"'] & B' \arrow[u, "\gamma(s)", "\cong"'] \arrow[r, "\gamma(q')"'] & B'/A' \arrow[u, "\cong", "\gamma(s/A')"'] \arrow[r, "\delta(j')"'] & \Sigma A'
\end{tikzcd}$$

The morphism

$$c \ = \ \gamma(s/A')^{-1} \circ \gamma((H \cup s\beta)/\alpha) \circ \gamma((jp \cup B)/A)^{-1} \ : \ B/A \longrightarrow B'/A'$$

is the desired filler.

In the general case we use Proposition 2.8 to write $a = \gamma(s)^{-1}\gamma(\alpha)$ where $\alpha : A \longrightarrow \bar{A}$ and $s : A' \longrightarrow \bar{A}$ are morphisms of symmetric spectra and s is an acyclic cofibration. We choose a pushout:

$$\begin{tikzcd}
\bar{A} \arrow[r, "\bar{j}"] & \bar{A} \cup_{A'} B' \\
A' \arrow[u, "s", "\simeq"'] \arrow[r, "j'"'] & B' \arrow[u, "\simeq", "s'"']
\end{tikzcd}$$

We write $\gamma(s')b = \gamma(t)^{-1}\gamma(\beta) : A \longrightarrow \bar{A} \cup_{A'} B'$ for suitable morphisms of symmetric spectra $\beta : B \longrightarrow \bar{B}$ and $t : \bar{A} \cup_{A'} B' \longrightarrow \bar{B}$ such that t is an acyclic cofibration. We then have

$$\gamma(t\bar{j})\gamma(\alpha) \ = \ \gamma(t\bar{j})\gamma(s)a \ = \ \gamma(ts')\gamma(j')a \ = \ \gamma(ts')b\gamma(j) \ = \ \gamma(\beta)\gamma(j) \ ,$$

so by the special case, applied to the monomorphisms $j : A \longrightarrow B$ and $t\bar{j} : \bar{A} \longrightarrow \bar{B}$ and the morphisms $\alpha : A \longrightarrow \bar{A}$ and $\beta : B \longrightarrow \bar{B}$, there exists a morphism $c : B/A \longrightarrow \bar{B}/\bar{A}$ in the stable homotopy category making the diagram

$$\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, "\gamma(\alpha)"'] & B \arrow[r, "\gamma(q)"] \arrow[d, "\gamma(\beta)"'] & B/A \arrow[r, "\delta(j)"] \arrow[d, dotted, "c"'] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\
\bar{A} \arrow[r, "\gamma(t\bar{j})"] & \bar{B} \arrow[r, "\gamma(\bar{q})"] & \bar{B}/\bar{A} \arrow[r, "\delta(t\bar{j})"] & \Sigma\bar{A} \\
A' \arrow[u, "\gamma(s)"] \arrow[r, "\gamma(j')"'] & B' \arrow[u, "\gamma(ts')"] \arrow[r, "\gamma(q')"'] & B'/A' \arrow[u, "\gamma(ts'/s)"'] \arrow[r, "\delta(j')"'] & \Sigma A' \arrow[u, "\Sigma\gamma(s)"']
\end{tikzcd}$$

commute (the lower part commutes by naturality of connecting morphisms). Since s is a monomorphism and stable equivalence, so is its cobase change s' (see Proposition I.4.31 (v)). Similarly, the stable equivalences

$s : A' \longrightarrow \bar{A}$ and $ts' : B' \longrightarrow \bar{B}$ induce a stable equivalence $ts'/s : B'/A' \longrightarrow \bar{B}/\bar{A}$ on quotients and the composite

$$B/A \xrightarrow{\ c\ } \bar{B}/\bar{A} \xrightarrow{\gamma(ts'/s)^{-1}} B'/A'$$

in $\mathcal{SHC}$ thus solves the original problem.

(T4 - Octahedral axiom) We start with the special case where $f = \gamma(i)$ and $f' = \gamma(j)$ for monomorphisms $i : A \longrightarrow B$ and $j : B \longrightarrow D$. Then the composite $ji : A \longrightarrow D$ is a monomorphism with $\gamma(ji) = f'f$. The diagram

$$\begin{tikzcd}
A \arrow[r, "\gamma(i)"] \arrow[d, equal] & B \arrow[r, "\gamma(q_i)"] \arrow[d, "\gamma(j)"'] & B/A \arrow[r, "\delta(i)"] \arrow[d, "\gamma(j/A)"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(ji)"'] & D \arrow[r, "\gamma(q_{ji})"'] \arrow[d, "\gamma(q_j)"'] & D/A \arrow[r, "\delta(ji)"'] \arrow[d, "\gamma(D/i)"] & \Sigma A \arrow[d, "\Sigma\gamma(i)"] \\
 & D/B \arrow[r, equal] \arrow[d, "\delta(j)"'] & D/B \arrow[r, "\delta(j)"'] \arrow[d, "\delta(j/A)=(\Sigma\gamma(q_i))\delta(j)"] & \Sigma B \\
 & \Sigma B \arrow[r, "\Sigma\gamma(q_i)"'] & \Sigma B/A &
\end{tikzcd}$$

then commutes by naturality of connecting morphisms. Moreover, the four triangles in question are the elementary distinguished triangles of the cofibrations i, j, ji and $j/A : B/A \longrightarrow D/A$.

In the general case we write $f = \gamma(s)^{-1}\gamma(a)$ for a morphism of symmetric spectra $a : A \longrightarrow B'$ and a stable equivalence $s : B \longrightarrow B'$. Then a can be factored as $a = pi$ for a monomorphism $i : A \longrightarrow \bar{B}$ and a stable equivalence $p : \bar{B} \longrightarrow B'$. Altogether we then have $f = \varphi \circ \gamma(i)$ where $\varphi = \gamma(s)^{-1} \circ \gamma(p) : \bar{B} \longrightarrow B$ is an isomorphism in $\mathcal{SHC}$. We can apply the same reasoning to the morphism $f'\varphi : \bar{B} \longrightarrow D$ and write it as $f' \circ \varphi = \psi \circ \gamma(j)$ for a monomorphism $j : \bar{B} \longrightarrow \bar{D}$ of symmetric spectra and an isomorphism $\psi : \bar{D} \longrightarrow D$ in $\mathcal{SHC}$. The special case can then be applied to the monomorphisms $i : A \longrightarrow \bar{B}$ and $j : \bar{B} \longrightarrow \bar{D}$. The resulting commutative diagram that solves (T4) for $(\gamma(i), \gamma(j))$ can then be translated back into a commutative diagram that solves (T4) for (f, f') by conjugating with the isomorphisms $\varphi : \bar{B} \longrightarrow B$ and $\psi : \bar{D} \longrightarrow D$. This completes the proof of the octahedral axiom (T4), and hence the proof of Theorem 2.9. □

The arguments involved in the verification of the axioms (T0) – (T4) are quite general and can be axiomatized to produce triangulated categories more generally. We do this in the axiomatic framework of *cofibration categories* and show that the homotopy category of any stable cofibration category is triangulated in a natural way. Besides symmetric spectra, we wil later also apply this to modules spectra over a symmetric ring spectrum. We also give various exercises that show how more triangulations can be constructed using the cofibration category framework. [ref to later]

Proposition 2.10. *Let $\mathcal{T}$ be a triangulated category. Then the following properties hold.*

(i) *For every distinguished triangle (f, g, h) and every object X of $\mathcal{T}$, the two sequences of abelian groups*

$$\mathcal{T}(\Sigma A, X) \xrightarrow{\mathcal{T}(h,X)} \mathcal{T}(C, X) \xrightarrow{\mathcal{T}(g,X)} \mathcal{T}(B, X) \xrightarrow{\mathcal{T}(f,X)} \mathcal{T}(A, X)$$

and

$$\mathcal{T}(X, A) \xrightarrow{\mathcal{T}(X,f)} \mathcal{T}(X, B) \xrightarrow{\mathcal{T}(X,g)} \mathcal{T}(X, C) \xrightarrow{\mathcal{T}(X,h)} \mathcal{T}(X, \Sigma A)$$

are exact.

(ii) *Let (a, b, c) be a morphism of distinguished triangles. If two out of the three morphisms are isomorphisms, then so is the third.*

(iii) *Let (f, g, h) be a triangle such that the triangle $(g, h, -\Sigma f)$ is distinguished. Then the triangle (f, g, h) is distinguished.*

(iv) *Let (f_1, g_1, h_1), (f_2, g_2, h_2) and (f_3, g_3, h_3) be three distinguished triangles such that f_1 and f_2 are composable and $f_3 = f_2 f_1$. Then there exist morphisms $\bar{x}$ and $\bar{y}$ such that $(\bar{x}, \bar{y}, (\Sigma g_1) \circ h_2)$ is a distinguished triangle and the following diagram commutes:*

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r, "g_1"] \arrow[d, "f_2"'] & \bar{C} \arrow[r, "h_1"] \arrow[d, dashed, "\bar{x}"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f_3"'] & D \arrow[r, "g_3"'] \arrow[d, "g_2"'] & \bar{E} \arrow[r, "h_3"'] \arrow[d, dashed, "\bar{y}"] & \Sigma A \arrow[d, "\Sigma f_1"] \\
 & \bar{F} \arrow[r, equal] \arrow[d, "h_2"'] & \bar{F} \arrow[r, "h_2"'] \arrow[d, "(\Sigma g_1)\circ h_2"] & \Sigma B \\
 & \Sigma B \arrow[r, "\Sigma g_1"'] & \Sigma C &
\end{tikzcd}$$

(v) *For every distinguished triangle (f, g, h) the following three conditions are equivalent:*
- *The morphism $f : A \longrightarrow B$ has a retraction, i.e., there is a morphism r such that $rf = \mathrm{Id}_A$.*
- *The morphism $g : B \longrightarrow C$ has a section, i.e., there is a morphism s such that $gs = \mathrm{Id}_C$.*
- *The morphism $h : C \longrightarrow \Sigma A$ is zero.*

(vi) *Let (f, g, h) be a distinguished triangle and $s : C \longrightarrow B$ a morphism such that $gs = \mathrm{Id}_C$. Then the morphisms $f : A \longrightarrow B$ and $s : C \longrightarrow B$ make B into a coproduct of A and C.*

(vii) *Let I be a set and let (f_i, g_i, h_i) be a distinguished triangle for every $i \in I$. Then the triangles*

$$\bigoplus\nolimits_I A_i \xrightarrow{\oplus f_i} \bigoplus\nolimits_I B_i \xrightarrow{\oplus g_i} \bigoplus\nolimits_I C_i \xrightarrow{\kappa \circ (\oplus h_i)} \Sigma(\bigoplus\nolimits_I A_i)$$

and

$$\prod\nolimits_I A_i \xrightarrow{\prod f_i} \prod\nolimits_I B_i \xrightarrow{\prod g_i} \prod\nolimits_I C_i \xrightarrow{\kappa^{-1} \circ (\prod h_i)} \Sigma(\prod\nolimits_I A_i)$$

are distinguished, whenever the respective coproducts and products exist. Here $\kappa : \bigoplus_I \Sigma A_i \longrightarrow \Sigma(\bigoplus_I A_i)$ and $\kappa : \Sigma(\prod_I A_i) \longrightarrow \prod_I \Sigma A_i$ are the canonical isomorphism.

(viii) *Let $A \oplus B$ be a coproduct of two objects A and B of $\mathcal{T}$ with respect to the morphisms $i_A : A \longrightarrow A \oplus B$ and $i_B : B \longrightarrow A \oplus B$. Then the triangle*

$$A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \xrightarrow{0} \Sigma A$$

is distinguished, where p_B is the morphism determined by $p_B i_A = 0$ and $p_B i_B = \mathrm{Id}_B$.

Proof. We start by showing that for every distinguished triangle (f, g, h) the composite gf is zero. Indeed, by (T3) applied to the pair (Id, f) there is a (necessarily unique) morphism from any zero object to C such that the diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{Id}"] \arrow[d, equal] & A \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d, dotted] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma A
\end{tikzcd}$$

commutes, so $gf = 0$ (the upper row is distinguished by (T1) and (T2)).

(i) Since $gf = 0$ the image of $\mathcal{T}(g, X)$ is contained in the kernel of $\mathcal{T}(f, X)$. Conversely, let $\psi : B \longrightarrow X$ be a morphism in the kernel of $\mathcal{T}(f, X)$, i.e., such that $\psi f = 0$. Applying (T3) to the pair $(0, \psi)$ gives a morphism $\varphi : C \longrightarrow X$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d, "\psi"'] & C \arrow[r, "h"] \arrow[d, dotted, "\varphi"] & \Sigma A \arrow[d] \\ 0 \arrow[r] & X \arrow[r, "\mathrm{Id}"'] & X \arrow[r] & 0 \end{tikzcd}$$

commutes (the lower row is distinguished by (T1)). So the first sequence is exact at $\mathcal{T}(B, X)$. Applying this to the triangle $(g, h, -\Sigma f)$ (which is distinguished by (T2)), we deduce that the first sequence is also exact at $\mathcal{T}(X, C)$.

The argument for the other sequence is similar, but slightly more involved and depends on the assumption that the functor Σ is fully faithful. Since $gf = 0$, the image of $\mathcal{T}(X, f)$ is contained in the kernel of $\mathcal{T}(X, g)$. Conversely, let $\psi : X \longrightarrow B$ be a morphism in the kernel of $\mathcal{T}(X, g)$, i.e., such that $g\psi = 0$. Applying (T3) to the pair $(\psi, 0)$ gives a morphism $\bar{\varphi} : \Sigma X \longrightarrow \Sigma A$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \Sigma X \arrow[r, "-\mathrm{Id}"] \arrow[d, dotted, "\bar{\varphi}"] & \Sigma X \arrow[d, "\Sigma\psi"] \\ B \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma A \arrow[r, "-\Sigma f"'] & \Sigma B \end{tikzcd}$$

commutes (both rows are distinguished by (T1) and (T2)). Since shifting is full, there exists a morphism $\varphi : X \longrightarrow A$ such that $\bar{\varphi} = \Sigma\varphi$, and hence $\Sigma(f\varphi) = (\Sigma f)(\Sigma\varphi) = \Sigma\psi$. Since shifting is faithful we have $f\varphi = \psi$, so ψ is in the image of $\mathcal{T}(X, f)$. Altogether, the first sequence is exact at $\mathcal{T}(X, B)$. If we apply this to the triangle $(g, h, -\Sigma f)$ (which is distinguished by (T2) which we assume), we deduce that the first sequence is also exact at $\mathcal{T}(X, C)$.

(ii) We first treat the case where a and b are isomorphisms. If X is any object of $\mathcal{T}$ we have a commutative diagram

$$\begin{tikzcd} \mathcal{T}(X,A) \arrow[r, "f_*"] \arrow[d, "a_*"'] & \mathcal{T}(X,B) \arrow[r, "g_*"] \arrow[d, "b_*"'] & \mathcal{T}(X,C) \arrow[r, "h_*"] \arrow[d, "c_*"'] & \mathcal{T}(X,\Sigma A) \arrow[r, "(-\Sigma f)_*"] \arrow[d, "(\Sigma a)_*"] & \mathcal{T}(X,\Sigma B) \arrow[d, "(\Sigma b)_*"] \\ \mathcal{T}(X,A') \arrow[r, "f'_*"'] & \mathcal{T}(X,B') \arrow[r, "g'_*"'] & \mathcal{T}(X,C') \arrow[r, "h'_*"'] & \mathcal{T}(X,\Sigma A') \arrow[r, "(-\Sigma(f'))_*"'] & \mathcal{T}(X,\Sigma B') \end{tikzcd}$$

where we write f_* for $\mathcal{T}(X, f)$, etc. The top row is exact by part (i) applied to the distinguished triangles (f, g, h) and $(g, h, -\Sigma f)$. Similarly, the bottom row is exact. Since a and b (and hence Σa and Σb) are isomorphisms, all vertical maps except possibly the middle one are isomorphisms of abelian groups. So the five lemma says that c_* is an isomorphism. Since this holds for all objects X, the morphism $c : C \longrightarrow C'$ is an isomorphism.

If b and c are isomorphisms, we apply the previous argument to the triple $(b, c, \Sigma a)$. This is a morphism from the distinguished (by (T2)) triangle $(g, h, -\Sigma f)$ to the distinguished triangle $(g', h', -\Sigma f')$. By the above, Σa is an isomorphism, hence so is a since shifting is an equivalence of categories. The third case is similar.

(iii) If the triangle $(g, h, -\Sigma f)$ is distinguished, then so is $(-\Sigma f, -\Sigma g, -\Sigma h)$ by two applications of (T2). Axiom (T4) lets us choose a distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{\bar{g}} \bar{C} \xrightarrow{\bar{h}} \Sigma A$$

and by three applications of (T2), the triangle $(-\Sigma f, -\Sigma\bar{g}, -\Sigma\bar{h})$ is distinguished. By (T3) there is a morphism $\bar{c} : \Sigma C \longrightarrow \Sigma\bar{C}$ such that the diagram

$$\begin{tikzcd}
\Sigma A \arrow[r, "-\Sigma f"] \arrow[d, equal] & \Sigma B \arrow[r, "-\Sigma g"] \arrow[d, equal] & \Sigma C \arrow[r, "-\Sigma h"] \arrow[d, "\bar{c}"] & \Sigma^2 A \arrow[d, equal] \\
\Sigma A \arrow[r, "-\Sigma f"'] & \Sigma B \arrow[r, "-\Sigma\bar{g}"'] & \Sigma\bar{C} \arrow[r, "-\Sigma\bar{h}"'] & \Sigma^2 A
\end{tikzcd}$$

commutes. By part (ii), c is an isomorphism. Since suspension is an equivalence of categories, we have $\bar{c} = \Sigma c$ for a unique isomorphism $c : C \longrightarrow \bar{C}$. Then $(\mathrm{Id}_A, \mathrm{Id}_B, c)$ is an isomorphism from the triangle (f, g, h) to the distinguished triangle $(f, \bar{g}, \bar{h})$. So the triangle (f, g, h) is itself distinguished.

(iv) Axiom (T4) provides a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "f_2"'] & C \arrow[r, "h"] \arrow[d, "x"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f_3"'] & D \arrow[r, "g''"'] \arrow[d, "g'"'] & E \arrow[r, "h''"'] \arrow[d, "y"] & \Sigma A \arrow[d, "\Sigma f_1"] \\
& F \arrow[r, equal] \arrow[d, "h'"'] & F \arrow[r, "h'"'] \arrow[d, "(\Sigma g)\circ h'"] & \Sigma B \\
& \Sigma B \arrow[r, "\Sigma g"'] & \Sigma C &
\end{tikzcd}$$

such that the triangles (f_1, g, h), (f_2, g', h'), (f_3, g'', h'') and $(x, y, (\Sigma g) \circ h')$ are distinguished. By (T3) there is a morphism $\varphi : \bar{C} \longrightarrow C$ that makes $(\mathrm{Id}_A, \mathrm{Id}_B, \varphi)$ a morphism of triangles from (f_1, g_1, h_1) to (f_1, g, h); this morphism is an isomorphism by part (ii). Similarly, there is an morphism $\psi : \bar{F} \longrightarrow F$ such that $(\mathrm{Id}_B, \mathrm{Id}_D, \psi)$ an isomorphism of triangles from (f_2, g_2, h_2) to (f_2, g', h'). Finally, there is an morphism $\nu : \bar{E} \longrightarrow E$ such that $(\mathrm{Id}_A, \mathrm{Id}_D, \nu)$ an isomorphism of triangles from (f_3, g_3, h_3) to (f_3, g'', h''). If we set

$$\bar{x} \;=\; \nu^{-1} x \varphi \;:\; \bar{C} \;\longrightarrow\; \bar{E} \qquad \text{and} \qquad \bar{y} \;=\; \psi^{-1} y \nu \;:\; \bar{E} \;\longrightarrow\; \bar{F}\ ,$$

then the desired diagram commutes. Moreover, the triple (φ, ν, ψ) is an isomorphism from the triangle $(\bar{x}, \bar{y}, (\Sigma g_1)h_2)$ to the triangle $(x, y, (\Sigma g)h')$. Since the latter triangle is distinguished, so is the former.

(v) By part (i), the composite of two adjacent morphism in any distinguished triangle is zero. So if s is a section to g, then $h = hgs = 0$. Similarly, if r is a retraction to f, then $h = (-\Sigma r)(-\Sigma f)h = 0$ because the triangle $(g, h, -\Sigma f)$ is distinguished. Conversely, if $h = 0$, then the sequence

$$\mathcal{T}(C, B) \xrightarrow{\mathcal{T}(C,g)} \mathcal{T}(C, C) \longrightarrow 0$$

is exact by part (i), and any preimage of the identity of C is a section to g. Similarly, the sequence

$$\mathcal{T}(\Sigma B, \Sigma A) \xrightarrow{\mathcal{T}(-\Sigma f, \Sigma A)} \mathcal{T}(\Sigma A, \Sigma A) \longrightarrow 0$$

is exact because the triangle $(g, h, -\Sigma f)$ is distinguished. So there is a morphism $\bar{r} : \Sigma B \longrightarrow \Sigma A$ such that $-\bar{r} \circ \Sigma f = \mathrm{Id}_{\Sigma A}$. Since Σ is full there is a morphism $r : B \longrightarrow A$ such that $\Sigma r = -\bar{r}$, hence $\Sigma(rf) = (\Sigma r)(\Sigma f) = \mathrm{Id}_{\Sigma A}$. Since Σ is faithful, r is a retraction to f.

(vi) Since s is a section to g, the morphism $\mathcal{T}(g, X)$ is injective. By part (v) the morphism f has a retraction, so $\mathcal{T}(f, X)$ is surjective. The first exact sequence of part (i) thus becomes a short exact sequence of abelian groups

$$0 \longrightarrow \mathcal{T}(C, X) \xrightarrow{\mathcal{T}(g,X)} \mathcal{T}(B, X) \xrightarrow{\mathcal{T}(f,X)} \mathcal{T}(A, X) \longrightarrow 0\ .$$

Because $\mathcal{T}(s, X)$ is a section to the first map, the map $(\mathcal{T}(f, X), \mathcal{T}(s, X)) : \mathcal{T}(B, X) \longrightarrow \mathcal{T}(A, X) \times \mathcal{T}(C, X)$ is bijective, i.e., the morphisms f and s make B a coproduct of A and B.

(vii) We choose a distinguished triangle:

$$\bigoplus_I A_i \xrightarrow{\oplus f_i} \bigoplus_I B_i \xrightarrow{g} C \xrightarrow{h} \Sigma(\bigoplus_I A_i) .$$

We apply axiom (T3) to the canonical morphisms $\kappa_j : A_j \longrightarrow \bigoplus_I A_i$ and $\kappa'_j : B_j \longrightarrow \bigoplus_I B_i$ and obtain a morphism $\varphi_j : C_j \longrightarrow C$ such that the diagram

$$\begin{tikzcd}
A_j \arrow[r, "f_j"] \arrow[d, "\kappa_j"'] & B_j \arrow[r, "g_j"] \arrow[d, "\kappa'_j"'] & C_j \arrow[r, "h_j"] \arrow[d, dotted, "\varphi_j"'] & \Sigma A_j \arrow[d, "\Sigma\kappa_j"] \\
\bigoplus_I A_i \arrow[r, "\oplus f_i"'] & \bigoplus_I B_i \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma(\bigoplus_I A_i)
\end{tikzcd}$$

commutes. We claim that then the morphisms $\varphi_i : C_i \longrightarrow C$ make C into a coproduct of the objects C_i. For this we observe that the diagram

$$\begin{tikzcd}
\mathcal{T}(\Sigma(\bigoplus_I B_i), X) \arrow[r, "-(\Sigma\oplus f_i)^*"] \arrow[d, "((\Sigma\kappa_i)^*)"'] & \mathcal{T}(\Sigma(\bigoplus_I A_i), X) \arrow[r, "h^*"] \arrow[d, "((\Sigma\kappa'_i)^*)"'] & \mathcal{T}(C, X) \arrow[r, "g^*"] \arrow[d, "(\varphi_i^*)"'] & \mathcal{T}(\bigoplus_I B_i, X) \arrow[r, "(\oplus f_i)^*"] \arrow[d, "((\kappa'_i)^*)"] & \mathcal{T}(\bigoplus_I A_i, X) \arrow[d, "(\kappa_i^*)"] \\
\prod_I \mathcal{T}(\Sigma B_i, X) \arrow[r, "-\prod \Sigma f_i^*"'] & \prod_I \mathcal{T}(\Sigma A_i, X) \arrow[r, "\prod h_i^*"'] & \prod_I \mathcal{T}(C_i, X) \arrow[r, "\prod g_i^*"'] & \prod_I \mathcal{T}(B_i, X) \arrow[r, "\prod f_i^*"'] & \prod_I \mathcal{T}(A_i, X)
\end{tikzcd}$$

commutes by construction of the morphisms φ_i. The top row is exact by part (i), the bottom row is exact as a product of exact sequences. The four outer vertical maps are isomorphisms by the universal property of coproducts, so the middle vertical map is an isomorphism by the 5-lemma. This shows that C is a coproduct of the C_i's in way that makes $g = \oplus g_i : \bigoplus_I B_i \longrightarrow C$ and $h = \kappa \circ (\oplus h_i) : C \longrightarrow \Sigma(\bigoplus_I A_i)$.

The statement about products of triangles can be proved in an analogous fashion. Alternatively, one can reduce to the first case by exploiting that products in $\mathcal{T}$ are coproducts in the opposite category $\mathcal{T}^{\mathrm{op}}$, which is triangulated with respect to the opposite triangulation (compare Exercise E.II.5).

(viii) This the special case of part (vii) for the two exact triangles

$$A \xrightarrow{\mathrm{Id}_A} A \longrightarrow 0 \longrightarrow \Sigma A \qquad \text{and} \qquad 0 \longrightarrow B \xrightarrow{\mathrm{Id}_B} B \longrightarrow 0$$

whose sum is the triangle in question, which is thus distinguished. □

Proposition 2.11. *Let*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r, "j"'] & D
\end{tikzcd}$$

be a pullback square of sequential spectra of spaces or simplicial sets. If j or g is levelwise a Serre fibration (respectively Kan fibration), then the induced morphism of mapping cones $C(i,j) : C(f) \longrightarrow C(g)$ is a $\hat{\pi}_*$-isomorphism.

Now suppose that $\varphi : X \longrightarrow Y$ is a morphism of symmetric spectra of simplicial sets that is levelwise a Kan fibrations. We denote by $\iota : F \longrightarrow X$ the inclusion of the fiber of φ. By Proposition 2.11 the pullback square

$$\begin{tikzcd}
F \arrow[r, "\iota"] \arrow[d] & X \arrow[d, "\varphi"] \\
* \arrow[r] & Y
\end{tikzcd}$$

induces a $\hat{\pi}_*$-isomorphism of mapping cones $l : S^1 \wedge F = C(F \to *) \longrightarrow C(\varphi)$. This $\hat{\pi}_*$-isomorphism becomes an isomorphism $\gamma(l) : \Sigma F \longrightarrow C(\varphi)$ in the stable homotopy category.

The next proposition shows that homotopy fiber sequences also give rise to distinguished triangles in the stable homotopy category. The homotopy fiber $F(\varphi)$ of a morphism $\varphi : X \longrightarrow Y$ between symmetric spectra of spaces was defined in (2.14) of Chapter I; it comes with natural morphisms

$$\Omega Y \xrightarrow{\ i\ } F(\varphi) \xrightarrow{\ p\ } X \xrightarrow{\ \varphi\ } Y$$

such that the composite pi is the trivial map and the composite φp is null homotopic. [clash notation wrt i and p]

Proposition 2.12. *Let $\varphi : X \longrightarrow Y$ be a morphism of symmetric spectra of simplicial sets.*

(i) *The sequence*

$$X \xrightarrow{\gamma(\varphi)} Y \xrightarrow{\gamma(i)} C(\varphi) \xrightarrow{\gamma(p)} \Sigma X$$

is a distingushed triangle in the stable homotopy category.

(ii) *If φ is levelwise a Kan fibration, then the triangle*

$$F \xrightarrow{\quad\gamma(\iota)\quad} X \xrightarrow{\quad\gamma(\varphi)\quad} Y \xrightarrow{-\gamma(l)^{-1}\gamma(i)} \Sigma F$$

is distinguished, where F is the fiber of φ and $\iota : F \longrightarrow X$ is the inclusion.

(iii) *If X and Y are levelwise Kan, then the triangle*

$$F(\varphi) \xrightarrow{\quad\gamma(\bar{p})\quad} X \xrightarrow{\quad\gamma(\varphi)\quad} Y \xrightarrow{(\Sigma\gamma(\bar{i}))\gamma(\epsilon)^{-1}} \Sigma F(\varphi)$$

is distinguished, where $\epsilon : S^1 \wedge \Omega Y \longrightarrow Y$ is the adjunction counit.

Proof. (i) The inclusion $i : Y \longrightarrow C(\varphi)$ into the mapping cone is a monomorphism, and the triangle

$$Y \xrightarrow{\ \gamma(i)\ } C(\varphi) \xrightarrow{\ \gamma(p)\ } \Sigma X \xrightarrow{\ \delta(i)\ } \Sigma Y$$

is the associated elementary distinguished triangle. By (2.7) we have $\delta(i) = -\Sigma\gamma(\varphi)$, so by Proposition 2.10 (iii) the triangle $(\gamma(\varphi), \gamma(i), \gamma(p))$ is distinguished.

(ii) We have $pl = S^1 \wedge \iota : S^1 \wedge F \longrightarrow S^1 \wedge X$, so the diagram of triangles

$$\begin{tikzcd}
X \arrow[r, "\gamma(\varphi)"] \arrow[d, equal] & Y \arrow[r, "-\gamma(l)^{-1}\gamma(i)"] \arrow[d, equal] & \Sigma F \arrow[r, "-\Sigma\gamma(\iota)"] \arrow[d, "-\gamma(l)"', "\cong"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "\gamma(\varphi)"'] & Y \arrow[r, "\gamma(i)"'] & C(\varphi) \arrow[r, "\gamma(p)"'] & \Sigma X
\end{tikzcd}$$

commutes. The lower triangle is distinguished by part (i), and all vertical morphisms are isomorphisms; so the upper triangle is distinguished. Proposition 2.10 (iii) let us rotate to the left, so that the triangle $(\gamma(\iota), \gamma(\varphi), -\gamma(l)^{-1}\gamma(i))$ is distinguished.

(iii) If X and Y are levelwise Kan, then the morphism $\bar{p} : F(\varphi) \longrightarrow X$ is levelwise a Kan fibration. The fiber of $\bar{p}$ is isomorphic to ΩY, via the morphism $\bar{i} : \Omega Y \longrightarrow F(\varphi)$. So by part (ii) the triangle

$$\Omega Y \xrightarrow{\quad\gamma(\bar{i})\quad} F(\varphi) \xrightarrow{\quad\gamma(\bar{p})\quad} X \xrightarrow{-\gamma(l\langle\bar{p}\rangle)^{-1}\gamma(i\langle\bar{p}\rangle)} \Sigma\Omega Y$$

is distinguished. Rotating to the right gives the upper disinguished triangle in the diagram

$$\begin{tikzcd}
F(\varphi) \arrow[r, "\gamma(\bar{p})"] \arrow[d, equal] & X \arrow[r, "-\gamma(l\langle\bar{p}\rangle)^{-1}\gamma(i\langle\bar{p}\rangle)"] \arrow[d, equal] & \Sigma\Omega Y \arrow[r, "-\Sigma\gamma(\bar{i})"] \arrow[d, "\cong"', "-\gamma(\epsilon)"] & \Sigma F(\varphi) \arrow[d, equal] \\
F(\varphi) \arrow[r, "\gamma(\bar{p})"'] & X \arrow[r, "\gamma(\varphi)"'] & Y \arrow[r, "\gamma(p)"'] & \Sigma F(\varphi)
\end{tikzcd}$$

[show the middle square commutes... sign?] □

3. Derived smash product

The main result of this section is that the pointset level smash product of symmetric spectra (see Section I.5) descends to a closed symmetric monoidal product on the stable homotopy category. Recall that $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ denotes the universal functor from symmetric spectra to the stable homotopy category which inverts stable equivalences (see Theorem 1.6).

Theorem 3.1. *The stable homotopy category admits a closed symmetric monoidal product*

$$\wedge^L \ : \ \mathcal{SHC} \times \mathcal{SHC} \ \longrightarrow \ \mathcal{SHC} \ ,$$

called the derived smash product *with the sphere spectrum $\mathbb{S}$ as strict unit object. Moreover, there is a natural transformation $\psi_{A,B} : \gamma(A) \wedge^L \gamma(B) \longrightarrow \gamma(A \wedge B)$ which makes the localization functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ into a lax symmetric monoidal functor and which is an isomorphism whenever A or B is flat.*

Before we go into the actual construction of the derived smash product we want to make some comments on what is important here. The construction of the derived smash product is really a largely formal consequence of the following facts which we have already established:

- there is a symmetric monoidal smash product for symmetric spectra,
- the smash product is homotopically well-behaved whenever at least one factor is flat,
- every symmetric spectrum is stably equivalent (even level equivalent) to a flat symmetric spectrum.

To make the construction of the derived smash product more transparent we first define it on the full subcategory $\mathcal{SHC}^\flat$ of the stable homotopy category whose objects are the flat symmetric spectra. On this subcategory we can define $\wedge^L$ on objects by the pointset level smash product, and there is a canonical way to extend $\wedge^L$ to morphisms in $\mathcal{SHC}^\flat$. Since there are no choices involved, the coherence properties are then fairly formal.

Every symmetric spectrum is level equivalent, thus isomorphic in $\mathcal{SHC}$, to a flat symmetric spectrum (compare Construction I.5.53), and so the inclusion $\mathcal{SHC}^\flat \longrightarrow \mathcal{SHC}$ is an equivalence of categories. A choice of inverse equivalence (which amounts to choices of 'flat resolutions') gives us an extension of the derived smash product from the category $\mathcal{SHC}^\flat$ to all of $\mathcal{SHC}$, with all necessary coherence for free.

We denote by $\mathcal{S}p^\flat$ the full subcategory of $\mathcal{S}p_{\mathbf{sS}}$ consisting of flat symmetric spectra; since this category is closed under smash product and contains the sphere spectrum, it is symmetric monoidal by restriction of all the structure from the larger category of symmetric spectra. We denote by $\gamma^\flat : \mathcal{S}p^\flat \longrightarrow \mathcal{SHC}^\flat$ the restriction of the localization functor to flat symmetric spectra.

We want to point out a potentially confusing issue: instead of the full subcategory $\mathcal{SHC}^\flat$ of the stable homotopy category $\mathcal{SHC}$ we could also consider the localization of the full subcategory $\mathcal{S}p^\flat$ of $\mathcal{S}p$ at the class of stable equivalences between flat symmetric spectra, which we denote by $(\mathrm{st.\,eq.})^{-1}\mathcal{S}p^\flat$ for the moment. The inclusion $\mathcal{S}p^\flat \longrightarrow \mathcal{S}p$ passes to a functor

$$(\mathrm{st.\,eq.})^{-1}\mathcal{S}p^\flat \ \longrightarrow \ (\mathrm{st.\,eq.})^{-1}\mathcal{S}p = \mathcal{SHC}$$

whose image lands in $\mathcal{SHC}^\flat$. The resulting functor $(\mathrm{st.\,eq.})^{-1}\mathcal{S}p^\flat \longrightarrow \mathcal{SHC}^\flat$ is the identity on objects, but it is not a priori clear that this is an isomorphism of categories. Conceivably, two flat symmetric spectra could be related by a zigzag of stable equivalences, but not by a zigzag of stable equivalences *through flat spectra.* However, the next proposition takes care of this and ensures that the functor $(\mathrm{st.\,eq.})^{-1}\mathcal{S}p^\flat \longrightarrow \mathcal{SHC}^\flat$ is full (it is in fact also faithful, hence an isomorphism of categories).

As we saw in (1.13), every morphism from X to Y in the stable homotopy category allows a presentation as a 'left fraction'

$$[f] \ = \ \gamma(p_Y)^{-1} \circ \gamma(f) \ .$$

We now explain that we can similarly express morphisms in $\mathcal{SHC}$ as 'right fractions' (with more terms) with the special property that all intermediate symmetric spectra are flat.

Proposition 3.2. *For every morphism $a : X \longrightarrow Y$ in the stable homotopy category there exist flat symmetric spectra A, B and C and morphisms of symmetric spectra*

$$X \xleftarrow{f} A \xrightarrow{g} B \xleftarrow{f'} C \xrightarrow{g'} Y$$

such that f and f' are stable equivalences and

$$a \;=\; \gamma(g') \circ \gamma(f')^{-1} \circ \gamma(g) \circ \gamma(f)^{-1} \;.$$

Proof. Suppose that $a = [\alpha]$ for a morphism of symmetric spectra $\alpha : X \longrightarrow \omega Y$. The flat resolution functor (see Construction I.5.53) provides a commutative diagram of symmetric spectra

$$\begin{tikzcd}
X^\flat \arrow[r, "\alpha^\flat"] \arrow[d, "r_X"'] & (\omega Y)^\flat \arrow[d, "r_{\omega Y}"] & Y^\flat \arrow[l, "(p_Y)^\flat"'] \arrow[d, "r_Y"] \\
X \arrow[r, "\alpha"'] & \omega Y & Y \arrow[l, "p_Y"]
\end{tikzcd}$$

in which $X^\flat$, $(\omega Y)^\flat$ and $Y^\flat$ are flat and the three vertical morphisms are level equivalences, hence stable equivalences. Moreover, $(p_Y)^\flat$ is a stable equivalence because the other three morphisms in the right square are. The desired factorization then is

$$\begin{aligned}
a \;&=\; \gamma(p_Y)^{-1} \circ \gamma(f) \\
&=\; (\gamma(r_{\omega Y}) \circ \gamma((p_Y)^\flat) \circ \gamma(r_Y)^{-1})^{-1} \circ (\gamma(r_{\omega Y}) \circ \gamma(f^\flat) \circ \gamma(r_X)^{-1}) \\
&=\; \gamma(r_Y) \circ \gamma((p_Y)^\flat)^{-1} \circ \gamma(f^\flat) \circ \gamma(r_X)^{-1} \;.
\end{aligned}$$ □

Remark 3.3. The last proposition is not optimal and one can get away with a shorter zigzag of morphisms. In fact, with a little more work one can represent any morphism in $\mathcal{SHC}$ as $a = \gamma(g) \circ \gamma(f)^{-1}$ for a stable equivalence $f : Z \longrightarrow X$ and a morphism $g : Z \longrightarrow Y$ where Z is flat. We refer to Exercise E.II.3 for details.

Proposition 3.4. *Consider two functors*

$$F, G \;:\; (\mathcal{SHC}^\flat)^n \;\longrightarrow\; \mathcal{C}$$

of n variables for some $n \geq 1$. Then for every natural transformation $\tau : F \circ (\gamma^\flat)^n \longrightarrow G \circ (\gamma^\flat)^n$ of functors $(\mathcal{S}p^\flat)^n \longrightarrow \mathcal{C}$ there is a unique natural transformation $\bar\tau : F \longrightarrow G$ such that $\bar\tau \circ (\gamma^\flat)^n = \tau$. If τ is a natural isomorphism, so is $\bar\tau$.

Proof. Since the functor $\gamma^\flat : \mathcal{S}p^\flat \longrightarrow \mathcal{SHC}^\flat$ is the identity on objects, there can be at most one natural transformation $\bar\tau : F \longrightarrow G$ such that $\bar\tau \circ (\gamma^\flat)^n = \tau$; more precisely, at every n-tupel of flat symmetric spectra $X_1, X_2, \dots, X_n$ the $\mathcal{C}$-morphism $\bar\tau_{X_1,\dots,X_n} : F(X_1, \dots, X_n) \longrightarrow G(X_1, \dots, X_n)$ has to be equal to $\tau_{X_1,\dots,X_n}$.

The substance of the proposition is in the proof that naturality of τ implies naturality of $\bar\tau$. To show naturality of $\bar\tau$ as a transformation of n variables it suffices to show naturality in each variable separately, while the other $n-1$ variables are fixed. We show naturality in the first variable. So we consider flat symmetric spectra $X_2, \dots, X_{n-1}, Y$ and Y' and a morphism $a : Y \longrightarrow Y'$ in $\mathcal{SHC}^\flat$. If a is of the form $a = \gamma(\varphi)$ for some morphism of symmetric spectra $\varphi : Y \longrightarrow Y'$, then the naturality square

$$\begin{tikzcd}
F(Y, X_2, \dots, X_n) \arrow[r, "\bar\tau_{Y,X_2,\dots,X_n}"] \arrow[d, "F(\gamma(\varphi),\mathrm{Id},\dots,\mathrm{Id})"'] & G(Y, X_2, \dots, X_n) \arrow[d, "G(\gamma(\varphi),\mathrm{Id},\dots,\mathrm{Id})"] \\
F(Y', X_2, \dots, X_n) \arrow[r, "\bar\tau_{Y,X_2,\dots,X_n}"'] & G(Y', X_2, \dots, X_n)
\end{tikzcd}$$

commutes since $\bar\tau = \tau$ objectwise and τ is natural. If $\bar\tau$ is natural for a particular morphism a in the first variable and a is invertible, then $\bar\tau$ is also natural for the inverse a^{-1} in the first variable.

By Proposition 3.2 an arbitrary morphism a from Y to Y' in $\mathcal{SHC}^\flat$ can be written as a composite of morphisms of the form $\gamma(g)$ and $\gamma(f)^{-1}$ for suitable morphisms between flat symmetric spectra such that f is a stable equivalence. By the above, $\bar\tau$ is natural for both $\gamma(g)$ and $\gamma(f)^{-1}$, so $\bar\tau$ is natural for an arbitrary morphism in the first variable, hence natural in general.

If τ is a natural isomorphism, then we apply the previous argument to the inverse transformation τ^{-1} and obtain a natural transformation $\overline{\tau^{-1}}$ satisfying $\overline{\tau^{-1}} \circ (\gamma^\flat)^n = \tau^{-1}$. Both composites of $\overline{\tau^{-1}}$ with $\bar\tau$ restrict to identity transformations along $(\gamma^\flat)^n$, so by the uniqueness, the transformations $\overline{\tau^{-1}}$ and $\bar\tau$ are inverse to each other. □

Now we can construct the (restricted) derived smash product on the category $\mathcal{SHC}^\flat$, see the following proposition. We leave it as Exercise E.II.9 to show that there is only one functor $\wedge^L : \mathcal{SHC}^\flat \times \mathcal{SHC}^\flat \longrightarrow \mathcal{SHC}^\flat$ which satisfies $\wedge^L \circ (\gamma^\flat \times \gamma^\flat) = \gamma^\flat \circ \wedge$ and there is only one way to extend this functor to a symmetric monoidal structure on $\mathcal{SHC}^\flat$ for which $\gamma^\flat : \mathcal{S}p^\flat \longrightarrow \mathcal{SHC}^\flat$ is strong symmetric monoidal. In other words, on the subcategory $\mathcal{SHC}^\flat$, the derived smash product to be constructed now is very canonical.

Proposition 3.5. *There is a functor*

$$\wedge^L \ : \ \mathcal{SHC}^\flat \times \mathcal{SHC}^\flat \ \longrightarrow \ \mathcal{SHC}^\flat$$

on the stable homotopy category of flat symmetric spectra that satisfies

$$\wedge^L \circ (\gamma^\flat \times \gamma^\flat) \ = \ \gamma^\flat \circ \wedge$$

as functors $\mathcal{S}p^\flat \times \mathcal{S}p^\flat \longrightarrow \mathcal{SHC}^\flat$. Moreover, the sphere spectrum is a strict unit for $\wedge^L$ and the functor $\wedge^L$ can be extended to a symmetric monoidal structure on the category $\mathcal{SHC}^\flat$ in such a way that the functor $\gamma^\flat : \mathcal{S}p^\flat \longrightarrow \mathcal{SHC}^\flat$ is strict symmetric monoidal.

Proof. We start with some preparations. If X is a flat symmetric spectrum, then smashing with X preserves stable equivalences (Proposition I.5.50), and so the composite functor $\gamma \circ (X \wedge -) : \mathcal{S}p \longrightarrow \mathcal{SHC}$ takes stable equivalences to isomorphisms. By the universal property of the functor γ (see Theorem 1.6 (ii)) there is a unique functor

$$lX \ : \ \mathcal{SHC} \ \longrightarrow \ \mathcal{SHC}$$

satisfying $(lX) \circ \gamma = \gamma \circ (X \wedge -)$. On objects, this functor is then given by $lX(Y) = X \wedge Y$, the pointset level smash product of symmetric spectra, and the proof of Theorem 1.6 (ii) reveals how to extend this definition to morphisms. If Y is another flat symmetric spectrum, then by the same argument there is a unique functor $rY : \mathcal{SHC} \longrightarrow \mathcal{SHC}$ satisfying $(rY) \circ \gamma = \gamma \circ (- \wedge Y)$.

We show that these two constructions have a certain compatibility in the range where both are defined. More precisely, suppose we are given four flat symmetric spectra X, X', Y and Y' and morphisms $a : X \longrightarrow X'$ and $b : Y \longrightarrow Y'$ in $\mathcal{SHC}^\flat$, then we claim the relation

$$(rY')(a) \circ (lX)(b) \ = \ (lX')(b) \circ (rY)(a) \tag{3.6}$$

as morphisms from $X \wedge Y$ to $X' \wedge Y'$ in $\mathcal{SHC}^\flat$. We will see below that this relations is exactly what is needed to combine the various individual functors into a two-variable functor.

To prove relation (3.6) we go through a sequence of four steps. In Step 1 we suppose that $a = \gamma(\alpha)$ and $b = \gamma(\beta)$ are images of morphisms of symmetric spectra $\alpha : X \longrightarrow X'$ and $\beta : Y \longrightarrow Y'$ under the localization functor. Then we have

$$\begin{aligned}(rY')(a) \circ (lX)(b) \ &= \ (rY')(\gamma(\alpha)) \circ (lX)(\gamma(\beta)) \ = \ \gamma(\alpha \wedge Y') \circ \gamma(X \wedge \beta) \ = \ \gamma(\alpha \wedge \beta) \\ &= \ \gamma(X' \wedge \beta) \circ \gamma(\alpha \wedge Y) \ = \ (lX')(\gamma(\beta)) \circ (rY)(\gamma(\alpha)) \ = \ (lX')(b) \circ (rY)(a) \ .\end{aligned}$$

Step 2: if the relation (3.6) holds for a pair of morphisms (a, b) in the stable homotopy category and a (respectively b) is an isomorphism, then the relation also holds for the pair (a^{-1}, b) (respectively (a, b^{-1})). For example, in the first case we simply take the equation (3.6) for (a, b), composite it from the left with $(rY')(a)^{-1}$ and from the right with $(rY)(a)^{-1}$ and obtain the relation for (a^{-1}, b).

Step 3: clearly, if $a : X \longrightarrow X'$ and $a' : X' \longrightarrow X''$ are composable morphisms in the stable homotopy category and the relation (3.6) holds for the pairs (a, b) and (a', b), then it also holds for the pair $(a' \circ a, b)$, and similarly for composable morphisms in the second variable.

Step 4: by Proposition 3.2 the morphism a and b can be written as a composite of morphisms of the form $\gamma(g)$ and $\gamma(f)^{-1}$ for suitable morphisms between flat symmetric spectra such that f is a stable equivalence. So steps 1-3 combine to prove the general case.

Now we are ready to define the derived smash product on the stable homotopy category of flat symmetric spectra. The requirement $\wedge^L \circ (\gamma^\flat \times \gamma^\flat) = \gamma^\flat \circ \wedge$ forces us to define the derived smash product on pairs of objects by the pointset level smash product. For morphisms $a : X \longrightarrow X'$ and $b : Y \longrightarrow Y'$ in $\mathcal{SHC}^\flat$ we define $a \wedge^L b : X \wedge^L Y \longrightarrow X' \wedge^L Y'$ as the common value of the the equation (3.6). To see that this is indeed a functor we consider further morphisms $a' : X' \longrightarrow X''$ and $b' : Y' \longrightarrow Y''$ in $\mathcal{SHC}^\flat$ and calculate

$$\begin{aligned}
(a'a) \wedge^L (b'b) &= (rY'')(a'a) \circ (lX)(b'b) \\
&= (rY'')(a') \circ (rY'')(a) \circ (lX)(b'b) \\
&= (rY'')(a') \circ (lX')(b'b) \circ (rY)(a) \\
&= (rY'')(a') \circ (lX')(b') \circ (lX')(b) \circ (rY)(a) \\
&= (a' \wedge^L b') \circ (a \wedge^L b)
\end{aligned}$$

It is clear that the derived smash product preserves identity morphisms.

For morphisms $\varphi : X \longrightarrow X'$ and $\psi : Y \longrightarrow Y'$ between flat symmetric spectra we have

$$\begin{aligned}
\gamma(\varphi) \wedge^L \gamma(\psi) &= (rY')(\gamma(\varphi)) \circ (lX)(\gamma(\psi)) = \gamma(\varphi \wedge Y') \circ \gamma(X \wedge \psi) \\
&= \gamma((\varphi \wedge Y') \circ (X \wedge \psi)) = \gamma(\varphi \wedge \psi) ,
\end{aligned} \tag{3.7}$$

so we have the equality $\wedge^L \circ (\gamma^\flat \times \gamma^\flat) = \gamma^\flat \circ \wedge$ as functors.

The sphere spectrum $\mathbb{S}$ is flat and a strict unit for the smash product of symmetric spectra. So we have $(l\mathbb{S}) \circ \gamma = \gamma \circ (\mathbb{S} \wedge -) = \gamma$ as functors $\mathcal{S}p \longrightarrow \mathcal{SHC}$. So $l\mathbb{S}$ must be the identity functor, i.e., the sphere spectrum is a strict left unit for $\wedge^L$. The sphere spectrum is a strict right unit by the same argument.

Now we define coherence isomorphisms for the functor $\wedge^L$ which make it into a symmetric monoidal structure on the category $\mathcal{SHC}^\flat$. To obtain the derived associativity isomorphism we apply Proposition 3.4 to the functors

$$\wedge^L \circ (\wedge^L \times \mathrm{Id}) ,\ \wedge^L \circ (\mathrm{Id} \times \wedge^L) \ : \ \mathcal{SHC}^\flat \times \mathcal{SHC}^\flat \times \mathcal{SHC}^\flat \ \longrightarrow \ \mathcal{SHC}^\flat .$$

The pointset level associativity isomorphism $\alpha_{X,Y,Z} : (X \wedge Y) \wedge Z \longrightarrow X \wedge (Y \wedge Z)$ restricts to a natural isomorphism between the functors $\wedge^L \circ (\wedge^L \times \mathrm{Id}) \circ (\gamma^\flat)^3$ and $\wedge^L \circ (\mathrm{Id} \times \wedge^L) \circ (\gamma^\flat)^3$, so there is a unique natural isomorphism $\bar{\alpha} : \wedge^L \circ (\wedge^L \times \mathrm{Id}) \longrightarrow \wedge^L \circ (\mathrm{Id} \times \wedge^L)$, which we define to be the derived associativity isomorphism. The construction of the derived symmetry isomorphism is similar: the pointset level symmetry isomorphism $\tau_{X,Y} : X \wedge Y \longrightarrow Y \wedge X$ restricts to a natural isomorphism between the functors $\wedge^L \circ (\gamma^\flat)^2$ and $\wedge^L \circ T \circ (\gamma^\flat)^2$, where $T : \mathcal{SHC}^\flat \times \mathcal{SHC}^\flat \longrightarrow \mathcal{SHC}^\flat \times \mathcal{SHC}^\flat$ is the automorphism that interchanges the two factors. So there is a unique natural isomorphism $\bar{\tau} : \wedge^L \longrightarrow \wedge^L \circ T$ satisfying $\bar{\tau} \circ (\gamma^\flat)^2 = \tau$, the derived symmetry isomorphism.

The various coherence conditions at the level of the stable homotopy category $\mathcal{SHC}^\flat$ are now a direct consequence of the corresponding coherence conditions at the level of symmetric spectra and the uniqueness statement in Proposition 3.4. We treat one case in detail and omit the other cases, which are very similar.

The two composites around the diagram

$$\begin{tikzcd}
(X \wedge^L Y) \wedge^L Z \arrow[r, "\bar{\alpha}_{X,Y,Z}"] \arrow[d, "\bar{\tau}_{X,Y} \wedge^L \mathrm{Id}"'] & X \wedge^L (Y \wedge^L Z) \arrow[r, "\bar{\tau}_{X, Y \wedge^L Z}"] & (Y \wedge^L Z) \wedge^L X \arrow[d, "\bar{\alpha}_{Y,Z,X}"] \\
(Y \wedge^L X) \wedge^L Z \arrow[r, "\bar{\alpha}_{Y,X,Z}"'] & Y \wedge^L (X \wedge^L Z) \arrow[r, "\mathrm{Id} \wedge^L \bar{\tau}_{X,Z}"'] & Y \wedge^L (Z \wedge^L X)
\end{tikzcd} \tag{3.8}$$

are two natural transformations from the functor

$$\wedge^L \circ (\wedge^L \times \mathrm{Id}) \ : \ \mathcal{SHC}^\flat \times \mathcal{SHC}^\flat \times \mathcal{SHC}^\flat \ \longrightarrow \ \mathcal{SHC}^\flat$$

to the functor $\wedge^L \circ (\mathrm{Id} \times \wedge^L) \circ C$ where C is the automorphism of $\mathcal{SHC}^\flat \times \mathcal{SHC}^\flat \times \mathcal{SHC}^\flat$ which cyclically permutes the factors. After composition with $(\gamma^\flat)^3$ the two natural transformations become equal since the corresponding diagram for the smash product in $\mathcal{S}p$ commutes. So the uniqueness statement in Proposition 3.4 guarantees that the diagram (3.8) commutes, and we have verified the coherence between associativity and symmetry isomorphisms.

There is one final claim which we have to check, namely that the identity transformation makes the functor $\gamma^\flat : \mathcal{S}p^\flat \longrightarrow \mathcal{SHC}^\flat$ into a strong symmetric monoidal functor. However, when we unravel what this means, we see that all conditions hold by construction. Indeed, if X and Y are flat symmetric spectra, then $\gamma(X) \wedge^L \gamma(Y)$ and $\gamma(X \wedge Y)$ are the same object (namely the smash product $X \wedge Y$) and for morphism $\varphi : X \longrightarrow X'$ and $\psi : Y \longrightarrow Y'$ of symmetric spectra the two morphisms $\gamma(\varphi) \wedge^L \gamma(\psi)$ and $\gamma(\varphi \wedge \psi)$ are equal, see (3.7) above. □

Now we can extend the derived smash product from the category $\mathcal{SHC}^\flat$ to the entire stable homotopy category and prove Theorem 3.1.

PROOF OF THEOREM 3.1. For every symmetric spectrum X we choose a flat resolution, i.e., a flat symmetric spectrum $X^\flat$ and a level equivalence $X^\flat \longrightarrow X$. We insist, however, that $X^\flat = X$ and $r = \mathrm{Id}$ if X is already flat. For example, for non-flat spectra we can take the flat resolution constructed in I.5.53. This provides a functor [sic!]

$$(-)^\flat \ : \ \mathcal{SHC} \ \longrightarrow \ \mathcal{SHC}^\flat$$

whose restriction to $\mathcal{SHC}^\flat$ is the identity. Moreover, applying the localization functor γ to the resolution morphisms $r : X^\flat \longrightarrow X$ provides an isomorphism $\gamma(r) : X^\flat \longrightarrow X$ which is natural on the level of the stable homotopy category. Hence the functor $(-)^\flat$ is an equivalence of categories.

We define the derived smash product on the stable homotopy category as the composite

$$\mathcal{SHC} \times \mathcal{SHC} \ \xrightarrow{(-)^\flat \times (-)^\flat} \ \mathcal{SHC}^\flat \times \mathcal{SHC}^\flat \ \xrightarrow{\wedge^L} \ \mathcal{SHC}^\flat \ \xrightarrow{\text{incl.}} \ \mathcal{SHC} \ .$$

[fill in details] We have to show that the monoidal structure given by the smash product is closed. For symmetric spectra Y and Z we define the *derived function spectrum* by

$$F(Y, Z) \ = \ \mathrm{Hom}(Y, \omega Z) \ ,$$

the internal symmetric function spectrum (see Example I.3.38) from Y to the chosen injective Ω-spectrum ωZ for Z. This spectrum comes with an evaluation morphism

$$\bar{\epsilon}_{Y,Z} \ : \ F(Y, Z) \wedge Y \ = \ \mathrm{Hom}(Y, \omega Z) \wedge Y \ \longrightarrow \ \omega Z$$

of symmetric spectra which is adjoint to the identity of $\mathrm{Hom}(Y, \omega Z)$; so $\bar{\epsilon}_{Y,Z}$ is the morphism corresponding to the bimorphism from $(\mathrm{Hom}(Y, \omega Z), Y)$ to ωZ with (p, q)-component

$$\mathrm{map}(Y, \mathrm{sh}^p(\omega Z)) \wedge Y_q \ \longrightarrow \ (\omega Z)_{p+q} \ , \quad f \wedge y \mapsto f_q(y) \ .$$

We define a morphism $\epsilon_{Y,Z} : F(Y, Z) \wedge^L Y \longrightarrow Z$ in the stable homotopy category as the composite

$$F(Y, Z) \wedge^L Y \ \xrightarrow{\psi_{F(Y,Z),Y}} \ F(Y, Z) \wedge Y \ \xrightarrow{[\bar{\epsilon}_{Y,Z}]} \ Z \tag{3.9}$$

where the second morphism is the the homotopy class of $\bar{\epsilon}_{Y,Z}$. We also call $\epsilon_{Y,Z}$ an evaluation morphism, possibly with adjective 'derived' if we need to distinguish it from $\bar{\epsilon}_{Y,Z}$.

To see that the monoidal structure given by the derived smash product is closed we have to show that for every triple of objects X, Y and Z of the stable homotopy category the map

$$\mathcal{SHC}(X, F(Y, Z)) \ \longrightarrow \ \mathcal{SHC}(X \wedge^L Y, Z) \ , \quad \alpha \mapsto \epsilon_{Y,Z} \circ (\alpha \wedge^L Y) \tag{3.10}$$

is bijective. In order to show this, we first make a reduction argument. The symmetric spectrum $F(Y, Z)$ is already a contravariant functor in Y (but not in Z) on the pointset level, and the map is natural in Y.

Moreover, the functor $F(-,Z) = \mathrm{Hom}(-,\omega Z)$ takes stable equivalences to level equivalences, and thus to isomorphism in $\mathcal{SHC}$, since ωZ is an injective Ω-spectrum (compare Proposition I.4.29). So we may replace Y by any stably equivalent object and can thus assume without loss of generality that Y is flat.

If Y is flat then by Corollary I.5.51 the function spectrum $\mathrm{Hom}(Y,\omega Z)$ is an injective Ω-spectrum. We have a commutative diagram

$$\begin{tikzcd} \mathcal{SHC}(X,F(Y,Z)) \arrow[r, "\alpha\mapsto\epsilon_{Y,Z}\circ(\alpha\wedge^L Y)"] \arrow[d, equal] & \mathcal{SHC}(X\wedge^L Y,Z) \arrow[r, equal] & {[X\wedge^L Y,\omega Z]} \arrow[d] \\ {[X,\mathrm{Hom}(Y,\omega Z)]} \arrow[rr, "\text{adjunction}"'] & & {[X\wedge Y,\omega Z]} \end{tikzcd}$$

in which all other maps are bijections (some even identities), which proves that (3.10) is bijective. $\square$

It is a formal consequence of the representability property (3.10) of the derived function spectrum that $F(Y,Z)$ is naturally a functor in the second variable. A morphism $\zeta : Z \longrightarrow Z'$ gives rise to a map

$$\mathcal{SHC}(X\wedge^L Y,\zeta)\ :\ \mathcal{SHC}(X\wedge^L Y,Z)\ \longrightarrow\ \mathcal{SHC}(X\wedge^L Y,Z')$$

which is natural in X and Y. Combining this with the adjunction bijections (3.10) gives a map

$$\mathcal{SHC}(X,F(Y,Z))\ \longrightarrow\ \mathcal{SHC}(X,F(Y,Z'))\ ,$$

still natural in X. By the Yoneda lemma, this transformation is induced by a unique morphism from $F(Y,Z)$ to $F(Y,Z')$ in the stable homotopy category, which we define to be $F(Y,\zeta)$. By the very construction, $F(Y,-)$ is right adjoint to $-\wedge^L Y$ with respect to the bijection (3.10). A similar representability arguments shows that altogether we obtain a functor

$$F\ :\ \mathcal{SHC}^{op}\times\mathcal{SHC}\ \longrightarrow\ \mathcal{SHC}\ .$$

In Section 5.3 we proved various relations between the pointset level smash product and other constructions with symmetric spectra. We see now that many of these relation descend to the stable homotopy category and have analogues for the derived smash product. For a Σ_m-simplicial set L and a Σ_n-simplicial set L' we have a natural isomorphism between semifree symmetric spectrum

$$G_{m+n}(\Sigma^+_{m+n}\wedge_{\Sigma_m\times\Sigma_n} L\wedge L')\ \cong\ G_mL\ \wedge^L G_nL'\ . \tag{3.11}$$

Since semifree symmetric spectra are flat, the derived smash product is given by the pointset level smash product. Thus the isomorphism is obtained from the corresponding pointset level isomorphism (3.11) of Chapter I by applying the localization functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$. As a special case we can consider smash products of free symmetric spectra. If K and K' are pointed simplicial sets then we have $F_mK = G_m(\Sigma^+_m\wedge K)$ and $F_nK' = G_n(\Sigma^+_n\wedge K')$, so the isomorphism (5.14) specializes to an associative, commutative and unital isomorphism

$$F_{m+n}(K\wedge K')\ \cong\ F_mK\ \wedge^L F_nK'\ .$$

As the even more special case for $m = n = 0$ we obtain a natural isomorphism of suspension spectra

$$(\Sigma^\infty K)\ \wedge^L\ (\Sigma^\infty L)\ \cong\ \Sigma^\infty(K\wedge L)$$

for all pairs of pointed simplicial sets K and L.

The pointset level composition morphisms (compare I.(5.20))

$$\circ\ :\ \mathrm{Hom}(Y,Z)\ \wedge\ \mathrm{Hom}(X,Y)\ \longrightarrow\ \mathrm{Hom}(X,Z)$$

'pass' to the stable homotopy category and admit a derived version. Indeed, the composite

$$(F(Y,Z)\wedge^L F(X,Y))\wedge^L X \xrightarrow{\bar\alpha} F(Y,Z)\wedge^L (F(X,Y)\wedge^L X) \xrightarrow{\mathrm{Id}\wedge\epsilon_{X,Y}} F(Y,Z)\wedge^L Y \xrightarrow{\epsilon_{Y,Z}} Z$$

has an adjoint

$$\circ\ :\ F(Y,Z)\ \wedge^L\ F(X,Y)\ \longrightarrow\ F(X,Z)\ . \tag{3.12}$$

We omit the verification that this 'derived composition map' is associative and unital.

The special case $X = \mathbb{S}$ of the adjunction bijection (3.10) yields a natural isomorphism between $\mathcal{SHC}(\mathbb{S}, F(Y,Z))$ and $\mathcal{SHC}(\mathbb{S} \wedge^L Y, Z) = \mathcal{SHC}(Y,Z)$. The former group is in turn isomorphic, via evaluation at the fundamental class $1 \in \pi_0\mathbb{S}$, to $\pi_0 F(Y,Z)$ (see Example 1.15). Combining this gives a natural isomorphism

$$\pi_0 F(Y,Z) \;\cong\; \mathcal{SHC}(Y,Z)\ .$$

[can we define this iso directly?] The 'derived composition map' (3.12) becomes composition in the category $\mathcal{SHC}$ when we apply the 0th homotopy group functor; in more detail, the diagram

$$\begin{tikzcd}
\pi_0 F(Y,Z) \times \pi_0 F(X,Y) \arrow[r, "\cdot"] \arrow[d, "\cong"'] & \pi_0\left(F(Y,Z) \wedge^L F(X,Y)\right) \arrow[r, "\pi_0(\circ)"] & \pi_0 F(X,Z) \arrow[d, "\cong"] \\
\mathcal{SHC}(Y,Z) \times \mathcal{SHC}(X,Y) \arrow[rr, "\circ"'] & & \mathcal{SHC}(X,Z)
\end{tikzcd}$$

commutes. [is the product $\cdot$ already defined ?] [justify: uniqueness property of product...]

Another useful fact about the derived smash product and its adjoint derived function spectrum is a compatibility with the triangulated structure. In fact, if we fix an symmetric spectrum X then the functors $X \wedge^L -$, $- \wedge^L X$, $F(X,-)$ and $F(-,X)$ are all *exact functors* on the stable homotopy category, as Proposition 3.19 below shows.

Definition 3.13. Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories. An *exact functor* is a pair (F,τ) consisting of a functor $F : \mathcal{T} \longrightarrow \mathcal{T}'$ and a natural isomorphism $\tau : F \circ \Sigma \cong \Sigma \circ F$ such that for every distinguished triangle (f,g,h) in $\mathcal{T}$ the triangle

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \xrightarrow{\tau_A \circ Fh} \Sigma(FA)$$

is distinguished in $\mathcal{T}'$.

When the natural isomorphism $\tau : F\Sigma \cong \Sigma F$ is understood, we often refer to the functor F as an exact functor. One should keep in mind, though, that 'exact' is not just a property of a functor, but extra structure.

A functor $F : \mathcal{A} \longrightarrow \mathcal{A}'$ between additive categories is *additive* if it satisfies the following equivalent conditions:

- the functor F preserves coproducts;
- the functor F preserves products;
- for all objects A and Z of $\mathcal{A}$ the map $\mathcal{A}(A,Z) \longrightarrow \mathcal{A}'(FA,FZ)$ is a group homomorphism.

Zero objects are characterized by the property that the identity morphism equals the zero morphism. An additive functor preserves identity morphisms and zero morphisms, so an additive functor takes zero objects to zero objects.

Proposition 3.14. *Every exact functor between triangulated categories is additive.*

Proof. We let A and B be objects of $\mathcal{T}$ and $A \oplus B$ a coproduct with respect to morphisms $i_A : A \longrightarrow A \oplus B$ and $i_B : B \longrightarrow A \oplus B$. By Proposition 2.10 (vii) the triangle

$$A \xrightarrow{i_A} A \oplus B \xrightarrow{p_B} B \xrightarrow{0} \Sigma A$$

is distinguished, where p_B is determined by $p_B i_A = 0$ and $p_B i_B = \mathrm{Id}_B$. Since F is exact the triangle

$$FA \xrightarrow{F(i_A)} F(A \oplus B) \xrightarrow{F(p_B)} FB \xrightarrow{\tau_A \circ F0} \Sigma(FA)$$

is distinguished in $\mathcal{T}'$. Since the morphism $F(i_B) : FB \longrightarrow F(A \oplus B)$ is a section to $F(p_B)$, the object $F(A \oplus B)$ is a biproduct of FA and FB with respect to the morphisms $F(i_A)$ and $F(i_B)$ (by Proposition 2.10 (vi)). So F preserves coproducts, and is thus an additive functor. $\square$

A useful fact is that adjoints of exact functors are again exact, in a specific way that we now explain. We let $F : \mathcal{T} \longrightarrow \mathcal{T}'$ be a functor between triangulated categories, with right adjoint G. Given a natural transformation of functors $\tau : F \circ \Sigma \longrightarrow \Sigma \circ F$ we define a natural transformation $\hat{\tau} : \Sigma \circ G \longrightarrow G \circ \Sigma$ at an object X of $\mathcal{T}'$ as the adjoint of the composite

$$F(\Sigma(GX)) \xrightarrow{\tau_{GX}} \Sigma(FGX) \xrightarrow{\Sigma\epsilon_X} \Sigma X \ ,$$

where $\epsilon : FG \longrightarrow \mathrm{Id}_{\mathcal{T}'}$ is the counit of the adjunction. More explicitly, $\hat{\tau}_X$ is the composite

$$\Sigma(GX) \xrightarrow{\eta_{\Sigma(GX)}} GF(\Sigma(GX)) \xrightarrow{G(\tau_{GX})} G(\Sigma(FGX)) \xrightarrow{G(\Sigma\epsilon_X)} G(\Sigma X) \ , \tag{3.15}$$

where $\eta : \mathrm{Id}_{\mathcal{T}} \longrightarrow GF$ is the unit of the adjunction.

Proposition 3.16. *Let*

$$\mathcal{T} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{T}'$$

be a pair of adjoint functors between triangulated categories, with adjunction unit $\eta : \mathrm{Id}_{\mathcal{T}} \longrightarrow GF$ *and counit* $\epsilon : FG \longrightarrow \mathrm{Id}_{\mathcal{T}'}$.

(i) *The assignment* (3.15)

$$(\tau : F \circ \Sigma \longrightarrow \Sigma \circ F) \quad \longmapsto \quad (\hat{\tau} : \Sigma \circ G \longrightarrow G \circ \Sigma)$$

from the class of natural transformations $F \circ \Sigma \longrightarrow \Sigma \circ F$ *to the class of natural transformations* $\Sigma \circ G \longrightarrow G \circ \Sigma$ *is bijective.*

(ii) *A transformation* $\tau : F \circ \Sigma \longrightarrow \Sigma \circ F$ *is a natural isomorphism if and only if the transformation* $\hat{\tau} : \Sigma \circ G \longrightarrow G \circ \Sigma$ *is a natural isomorphism.*

(iii) *The pair* (F, τ) *is an exact functor if and only if the pair* $(G, \hat{\tau}^{-1})$ *is an exact functor.*

Proof. (i) Given a natural transformation $\psi : \Sigma \circ G \longrightarrow G \circ \Sigma$ we define a natural transformation $\bar{\psi} : F \circ \Sigma \longrightarrow \Sigma \circ F$ at an object A of $\mathcal{T}$ as the adjoint of the composite

$$\Sigma A \xrightarrow{\Sigma\eta_A} \Sigma(GFA) \xrightarrow{\psi_{FA}} G(\Sigma(FA)) \ .$$

We omit the verfication that this construction is inverse to $\tau \mapsto \hat{\tau}$.

(ii) For every object A of $\mathcal{T}$ and every object X of $\mathcal{T}'$ the diagram

$$\begin{tikzcd}
\mathcal{T}'(FA, X) \arrow[r, "\Sigma"] \arrow[d, "\cong"'] & \mathcal{T}'(\Sigma(FA), \Sigma X) \arrow[r, "\mathcal{T}'(\tau_A, \Sigma X)"] & \mathcal{T}'(F(\Sigma A), \Sigma X) \arrow[d, "\cong"] \\
\mathcal{T}(A, GX) \arrow[r, "\Sigma"'] & \mathcal{T}(\Sigma A, \Sigma(GX)) \arrow[r, "\mathcal{T}(\Sigma A, \hat{\tau}_X)"'] & \mathcal{T}(\Sigma A, G(\Sigma X)
\end{tikzcd}$$

commutes by the definition of $\hat{\tau}$, where the two vertical maps are the adjunction bijections. Both suspension functors are fully faithful and all maps, except possibly $\mathcal{T}'(\tau_A, \Sigma X)$ and $\mathcal{T}(\Sigma A, \hat{\tau}_X)$, are bijective. So $\mathcal{T}'(\tau_A, \Sigma X)$ is bijective if and only if $\mathcal{T}(\Sigma A, \hat{\tau}_X)$ is bijective. So if τ is a natural isomorphism, then $\mathcal{T}(\Sigma A, \hat{\tau}_X)$ is bijective for all A and X of $\mathcal{T}$. Since every object of $\mathcal{T}$ is isomorphic to a suspension, $\hat{\tau}_X$ is an isomorphism for all objects X. The converse is analogous.

(iii) We assume that (F, τ) is exact and show that then $(G, \hat{\tau}^{-1})$ is exact; the other implication is similar. So we let (f, g, h) be a distinguished triangle in $\mathcal{T}'$. We choose a distinguished triangle

$$GA \xrightarrow{Gf} GB \xrightarrow{g'} C' \xrightarrow{h'} \Sigma(GA)$$

in $\mathcal{T}$. Then the upper triangle in the diagram

$$\begin{tikzcd} FGA \arrow[r, "FGf"] \arrow[d, "\epsilon_A"'] & FGB \arrow[r, "Fg'"] \arrow[d, "\epsilon_B"'] & FC' \arrow[r, "\tau_{GA}\circ Fh'"] \arrow[d, dotted, "\varphi"] & \Sigma(FGA) \arrow[d, "\Sigma\epsilon_A"] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma A \end{tikzcd}$$

is distinguished by hypothesis on F. By axiom (T3) there is a morphism $\varphi : FC' \longrightarrow C$ such that $(\epsilon_A, \epsilon_B, \varphi)$ is a morphism of triangles. We let $\hat{\varphi} : C' \longrightarrow GC$ be the adjoint of φ. The relation $g \circ \epsilon_B = \varphi \circ Fg'$ is then adjoint to the relation $Gg = \hat{\varphi} \circ g'$. Similarly, the relation $h \circ \varphi = \Sigma\epsilon_A \circ \tau_{GA} \circ Fh'$ is adjoint to the relation $Gh \circ \hat{\varphi} = \hat{\tau}_A \circ h'$ (because $\hat{\tau}_A$ was defined as the adjoint of $(\Sigma\epsilon_A) \circ \tau_{GA}$). In other words, the diagram of triangles in $\mathcal{T}$

$$\begin{tikzcd} GA \arrow[r, "Gf"] \arrow[d, equal] & GB \arrow[r, "g'"] \arrow[d, equal] & C' \arrow[r, "h'"] \arrow[d, dotted, "\hat{\varphi}"] & \Sigma(GA) \arrow[d, equal] \\ GA \arrow[r, "Gf"'] & GB \arrow[r, "Gg"'] & GC \arrow[r, "\hat{\tau}_A^{-1}\circ Gh"'] & \Sigma(GA) \end{tikzcd} \tag{3.17}$$

commutes. For every object X of $\mathcal{T}$, the sequence of abelian groups

$$\mathcal{T}(X, GA) \xrightarrow{\mathcal{T}(X,Gf)} \mathcal{T}(X, GB) \xrightarrow{\mathcal{T}(X,Gg)} \mathcal{T}(X, GC) \xrightarrow{\mathcal{T}(X,\hat{\tau}_A^{-1}\circ Gh)} \mathcal{T}(X, \Sigma GA) \xrightarrow{\mathcal{T}(X,-\Sigma Gf)} \mathcal{T}(X, \Sigma GB)$$

is isomorphic, by the adjunction, to the sequence

$$\mathcal{T}(FX, A) \xrightarrow{\mathcal{T}(FX,f)} \mathcal{T}(FX, B) \xrightarrow{\mathcal{T}(FX,g)} \mathcal{T}(FX, C) \xrightarrow{\mathcal{T}(FX,\hat{\tau}_A\circ h)} \mathcal{T}(FX, \Sigma A) \xrightarrow{\mathcal{T}(FX,-\Sigma f)} \mathcal{T}(FX, \Sigma B)\ .$$

The latter is exact by Proposition 2.10 (i), hence so is the former. So even though we do not yet know whether the lower triangle in the diagram (3.17) is distinguished, both triangles are taking to exact sequences by $\mathcal{T}(X, -)$. This is enough for the argument used in the proof of Proposition 2.10 (ii); so the argument of that proof shows that the morphism $\hat{\varphi}$ is then an isomorphism. Since the triangle (Gf, g', h') is distinguished, so is the isomorphic triangle $(Gf, Gg, \hat{\tau}_A^{-1} \circ Gh)$. This completes the proof that the pair $(G, \hat{\tau}^{-1})$ is an exact functor. □

For symmetric spectra A and Y the associativity isomorphism $(S^1 \wedge A) \wedge Y \cong S^1 \wedge (A \wedge Y)$ gives rise, by applying the localization functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ to a natural isomorphism [on flat spectra; derive...]

$$\kappa_{A,Y} \ : \ (\Sigma A) \wedge^L Y \ \longrightarrow \ \Sigma(A \wedge^L Y) \tag{3.18}$$

in the stable homotopy category. The composite

$$(\Sigma F(Y, A)) \wedge^L Y \ \xrightarrow{\kappa_{F(Y,A),Y}} \ \Sigma(F(Y, A) \wedge^L Y) \ \xrightarrow{\Sigma\epsilon_{Y,A}} \ \Sigma A$$

has an adjoint $\psi_{A,Y} : \Sigma F(Y, A) \longrightarrow F(Y, \Sigma A)$ which is an isomorphism by Proposition 3.16. Moreover, the composite

$$(\Sigma(F(\Sigma A, Y)) \wedge^L A \ \xrightarrow{\kappa_{F(\Sigma A,Y),Y}} \ F(\Sigma A, Y) \wedge^L (\Sigma A) \ \xrightarrow{\epsilon_{\Sigma A,Y}} \ Y$$

has an adjoint $\bar{\psi}_{A,Y} : \Sigma(F(\Sigma A, Y)) \longrightarrow F(A, Y)$ which is an isomorphism.

A *contravariant* functor F from $\mathcal{T}$ to $\mathcal{T}'$ is exact when equipped with a natural isomorphism $\psi : FA \cong \Sigma(F\Sigma A)$ such that such that for every distinguished triangle (f, g, h) in $\mathcal{T}$ the triangle

$$F(\Sigma A) \xrightarrow{Fh} FC \xrightarrow{Fg} FB \xrightarrow{\psi_A\circ Ff} \Sigma F(\Sigma A)$$

is distinguished in $\mathcal{T}'$. This can be said differently. There is a way to make the opposite of a triangulated category into a triangulated category, compare Example E.II.5. Then a contravariant exact functor F is the same as an exact functor from $\mathcal{T}^{\mathrm{op}}$ with respect to the opposite triangulation.

Proposition 3.19. *Let Y be a symmetric spectrum of simplicial sets and*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

a distinguished triangle in the stable homotopy category. Then the three triangles

$$A \wedge^L Y \xrightarrow{f \wedge^L Y} B \wedge^L Y \xrightarrow{g \wedge^L Y} C \wedge^L Y \xrightarrow{\kappa_{A,Y} \circ (h \wedge^L Y)} \Sigma(A \wedge^L Y)$$

$$F(Y,A) \xrightarrow{F(Y,f)} F(Y,B) \xrightarrow{F(Y,g)} F(Y,C) \xrightarrow{\psi^{-1}_{Y,A} \circ F(Y,h)} \Sigma F(Y,A)$$

$$F(\Sigma A, Y) \xrightarrow{F(h,Y)} F(C,Y) \xrightarrow{F(g,Y)} F(B,Y) \xrightarrow{\bar{\psi}^{-1}_{A,Y} \circ F(f,Y)} \Sigma F(\Sigma A, Y)$$

are distinguished. In other words, the pairs $(- \wedge^L Y, \kappa_{-,Y})$, $(F(Y,-), \psi^{-1}_{Y,-})$ and $(F(-,Y), \bar{\psi}^{-1}_{-,Y})$ are exact functors of triangulated categories.

Proof. It suffices to consider the elementary distinguished triangle $(\gamma(j), \gamma(q), \delta(j))$ associated a monomorphism $j : A \longrightarrow B$ of symmetric spectra of simplicial sets.

For the first claim we can assume without loss of generality that Y is flat (since every object of $\mathcal{SHC}$ is isomorphic to a flat spectrum). Then the derived smash product with Y is represented by the pointset level smash product. Since Y is flat, the morphism $j \wedge Y : A \wedge Y \longrightarrow B \wedge Y$ is again a monomorphism; since smashing with Y preserves colimits, the morphism $q \wedge Y : B \wedge Y \longrightarrow (B/A) \wedge Y$ can serve as the associated quotient morphism. We claim that the composite morphism

$$(B/A) \wedge^L Y \xrightarrow{\delta(j) \wedge^L Y} (\Sigma A) \wedge^L Y \xrightarrow{\kappa_{A,Y}} \Sigma(Y \wedge^L A)$$

coincides with the connecting morphism $\delta(j \wedge Y) : (B/A) \wedge Y \longrightarrow \Sigma(A \wedge Y)$ in the stable homotopy category. Indeed, the diagram

$$\begin{tikzcd} & C(j) \wedge Y \arrow[r, "p \wedge Y"] \arrow[dl, "(0 \cup q) \wedge Y"', "\sim"] \arrow[dd, "\cong"] & (S^1 \wedge A) \wedge Y \arrow[dd, "\cong"] \\ (B/A) \wedge Y & & \\ & C(j \wedge Y) \arrow[r, "p \wedge Y"'] \arrow[ul, "0 \cup (q \wedge Y)", "\sim"'] & S^1 \wedge (A \wedge Y) \end{tikzcd}$$

commutes on in the category of symmetric spectra, where the isomorphism $C(j) \wedge Y \longrightarrow C(j \wedge Y)$ uses that smashing with Y commutes with pushouts. Applying the localization functor γ to this commutative diagram yields the relation $\kappa_{A,Y} \circ (\delta(j) \wedge^L Y) = \delta(j \wedge Y)$. This shows that the triangle $(\gamma(j) \wedge^L Y, \gamma(q) \wedge^L Y, \kappa_{A,Y} \circ (\delta(j) \wedge^L Y))$ is the elementary distinguished triangle of the monomorphism $j \wedge Y : A \wedge Y \longrightarrow B \wedge Y$, and hence distinguished.

Proposition 3.16 makes the second case a formal consequence of the first since the functor $F(Y,-)$ is right adjoint to $- \wedge^L Y$.

In the third case we argue as follows. We can replace Y by any isomorphic spectrum and thereby assume that Y is an injective Ω-spectrum. Since Y is injective, the morphism $\mathrm{map}(j,Y) : \mathrm{map}(B,Y) \longrightarrow \mathrm{map}(A,Y)$ is a Kan fibration by Proposition I.4.4 (i). Since $\mathrm{Hom}(A,Y)_n = \mathrm{map}(A, \mathrm{sh}^n Y)$ and all shifts of Y are again injective, the morphism $\mathrm{Hom}(j,Y) : \mathrm{Hom}(B,Y) \longrightarrow \mathrm{Hom}(A,Y)$ is levelweise a Kan fibration. Moreover, the spectrum $\mathrm{Hom}(B/A,Y)$ is isomorphic to the fiber of $\mathrm{Hom}(j,Y)$ with $\mathrm{Hom}(q,Y) : \mathrm{Hom}(B/A,Y) \longrightarrow \mathrm{Hom}(B,Y)$ corresponding to the inclusion of the fiber. Proposition 2.12 (ii) applies to $\mathrm{Hom}(j,Y)$ and shows that the triangle

$$\mathrm{Hom}(B/A,Y) \xrightarrow{\gamma(\mathrm{Hom}(q,Y))} \mathrm{Hom}(B,Y) \xrightarrow{\gamma(\mathrm{Hom}(j,Y))} \mathrm{Hom}(A,Y) \xrightarrow{-\gamma(l)^{-1}\gamma(i)} \Sigma\,\mathrm{Hom}(B/A,Y)$$

is distinguished. The morphism $S^1 \wedge \mathrm{Hom}(S^1 \wedge A, Y) \longrightarrow \mathrm{Hom}(A,Y)$ is a stable equivalence, and it represents the isomorphism $\bar{\psi}_{A,Y} : \Sigma\,\mathrm{Hom}(\Sigma A, Y) \longrightarrow \mathrm{Hom}(A,Y)$ in $\mathcal{SHC}$. So we can replace the spectrum $\mathrm{Hom}(A,Y)$ by $\Sigma\,\mathrm{Hom}(\Sigma A, Y)$ and deduce that the triangle

$$\mathrm{Hom}(B/A,Y) \xrightarrow{\gamma(\mathrm{Hom}(q,Y))} \mathrm{Hom}(B,Y) \xrightarrow{\gamma(\mathrm{Hom}(j,Y))} \Sigma\,\mathrm{Hom}(\Sigma A,Y) \xrightarrow{-\Sigma ?} \Sigma\,\mathrm{Hom}(B/A,Y)$$

is distinguished. Rotating to the left gives the desired distinguished triangle

$$F(\Sigma A,Y)\xrightarrow{F(\delta(j),Y)} F(B/A,Y)\xrightarrow{F(\gamma(q),Y)} F(B,Y)\xrightarrow{\bar\psi^{-1}_{A,Y}\circ F(\gamma(j),Y)} \Sigma F(\Sigma A,Y)\ . \qquad \square$$

The functor $Y\wedge^L -$ is isomorphic to the functor $-\wedge^L Y$, by the derived symmetric isomorphism. So we can also make smash product with Y in the first variable into an exact functor, where we use the natural isomorphism $\bar\kappa_{Y,A} : Y\wedge^L(\Sigma A)\longrightarrow \Sigma(Y\wedge^L A)$ defined as the composite

$$Y\wedge^L(\Sigma A)\xrightarrow{\bar\tau_{Y,\Sigma A}} (\Sigma A)\wedge^L Y\xrightarrow{\kappa_{A,Y}} \Sigma(A\wedge^L Y)\xrightarrow{\Sigma(\bar\tau_{A,Y})} \Sigma(Y\wedge^L A)\ . \tag{3.20}$$

Then for every distinguished triangle (f,g,h) the triangle

$$Y\wedge^L A\xrightarrow{Y\wedge^L f} Y\wedge^L B\xrightarrow{Y\wedge^L g} Y\wedge^L C\xrightarrow{\bar\kappa_{Y,A}\circ(Y\wedge^L h)} \Sigma(Y\wedge^L A)$$

is isomorphic to the triangle $(f\wedge^L Y, g\wedge^L Y, \kappa_{A,Y}\circ(h\wedge^L Y))$, and hence distinguished.

4. Grading

The spheres S^n represent the unstable homotopy groups and are related by the homeomorphisms $S^m\wedge S^n\cong S^{m+n}$ which are one-point compactification of the 'canonical' linear isomorphism

$$\mathbb{R}^m\times\mathbb{R}^n\longrightarrow\mathbb{R}^{m+n}\ ,\quad ((x_1,\dots,x_m),(y_1,\dots,y_n))\ \mapsto\ (x_1,\dots,x_m,y_1,\dots,y_n)\ .$$

These homeomorphisms are obviously associative and unital. In the stable homotopy category, all objects can be desuspended. So we have 'stable spheres' $\mathbb{S}^n$ for all integers n which represent stable homotopy groups, compare Example 1.15. However, the symmetric spectra $\mathbb{S}^n$ can *not* be chosen so that $\mathbb{S}^m\wedge\mathbb{S}^n$ and $\mathbb{S}^{m+n}$ are isomorphic as symmetric spectra for all $m,n\in\mathbb{Z}$. However, when we pass to the stable homotopy category we can consistently choose such isomorphisms. We want these isomorphisms in $\mathcal{SHC}$ to have suitable associativity properties, which is a bit less obvious than in the case of spaces.

In more detail we proceed as follows. For an integer n we define the *n-dimensional sphere spectrum* $\mathbb{S}^n$ by

$$\mathbb{S}^n\ =\ \begin{cases}\Sigma^\infty S^n & \text{for } n\ge 1,\\ \mathbb{S} & \text{for } n=0, \text{ and}\\ F_{-n} & \text{for } n\le -1.\end{cases} \tag{4.1}$$

[explain F_k without an argument] If m and n are both positive or both negative, then the smash product $\mathbb{S}^n\wedge\mathbb{S}^m$ is isomorphic, as a symmetric spectrum, to $\mathbb{S}^{n+m}$, but if n and m have opposite signs, the spectra $\mathbb{S}^n\wedge\mathbb{S}^m$ and $\mathbb{S}^{n+m}$ are not isomorphic in $\mathcal{S}p$. However, the next proposition shows that in the stable homotopy category, we can consistently choose such isomorphisms for all integers n and m. The next proposition in particular shows that the sphere spectra $\mathbb{S}^n$ are *invertible* for all integers n, i.e., they have inverses in $\mathcal{SHC}$ (up to isomorphism) for the derived smash product. We shall see in Proposition 7.10 that conversely, every invertible object of the stable homotopy category is isomorphic to $\mathbb{S}^n$ for an integer n.

Let us recall the definition of the fundamental classes $\iota_n\in\pi_n\mathbb{S}^n$ from (6.4) of Chapter I. For $n\ge 0$ the identity of the sphere S^n is a based map $S^n\longrightarrow(\Sigma^\infty S^n)_0$ whose homotopy class is the naive fundamental class $\hat\iota_n\in\hat\pi_n\mathbb{S}^n$ (we used the notation $\hat\iota_0^n$ earlier). For $n<0$ the identity permutation of Σ_{-n} specifies a point in $\Sigma^+_{-n}=(F_{-n})_{-n}$ whose homotopy class represents the naive fundamental class $\iota_n\in\hat\pi_n\mathbb{S}^n$. The true fundamental class is the image of this naive fundamental class $\hat\iota_n\in\hat\pi_n\mathbb{S}^n$ under the map $c:\hat\pi_n\mathbb{S}^n\longrightarrow\pi_n\mathbb{S}^n$. By Example 1.15 the evaluation map

$$\mathcal{SHC}(\mathbb{S}^n,X)\ \longrightarrow\ \pi_n X\ ,\quad \alpha\longmapsto(\pi_n\alpha)(\iota_n)$$

is an isomorphism of abelian groups for every symmetric spectrum X.

Since the spectrum $\mathbb{S}^{1+n}$ represents the homotopy group functor π_{1+n}, there is a unique morphism

$$\beta_n\ :\ \mathbb{S}^{1+n}\ \longrightarrow\ S^1\wedge\mathbb{S}^n \tag{4.2}$$

in the stable homotopy category such that $\pi_{1+n}(\beta_n) : \pi_{1+n}\mathbb{S}^{1+n} \longrightarrow \pi_{1+n}(S^1 \wedge \mathbb{S}^n)$ takes the fundamental class ι_{1+n} of $\mathbb{S}^{1+n}$ to $S^1 \wedge \iota_n$, the suspension of the previous fundamental class. For every symmetric spectrum X, the diagram

$$\begin{tikzcd}
\mathcal{SHC}(\mathbb{S}^n, X) \arrow[r, "\Sigma"] \arrow[d, "\cong"'] & \mathcal{SHC}(S^1 \wedge \mathbb{S}^n, S^1 \wedge X) \arrow[r, "\beta_n^*"] \arrow[dr, "\cong"'] & \mathcal{SHC}(\mathbb{S}^{1+n}, S^1 \wedge X) \arrow[d, "\cong"] \\
\pi_n X \arrow[rr, "S^1 \wedge -"'] & & \pi_{1+n}(S^1 \wedge X)
\end{tikzcd}$$

commutes; the vertical maps are evaluation at the fundamental class ι_n, at $S^1 \wedge \iota_n$ and at ι_{1+n} respectively. Since the suspension functor is fully faithful and evaluation at ι_n and ι_{1+n} and the suspension homomorphism are isomorphisms, the map $\beta_n^* : \mathcal{SHC}(S^1 \wedge \mathbb{S}^n, S^1 \wedge X) \longrightarrow \mathcal{SHC}(\mathbb{S}^{1+n}, S^1 \wedge X)$ is bijective. Since every object in the stable homotopy category is isomorphic to a suspension, the morphism β_n is thus an isomorphism.

In Theorem I.6.16 we constructed a natural pairing of true homotopy groups, which in the special case of sphere spectra is a biadditive map

$$\cdot \ : \ \pi_m\mathbb{S}^m \times \pi_n\mathbb{S}^n \ \longrightarrow \ \pi_{m+n}(\mathbb{S}^m \wedge \mathbb{S}^n) \ .$$

The sphere spectra are flat, so the pointset smash product which appears in the target is also the derived smash product. Since the spectrum $\mathbb{S}^{m+n}$ represents the homotopy group functor π_{m+n}, there is a unique morphism

$$\alpha_{m,n} \ : \ \mathbb{S}^{m+n} \ \longrightarrow \ \mathbb{S}^m \wedge^L \mathbb{S}^n \tag{4.3}$$

in the stable homotopy category such that $(\alpha_{m+n})_* : \pi_{m+n}\mathbb{S}^{m+n} \longrightarrow \pi_{m+n}(\mathbb{S}^m \wedge \mathbb{S}^n)$ takes the fundamental class of $\mathbb{S}^{m+n}$ to $\iota_m \cdot \iota_n$.

Proposition 4.4. *For all $m, n \in \mathbb{Z}$ the morphism $\alpha_{m,n} : \mathbb{S}^{m+n} \longrightarrow \mathbb{S}^m \wedge^L \mathbb{S}^n$ is an isomorphism in the stable homotopy category. Moreover, these isomorphisms satisfy the following properties:*

- (Normalization) *The morphisms $\alpha_{m,0}$ and $\alpha_{0,n}$ are identities;*
- (Associativity) *The square*

$$\begin{tikzcd}
\mathbb{S}^{l+m+n} \arrow[rr, "\alpha_{l,m+n}"] \arrow[d, "\alpha_{l+m,n}"'] & & \mathbb{S}^l \wedge^L \mathbb{S}^{m+n} \arrow[d, "\mathbb{S}^l \wedge^L \alpha_{m,n}"] \\
\mathbb{S}^{l+m} \wedge^L \mathbb{S}^n \arrow[r, "\alpha_{l,m} \wedge^L \mathbb{S}^n"'] & (\mathbb{S}^l \wedge^L \mathbb{S}^m) \wedge^L \mathbb{S}^n \arrow[r, "\bar{\alpha}_{\mathbb{S}^l, \mathbb{S}^m, \mathbb{S}^n}"'] & \mathbb{S}^l \wedge^L (\mathbb{S}^m \wedge^L \mathbb{S}^n)
\end{tikzcd}$$

  commutes in the stable homotopy category for all integers l, m and n.
- (Commutativity) *The square*

$$\begin{tikzcd}
\mathbb{S}^{m+n} \arrow[r, "(-1)^{mn}"] \arrow[d, "\alpha_{m,n}"'] & \mathbb{S}^{n+m} \arrow[d, "\alpha_{n,m}"] \\
\mathbb{S}^m \wedge^L \mathbb{S}^n \arrow[r, "\bar{\tau}_{\mathbb{S}^m, \mathbb{S}^n}"'] & \mathbb{S}^n \wedge^L \mathbb{S}^m
\end{tikzcd}$$

  commutes in the stable homotopy category for all integers m and n.

Proof. The sources of the associativity and commutativity squares are sphere spectra; since these represent homotopy groups, the diagrams commute if we can show that the fundamental classes have the same image both ways around the squares. But this amounts to the associativity respectively commutativity properties of the smash product pairing and the defining properties of the morphisms $\alpha_{m,n}$.

Now we argue that the morphism $\alpha_{m,n}$ are isomorphisms. Since $\mathbb{S}^1 = \Sigma^\infty S^1$ we have a natural isomorphism $\mathbb{S}^1 \wedge^L X = (\Sigma^\infty S^1) \wedge^L X \cong S^1 \wedge X$ [ref]. In the special case $X = \mathbb{S}^n$, the composite of this isomorphism with $\alpha_{1,n} : \mathbb{S}^{1+n} \longrightarrow \mathbb{S}^1 \wedge^L \mathbb{S}^n$ equals the morphism $\beta_n : \mathbb{S}^{1+n} \longrightarrow S^1 \wedge \mathbb{S}^n$ since both have the same effect on the fundamental class ι_{1+n}. So $\alpha_{1,n}$ is an isomorphism in $\mathcal{SHC}$ since the other two morphisms are. The commutativity property then implies that $\alpha_{m,1}$ is also an isomorphism. The associativity property for $m = 1$ then shows that $\alpha_{l+1,n}$ is an isomorphism if and only if $\alpha_{l,1+n}$ is. Since $\alpha_{0,n}$ is an isomorphism for all integers n, an induction on the absolute values of m shows that $\alpha_{m,n}$ is an isomorphism for all m and n. □

Remark 4.5. One cannot define associative stable equivalences $\mathbb{S}^m \wedge \mathbb{S}^n \longrightarrow \mathbb{S}^{m+n}$ in the category of symmetric spectra for arbitrary m and n. For example, $A(1, -1, 1)$ is a problem. In $\mathcal{SHC}$ we can add a sign to compensate for the twist permutation of S^2.

We now define graded homomorphism groups in the stable homotopy category by

$$[X, Y]_n \;=\; \mathcal{SHC}(\mathbb{S}^n \wedge X, Y)$$

for $n \in \mathbb{Z}$. We recall that all the sphere spectra $\mathbb{S}^n$ are flat, so the pointset level smash product with $\mathbb{S}^n$ is the derived smash product. Since the sphere spectrum $\mathbb{S}^0 = \mathbb{S}$ is a strict unit for the derived smash product we have $[X, Y]_0 = \mathcal{SHC}(X, Y)$, the homomorphism group from X to Y in the stable homotopy category. We can also define a graded composition

$$\begin{aligned} \circ \;:\; [Y, Z]_m \otimes [X, Y]_n \;&\longrightarrow\; [X, Z]_{m+n} \\ f \otimes g \;&\longmapsto\; f \circ (\mathbb{S}^m \wedge g) \circ (\alpha_{m,n} \wedge X) \end{aligned} \tag{4.6}$$

which is unital and associative by Proposition 4.4 and the associativity of composition (strictly speaking we also have to throw in the associativity isomorphism $\bar{\alpha}_{\mathbb{S}^m, \mathbb{S}^n, X} : (\mathbb{S}^m \wedge \mathbb{S}^n) \wedge X \cong \mathbb{S}^m \wedge (\mathbb{S}^n \wedge X)$, but the notation is already complicated enough). [compare $\mathbb{S}^1 \wedge^L X$ with $\Sigma X = S^1 \wedge X$] There is also a graded extension of the smash product pairing

$$\wedge \;:\; [X, Y]_n \otimes [X', Y']_{n'} \;\longrightarrow\; [X \wedge^L X',\, Y \wedge^L Y']_{n+n'} \tag{4.7}$$

that takes $g \otimes g'$ to the composite

$$\begin{aligned} \mathbb{S}^{n+n'} \wedge^L X \wedge^L X' \xrightarrow{\alpha_{n,n'} \wedge X \wedge X'} &\; \mathbb{S}^n \wedge^L \mathbb{S}^{n'} \wedge^L X \wedge X' \\ \xrightarrow{\mathbb{S}^n \wedge \bar{\tau}_{\mathbb{S}^{n'}, X} \wedge X'} &\; \mathbb{S}^n \wedge^L X \wedge^L \mathbb{S}^{n'} \wedge X' \xrightarrow{g \wedge^L g'} Y \wedge^L Y' \;. \end{aligned}$$

This pairing is also unital and associative by Proposition 4.4. It is also commutative in the sense of the relation

$$g' \wedge g \;=\; (-1)^{nn'} \cdot \tau_{Y, Y'} \circ (g \wedge g') \circ \tau_{X', X} \;.$$

The composition and smash product pairing commute in the following sense:

Proposition 4.8. *For morphisms $f \in [Y, Z]_m$, $f' \in [Y', Z']_{m'}$, $g \in [X, Y]_n$ and $g' \in [X', Y']_{n'}$ the relation*

$$(f \wedge f') \circ (g \wedge g') \;=\; (-1)^{m' \cdot n} \cdot (f \circ g) \wedge (f' \circ g')$$

holds in $[X \wedge^L X',\, Z \wedge^L Z']_{m+m'+n+n'}$.

Proof. We start by considering two special cases of the relations, namely

$$(g \wedge Z') \circ (X \wedge f') \;=\; g \wedge f' \;=\; (-1)^{m' \cdot n} \cdot (Y \wedge f') \circ (g \wedge Y') \;.$$

The second relation is obtained as:

$$\begin{aligned} (Y \wedge f') \circ (g \wedge Y') \;&=\; (Y \wedge f') \circ (\tau_{\mathbb{S}^{m'}, Y} \wedge Y') \circ (\mathbb{S}^{m'} \wedge g \wedge Y') \circ (\alpha_{m',n} \wedge X \wedge Y') \\ &=\; (Y \wedge f') \circ (g \wedge \mathbb{S}^{m'} \wedge Y') \circ (\tau_{\mathbb{S}^{m'}, \mathbb{S}^n \wedge X} \wedge Y') \circ (\alpha_{m',n} \wedge X \wedge Y') \\ &=\; (g \wedge f') \circ (\mathbb{S}^n \wedge \tau_{\mathbb{S}^{m'}, X} \wedge Y') \circ (\tau_{\mathbb{S}^{m'}, \mathbb{S}^n} \wedge X \wedge Y') \circ (\alpha_{m',n} \wedge X \wedge Y') \\ &=\; (-1)^{m' \cdot n} (g \wedge f') \circ (\mathbb{S}^n \wedge \tau_{\mathbb{S}^{m'}, X} \wedge Y') \circ (\alpha_{n,m'} \wedge X \wedge Y') \\ &=\; (-1)^{m' \cdot n} (g \wedge f') \;. \end{aligned}$$

The signs comes from the commutativity relation in Proposition 4.4. The proof of the first relation is very similar, but slightly easier because no spheres move past each other, so no sign occurs. The general case is then a combination of the special cases:

$$\begin{aligned}(f\wedge f')\circ(g\wedge g') &= (f\wedge Z')\circ(Y\wedge f')\circ(g\wedge Y')\circ(X\wedge g')\\ &= (-1)^{m'\cdot n}\cdot(f\wedge Z')\circ(g\wedge Z')\circ(X\wedge f')\circ(X\wedge g')\\ &= (-1)^{m'\cdot n}\cdot((f\circ g)\wedge Z')\circ(X\wedge(f'\circ g'))\\ &= (-1)^{m'\cdot n}\cdot(f\circ g)\wedge(f'\circ g')\end{aligned}$$

□

The graded homomorphism groups act on the graded homotopy groups through maps

$$\begin{aligned}[X,Y]_n\ \otimes\ \pi_m X &\longrightarrow \pi_{n+m}Y\\ g\otimes x &\longmapsto g_*(x)=(\pi_{n+m}g)(\iota_n\cdot x)\ .\end{aligned}$$

Here $\iota_n\in\pi_n\mathbb{S}^n$ is the fundamental class. This action is strictly associative and unital.

We can trade suspension in the source or target of $[X,Y]_n$ for a change in grading. More precisely, we define isomorphisms

$$\nu^*\ :\ [X,Y]_{n+1}=[\mathbb{S}^{n+1}\wedge X,Y]\ \xrightarrow{\cong}\ [\mathbb{S}^n\wedge\Sigma X,Y]=[\Sigma X,Y]_n \tag{4.9}$$

where $\nu:\mathbb{S}^n\wedge\Sigma X\longrightarrow\mathbb{S}^{n+1}\wedge X$ is the isomorphism that satisfies $\nu_*(\iota_n\cdot(S^1\wedge x))=\iota_{n+1}\cdot x$ for all homotopy classes x of X. Similarly, we define an isomorphism

$$[X,Y]_n=[\mathbb{S}^n\wedge X,Y]\ \xrightarrow{\Sigma}\ [\Sigma(\mathbb{S}^n\wedge X),\Sigma Y]\xrightarrow{\bar\nu^*}[\mathbb{S}^{1+n}\wedge X,\Sigma Y]=[X,\Sigma Y]_{1+n}$$

where $\bar\nu:\mathbb{S}^{1+n}\wedge X\longrightarrow\Sigma(\mathbb{S}^n\wedge X)$ is the isomorphism that satisfies $\bar\nu_*(\iota_{1+n}\cdot x)=S^1\wedge(\iota_n\cdot x)$ for all homotopy classes x of X. We then have the relations

$$(\nu^*g)_*(S^1\wedge x)\ =\ g_*(x)\qquad\text{and}\qquad(\bar\nu^*g)_*(x)\ =\ S^1\wedge g_*(x) \tag{4.10}$$

in π_*Y.

Proposition 4.11. *For morphisms $f\in[Y,Z]_m$, $f'\in[Y',Z']_{m'}$ and homotopy classes $y\in\pi_nY$ and $y'\in\pi_{n'}Y'$ the relation*

$$(f\wedge f')_*(y\cdot y')\ =\ (-1)^{m'\cdot n}\cdot f_*(y)\cdot f'_*(y') \tag{4.12}$$

holds in $\pi_{m+m'+n+n'}(Z\wedge^L Z')$.

Proof. The proof is straightforward:

$$\begin{aligned}(f\wedge f')_*(y\cdot y') &= \pi_{m+m'+n+n'}((f\wedge f')\circ(\mathbb{S}^m\wedge\tau_{S^{m'},Y}\wedge Y')\circ(\alpha_{m,m'}\wedge Y\wedge Y'))(\iota_{m+m'}\cdot y\cdot y')\\ &= \pi_{m+m'+n+n'}((f\wedge f')\circ(\mathbb{S}^m\wedge\tau_{S^{m'},Y}\wedge Y'))(\iota_m\cdot\iota_{m'}\cdot y\cdot y')\\ &= (-1)^{m'\cdot n}\cdot\pi_{m+m'+n+n'}(f\wedge f')(\iota_m\cdot y\cdot\iota_{m'}\cdot y')\\ &= (-1)^{m'\cdot n}\cdot(\pi_{m+n}f)(\iota_m\cdot y)\cdot(\pi_{m'+n'}f')(\iota_{m'}\cdot y')\\ &= (-1)^{m'\cdot n}\cdot f_*(y)\cdot f'_*(y')\end{aligned}$$

□

In the special case $X=\bar X=\mathbb{S}$, the graded smash product pairing (4.7) specializes to a pairing

$$\wedge\ :\ [\mathbb{S},Y]_n\ \otimes\ [\mathbb{S},Y']_{n'}\ \longrightarrow\ [\mathbb{S},Y\wedge^L Y']_{n+n'}$$

that takes $g\otimes g'$ to the composite

$$\mathbb{S}^{n+n'}\ \xrightarrow{\alpha_{n,n'}}\ \mathbb{S}^n\wedge^L\mathbb{S}^{n'}\ \xrightarrow{g\wedge^L g'}\ Y\wedge^L Y'\ .$$

We can turn this into a pairing

$$\cdot\ :\pi_nY\ \times\ \pi_{n'}Y'\ \longrightarrow\ \pi_{n+n'}(Y\wedge^L Y') \tag{4.13}$$

to the true homotopy groups of the derived smash product by composing with the evaluation isomorphism $[\mathbb{S}, Y]_n = \mathcal{SHC}(\mathbb{S}^n, Y) \cong \pi_n Y$ at the fundamental class, and similarly for $[\mathbb{S}, Y']_{n'}$ and $[\mathbb{S}, Y \wedge^L Y']_{n+n'}$. This pairing is the 'derived version' of the homotopy group pairing of Theorem I.6.16, i.e., for the derived smash product (as opposed to the actual smash product). More precisely, the triangle

$$\begin{tikzcd}
 & \pi_{n+n'}(Y \wedge^L Y') \arrow[d, "\pi_{n+n'}(\psi_{Y,Y'})"] \\
\pi_n Y \otimes \pi_{n'} Y' \arrow[ur, "\cdot"] \arrow[r, "\cdot"'] & \pi_{n+n'}(Y \wedge Y')
\end{tikzcd}$$

commutes for all integers m, n, where the lower product is as in Theorem I.6.16, the upper product is (4.13) and the vertical map is induced by the transformation $\psi_{X,Y} : X \wedge^L Y \longrightarrow X \wedge Y$ that comes with the derived smash product (compare Theorem 3.1). Indeed, after spelling out all definitions, this comes down to the relation

$$(\psi_{Y,Y'})_*((g \wedge^L g')_*(\iota_n \cdot \iota_{n'})) \;=\; g_*(\iota_n) \cdot g'_*(\iota_{n'})$$

for morphisms $g : \mathbb{S}^n \longrightarrow Y$ and $g' : \mathbb{S}^{n'} \longrightarrow Y'$ in $\mathcal{SHC}$. It suffices to show this when $g = \gamma(a)$ and $g' = \gamma(a')$ for morphisms of symmetric spectra $a : \mathbb{S}^n \longrightarrow Y$ and $a' : \mathbb{S}^{n'} \longrightarrow Y'$. In this case the desired relation follows from the fact that $\psi_{Y,Y'} \circ (\gamma(a) \wedge^L \gamma(a')) = \gamma(a \wedge a')$, by naturality of ψ and the fact that $\psi_{\mathbb{S}^n, \mathbb{S}^{n'}} : \mathbb{S}^n \wedge^L \mathbb{S}^{n'} \longrightarrow \mathbb{S}^n \wedge \mathbb{S}^{n'}$ is the identity because $\mathbb{S}^n$ and $\mathbb{S}^{n'}$ are flat.

[iso $\pi_m F(X,Y) \cong [X,Y]_m$] Also, for all integers m and n the following commutes:

$$\begin{tikzcd}
\pi_m F(Y,Z) \otimes \pi_n F(X,Y) \arrow[r, "\cdot"] \arrow[d, "\cong"'] & \pi_{m+n}\left(F(Y,Z) \wedge^L F(X,Y)\right) \arrow[r, "\pi_{m+n}(\circ)"] & \pi_{m+n} F(X,Z) \arrow[d, "\cong"] \\
{[Y,Z]_m \otimes [X,Y]_n} \arrow[rr, "\circ"'] & & {[X,Z]_{m+n}}
\end{tikzcd}$$

4.1. Homotopy ring spectra. Now that we constructed the derived smash product we can consider monoid objects in the stable homotopy category with respect to the derived smash product. For us a *homotopy ring spectrum* or *ring spectrum up to homotopy* is a symmetric spectrum E together with morphisms $\mu : E \wedge^L E \longrightarrow E$ and $\iota : \mathbb{S} \longrightarrow E$ in the stable homotopy category which are associative and unital in the sense that the following diagrams commute:

$$\begin{tikzcd}
(E \wedge^L E) \wedge^L E \arrow[r, "\bar{\alpha}_{E,E,E}"] \arrow[d, "\mu \wedge^L E"'] & E \wedge^L (E \wedge^L E) \arrow[r, "E \wedge^L \mu"] & E \wedge^L E \arrow[d, "\mu"] \\
E \wedge^L E \arrow[rr, "\mu"'] & & E
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S} \wedge^L E \arrow[r, "\iota \wedge^L E"] \arrow[dr, equal] & E \wedge^L E \arrow[d, "\mu"] & E \wedge^L \mathbb{S} \arrow[l, "E \wedge^L \iota"'] \arrow[dl, equal] \\
 & E &
\end{tikzcd}$$

A homotopy ring spectrum (E, μ, ι) is *homotopy commutative* if the multiplication is unchanged when composed with the derived symmetry isomorphism, i.e., if the relation $\mu \circ \bar{\tau}_{E,E} = \mu$ holds in the stable homotopy category.

A *(left) homotopy module* over a homotopy ring spectrum is a symmetric spectrum M together with a morphism $a : E \wedge^L M \longrightarrow M$ in the stable homotopy category which is associative and unital in the sense that the following diagrams commute:

$$\begin{tikzcd}
(E \wedge^L E) \wedge^L M \arrow[r, "\bar{\alpha}_{E,E,E}"] \arrow[d, "\mu \wedge^L M"'] & E \wedge^L (E \wedge^L M) \arrow[r, "E \wedge^L a"] & E \wedge^L M \arrow[d, "a"] \\
E \wedge^L M \arrow[rr, "a"'] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S} \wedge^L M \arrow[r, "\iota \wedge^L M"] \arrow[dr, equal] & E \wedge^L M \arrow[d, "a"] \\
 & M
\end{tikzcd}$$

The definition of the derived smash product was such that the universal functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ is lax symmetric monoidal (with respect to the universal transformation $\psi : \wedge^L \circ (\gamma \times \gamma) \longrightarrow \gamma \circ \wedge$). A formal consequence is that γ takes symmetric ring spectra to homotopy ring spectra. Indeed, if $(R, \mu : R \wedge R \longrightarrow R, \iota : \mathbb{S} \longrightarrow R)$ is a symmetric ring spectrum, then R becomes a ring spectrum up to homotopy with respect to the multiplication map

$$\gamma(R) \wedge^L \gamma(R) \xrightarrow{\psi_{R,R}} \gamma(R \wedge R) \xrightarrow{\gamma(\mu)} \gamma(R)$$

and the unit map $\gamma(\iota) : \mathbb{S} \longrightarrow R$.

Suppose that (E, μ, ι) is a homotopy ring spectrum. The derived smash product of morphisms can be used to make the graded abelian group $[\mathbb{S}, E]_*$ into a graded ring. The unit is given by the unit map $\iota : \mathbb{S} \longrightarrow E$ which is an element of $[\mathbb{S}, E]_0$, and the multiplication is the composite

$$[\mathbb{S}, E]_m \otimes [\mathbb{S}, E]_n \xrightarrow{\wedge} [\mathbb{S}, E \wedge^L E]_{m+n} \xrightarrow{\mu_*} [\mathbb{S}, E]_{m+n} .$$

If the homotopy ring spectrum comes from a symmetric ring spectrum R then evaluation at the fundamental classes is an isomorphism of graded rings $[\mathbb{S}, R]_* \cong \pi_* R$. If the homotopy ring spectrum E is commutative, then this multiplication is graded commutative.

Remark 4.14 (Obstructions to rigidifying a homotopy ring spectrum)**.** As we just explained, every symmetric ring spectrum gives rise to a homotopy ring spectrum, but the converse is far from being true. More precisely, given a ring spectrum up to homotopy E one can ask if there is a symmetric ring spectrum R such that $\gamma(R)$ is isomorphic to E as a homotopy ring spectrum. There is an infinite sequence of coherence obstructions for the associativity to get a positive answer, and we'll exhibit the first obstruction now. By replacing E by an isomorphic object in $\mathcal{SHC}$, if necessary, we may assume that the symmetric spectrum E is flat and a flat fibrant Ω-spectrum. The pointset level smash product $E \wedge E$ is then also flat, and so the map

$$[E \wedge E, E] \longrightarrow \mathcal{SHC}(E \wedge E, E)$$

induced by the localization functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ is bijective [ref]. Hence the multiplication map $\mu : E \wedge^L E \longrightarrow E$ in the stable homotopy category is of the form $\mu = \gamma(\bar{\mu})$ for a morphism $\bar{\mu} : E \wedge E \longrightarrow E$ of symmetric spectra which is unique up to homotopy.

As above, the map $[E \wedge E \wedge E, E] \longrightarrow \mathcal{SHC}(E \wedge E \wedge E, E)$ induced by the localization functor is also bijective. Since the multiplication μ is homotopy associative, the two morphisms of symmetric spectra

$$\bar{\mu} \circ (\bar{\mu} \wedge \mathrm{Id}) \, , \, \bar{\mu} \circ (\mathrm{Id} \wedge \bar{\mu}) \; : \; E \wedge E \wedge E \longrightarrow E$$

are thus homotopic. Let us choose a homotopy

$$H \; : \; I^+ \wedge E \wedge E \wedge E \longrightarrow E$$

from $\bar{\mu} \circ (\bar{\mu} \wedge \mathrm{Id})$ to $\bar{\mu} \circ (\mathrm{Id} \wedge \bar{\mu})$. If we consider product of four factors we arrive at a *pentagon condition* that we visualize as follows:

$$\begin{tikzcd}
 & & ((ab)c)d \arrow[dll, "\bar{\mu}(H \wedge \mathrm{Id})"'] \arrow[drr, "H(\bar{\mu} \wedge \mathrm{Id} \wedge \mathrm{Id})"] & & \\
(a(bc))d \arrow[dr, "H(\mathrm{Id} \wedge \bar{\mu} \wedge \mathrm{Id})"'] & & & & (ab)(cd) \arrow[dl, "H(\mathrm{Id} \wedge \mathrm{Id} \wedge \bar{\mu})"] \\
 & a((bc)d) \arrow[rr, "\bar{\mu}(\mathrm{Id} \wedge H)"'] & & a(b(cd)) &
\end{tikzcd}$$

Here each vertex of the pentagon represents a morphism of symmetric spectra from $E \wedge E \wedge E \wedge E$ (four factors) to E obtained by composing smash products of $\bar{\mu}$ and identity maps. The way we put parentheses should make clear which morphism is intended; for example, the leftmost corner $(a(bc))d$ represents the composite of first the morphism $\mathrm{Id} \wedge \bar{\mu} \wedge \mathrm{Id}$ corresponding to the inner pair of parenthesis, then $\bar{\mu} \wedge \mathrm{Id}$ corresponding to the other pair of parenthesis, and then $\bar{\mu}$. [associativity isomorphism for smash...]

In this symbolic notation the homotopy H goes from $(ab)c$ to $a(bc)$. Each of the five edges represents a morphism of symmetric spectra from $I^+ \wedge E \wedge E \wedge E \wedge E$ to E obtained by composing smash products of the homotopy H, the morphism $\bar{\mu}$ and identity maps.

So altogether the chosen spectrum morphisms $\bar{\mu}$ and H provide a morphism of symmetric spectra

$$\Omega \;:\; P^+ \wedge E \wedge E \wedge E \wedge E \;\longrightarrow\; E$$

where P is a simplicial pentagon, i.e., five copies of the simplicial 1-simplex cyclically glued together at their vertices. We claim that the homotopy class of this morphism Ω is the first obstruction to rigidifying the given homotopy ring spectrum into a symmetric ring spectrum. Indeed, if E is isomorphic, as a homotopy ring spectrum, to $\gamma(R)$ for a symmetric ring spectrum R, then by the model category arguments of [...] we can assume that R itself is flat and a flat fibrant Ω-spectrum. Then the multiplication of R can serve as the morphism $\bar{\mu}$. Since $\bar{\mu}$ is strictly associative, we can choose H as the constant homotopy. But then the obstruction morphism Ω is also constant on the pentagon P, so its homotopy class is trivial. (We should remember here that the construction of the morphism Ω involved some choices, and in fact the homotopy class of Ω is only well-defined up to some indeterminacy that we don't want to discuss here. The true obstruction thus lies in the factor group of $[P^+ \wedge E \wedge E \wedge E \wedge E, E]$ by a suitable indeterminacy subgroup.)

The pentagon condition is not the end of the story. The vanishing of the pentagon obstruction means that $\bar{\mu}$ and H can be chosen in such a way that the morphism Ω extends over the cone of the spectrum $P^+ \wedge E \wedge E \wedge E \wedge E$. A choice of extension determines the next in the sequence of obstructions, and so on. Pursuing this line of investigation systematically leads to the notion of an *A_∞-ring spectrum*, a concept whose space level analog is due to Stasheff.

The question of when a homotopy commutative homotopy ring spectrum is represented by a commutative symmetric ring spectrum is even more subtle. We hope to get back to this later, and discuss some of the obstruction theories available to tackle such 'rigidification' questions.

A specific example is the mod-p Moore spectrum $\mathbb{S}/p$ (see Section 6.3 of this chapter) for a prime p. The mod-2 Moore spectrum has no multiplication map in the stable homotopy category (compare Exercise E.II.29) and the mod-3 Moore spectrum has a product which however is *not* homotopy associative (compare Exercise E.II.30). For $p \geq 5$, the Moore spectrum $\mathbb{S}/p$ has a homotopy associative and homotopy commutative multiplication in the stable homotopy category, [ref] but there is no symmetric ring spectrum whose underlying spectrum is a mod-p Moore spectrum. We hope to get back to this.

Remark 4.15. If we specialize the derived composition pairing (3.12) to $X = Y = Z$, we see that the derived function spectrum $F(X,X)$ is a homotopy ring spectrum. However, these particular 'endomorphism' homotopy ring spectra always arise from symmetric ring spectra. Indeed, if we replace X by a stably equivalent symmetric spectrum, then $F(X,X)$ changes to an isomorphic homotopy ring spectrum. So we can assume that X is an injective Ω-spectrum. For such X we have $F(X,X) = \gamma(\mathrm{Hom}(X,X))$ as homotopy ring spectra, where $\mathrm{Hom}(X,X)$ is the symmetric endomorphism ring spectrum as defined in Example I.3.41.

5. Generators

Definition 5.1. Let $\mathcal{T}$ be a triangulated category which has infinite sums. An object G of a triangulated category $\mathcal{T}$ is called *compact* (sometimes called *finite* or *small*) if for every family $\{X^i\}_{i \in I}$ of objects the canonical map

$$\bigoplus_{i \in I} [G, X^i] \;\longrightarrow\; [G, \bigoplus_{i \in I} X^i]$$

is an isomorphism.

A set $\mathcal{G}$ of objects of $\mathcal{T}$ is called a *weak generating set* if the following condition holds: if X is an object such that the groups $[\Sigma^k G, X]$ are trivial for all $k \in \mathbb{Z}$ and all $G \in \mathcal{G}$, then X is a zero object. An individual object G is a *weak generator* if the set $\{G\}$ is a weak generating set.

So weak generating sets detect whether objects are trivial. But they also detect isomorphisms: let $\mathcal{G}$ be a weak generating set and $f : A \longrightarrow B$ a morphism such that the map $[\Sigma^k G, f] : [\Sigma^k G, A] \longrightarrow [\Sigma^k G, B]$

is bijective for all integers k and all $G \in \mathcal{G}$. We choose distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow \Sigma A ;$$

applying $[\Sigma^k G, -]$ for varying k results in a long exact sequence, so the groups $[\Sigma^k G, C]$ vanish for all integers k and all G in $\mathcal{G}$. Since $\mathcal{G}$ is a weak generating set, C is a zero object, and so $f : A \longrightarrow B$ is an isomorphism.

Proposition 5.2. *The sphere spectrum $\mathbb{S}$ is compact and a weak generator of the stable homotopy category.*

Proof. If X is a symmetric spectrum for which the graded abelian group $[\mathbb{S}, X]_*$ is trivial, then the true homotopy groups of X are trivial by Example 1.15. Thus X is stably equivalent to the trivial spectrum, hence a zero object in $\mathcal{SHC}$. This proves that the sphere spectrum is a weak generator of the stable homotopy category.

According to Proposition 1.10 (i), the coproduct in $\mathcal{SHC}$ of a family $\{X^i\}_{i \in I}$ of symmetric spectra is given by the wedge. We have a commutative square

$$\begin{tikzcd}
\bigoplus_{i\in I}[\mathbb{S}, X^i] \arrow[r] \arrow[d] & {[\mathbb{S}, \bigoplus_{i\in I} X^i]} \arrow[d] \\
\bigoplus_{i\in I}\pi_0(X^i) \arrow[r] & \pi_0\left(\bigvee_{i\in I} X^i\right)
\end{tikzcd}$$

in which the vertical maps are evaluation at the fundamental class, which are isomorphisms by the above. The lower horizontal map is an isomorphism by Proposition I.6.12 (i), hence so is the upper horizontal map, which shows that the sphere spectrum is compact. □

Now we want to show that the sphere spectrum also generates the stable homotopy category in another sense, namely that every triangulated subcategory of $\mathcal{SHC}$ that contains $\mathbb{S}$ and is closed under sums is already the entire stable homotopy category. This is really a special case of a general fact about triangulated categories, Proposition 5.17 below. For the proof we need the notion of *homotopy colimits* in triangulated categories.

Definition 5.3 (Homotopy colimit)**.** Let $\mathcal{T}$ be a triangulated category with infinite sums. We consider a countably infinite sequence

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \cdots$$

of morphisms in $\mathcal{T}$. A *homotopy colimit* of the sequence consists of an object $\bar{X}$ together with morphisms $\varphi_n : X_n \longrightarrow \bar{X}$ satisfying $\varphi_{n+1} f_n = \varphi_n$ such that there exists a distinguished triangle

$$\bigoplus_{n\geq 0} X_n \xrightarrow{1-f} \bigoplus_{n\geq 0} X_n \xrightarrow{\oplus \varphi_n} \bar{X} \longrightarrow \Sigma(\bigoplus_{n\geq 0} X_n) .$$

Here we denote by $1 - f : \bigoplus_{n\geq 0} X_n \longrightarrow \bigoplus_{n\geq 0} X_n$ the morphism whose restriction to the ith summand is the difference of the canonical morphism $X_i \longrightarrow \bigoplus_{n\geq 0} X_n$ and the composition of $f_i : X_i \longrightarrow X_{i+1}$ with the canonical morphism $X_{i+1} \longrightarrow \bigoplus_{n\geq 0} X_n$.

Triangulated categories typically do not have limits or colimits (except for sums and products), and the homotopy colimit just introduced is *not* a colimit in the triangulated category. In fact, the homotopy colimit is less functorial and canonical than a categorical colimit since it does not enjoy the universal property of a colimit. We will see in Exercise E.II.6 below that the homotopy colimit is unique up to isomorphism; however, in contrast to ordinary colimits there is usually no preferred isomorphism between two homotopy colimits.

We now see how to calculate maps from and to a homotopy colimit, in the latter case from compact objects.

Definition 5.4. Let $\mathcal{T}$ be a triangulated category. A covariant functor E from $\mathcal{T}$ to the category of abelian groups is called *homological* if it takes sums in $\mathcal{T}$ to sums of abelian groups and if for every distinguished triangle (f,g,h) in $\mathcal{T}$ the sequence of abelian groups

$$E(A) \xrightarrow{E(f)} E(B) \xrightarrow{E(g)} E(C) \xrightarrow{E(h)} E(\Sigma A)$$

is exact. A contravariant functor E from $\mathcal{T}$ to the category of abelian groups is called *cohomological* if it takes sums in $\mathcal{T}$ to products of abelian groups and if for every distinguished triangle (f,g,h) in $\mathcal{T}$ the sequence of abelian groups

$$E(\Sigma A) \xrightarrow{E(h)} E(C) \xrightarrow{E(g)} E(B) \xrightarrow{E(f)} E(A)$$

is exact.

Since distinguished triangles can be rotated, the 4-term exact sequence produced by a homological or cohomological functor can be extended indefinitely in both directions resulting in a long exact sequence.

Example 5.5. Examples of homological and cohomological functors are given by (certain) representable functors. Let Y be any object of a triangulated category $\mathcal{T}$. The covariant represented functor $\mathcal{T}(Y,-)$ takes distinguished triangles to exact sequences. If in addition Y is compact, then $\mathcal{T}(Y,-)$ takes sums to sums, hence it is homological. The functor π_k, the k-th homotopy group, is a homological functor from the triangulated stable homotopy category.

The covariant represented functor $\mathcal{T}(-,Y)$ takes distinguished triangles to exact sequences and sums to product, without further hypothesis on Y. So $\mathcal{T}(-,Y)$ is cohomological.

Lemma 5.6. *Let $\mathcal{T}$ be a triangulated category, $f_n : X_n \longrightarrow X_{n+1}$ a sequence of composable morphisms and $(\bar{X},\varphi_n)$ a homotopy colimit of the sequence $\{f_n\}$.*

(i) *For every homological functor $E : \mathcal{T} \longrightarrow \mathcal{A}b$ the natural map*

$$\mathrm{colim}_n\, E(X_n) \longrightarrow E(\bar{X})$$

induced from the compatible morphisms $E(\varphi_n) : E(X_n) \longrightarrow E(\bar{X})$ is an isomorphism. In particular, for every compact object Y of $\mathcal{T}$ the map

$$\mathrm{colim}_n\, [Y,X_n] \longrightarrow [Y,\bar{X}]$$

induced by $[Y,\varphi_n]$ is an isomorphism.

(ii) *For every cohomological functor $E : \mathcal{T}^{op} \longrightarrow \mathcal{A}b$ the short sequence of abelian groups*

$$0 \longrightarrow \lim^1_n E(\Sigma X_n) \longrightarrow E(\bar{X}) \longrightarrow \lim_n E(X_n) \longrightarrow 0 \tag{5.7}$$

is exact, where the right map arises from the system of compatible homomorphisms $E(\varphi_n) : E(\bar{X}) \longrightarrow E(X_n)$. In particular, for every object Y of $\mathcal{T}$ the short sequence of abelian groups

$$0 \longrightarrow \lim^1_n [\Sigma X_n, Y] \longrightarrow [\bar{X},Y] \longrightarrow \lim_n [X_n,Y] \longrightarrow 0 \tag{5.8}$$

is exact.

Proof. (i) By definition the homological functor E takes sums to sums. So applying E to the defining triangle of the homotopy colimit gives an exact sequence

$$\bigoplus_{n\geq 0} E(X_n) \xrightarrow{1-E(f)} \bigoplus_{n\geq 0} E(X_n) \xrightarrow{\bigoplus E(\varphi_n)} E(\bar{X}) \longrightarrow \bigoplus_{n\geq 0} E(\Sigma X_n) \xrightarrow{1-E(f)} \bigoplus_{n\geq 0} E(\Sigma X_n) \ .$$

The map $1-E(f)$ is always injective and its cokernel is a colimit of the sequence of maps $E(f_n) : E(X_n) \longrightarrow E(X_{n+1})$, which proves the claim.

(ii) We apply E to the defining triangle of the homotopy colimit and use that E takes sums to products. We get an exact sequence

$$\prod_n E(\Sigma X_n) \xrightarrow{1-E(f)} \prod_n E(\Sigma X_n) \longrightarrow E(\bar{X}) \xrightarrow{\prod E(\varphi_n)} \prod_n E(X_n) \xrightarrow{1-E(f)} \prod_n E(X_n) \ .$$

Kernel respectively cokernel of the selfmap $1 - E(f)$ of the product $\prod_n E(X_n)$ are the limit respectively derived limit of the sequence of maps $E(f_n) : E(X_{n+1}) \longrightarrow E(X_n)$, which proves that (5.8) is exact. □

Remark 5.9. The short exact sequence (5.8) is often called the *Milnor exact sequence*. The surjectivity of the second map in the Milnor sequence says that the data $(\bar{X}, \varphi_n)$ is a *weak colimit*, i.e., it has 'half' of the universal property of a categorical colimit: given morphisms $g_n : X_n \longrightarrow Y$ in the triangulated category $\mathcal{T}$ which are compatible in the sense that we have $g_{n+1} f_n = g_n$, then the tuple $\{g_n\}_n$ is an element in the limit of the groups $[X_n, Y]$. So by surjectivity there is a morphism $g : \bar{X} \longrightarrow Y$ restricting to g_n on each X_n. However, when the $\lim^1$ term is non-trivial, there is more than one such morphism g. So the $\lim^1$ term measures to what extent the homotopy colimit lacks the uniqueness part of the universal property. In [...] we give an example of a Milnor sequence with non-trivial $\lim^1$ term.

The k-th homotopy group is a homological functor from the triangulated stable homotopy category, so as a special case of part (i) of the previous lemma we get:

Corollary 5.10. *Let $f_n : X_n \longrightarrow X_{n+1}$ be a sequence of composable morphisms in the stable homotopy category and $(\bar{X}, \varphi_n)$ a homotopy colimit of the sequence $\{f_n\}$. Then the natural map*

$$\operatorname{colim}_n \pi_k(X_n) \longrightarrow \pi_k \bar{X}$$

induced from the morphisms $\pi_k(\varphi_n) : \pi_k(X_n) \longrightarrow \pi_k \bar{X}$ is an isomorphism.

We now relate the abstract notion of homotopy colimit in the triangulated stable homotopy category to sequential colimits of symmetric spectra: the following lemma says that a homotopy colimit in $\mathcal{SHC}$ can essentially be calculated as the colimit, in the category of symmetric spectra, over arbitrary choices of morphisms which represent the given homotopy classes.

Proposition 5.11. *Let $f_n : X_n \longrightarrow X_{n+1}$ be morphisms of symmetric spectra of simplicial sets for $n \geq 0$. Then every colimit, in the category of symmetric spectra, of the sequence of morphisms f_n is a homotopy colimit in the stable homotopy category of the sequence of morphisms $\gamma(f_n)$.*

Proof. Let

$$\bigoplus_{n \geq 0} X_n \xrightarrow{\operatorname{Id} - \gamma(f)} \bigoplus_{n \geq 0} X_n \xrightarrow{\oplus \varphi_n} \bar{X} \longrightarrow \Sigma \left(\bigoplus_{n \geq 0} X_n \right)$$

be a distinguished triangle in the stable homotopy category, so that $(\bar{X}, \varphi_n)$ is a homotopy colimit of the sequence $\{\gamma(f_n)\}$. We let $\kappa_n : X_n \longrightarrow \operatorname{colim}_m X_m$ denote the canonical morphisms to a colimit in the category of symmetric spectra. Then the map $\bigoplus_n \gamma(\kappa_n) : \bigoplus_n X_n \longrightarrow \operatorname{colim}_m X_m$ in $\mathcal{SHC}$ becomes zero when composed with $1 - f$, so there is a morphism $\psi : \bar{X} \longrightarrow \operatorname{colim}_m X_m$ in $\mathcal{SHC}$ such that $\psi \circ \varphi_n = \gamma(\kappa_n)$. For every integer k the triangle

$$\begin{tikzcd} & \pi_k(\operatorname{colim}_m X_m) \arrow[d, "\pi_k(\psi)"] \\ \operatorname{colim}_n \pi_k(X_n) \arrow[ur] \arrow[r] & \pi_k \bar{X} \end{tikzcd}$$

commutes where the diagonal maps are induced on π_k by the morphisms $\kappa_n : X_n \longrightarrow \operatorname{colim}_n X_n$ respectively $\varphi_n : X_n \longrightarrow \bar{X}$. The upper diagonal map is an isomorphism by part (iv) of Proposition I.6.12 and the lower diagonal map is an isomorphism by Corollary 5.10. So $\psi : \operatorname{colim}_m X_m \longrightarrow \bar{X}$ is an isomorphism in $\mathcal{SHC}$, and hence $\operatorname{colim}_m X_m$ is also a homotopy colimit. □

Now we show that in the stable homotopy category every symmetric spectrum is a homotopy colimit of desuspended suspension spectra. For a symmetric spectrum X and $m \geq 0$ we denote by $\lambda_m : F_{m+1}(X_m \wedge S^1) \longrightarrow F_m X_m$ the morphism that is freely generated by the map $1 \wedge - : X_m \wedge S^1 \longrightarrow \Sigma^+_{m+1} \wedge X_m \wedge S^1 =$

$(F_m X_m)_{m+1}$. Then λ_m is a stable equivalence, so its image in the stable homotopy category is invertible and we can form the composite

$$F_m X_m \xrightarrow{\gamma(\lambda_m)^{-1}} F_{m+1}(X_m \wedge S^1) \xrightarrow{\gamma(F_{m+1}\sigma_m)} F_{m+1} X_{m+1}$$

that we denote by $j_m : F_m X_m \longrightarrow F_{m+1} X_{m+1}$. For every $m \geq 0$, the identity of X_m is adjoint to a morphism of symmetric spectra

$$i_m \; : \; F_m X_m \longrightarrow X \; .$$

Since the square of morphisms of symmetric spectra

$$\begin{tikzcd} F_{m+1}(X_m \wedge S^1) \arrow[r, "F_{m+1}\sigma_m"] \arrow[d, "\lambda_m"', "\simeq"] & F_{m+1} X_{m+1} \arrow[d, "i_{m+1}"] \\ F_m X_m \arrow[r, "i_m"'] & X \end{tikzcd}$$

commutes, the relation $\gamma(i_{m+1}) \circ j_m = \gamma(i_m)$ holds as morphism from $F_m X_m$ to X in $\mathcal{SHC}$. For any choice of homotopy colimit of the sequence $j_m : F_m X_m \longrightarrow F_{m+1} X_{m+1}$, there is thus a morphism

$$j \; : \; \mathrm{hocolim}_m \; F_m X_m \longrightarrow X$$

in the stable homotopy category (not necessarily unique) such that $j \circ \varphi_m = \gamma(i_m)$.

Proposition 5.12. *For every semistable symmetric spectrum X the map*

$$j \; : \; \mathrm{hocolim}_m \; F_m X_m \longrightarrow X$$

is an isomorphism in the stable homotopy category.

Proof. We fix an integer k and a natural number m with $k + m \geq 0$. We define a map $\alpha_m : \pi_{k+m} X_m \longrightarrow \pi_k(F_m X_m)$ from the unstable homotopy group of the simplicial set X_m to the true homotopy groups as the composite

$$\pi_{k+m} X_m \xrightarrow{1\wedge -} \pi_{k+m}((F_m X_m)_m) \xrightarrow{\text{can.}} \hat{\pi}_k(F_m X_m) \xrightarrow{c} \pi_k(F_m X_m) \; .$$

In more detail: the first map is the effect on π_{k+m} of the based map $1 \wedge - : X_m \longrightarrow \Sigma^+_m \wedge X_m = (F_m X_m)_m$, the second map is the canonical map from an unstable homotopy group of a level to the naive homotopy group of a spectrum and the third is the tautological map from naive to true homotopy groups. We claim that the diagrams

$$(5.13) \qquad \begin{tikzcd} \pi_{k+m-1} X_{m-1} \arrow[r, "\iota"] \arrow[d, "\alpha_{m-1}"] & \pi_{k+m} X_m \arrow[r, "\text{can.}"] \arrow[d, "\alpha_m"] & \hat{\pi}_k X \arrow[d, "c"] \\ \pi_k(F_{m-1} X_{m-1}) \arrow[r, "\pi_k(j_{m-1})"'] & \pi_k(F_m X_m) \arrow[r, "\pi_k(i_m)"'] & \pi_k X \end{tikzcd}$$

commute. Indeed, the left square decomposes into subdiagrams

$$\begin{tikzcd}
\pi_{k+m-1}X_{m-1} \arrow[r, "-\wedge S^1"'] \arrow[d, "1\wedge -"] \arrow[rr, bend left, "\iota"] \arrow[dddd, bend right=60, "\alpha_{m-1}"'] & \pi_{k+m}(X_{m-1}\wedge S^1) \arrow[r, "\sigma_{m-1}"'] \arrow[dd, "1\wedge -"] & \pi_{k+m}X_m \arrow[dd, "1\wedge -"'] \arrow[dddd, bend left=60, "\alpha_m"] \\
\pi_{k+m-1}(F_{m-1}X_{m-1})_{m-1} \arrow[d, "\iota"] & & \\
\pi_{k+m}(F_{m-1}X_{m-1})_m \arrow[d, "\mathrm{can.}"] & \pi_{k+m}(F_m(X_{m-1}\wedge S^1))_m \arrow[l, "(\lambda_{m-1})_m"] \arrow[r, "F_m\sigma_{m-1}"'] \arrow[d, "\mathrm{can.}"'] & \pi_{k+m}(F_mX_m)_m \arrow[d, "\mathrm{can.}"'] \\
\hat{\pi}_k(F_{m-1}X_{m-1}) \arrow[d, "c"] & \hat{\pi}_k(F_m(X_{m-1}\wedge S^1)) \arrow[l, "\lambda_{m-1}"] \arrow[r, "F_m\sigma_{m-1}"'] \arrow[d, "c"] & \hat{\pi}_k(F_mX_m) \arrow[d, "c"'] \\
\pi_k(F_{m-1}X_{m-1}) \arrow[rr, bend right, "j_{m-1}"'] & \pi_k(F_m(X_{m-1}\wedge S^1)) \arrow[l, "\lambda_{m-1}"', "\cong"] \arrow[r, "F_m\sigma_{m-1}"] & \pi_k(F_mX_m)
\end{tikzcd}$$

Here the upper left square commutes by the definition of the morphism λ_{m-1}, and all other parts commute by naturality. The right square of (5.13) can also be decomposed into subdiagrams:

$$\begin{tikzcd}
\pi_{k+m}X_m \arrow[d, "1\wedge -"'] \arrow[dr, equal] \arrow[ddd, bend right=60, "\alpha_m"'] & \\
\pi_{k+m}(F_mX_m)_m \arrow[r, "(i_m)_m"'] \arrow[d, "\mathrm{can.}"] & \pi_{k+m}X_m \arrow[d, "\mathrm{can.}"] \\
\hat{\pi}_k(F_mX_m) \arrow[r, "i_m"'] \arrow[d, "c"] & \hat{\pi}_kX \arrow[d, "c"] \\
\pi_k(F_mX_m) \arrow[r, "i_m"'] & \pi_kX
\end{tikzcd}$$

Here again, the squares commute by naturality and the triangle by definition of the morphism i_m.

Since the left part of diagram (5.13) commutes, the maps α_m assemble into a homomorphism

$$\alpha_\infty = \mathrm{colim}_m\, \alpha_m \ : \ \hat{\pi}_kX = \mathrm{colim}_m\, \pi_{k+m}X_m \ \longrightarrow \ \mathrm{colim}_m\, \pi_k(F_mX_m) \ .$$

We claim that α_∞ is an isomorphism [...$\hat{\pi}_kX \cong \mathrm{colim}_m\, \pi^{\mathrm{s}}_{k+m}X_m$ and $\pi^{\mathrm{s}}_{k+m}X_m \cong \pi_k(F_mX_m)$...]

Since the right part of diagram (5.13) commutes, the composite

$$\hat{\pi}_kX \ \xrightarrow{\alpha_\infty} \ \mathrm{colim}_m\, \pi_k(F_mX_m) \ \xrightarrow{\cong} \ \pi_k(\mathrm{hocolim}_m\ F_mX_m) \ \xrightarrow{\pi_k(j)} \ \pi_kX$$

coincides with the tautological map $c : \hat{\pi}_kX \longrightarrow \pi_kX$. We have assumed that X is semistable, so c is an isomorphism for all integers k. Since α_∞ is an isomorphism by the above, and the second map is an isomorphism by Corollary 5.10, the map $\pi_k(j)$ is an isomorphism for all integers k. So the morphism j is an isomorphism in the stable homotopy category, and this finishes the proof. $\square$

Now we return to some more general theory of triangulated categories.

Proposition 5.14. *Let $\mathcal{T}$ be a triangulated category with infinite sums and let $\mathcal{C}$ be a set of compact objects of $\mathcal{T}$. Let $\langle\mathcal{C}\rangle_+$ denote the smallest class of objects of $\mathcal{T}$ which contains $\mathcal{C}$ and is closed under sums (possibly infinite) and 'extensions to the right' in the following sense: if*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

is a distinguished triangle such that A and B belong to the class, then so does C. Then for every cohomological functor $E : \mathcal{T}^{op} \longrightarrow \mathcal{A}b$ there exists an object R in the class $\langle\mathcal{C}\rangle_+$ and an element $u \in E(R)$ such *that for every object G of $\mathcal{C}$ the evaluation map*

$$\mathrm{ev}_u \ : \ [G,R] \ \longrightarrow \ E(G) \ , \quad f \mapsto E(f)(u)$$

is bijective.

Proof. By induction on n we construct objects R_n in $\langle\mathcal{C}\rangle_+$, morphisms $i_n : R_n \longrightarrow R_{n+1}$ and elements $u_n \in E(R_n)$ such that $E(i_n)(u_{n+1}) = u_n$. We start with

$$R_0 \ = \ \bigoplus_{G\in\mathcal{C}} \bigoplus_{x\in E(G)} G \ .$$

Since E is cohomological the canonical map

$$E(R_0) \ \longrightarrow \ \prod_{G\in\mathcal{C}} \prod_{x\in E(G)} E(G)$$

is bijective; so there is a tautological element $u_0 \in E(R_0)$ that restricts to $x \in E(G)$ on the summand indexed by x. We note that R_0 belongs to $\langle\mathcal{C}\rangle_+$ and $\mathrm{ev}_{u_0} : [G,R_0] \longrightarrow E(G)$ is surjective for all $G \in \mathcal{C}$.

In the inductive step we suppose that the pair (R_n,u_n) has already been constructed. We let $I_n(G)$ denote the kernel of the evaluation morphism $\mathrm{ev}_{u_n} : [G,R_n] \longrightarrow E(G)$ and consider

$$C_n \ = \ \bigoplus_{G\in\mathcal{C}} \bigoplus_{x\in I_n(G)} G \ ,$$

which comes with a tautological morphism $\tau : C_n \longrightarrow R_n$ which is given by x on the summand indexed by x. We choose a distinguished triangle

$$C_n \xrightarrow{\ \tau\ } R_n \xrightarrow{\ i_n\ } R_{n+1} \longrightarrow \Sigma C_n \ .$$

Since E is cohomological the sequence

$$E(\Sigma C_n) \ \longrightarrow \ E(R_{n+1}) \ \xrightarrow{E(i_n)} E(R_n) \ \xrightarrow{E(\tau)} E(C_n)$$

is exact. Under the isomorphism $E(C_n) \cong \prod_{G\in\mathcal{C}}\prod_{x\in I_n(G)} E(G)$ the map $E(\tau)$ takes $u_n \in E(R_n)$ to the family $\{\mathrm{ev}_{u_n}(x)\}$ which is zero by definition. So there exists an element $u_{n+1} \in E(R_{n+1})$ satisfying $E(i_n)(u_{n+1}) = u_n$.

Now we choose a homotopy colimit $(R, \{\varphi_n : R_n \longrightarrow R\}_n)$, in the sense of Definition 5.3, of the sequence of morphisms $i_n : R_n \longrightarrow R_{n+1}$. Since all the objects R_n are in $\langle\mathcal{C}\rangle_+$, so is R. The short exact sequence (5.7) shows that we can choose an element $u \in E(R)$ satisfying $E(\varphi_n)(u) = u_n$ in $E(R_n)$ for all $n \geq 0$. We claim that for any such pair (R,u) the evaluation map $\mathrm{ev}_u : [G,R] \longrightarrow E(G)$ is bijective.

Since $E(\varphi_0)(u) = u_0$ in $E(R_0)$, the composite $[G,R_0] \longrightarrow [G,R] \xrightarrow{\mathrm{ev}_u} E(G)$ is evaluation at u_0, which is surjective. Hence $\mathrm{ev}_u : [G,R] \longrightarrow R(G)$ is also surjective. To show that ev_u is injective we let $\alpha : G \longrightarrow R$ be an element in its kernel, i.e., such that $E(\alpha)(u) = 0$. Since G is compact, the functor $[G,-]$ is homological; we apply Lemma 5.6 (i) and conclude that there is an $n \geq 0$ and a morphism $\alpha' : G \longrightarrow R_n$ such that $\alpha = \varphi_n\alpha'$. Then $E(\alpha')(u_n) = E(\alpha')(E(\varphi_n)(u)) = E(\alpha)(u) = 0$. So α' lies in $I_n(G)$ and indexes one of the summands of C_n. So α' factors through the tautological morphism $\tau : C_n \longrightarrow R_n$ as $\alpha' = \tau\alpha''$, and hence

$$\alpha \ = \ \varphi_n\alpha' \ = \ \varphi_{n+1}i_n\tau\alpha'' \ = \ 0$$

since the morphisms i_n and τ are adjacent in a distinguished triangle, and so have trivial composite. Hence $\mathrm{ev}_u : [G,R] \longrightarrow E(G)$ is also injective, which finishes the proof. □

Now we show that for compact objects, the two meanings of 'generator' for a triangulated category coincide. We need another definition first.

Definition 5.15. A non-empty full subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is a *triangulated subcategory* if the following conditions holds: given a distinguished triangle

$$A \ \longrightarrow \ B \ \longrightarrow \ C \ \longrightarrow \ \Sigma A$$

in $\mathcal{T}$ such that two of the objects A, B and C belong to $\mathcal{S}$, then so does the third object. A full triangulated subcategory of a triangulated category is *localizing* if it is also closed under sums (indexed by arbitrary sets).

[triangulated subcategory contains all zero objects, and is closed under suspension, desuspension and isomorphism]

[next paragraphs out of place] For a set of not necessarily compact objects the two conditions are not generally equivalent; for arbitrary objects, condition (ii) in the next proposition implies condition (i), but not necessarily the other way around. In the special case of the stable homotopy category and the set $\mathcal{G} = \{\mathbb{S}\}$, the next proposition specializes to Proposition 5.16.

As we just saw, the sphere spectrum $\mathbb{S}$ is a compact weak generator of the stable homotopy category. But it also generates the stable homotopy category in the sense that the whole stable homotopy category is the smallest localizing subcategory containing $\mathbb{S}$.

Proposition 5.16. *Every localizing subcategory of the stable homotopy category which contains the sphere spectrum $\mathbb{S}$ is all of $\mathcal{SHC}$.*

Proposition 5.17. *Let $\mathcal{T}$ be a triangulated category with infinite sums and let $\mathcal{G}$ be a set of compact objects of $\mathcal{T}$. Then the following conditions are equivalent.*

(i) *$\mathcal{G}$ is a weak generating set.*
(ii) *Every localizing subcategory of $\mathcal{T}$ which contains $\mathcal{G}$ is all of $\mathcal{T}$.*

PROOF. Let us first assume that $\mathcal{G}$ is a weak generating set and let $\mathcal{X}$ be a localizing subcategory of $\mathcal{T}$ which contains $\mathcal{G}$. We let X be any object of $\mathcal{T}$.

We apply Proposition 5.14 to the set $\mathcal{C} = \{\Sigma^k G\}_{k\in\mathbb{Z},G\in\mathcal{G}}$ of all positive and negative suspensions of objects in $\mathcal{G}$. Since a localizing subcategory is closed under suspensions and desuspensions, the set $\mathcal{C}$ is contained in $\mathcal{X}$, and hence so is the class $\langle\mathcal{C}\rangle_+$. Proposition 5.14 applied to the representable functor $[-,X]$ provides a morphism $u : R \longrightarrow X$ such that $[\Sigma^k G, u] : [\Sigma^k G, R] \longrightarrow [\Sigma^k G, X]$ is bijective for all $k \in \mathbb{Z}$ and $G \in \mathcal{G}$. Since $\mathcal{G}$ weakly generates, u must be an isomorphism. Since R is contained in $\langle\mathcal{C}\rangle_+ \subset \mathcal{X}$, so is X. So the localizing subcategory $\mathcal{X}$ contains all objects of $\mathcal{T}$.

Now we assume condition (ii) and show that $\mathcal{G}$ is then a weak generating set. This implication does not need the assumption that the object in $\mathcal{G}$ are compact. We let X be an object of $\mathcal{T}$ such that the graded abelian group $[G,X]_*$ is trivial for every $G \in \mathcal{G}$. We let $\mathcal{X}$ be the class of all those objects A of $\mathcal{T}$ for which the graded abelian group $[A,X]_*$ is trivial. Then $\mathcal{X}$ is a localizing subcategory of $\mathcal{T}$ and contains $\mathcal{G}$, hence it contains all objects, in particular the object X itself. Thus the group $[X,X]$ is trivial, so X must be a zero object. □

Definition 5.18. A triangulated category is *compactly generated* if it has sums and a weak generating set consisting of compact objects.

By Proposition 5.17 we could replace the condition 'weak generators' by 'generators' for the triangulated category (as long as we insist of compact objects). The stable homotopy category is our main example of a compactly generated triangulated category, where we can take the sphere spectrum $\mathbb{S}$ as a single compact generator. More generally we will show in Chapter IV that the triangulated derived category of a symmetric ring spectrum is compactly generated, where the free module of rank one can be taken as a single compact generator.

Proposition 5.19 (Brown representability)**.** *Every cohomological functor defined on a compactly generated triangulated category is representable.*

PROOF. Let $\mathcal{T}$ be compactly generated and let $\mathcal{G}$ be a set of compact generators. Given a cohomological functor $E : \mathcal{T}^{op} \longrightarrow \mathcal{A}b$ we apply Proposition 5.14 with $\mathcal{C} = \{\Sigma^k G\}_{k\in\mathbb{Z},G\in\mathcal{G}}$ the set of all positive and negative suspensions of all objects in $\mathcal{G}$. We obtain an object R of $\mathcal{T}$ and an element $u \in E(R)$ such that $\mathrm{ev}_u : [\Sigma^k G, R] \longrightarrow E(\Sigma^k G)$ is bijective for all integers k and $G \in \mathcal{G}$.

We let $\mathcal{X}$ be the class of all objects X of $\mathcal{T}$ for which the evaluation morphism $\mathrm{ev}_u : [X,R] \longrightarrow E(X)$ is bijective. Since $[-,R]$ and E are both cohomological functors, the class $\mathcal{X}$ is localizing. By the above, it also contains the set of generators. Proposition 5.17 shows that $\mathcal{X} = \mathcal{T}$, so the pair (R,u) represents the functor E. □

Example 5.20. The representability result given by Proposition 5.19 can sometimes be used to construct spectra with prescribed homotopy groups. One example of this is the *Brown-Comenetz dual* (of the sphere spectrum). The construction uses the contravariant endofunctor of abelian groups

$$A \longmapsto A^\vee = \mathcal{A}b(A, \mathbb{Q}/\mathbb{Z})$$

which is sometimes called the *Pontryagin dual* of A. There is a natural evaluation homomorphism

$$A \longrightarrow (A^\vee)^\vee , \quad a \longmapsto (\varphi \mapsto \varphi(a))$$

which injects A into its double dual. If A is finite, then the evaluation is an isomorphism; in that case A is also isomorphic to its (single) dual $A^\vee$, but this isomorphism is not natural. The group $\mathbb{Q}/\mathbb{Z}$ is injective as an abelian group, i.e., the Pontryagin duality functor $\mathcal{A}b(-,\mathbb{Q}/\mathbb{Z})$ is exact.

We consider the contravariant functor

$$E : \mathcal{SHC}^{\mathrm{op}} \longrightarrow \mathcal{A}b , \quad X \longmapsto E(X) = (\pi_0 X)^\vee .$$

For every family $\{X^i\}_{i\in I}$ of symmetric spectra the natural map

$$\bigoplus_{i\in I} \pi_n(X^i) \longrightarrow \pi_n\left(\bigvee_{i\in I} X^i\right)$$

is an isomorphism by Proposition I.6.12 (i), and Pontryagin duality takes sums to products, so the functor E takes sums to products. Every distinguished triangle (f,g,h) in the stable homotopy category gives rise to a long exact sequence of homotopy groups

$$\pi_0 A \xrightarrow{\pi_0 f} \pi_0 B \xrightarrow{\pi_0 g} \pi_0 C \xrightarrow{\pi_0 h} \pi_0(\Sigma A) .$$

[ref] Since Pontryagin duality is exact, we get an exact sequence

$$(\pi_0 A)^\vee \xleftarrow{(\pi_0 f)^\vee} (\pi_0 B)^\vee \xleftarrow{(\pi_0 g)^\vee} (\pi_0 C)^\vee \xleftarrow{(\pi_0 h)^\vee} (\pi_0(\Sigma A))^\vee .$$

So E is a cohomological functor. The Brown-Comenetz dual $I\mathbb{S}$ is a representing spectrum for this cohomological functor; it comes with a universal element $u \in (\pi_0 I\mathbb{S})^\vee$, i.e., a homomorphism $u : \pi_0 I\mathbb{S} \longrightarrow \mathbb{Q}/\mathbb{Z}$. We can calculate the homotopy groups of $I\mathbb{S}$ as follows. For any integer k we compose the action of the stable homotopy groups of spheres with the universal homomorphism to a homomorphism of abelian groups

$$\pi_k I\mathbb{S} \otimes \pi_{-k}\mathbb{S} \xrightarrow{\cdot} \pi_0 I\mathbb{S} \xrightarrow{u} \mathbb{Q}/\mathbb{Z} .$$

This map is a perfect pairing in the sense that its adjoint

$$\hat{u} : \pi_k I\mathbb{S} \longrightarrow \mathcal{A}b(\pi_{-k}\mathbb{S}, \mathbb{Q}/\mathbb{Z}) = (\pi_{-k}\mathbb{S})^\vee$$

is an isomorphism. Indeed, we have a commutative square

$$\begin{tikzcd}
\mathcal{SHC}(\mathbb{S}^k, I\mathbb{S}) \arrow[r, "\mathrm{ev}_u"] \arrow[d, "\mathrm{ev}_{\iota_k}"'] & (\pi_0 \mathbb{S}^k)^\vee \arrow[d, "(S^k\wedge -)^\vee"] \\
\pi_k I\mathbb{S} \arrow[r, "\hat{u}"'] & (\pi_{-k}\mathbb{S})^\vee
\end{tikzcd}$$

in which the right vertical morphism is dual to the suspension isomorphism $S^k \wedge - : \pi_{-k}\mathbb{S} \longrightarrow \pi_0 \mathbb{S}^k$. The other three maps in the square are isomorphisms, hence so is $\hat{u}$. Since the stable homotopy groups of spheres are finite except in dimension zero (compare Theorem I.1.9), and thus self-dual (in a non-canonical way), one could say that the homotopy groups of the Brown-Comenetz dual $I\mathbb{S}$ are the stable homotopy groups of spheres 'turned upside down'. There are non-trivial stable homotopy groups of spheres in arbitrarily high dimensions, so the spectrum $I\mathbb{S}$ has non-trivial homotopy groups in arbitrarily low dimensions.

A symmetric spectrum X is *n-connected* if the true homotopy groups $\pi_k X$ are trivial for $k \leq n$. The spectrum X is *connective* if it is (-1)-connected, i.e., its true homotopy groups vanish in negative dimensions.

Proposition 5.21. *For an integer n, let $\langle \mathbb{S}^n \rangle_+$ denote the smallest class of symmetric spectra which contains the n-dimensional sphere spectrum $\mathbb{S}^n$ and is closed under sums (possibly infinite) and 'extensions to the right' in the following sense: if*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

is a distinguished triangle such that A and B belong to the class, then so does C. Then $\langle \mathbb{S}^n \rangle_+$ equals the class of $(n-1)$-connected spectra.

Proof. Since the sphere spectrum $\mathbb{S}$ is connective and suspensions shifts homotopy groups, the sphere spectrum $\mathbb{S}^n$ is $(n-1)$-connected. The class of $(n-1)$-connected spectra is closed under sums (since homotopy groups commute with sums) and extensions to the right (by the long exact sequence of homotopy groups associated to a distinguished triangle), so every spectrum in the class $\langle \mathbb{S}^n \rangle_+$ is $(n-1)$-connected.

For the converse we let X be any $(n-1)$-connected symmetric spectrum. We let $\mathcal{C} = \{\mathbb{S}^k\}_{k \geq n}$ be the set of sphere spectra of dimension at least n. Every class closed under extensions to the right is in particular closed under suspensions, so we in fact have $\langle \mathcal{C} \rangle_+ = \langle \mathbb{S}^n \rangle_+$. We apply Proposition 5.14 to the representable functor $[-, X]$. We obtain a symmetric spectrum R belonging to the class $\langle \mathbb{S}^n \rangle_+$ and a morphism $u : R \longrightarrow X$ such that $[\mathbb{S}^k, u] : [\mathbb{S}^k, R] \longrightarrow [\mathbb{S}^k, X]$ is bijective for all $k \geq n$. Since $\mathbb{S}^k$ represents the k-th homotopy group this means that u induces isomorphisms of homotopy groups in dimensions n and above. Since R (by the previous paragraph) and X are $(n-1)$-connected, the morphism u also induces isomorphisms of homotopy groups below dimension n, so u is an isomorphism in the stable homotopy category. Thus X belongs to $\langle \mathcal{C} \rangle_+$, which finishes the proof. □

In the special case $n = -1$ Proposition 5.21 says that a symmetric spectrum is connective if and only it belongs to $\langle \mathbb{S} \rangle_+$, the smallest class of objects of the stable homotopy category which contains the sphere spectrum $\mathbb{S}$ and is closed under sums and extensions to the right. Exercise E.II.10 is devoted to showing that a symmetric spectrum is connective if and only if it is stably equivalent to a symmetric spectrum of the form $A(\mathbb{S})$ for a $\mathbf{\Gamma}$-space A.

The previous characterization of connective (i.e., (-1)-connected) spectra as being generated by the sphere spectrum $\mathbb{S}$ under sums and extensions to the right can be useful for reducing claims about connective spectra to the special case of the sphere spectrum. The following result is an example where we use this strategy in the proof.

Proposition 5.22. *Let X and Y be symmetric spectra such that X is $(k-1)$-connected and Y is $(l-1)$-connected. Then the derived smash product $X \wedge^L Y$ is $(k+l-1)$-connected and the pairing* (4.13)

$$\cdot \;:\; \pi_k X \;\otimes\; \pi_l Y \;\longrightarrow\; \pi_{k+l}(X \wedge^L Y)$$

is an isomorphism of abelian groups.

Proof. We fix an $(l-1)$-connected spectrum Y and let $\mathcal{X}$ be the class of all $(k-1)$-connected spectra X for which the theorem is true.

The class $\mathcal{X}$ contains the sphere spectrum $\mathbb{S}^k$: for $i < k$ the group $\pi_{i+l}(\mathbb{S}^k \wedge^L Y)$ is isomorphic to $\pi_i Y$, hence trivial for $i < k$, so $\mathbb{S}^k \wedge^L Y$ is $(k+l-1)$-connected. Moreover the composite map

$$\pi_k \mathbb{S}^k \;\otimes\; \pi_l Y \;\xrightarrow{\quad\cdot\quad}\; \pi_{k+l}(\mathbb{S}^k \wedge^L Y) \;\underset{\cong}{\xrightarrow{(S^k \wedge -)^{-1}}}\; \pi_l Y$$

sends $\iota_k \otimes y$ to y. Since $\pi_k \mathbb{S}^k$ is freely generated, as an abelian group, by the fundamental class ι_k, this composite is an isomorphism. Hence the product map is an isomorphism for $X = \mathbb{S}^k$.

The class $\mathcal{X}$ is also closed under sums since both sides of the map commute with sums in X. Finally, $\mathcal{X}$ is closed under extensions to the right. Indeed, suppose that

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

is a distinguished triangle such that A and B belong to $\mathcal{X}$. Then the triangle

$$A \wedge^L Y \xrightarrow{f\wedge^L \mathrm{Id}} B \wedge^L Y \xrightarrow{g\wedge^L \mathrm{Id}} C \wedge^L Y \xrightarrow{\kappa_{A,Y}\circ(h\wedge^L \mathrm{Id})} \Sigma(A \wedge^L Y)$$

is distinguished by Proposition 3.19. The associated long exact sequence of homotopy groups shows that $C \wedge^L Y$ is $(k+l-1)$-connected because both $B \wedge^L Y$ and $\Sigma(A \wedge^L Y)$ are. In the critical dimension we get a commutative diagram

$$\begin{tikzcd}
\pi_k A \otimes \pi_l Y \arrow[r, "\pi_k f\otimes \mathrm{Id}"] \arrow[d, "\cdot"'] & \pi_k B \otimes \pi_l Y \arrow[r, "\pi_k g\otimes \mathrm{Id}"] \arrow[d, "\cdot"'] & \pi_k C \otimes \pi_l Y \arrow[r] \arrow[d, "\cdot"] & 0 \\
\pi_{k+l}(A \wedge^L Y) \arrow[r, "\pi_{k+l}(f\wedge \mathrm{Id})"'] & \pi_{k+l}(B \wedge^L Y) \arrow[r, "\pi_{k+l}(g\wedge \mathrm{Id})"'] & \pi_{k+l}(C \wedge^L Y) \arrow[r] & 0
\end{tikzcd}$$

The upper row is exact since A is $(k-1)$-connected, thus $\pi_k(\Sigma A) \cong \pi_{k-1}A = 0$, and tensoring with $\pi_l Y$ is right exact. The lower row is exact since $\pi_{k+l}(\Sigma(A \wedge^L Y))$ is isomorphic to $\pi_{k+l}(A \wedge^L (\Sigma Y))$, which is trivial since ΣY is l-connected and A belongs to $\mathcal{X}$. Since both rows are exact and the left and middle vertical map are isomorphisms, the right vertical map is an isomorphism and thus $C \in \mathcal{X}$. Proposition 5.21 now applies and shows that every $(k-1)$-connected spectrum belongs to the class $\mathcal{X}$, which is what we claimed. □

Remark 5.23. The previous theorem about the lowest potentially non-trivial homotopy group of a smash product immediately implies a similar result for the pointset level smash product whenever at least one factor is flat. Indeed, if X and Y are symmetric spectra at least one of which is flat, then by Theorem 3.1 the natural map $\psi_{X,Y} : X \wedge^L Y \longrightarrow X \wedge Y$ from the derived to the pointset level smash product is an isomorphism in $\mathcal{SHC}$. So if X is $(k-1)$-connected and Y is $(l-1)$-connected and one of them is flat, then $X \wedge Y$ is $(k+l-1)$-connected and the pairing

$$\cdot \ : \ \pi_k X \ \otimes \ \pi_l Y \ \longrightarrow \ \pi_{k+l}(X \wedge Y)$$

of Theorem I.6.16 is an isomorphism of abelian groups.

Proposition 5.24. *Let X be a coconnective symmetric spectrum, i.e., the homotopy group $\pi_n X$ is trivial for all $n \geq 1$, and let A be a connective spectrum. Then the map*

$$\pi_0 \ : \ \mathcal{SHC}(A, X) \ \longrightarrow \ \mathrm{Hom}_{\mathcal{A}b}(\pi_0 A, \pi_0 X)$$

is an isomorphism of abelian groups.

Proof. We consider the class $\mathcal{X}$ of all connective spectra A such that for all coconnective X the map $\pi_0 : \mathcal{SHC}(A, X) \longrightarrow \mathrm{Hom}_{\mathcal{A}b}(\pi_0 A, \pi_0 X)$ is an isomorphism. The map

$$\pi_0 \ : \ \mathcal{SHC}(\mathbb{S}, X) \ \longrightarrow \ \mathrm{Hom}_{\mathcal{A}b}(\pi_0 \mathbb{S}, \pi_0 X)$$

is an isomorphism of abelian groups because $\pi_0 \mathbb{S}$ is free abelian of rank 1, generated by the unit $1 \in \pi_0 \mathbb{S}$. So the sphere spectrum $\mathbb{S}$ belongs to $\mathcal{X}$.

Now consider a family $\{A_i\}_{i\in I}$ of objects from $\mathcal{X}$. We have a commutative square

$$\begin{tikzcd}
\mathcal{SHC}(\bigoplus_I A^i, X) \arrow[r, "\pi_0"] \arrow[dd, "\cong"'] & \mathrm{Hom}_{\mathcal{A}b}(\pi_0\left(\bigoplus_I A^i\right), \pi_0 X) \arrow[d, "\cong"] \\
 & \mathrm{Hom}_{\mathcal{A}b}(\bigoplus_I \pi_0(A^i), \pi_0 X) \arrow[d, "\cong"] \\
\prod_I \mathcal{SHC}(A_i, X) \arrow[r, "\prod_I \pi_0"'] & \prod_I \mathrm{Hom}_{\mathcal{A}b}(\pi_0(A^i), \pi_0 X)
\end{tikzcd}$$

in which the vertical maps are isomorphisms by the universal property of sums and because homotopy groups commute with sums. The lower horizontal map is an isomorphism by the assumption on the objects A_i, hence the upper map is an isomorphism; this proves that the class $\mathcal{X}$ is closed under sums.

Now consider a distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

such that A and B belong to $\mathcal{X}$. We consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{SHC}(C, X) \arrow[r] \arrow[d, "\pi_0"'] & \mathcal{SHC}(B, X) \arrow[r] \arrow[d, "\pi_0"', "\cong"] & \mathcal{SHC}(A, X) \arrow[d, "\pi_0"', "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}b}(\pi_0 C, \pi_0 X) \arrow[r] & \mathrm{Hom}_{\mathcal{A}b}(\pi_0 B, \pi_0 X) \arrow[r] & \mathrm{Hom}_{\mathcal{A}b}(\pi_0 A, \pi_0 X)
\end{tikzcd}$$

The upper row is exact since $\Sigma^{-1}X$ is coconnective, thus $\pi_0(\Sigma^{-1}X) = 0$ and so the group $\mathcal{SHC}(\Sigma A, X) \cong \mathcal{SHC}(A, \Sigma^{-1}X) \cong \mathrm{Hom}_{\mathcal{A}b}(\pi_0 A, \pi_0(\Sigma^{-1}X))$ is trivial. The lower row is exact since $\pi_0 A \longrightarrow \pi_0 B \longrightarrow \pi_0 C \longrightarrow \pi_{-1}A = 0$ is. Since the vertical maps for B and A are isomorphisms, so is the one for C. Thus C also belongs to $\mathcal{X}$ which means that the class $\mathcal{X}$ is closed under extensions to the right. So by Proposition 5.21 the class $\mathcal{X}$ contains all connective spectra. □

Theorem 5.25 (Uniqueness of Eilenberg-Mac Lane spectra)**.** (i) *Let X be a connective symmetric spectrum and A an abelian group. Then the map*

$$\pi_0 \ :\ \mathcal{SHC}(X, HA) \ \longrightarrow\ \mathrm{Hom}_{\mathcal{A}b}(\pi_0 X, A)$$

is an isomorphism of abelian groups.

(ii) *Let X be a symmetric spectrum whose homotopy groups are trivial in dimensions different from 0. Then there is a unique morphism in the stable homotopy category from the Eilenberg-Mac Lane spectrum $H(\pi_0 X)$ to X which induces the isomorphism $\pi_0(H\pi_0 X) \cong \pi_0 X$ on homotopy.*

(iii) *The restriction of the functor $\pi_0 : \mathcal{SHC} \longrightarrow \mathcal{A}b$ to the full subcategory of spectra with homotopy concentrated in dimension 0 is an equivalence of categories.*

Proof. (i) Since Eilenberg-Mac Lane spectra are coconnective this is a special case of Proposition 5.24.

(ii) Part (i) gives a morphism $f : X \longrightarrow HA$, unique in the stable homotopy category, which induces induces the isomorphism $\pi_0(H\pi_0 X) \cong \pi_0 X$. Since source and target of f have no homotopy in dimensions other than 0, f is an isomorphism in $\mathcal{SHC}$.

(iii) The restriction of π_0 to the full subcategory of spectra with homotopy concentrated in dimension 0 is fully faithful by (i) and essentially surjective since every abelian group has an Eilenberg-Mac Lane spectrum. □

Remark 5.26. In the theory of triangulated categories, the notion of a *t-structure* formalizes the behavior of 'connective' and 'co-connective' objects. What we have shown can be summarized in this language as saying that in the situation of the classes of connective and co-connective spectra provide a t-structure on the stable homotopy category. Moreover, the objects with homotopy groups concentrated in dimension 0 form the so-called *heart* of the t-structure. So Theorem 5.25 can be rephrased in a fancy way as saying that the functor π_0 is an equivalence of abelian categories from the heart of this t-structure to the category of abelian groups.

Lemma 5.27. *Let n be an integer and $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ composable morphisms in the stable homotopy category such that X is $(n-1)$-connected, Z is $(n+1)$-coconnected and*

$$0 \longrightarrow \pi_k X \xrightarrow{\pi_k f} \pi_k Y \xrightarrow{\pi_k g} \pi_k Z \longrightarrow 0$$

is exact for all $k \in \mathbb{Z}$. Then there is a unique morphism $h : Z \longrightarrow \Sigma X$ such that (f, g, h) is a distinguished triangle.

Proof. Since Z is $(n+1)$-coconnected, the exactness of the above sequence implies that $\pi_k f$ is bijective for $k > n$. Similarly, $\pi_k g$ is bijective for $k < n$.

We choose a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X$$

which extends f. We contemplate the long exact sequence of homotopy groups of this triangle: since $\pi_k f$ is bijective for $k > n$ and injective for $k = n$, the spectrum Z' has trivial homotopy groups above dimension n. Since X is $(n-1)$-connected, the map $\pi_k g' : \pi_k Y \longrightarrow \pi_k Z'$ is an isomorphism for $k < n$ and passes to an isomorphism from the cokernel of $\pi_n f : \pi_n X \longrightarrow \pi_n Y$ to $\pi_n Z'$.

The composite of $gf : X \longrightarrow Z$ is trivial in $\mathcal{SHC}$ by Proposition 5.24 (or rather its n-fold shifted version). So there exists a morphism $\varphi : Z' \longrightarrow Z$ satisfying $\varphi g' = g$. By the assumptions on the homotopy groups of Z and the above calculation of the homotopy groups of Z', the morphism φ induces an isomorphism of all homotopy groups. So φ is an isomorphism in the stable homotopy category and we can replace Z' by the isomorphic spectrum Z to obtain a distinguished triangle of the desired form with $h = h'\varphi^{-1}$.

For the uniqueness statement we let $\bar{h} : Z \longrightarrow \Sigma X$ be another morphism which extends (f, g) to a distinguished triangle. Axiom (T3) of the triangulated category allows us to choose an endomorphism $\varphi : Z \longrightarrow Z$ which makes the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, dotted, "\varphi"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "\bar{h}"'] & \Sigma X
\end{tikzcd}$$

commute. We have $(\mathrm{Id}_Z - \varphi)g = 0$, so there exists a morphism $\psi : \Sigma X \longrightarrow Z$ such that $\psi h = (\mathrm{Id} - \varphi)$ by exactness of (f, g, h). Since ΣX is n-connected and Z is $(n+1)$-coconnected, the morphism ψ is trivial by Proposition 5.24 (or rather its n-fold shifted version). So φ equals the identity of Z and thus $\bar{h} = h$. □

By an *extension* of abelian groups we mean a pair of homomorphisms $i : A \longrightarrow B$ and $j : B \longrightarrow C$ of abelian groups such that i is injective, j is surjective and the image of i equals the kernel of B. Equivalently, (i, j) is an extension if the sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is exact.

Proposition 5.28. (i) *For every extension of abelian groups $A \xrightarrow{i} B \xrightarrow{j} C$ there is a unique morphism $\delta_{i,j} : HC \longrightarrow \Sigma(HA)$ in the stable homotopy category such that the diagram*

$$HA \xrightarrow{Hi} HB \xrightarrow{Hj} HC \xrightarrow{\delta_{i,j}} \Sigma(HA)$$

is a distinguished triangle. This morphism $\delta_{i,j}$ will be called the Bockstein morphism *associated to the extension (i, j).*

(ii) *The Bockstein morphism is natural for morphisms of extensions in the following sense. Given a commutative diagram of abelian groups*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"'] & B \arrow[r, "j"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"] \\ A' \arrow[r, "i'"'] & B' \arrow[r, "j'"'] & C' \end{tikzcd}$$

in which both rows are extensions, then the relation

$$\Sigma(H\alpha) \circ \delta_{i,j} \;=\; \delta_{i',j'} \circ H\gamma$$

holds as morphisms from HC to $\Sigma(HA')$ in the stable homotopy category.

(iii) *The Bockstein morphism only depends on the Yoneda class of the extensions and the assignment*

$$\mathrm{Ext}(C, A) \longrightarrow \mathcal{SHC}(HC, \Sigma(HA)) , \quad [i, j] \longmapsto \delta_{i,j}$$

is a group isomorphism for all abelian groups A and C.

(iv) *The composite of every composable pair of Bockstein operations is zero.*

Proof. (i) This is the special case of Lemma 5.27 with $n = 0$ for the composable morphisms $Hi : HA \longrightarrow HB$ and $Hj : HB \longrightarrow HC$.

(ii) Suppose we are given extensions (i, j) and (i', j') and a morphism (α, β, γ) from (i, j) to (i', j'). Let $\delta_{i,j} : HC \longrightarrow \Sigma(HA)$ and $\delta_{i',j'} : HC' \longrightarrow \Sigma(HA')$ be the Bockstein operations as in (i). Axiom (T3) of the triangulated category lets us choose a morphism $\varphi : HC \longrightarrow HC'$ which makes the diagram

$$\begin{tikzcd} HA \arrow[r, "Hi"] \arrow[d, "H\alpha"'] & HB \arrow[r, "Hj"] \arrow[d, "H\beta"'] & HC \arrow[r, "\delta_{i,j}"] \arrow[d, dotted, "\varphi"] & \Sigma(HA) \arrow[d, "\Sigma(H\alpha)"] \\ HA' \arrow[r, "Hi'"'] & HB \arrow[r, "Hj'"'] & HC \arrow[r, "\delta_{i',j'}"'] & \Sigma(HA) \end{tikzcd}$$

commute. The relation $\pi_0(\varphi) \circ \pi_0(Hj) = \pi_0(Hj') \circ \pi_0(H\beta) = \pi_0(H\gamma) \circ \pi_0(Hj)$ holds since $j'\beta = \gamma j$. Since Hj is surjective on π_0, we deduce that φ and $H\gamma$ induce the same map π_0. Then $\varphi = H\gamma$ by Theorem 5.25, and thus $\Sigma(H\alpha) \circ \delta_{i,j} = \delta_{i',j'} \circ \varphi = \delta_{i',j'} \circ H\gamma$, as we claimed.

(iii) Two extensions (i, j) and (i', j') represent the same Yoneda class if and only if there is a homomorphism $f : B \longrightarrow B'$ (necessarily an isomorphism) satisfying $fi = i'$ and $j'f = j$. We can apply (ii) to the morphism $(\mathrm{Id}_A, f, \mathrm{Id}_C)$ from (i, j) to (i', j'). The naturality statement then boils down to the equation $\delta_{i,j} = \delta_{i',j'}$. So the Bockstein morphism only depends on the Yoneda class of the extensions.

For the additivity of the Bockstein morphism construction we choose an epimorphism $\epsilon : F \longrightarrow C$ from a free abelian group onto C and let K denote the kernel of ϵ. We let $\delta_u : HC \longrightarrow \Sigma(HK)$ be the Bockstein homomorphism associated to the extension $(\mathrm{incl} : K \longrightarrow F, \epsilon)$. This extension has an associated 6-term exact sequence of Ext groups whose connecting morphism $c : \mathrm{Hom}(K, A) \longrightarrow \mathrm{Ext}(C, A)$ takes $\alpha : K \longrightarrow A$ to the lower extension in the commutative diagram

$$\begin{tikzcd} K \arrow[r, "\mathrm{incl}"] \arrow[d, "\alpha"'] & F \arrow[r, "\epsilon"] \arrow[d, "\beta"'] & C \arrow[d, equal] \\ A \arrow[r, "i"'] & B \arrow[r, "j"'] & C \end{tikzcd} \tag{5.29}$$

where the left square is a pushout. So the square of abelian groups

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{A}b}(K, A) \arrow[r, "c"] \arrow[d, "H"'] & \mathrm{Ext}(C, A) \arrow[d, "{[i,j] \mapsto \delta_{i,j}}"] \\ \mathcal{SHC}(HK, HA) \arrow[r, "f \,\longmapsto\, \Sigma f \circ \delta_u"'] & \mathcal{SHC}(HC, \Sigma(HA)) \end{tikzcd}$$

commutes by naturality of the Bockstein morphism. In this square the other three maps are additive and the connective morphism c is surjective; so the right vertical map is also additive.

To prove injectivity we consider an extension (i,j) whose Bockstein morphism is zero. Then the distinguished triangle

$$HA \xrightarrow{Hi} HB \xrightarrow{Hj} HC \xrightarrow{\delta_{i,j}} \Sigma(HA)$$

splits, i.e., there is a morphism $\psi : HC \longrightarrow HB$ in the stable homotopy category such that $Hj \circ \psi = \mathrm{Id}_{HC}$. This implies that $\pi_0(Hj)$ and thus $j : B \longrightarrow C$ is split surjective, so (i,j) represents the zero element in the group $\mathrm{Ext}(C,A)$.

For surjectivity we consider any morphism $\delta : HC \longrightarrow \Sigma(HA)$ and embed it in a distinguished triangle

$$HA \xrightarrow{f} Y \xrightarrow{g} HC \xrightarrow{\delta} \Sigma(HA)\ .$$

The long exact sequence of homotopy groups associated to this triangle reduces shows that the homotopy groups of Y are concentrated in dimension zero and gives an extension

$$A = \pi_0 HA \xrightarrow{\pi_0 f} \pi_0 Y \xrightarrow{\pi_0 g} \pi_0 HC = C\ . \tag{5.30}$$

Theorem 5.25 (ii) constructs a preferred isomorphism in $\mathcal{SHC}$ from the spectrum Y to the Eilenberg-Mac Lane spectrum $H(\pi_0 Y)$. So we can replace Y in the above distinguished triangle by the isomorphic object $H(\pi_0 Y)$ in a way which turns f into $H(\pi_0 f)$ and g into $H(\pi_0 g)$. The uniqueness of Bockstein morphisms then shows that the original morphism δ is the Bockstein associated to the extension (5.30).

(iv) The reason behind this property is the fact that the category of abelian groups has homological dimension 1, i.e., there are no non-trivial Ext-groups beyond dimension one. In more detail we argue as follows. We let

$$Y \xrightarrow{i'} Z \xrightarrow{j'} A \qquad \text{and} \qquad A \xrightarrow{i} B \xrightarrow{j} C$$

be two 'composable' extensions. We wish to show that the composite

$$HC \xrightarrow{\delta_{i,j}} \Sigma(HA) \xrightarrow{\Sigma(\delta_{i',j'})} \Sigma^2(HY)$$

is trivial. We choose an epimorphism $\epsilon : F \longrightarrow C$ from a free abelian group and a lift $\beta : F \longrightarrow B$ satisfying $j\beta = \epsilon$. This data yields a morphism of extensions (5.29) where K denotes the kernel of ϵ. Since the kernel K is again a free abelian group, we can choose a lift $\lambda : K \longrightarrow Z$ satisfying $j'\lambda = \alpha$ and get another morphism of extensions:

$$\begin{tikzcd} Y \arrow[r, "{(1,0)}"] \arrow[d, equal] & Y \oplus K \arrow[r, "\binom{0}{1}"] \arrow[d, "\binom{i'}{\lambda}"] & K \arrow[d, "\alpha"] \\ Y \arrow[r, "i'"'] & Z \arrow[r, "j'"'] & A \end{tikzcd}$$

By naturality of the Bockstein morphism, the composite $\Sigma(\delta_{i',j'}) \circ \delta_{i,j}$ equals the composite $\Sigma(\delta_{(1,0),\binom{0}{1}}) \circ \delta_{\mathrm{incl},\epsilon}$. Since the Bockstein of a split extension is zero, this proves $\Sigma(\delta_{i',j'}) \circ \delta_{i,j} = 0$. □

An important example is the *mod-p Bockstein* $\beta : H\mathbb{Z}/p \longrightarrow \Sigma(H\mathbb{Z}/p)$ associated to the extension

$$\mathbb{Z}/p \xrightarrow{\cdot p} \mathbb{Z}/p^2 \xrightarrow{\text{proj}} \mathbb{Z}/p \tag{5.31}$$

where the first map sends $n + p\mathbb{Z}$ to $np + p^2\mathbb{Z}$ and the second map sends $n + p^2\mathbb{Z}$ to $n + p\mathbb{Z}$.

6. Homology and cohomology

6.1. Generalized homology and cohomology. In this section we discuss how a symmetric spectrum E determines a 'generalized homology theory' E_* and a 'generalized cohomology theory' E^*, both for spaces (or simplicial sets) and spectra. In the special case $E = HA$ of Eilenberg-Mac Lane spectra, this yields 'ordinary' homology and cohomology groups with coefficients in an abelian group A, which generalizes singular (co-)homology of spaces and simplicial sets. Then we discuss various isomorphic descriptions of these spectrum homology and cohomology groups.

Definition 6.1. Let E be a symmetric spectrum. For any other symmetric spectrum X and an integer k, we define the k-th *E-homology group of X* as

$$E_k(X) \ = \ \pi_k(E \wedge^L X) \ ,$$

the k-th true homotopy group of the derived smash product of E and X. The k-th *E-cohomology group* of X is defined as

$$E^k(X) \ = [X, E]_{-k} \ = \ [\mathbb{S}^{-k} \wedge X, E] \ ,$$

the group of maps in the stable homotopy category from $\mathbb{S}^{-k} \wedge X$ to E.

The notation offers the possibility of confusion since the symbol E_k also refers to the k-th level of the symmetric spectrum. However, the notation E_k for the generalized E-homology is so standard and convenient that we use it despite the clash of notation. Moreover, in the role as the k-th E-homology group, E_k typically comes with an argument in parenthesis, which it usually does not have as the k-th level of E. We hope that in every case the symbol E_k appears, it is clear from the context whether we mean the k-th E-homology group or the k-th level of the symmetric spectrum E.

We have defined E-homology and E-cohomology as functors on the stable homotopy category. However, we can (and will) consider $E_k(-)$ and $E^k(-)$ as functors of symmetric spectra by precomposing with the localization functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$.

If the symmetric spectrum E is flat, then we can dispose of the decoration '$\ ^L$' on the smash product in the definition of E-homology. Indeed, in this case the natural map $E \wedge^L X \longrightarrow E \wedge X$ in $\mathcal{SHC}$ from the derived to the pointset level smash product is a stable equivalence [ref], so it induces isomorphisms of homotopy groups. This yields an isomorphism $E_k(X) = \pi_k(E \wedge^L X) \longrightarrow \pi_k(E \wedge X)$ which is natural in X.

As a special case we get the 'ordinary' spectrum homology and cohomology with coefficients in an abelian group A, see Definition 6.21 below. For this we take $E = HA$ to be the Eilenberg-Mac Lane spectrum of A.

The *suspension isomorphisms*

$$\Sigma : \ E_k(X) \ \longrightarrow \ E_{1+k}(\Sigma X) \qquad \text{and} \qquad \Sigma : \ E^k(X) \ \longrightarrow \ E^{1+k}(\Sigma X)$$

are defined as follows. The homological suspension isomorphism is the composite

$$E_k(X) = \pi_k(E \wedge^L X) \ \xrightarrow{S^1 \wedge -} \ \pi_{1+k}(\Sigma(E \wedge^L X)) \ \xrightarrow{(\bar\kappa_{E,X}^{-1})_*} \ \pi_{1+k}(E \wedge^L (\Sigma X)) = E_{1+k}(\Sigma X) \ ,$$

where $\bar\kappa_{E,X} : E \wedge^L (\Sigma X) \longrightarrow \Sigma(E \wedge^L X)$ is the isomorphism defined in (3.20). The cohomological suspension isomorphism is the morphism

$$\nu^* \ : \ E^k(X) = [X, E]_{-k} \ \longrightarrow \ [\Sigma X, E]_{-k-1} = E^{k+1}(\Sigma X)$$

defined in (4.9).

As the terminology suggests, E-homology is a homological functor and E-homology is a cohomological functor, in the sense of Definition 5.4. We spell this out in the next proposition:

Proposition 6.2. *Let E be a symmetric spectrum and k any integer.*

(i) *For every distinguished triangle (f, g, h) in the stable homotopy category the sequences*

$$E_k(A) \ \xrightarrow{E_k(f)} \ E_k(B) \ \xrightarrow{E_k(g)} \ E_k(C) \ \xrightarrow{E_k(h)} \ E_k(\Sigma A)$$

and

$$E^k(\Sigma A) \ \xrightarrow{E^k(h)} \ E^k(C) \ \xrightarrow{E^k(g)} \ E^k(B) \ \xrightarrow{E^k(f)} \ E^k(A)$$

are exact.

(ii) *For every family* $\{X_i\}_{i\in I}$ *of symmetric spectra the natural maps*

$$\bigoplus_{i\in I} E_k(X_i) \longrightarrow E_k\left(\bigoplus_{i\in I} X_i\right) \quad \text{and} \quad E^k\left(\prod_{i\in I} X_i\right) \longrightarrow \prod_{i\in I} E^k(X_i)$$

are isomorphisms.

Lemma 5.6 allows us to deduce the behaviour of E-homology and cohomology on homotopy colimits: If $f_n : X_n \longrightarrow X_{n+1}$ is a composable sequence of morphisms in the stable homotopy category and $(\bar{X}, \varphi_n)$ a homotopy colimit of the sequence $\{f_n\}$, then for every integer k the natural map

$$\operatorname{colim}_n E_k(X_n) \longrightarrow E_k(\bar{X}) \tag{6.3}$$

induced from the compatible morphisms $E_k(\varphi_n) : E_k(X_n) \longrightarrow E_k(\bar{X})$ is an isomorphism. Moreover, we have a short exact sequence

$$0 \longrightarrow \lim\nolimits^1_n E^{k-1}(X_n) \longrightarrow E^k(\bar{X}) \longrightarrow \lim\nolimits_n E^k(X_n) \longrightarrow 0 \tag{6.4}$$

where the right map arises from the system of compatible homomorphisms $E^k(\varphi_n) : E^k(\bar{X}) \longrightarrow E^k(X_n)$ (and where we used the suspension isomorphism to identify $E^k(\Sigma X_n)$ with $E^{k-1}(X_n)$).

In good cases (where 'good' in this context means 'semistable'), the E-homology and E-cohomology of a symmetric spectrum X can be calculated from the E-homology respectively E-cohomology of the levels X_n of X. Here we define the generalized (co-)homology of a based simplicial set as the generalized (co-)homology of its suspension spectrum:

$$E_k(X) \;=\; E_k(\Sigma^\infty X) \qquad \text{and} \qquad E^k(X) \;=\; E^k(\Sigma^\infty X)\ .$$

These (co-)homology theories for simplicial sets come with suspension isomorphisms

$$-\wedge S^1 \;:\; E_k(X) \longrightarrow E_{k+1}(X\wedge S^1) \qquad \text{and} \qquad -\wedge S^1 \;:\; E^k(X) \longrightarrow E^{k+1}(X\wedge S^1)$$

defined as [...]

Proposition 6.5. *Let E be a symmetric spectrum, X a semistable symmetric spectrum and k any integer.*

(i) *The map [...]*

$$\operatorname{colim}_n E_{k+n}(X_n) \longrightarrow E_k(X)$$

is an isomorphism, where the colimit is taken over the sequence

$$E_{k+n}(X_n) \xrightarrow{-\wedge S^1} E_{k+n+1}(X_n\wedge S^1) \xrightarrow{E_{k+n+1}(\sigma_n)} E^{k+n+1}(X_{n+1})\ .$$

(ii) *There is a natural short exact sequence*

$$0 \longrightarrow \lim\nolimits^1_n E^{k+n-1}(X_n) \longrightarrow E^k(X) \longrightarrow \lim\nolimits_n E^{k+n}(X_n) \longrightarrow 0$$

where the inverse limit and derivd limit are taken over the maps

$$E^{k+n+1}(X_{n+1}) \xrightarrow{E^{k+n+1}(\sigma_n)} E^{k+n+1}(X_n\wedge S^1) \xrightarrow{(-\wedge S^1)^{-1}} E^{k+n}(X_n)\ .$$

Proof. Since X is semistable, then by Proposition 5.12 it is a homotopy colimit in $\mathcal{SHC}$ of the sequence of morphisms

$$j_n = \gamma(F_{n+1}X_{n+1})\circ\gamma(\lambda_n)^{-1} \;:\; F_nX_n \longrightarrow F_{n+1}X_{n+1}\ .$$

(i) The isomorphism (6.3) specializes to an isomorphism between the colimit of the groups $E_k(F_nX_n)$ and $E_k(X)$. We use the suspension isomorphism and the stable equivalence $\Sigma^n(F_nX_n) \longrightarrow \Sigma^\infty X$ to identify the group $E_k(F_nX_n)$ with $E_{k+n}(\Sigma^n(F_nX_n))$ and then with $E_{k+n}(\Sigma^\infty X_n) = E_{k+n}(X_n)$. The diagram

$$\begin{tikzcd} E_k(F_nX_n) \arrow[r, "E_k(\lambda_n)^{-1}"'] \arrow[d, "\cong"'] \arrow[rr, bend left, "E_k(j_n)"] & E_k(F_{n+1}(X_n \wedge S^1)) \arrow[r, "E_k(F_{n+1}\sigma_n)"'] \arrow[d, "\cong"] & E_k(F_{n+1}X_{n+1}) \arrow[d, "\cong"] \\ E_{k+n}(X_n) \arrow[r, "-\wedge S^{-1}"'] & E_{k+n+1}(X_n \wedge S^1) \arrow[r, "E_k(\sigma_n)"'] & E_{k+n+1}(X_{n+1}) \end{tikzcd}$$

commutes by naturality of this isomorphism and [...], and the result follows, and allows us to identity the colimit of the groups $E_k(F_nX_n)$ along the maps $E_k(j_n)$ with the colimit of the proposition.

(ii) So the Milnor sequence (6.4) becomes a short exact sequence

$$0 \longrightarrow \lim^1_n E^{k-1}(F_nX_n) \longrightarrow E^k(X) \longrightarrow \lim_n E^k(F_nX_n) \longrightarrow 0 .$$

We use the suspension isomorphism and the stable equivalence $\Sigma^n(F_nX_n) \longrightarrow \Sigma^\infty X$ to identify the group $E^k(F_nX_n)$ with $E^{k+n}(\Sigma^n(F_nX_n))$ and then with $E^{k+n}(\Sigma^\infty X_n) = E^{k+n}(X_n)$. The result then follows from the commutativity of the diagram

$$\begin{tikzcd} E^k(F_{n+1}X_{n+1}) \arrow[r, "E^k(F_{n+1}\sigma_n)"'] \arrow[d, "\cong"'] \arrow[rr, bend left, "E^k(j_n)"] & E^k(F_{n+1}(X_n \wedge S^1)) \arrow[r, "E^k(\lambda_n)^{-1}"'] \arrow[d, "\cong"] & E^k(F_nX_n) \arrow[d, "\cong"] \\ E^{k+n+1}(X_{n+1}) \arrow[r, "E^k(\sigma_n)"'] & E^{k+n+1}(X_n \wedge S^1) \arrow[r, "-\wedge S^{-1}"'] & E^{k+n}(X_n) \end{tikzcd}$$

commutes by naturality of this isomorphism and [...], and the result follows, and allows us to identify the limit and derived limit of the system of groups $E^k(F_nX_n)$ along the maps $E^k(j_n)$ with the limit and derived limit of the proposition. □

Now we discuss pairings and products on generalized homology and cohomology groups. For this we assume that E is a homotopy ring spectrum in the sense of Section 4.1. So E is equipped with a multiplication morphism $\mu : E \wedge^L E \longrightarrow E$ and a unit morphism $\iota : \mathbb{S} \longrightarrow E$ in the stable homotopy category which are associative and unital (in $\mathcal{SHC}$). As we explained in [...] the homotopy groups of a homotopy ring spectrum form a graded ring $\pi_*E = [\mathbb{S}, E]_*$.

Construction 6.6. We let E be a homotopy ring spectrum, X and Y symmetric spectra and k and l integers. Then we define *exterior products*

$$\times \ : \ E_k(X) \otimes E_l(Y) \longrightarrow E_{k+l}(X \wedge^L Y) \tag{6.7}$$

in E-homology and

$$\times \ : \ E^k(X) \otimes E^l(Y) \longrightarrow E^{k+l}(X \wedge^L Y) \tag{6.8}$$

in E-cohomology as follows. The E-homology product (6.7) is the composite

$$E_k(X) \otimes E_l(Y) = \pi_k(E \wedge^L X) \otimes \pi_l(E \wedge^L Y) \xrightarrow{\ \cdot\ } \pi_{k+l}(E \wedge^L X \wedge^L E \wedge^L Y)$$
$$\xrightarrow{(E\wedge \bar\tau_{X,E}\wedge Y)_*} \pi_{k+l}(E \wedge^L E \wedge^L X \wedge^L Y) \xrightarrow{(\mu\wedge X\wedge Y)_*} \pi_{k+l}(E \wedge^L X \wedge^L Y) = E_{k+l}(X \wedge^L Y) .$$

For cohomology classes $f \in E^k(X)$ and $g \in E^l(Y)$, the exterior product (6.8) is the composite

$$[X, E]_{-k} \otimes [Y, E]_{-l} \xrightarrow{-\wedge-} [X \wedge^L Y,\, E \wedge^L E]_{-k-l} \xrightarrow{[X\wedge^L Y, \mu]} [X \wedge^L Y,\, E]_{-(k+l)} .$$

Both exterior products are associative and unital, and they are commutative whenver E is homotopy commutative. More precisely, this means that the relations

$$x \times y \;=\; (-1)^{kl} \cdot (\tau_{Y,X})_*(y \times x) \qquad \text{and} \qquad f \times g \;=\; (-1)^{kl} \cdot (\tau_{X,Y})^*(g \times f) \tag{6.9}$$

hold in the respective E-homology or E-cohomology group.

The behaviour of exterior product with respect to suspension isomorphisms is as follows. In E-homology we have

$$(\kappa_{X,Y})_*((\Sigma x) \times y) \;=\; \Sigma(x \times y) \;=\; (-1)^k \cdot (\bar{\kappa}_{X,Y})_*(x \times (\Sigma y)) \tag{6.10}$$

in $E_{1+k+l}(\Sigma(X \wedge^L Y))$, where $\kappa_{X,Y} : (\Sigma X) \wedge^L Y \longrightarrow \Sigma(X \wedge^L Y)$ and $\bar{\kappa}_{X,Y} : X \wedge^L (\Sigma Y) \longrightarrow \Sigma(X \wedge^L Y)$ are the isomorphisms defined (3.18) respectively (3.20). In E-cohomology we similarly have

$$(\kappa_{X,Y})^*(\Sigma(f \times g)) \;=\; (\Sigma f) \times g \qquad \text{and} \qquad (\bar{\kappa}_{X,Y})^*(\Sigma(f \times g)) \;=\; (-1)^k \cdot f \times (\Sigma g)\ . \tag{6.11}$$

In the special case where $Y = \mathbb{S}$ is the sphere spectrum we have $E_l(\mathbb{S}) = \pi_l E$ and $X \wedge^L \mathbb{S} = X$; so the exterior product (6.7) specializes to a map

$$\times \;:\; E_k(X) \otimes \pi_l E \;\longrightarrow\; E_{k+l}(X)\ . \tag{6.12}$$

These products make the E-homology of X into a graded right module over the graded ring $\pi_* E$. In the special case where $X = \mathbb{S}$ is the sphere spectrum the exterior product (6.8) specializes to a map

$$\times \;:\; E^k(\mathbb{S}) \otimes E^l(Y) \;\longrightarrow\; E^{k+l}(Y)\ .$$

These products make the E-cohomology of Y into a graded left module over the graded ring $E^*(\mathbb{S})$. This graded ring is isomorphic to the homotopy ring $\pi_* E$, with grading reversed.

Construction 6.13. The *Kronecker pairing* is a pairing between E-homology and E-cohomology which exists whenever E is a homotopy ring spectrum (see Section 4.1), for example one arising from a symmetric ring spectrum. Then the Kronecker pairing is a map

$$\langle -, - \rangle \;:\; E^k(X) \;\otimes\; E_{k+m}(X \wedge^L Y) \;\longrightarrow\; E_m(Y) \tag{6.14}$$

defined as follows. Given a cohomology class $f \in E^k(X)$ and a homology class $w \in E_{k+m}(X \wedge^L Y) = \pi_{k+m}(E \wedge^L X \wedge^L Y)$ we define

$$\langle f, w \rangle \;=\; ((E \times f) \wedge Y)_*(w) \;\in\; \pi_m(E \wedge Y) \;=\; E_m(Y)\ ,$$

where $E \times f = \mathrm{Id}_E \times f \in E^k(E \wedge X) = [E \wedge X, E]_{-k}$ and $(E \times f) \wedge Y \in [E \wedge^L X \wedge^L Y,\, E \wedge^L Y]_{-k}$.

We call a cohomology class $f \in E^k(X)$ *central* if the relation

$$f \times E \;=\; \bar{\tau}^*_{X,E}(E \times f)$$

holds in the group $E^k(E \wedge X)$. Explicitly, this means that the diagram

$$\begin{tikzcd}
\mathbb{S}^{-k} \wedge E \wedge^L X \arrow[r, "\bar{\tau}_{\mathbb{S}^{-k},E} \wedge X"] \arrow[d, "\mathbb{S}^{-k} \wedge \bar{\tau}_{E,X}"'] & E \wedge^L \mathbb{S}^{-k} \wedge X \arrow[r, "E \wedge f"] & E \wedge^L E \arrow[d, "\mu"] \\
\mathbb{S}^{-k} \wedge X \wedge^L E \arrow[r, "f \wedge E"'] & E \wedge^L E \arrow[r, "\mu"'] & E
\end{tikzcd}$$

commutes in the stable homotopy category. For example, the unit $\iota : \mathbb{S} \longrightarrow E$ is central as a cohomology class $\iota \in E^0(\mathbb{S})$. If E is homotopy commutative, then every cohomology class is central.

Proposition 6.15. *The Kronecker pairing has the following properties.*

(i) *For a morphism $\varphi : X \longrightarrow X'$ in $\mathcal{SHC}$, $f' \in E^k(X')$ and $w \in E_{k+m}(X \wedge^L Y)$ the relation*

$$\langle \varphi^*(f'), w \rangle \;=\; \langle f' \circ \varphi, w \rangle \;=\; \langle f', (\varphi \wedge Y)_*(w), \rangle$$

holds in $E_m(Y)$.

(ii) *For a morphism $\psi : Y \longrightarrow Y'$ in $\mathcal{SHC}$, $f \in E^k(X)$ and $w \in E_{k+m}(X \wedge^L Y)$ the relation*

$$\psi_* \langle f, w\rangle \;=\; \langle f, (X \wedge \psi)_*(w)\rangle$$

holds in $E_m(Y')$.

(iii) *For $f \in E^k(X)$ and $w \in E_{k+m}(X \wedge^L Y)$ the relation*

$$\langle \Sigma f,\, (\kappa_{X,Y})_*^{-1}(\Sigma w)\rangle \;=\; \langle f, w\rangle$$

holds in $E_m(Y)$, where $\kappa_{X,Y} : (\Sigma X) \wedge^L Y \longrightarrow \Sigma(X \wedge^L Y)$ is the isomorphism (3.18).

(iv) *For all E-cohomology classes $f \in E^k(X)$, $g \in E^l(Y)$ and E-homology classes $w \in E_{k+l+m}(X \wedge^L Y \wedge^L Z)$ the relation*

$$\langle f \times g,\, w\rangle \;=\; (-1)^{kl} \cdot \langle g,\, \langle f, w\rangle\rangle$$

holds in $E_m(Z)$.

(v) *For all E-homology class $w \in E_{k+m}(X \wedge^L Y)$ and $z \in E_l(Z)$ and all* central *E-cohomology classes $f \in E^k(X)$, the relation*

$$\langle f,\, w \times z\rangle \;=\; \langle f, w\rangle \times z$$

holds in $E_{m+l}(Y \wedge^L Z)$.

(vi) *Suppose that E is homotopy commutative. Then for all classes $x \in E_k(X)$, $g \in E^l(Y)$ and $w \in E_{l+m}(Y \wedge^L Z)$ we have*

$$x \times \langle g, w\rangle \;=\; (-1)^{kl} \cdot \langle g,\, (\bar\tau_{X,Y} \wedge Z)_*(x \times w)\rangle$$

in the group $E_{k+m}(X \wedge Z)$.

Proof. To simplify notation we write $\wedge$ for $\wedge^L$ throughout the proof. Parts (i) and (ii) are straightforward from the definition.

(iii) The relation

$$\begin{aligned}
(E \times (\Sigma f)) \wedge Y \quad {}_{(6.11)}\!&= (\bar\kappa^*_{E,X}(\Sigma(E \times f))) \wedge Y \;=\; (\Sigma(E \times f)) \circ \bar\kappa_{E,X}) \wedge Y\\
&= (\Sigma(E \times f) \wedge Y) \circ (\bar\kappa_{E,X} \wedge Y)\\
{}_{???}\!&= \Sigma((E \times f) \wedge Y) \circ \kappa_{E\wedge X,Y} \circ (\bar\kappa_{E,X} \wedge Y)\\
&= \Sigma((E \times f) \wedge Y) \circ \bar\kappa_{E,X\wedge Y} \circ (E \wedge \kappa_{X,Y})
\end{aligned}$$

holds in the group $[\Sigma(E \wedge X \wedge Y), E \wedge Y]_{-k-1}$. The fifth equation uses that the square of isomorphisms

$$\begin{tikzcd}
E \wedge (\Sigma X) \wedge Y \arrow[r, "\bar\kappa_{E,X} \wedge Y"] \arrow[d, "E \wedge \kappa_{X,Y}"'] & \Sigma(E \wedge X) \wedge Y \arrow[d, "\kappa_{E\wedge X,Y}"] \\
E \wedge \Sigma(X \wedge Y) \arrow[r, "\bar\kappa_{E,X\wedge Y}"'] & \Sigma(E \wedge X \wedge Y)
\end{tikzcd}$$

commutes. So we deduce

$$\begin{aligned}
\langle \Sigma f,\, (\kappa_{X,Y})_*^{-1}(\Sigma w)\rangle &= ((E \times (\Sigma f)) \wedge Y)_*((\kappa_{X,Y})_*^{-1}(\Sigma w))\\
&= (\Sigma((E \times f) \wedge Y) \circ \bar\kappa_{E,X\wedge Y} \circ (E \wedge \kappa_{X,Y}))_*((\kappa_{X,Y})_*^{-1}(\Sigma w))\\
&= (\Sigma((E \times f) \wedge Y) \circ \bar\kappa_{E,X\wedge Y})_*(\Sigma w)\\
&= (\Sigma((E \times f) \wedge Y))_*(S^1 \wedge w)\\
{}_{(4.10)}\!&= ((E \times f) \wedge Y)_*(w) \;=\; \langle f,\, w\rangle\;,
\end{aligned}$$

where Σw denotes the E-homology suspension and $S^1 \wedge w$ the homotopy suspension of the class w in $E_{k+m}(X \wedge Y) = \pi_{k+m}(E \wedge X \wedge Y)$.

(iv) We have

$$\begin{aligned}
E \times (f \times g) \;&=\; (E \times f) \times g \;=\; \mu \circ ((E \times f) \wedge g)\\
&= (-1)^{kl} \cdot \mu \circ (E \wedge g) \circ ((E \times f) \wedge Y) \;=\; (-1)^{kl} \cdot (E \times g) \circ ((E \times f) \wedge Y)\;,
\end{aligned}$$

in $E^{k+l}(E \wedge^L X \wedge^L Y)$, where the third equality is Proposition 4.8. Hence

$$\begin{aligned}\langle f \times g,\, w\rangle &= ((E \times (f \times g)) \wedge Z)_*(w)\\ &= (-1)^{kl} \cdot (((E \times g) \wedge Z) \circ ((E \times f) \wedge Y \wedge Z))_*(w))\\ &= (-1)^{kl} \cdot ((E \times g) \wedge Z)_*\langle w, f\rangle \;=\; \langle g,\, \langle f, w\rangle\rangle\ .\end{aligned}$$

(v) We have

$$\begin{aligned}(E \times f) \circ (\mu \wedge X) \circ (E \wedge \bar\tau_{X,E}) &= \mu \circ (E \wedge f) \circ (\mu \wedge X) \circ (E \wedge \bar\tau_{X,E})\\ &= \mu \circ (\mu \wedge E) \circ (E \wedge E \wedge f) \circ (E \wedge \bar\tau_{X,E})\\ &= \mu \circ (E \wedge \mu) \circ (E \wedge E \wedge f) \circ (E \wedge \bar\tau_{X,E})\\ &= \mu \circ (E \wedge (\mu \circ (E \wedge f) \circ \bar\tau_{X,E}))\\ &= \mu \circ (E \wedge (\mu \circ (f \wedge E))) \;=\; \mu \circ (E \wedge \mu) \circ (E \wedge f \wedge E)\\ &= \mu \circ (\mu \wedge E) \circ (E \wedge f \wedge E) \;=\; \mu \circ ((E \times f) \wedge E)\end{aligned}$$

in $E^k(E \wedge^L X \wedge^L E) = [E \wedge^L X \wedge^L E, E]_{-k}$, where the fourth relation is the centrality of f. [where is $\mathbb{S}^{-k}$?]

$$E \times f \times E \;=\; (E \wedge \tau_{X,E})^*(E \times E \times f)$$

$$\begin{aligned}\langle f,\, w \times z\rangle &= ((E \times f) \wedge Y \wedge Z)_*(w \times z)\\ &= ((E \times f) \wedge Y \wedge Z) \circ (\mu \wedge X \wedge Y \wedge Z) \circ (E \wedge \bar\tau_{X\wedge Y,E} \wedge Z))_*(w \cdot z)\\ &= (((E \times f) \circ (\mu \wedge X) \circ (E \wedge \bar\tau_{X,E})) \wedge Y \wedge Z)_*((E \wedge X \wedge \bar\tau_{Y,E} \wedge Z)_*(w \cdot z))\\ &= ((((E \times E \times f) \wedge Y) \circ (E \wedge \bar\tau_{X,E} \wedge Y) \circ (E \wedge X \wedge \bar\tau_{Y,E})) \wedge Z)_*(w \cdot z)\\ &= (((E \times E \times f) \circ (E \wedge \bar\tau_{X,E})) \wedge Y \wedge Z)_*((E \wedge X \wedge \bar\tau_{Y,E} \wedge Z)_*(w \cdot z))\\ &= ((E \times f \times E) \wedge Y \wedge Z)_*((E \wedge X \wedge \bar\tau_{Y,E} \wedge Z)_*(w \cdot z))\\ &= ((((E \times f \times E) \wedge Y) \circ (E \wedge X \wedge \bar\tau_{Y,E})) \wedge Z)_*(w \cdot z))\\ &= (((E \times f \times E) \wedge Y) \circ (E \wedge X \wedge \bar\tau_{Y,E}))_*(w) \cdot z\\ &= ((\mu \wedge Y \wedge Z) \circ ((E \times f) \wedge E \wedge Y \wedge Z))_*((E \wedge X \wedge \bar\tau_{Y,E} \wedge Z)_*(w \cdot z))\\ &= ((\mu \wedge Y \wedge Z) \circ (E \wedge \bar\tau_{Y,E} \wedge Z) \circ ((E \times f) \wedge Y \wedge E \wedge Z))_*(w \cdot z)\\ &= ((\mu \wedge Y \wedge Z) \circ (E \wedge \bar\tau_{Y,E} \wedge Z))_*(((E \times f) \wedge Y)_*(w) \cdot z)\\ &= ((\mu \wedge Y \wedge Z) \circ (E \wedge \bar\tau_{Y,E} \wedge Z))_*(\langle f, w\rangle \cdot z)\\ &= ((E \times f) \wedge Y)_*(w) \times z\\ &= \langle f, w\rangle \times z\end{aligned}$$

$$\begin{aligned}\langle f,\, w \times z\rangle &= ((E \times f) \wedge Y \wedge Z)_*(w \times z)\\ &= ((E \times f) \wedge Y \wedge Z) \circ (\mu \wedge X \wedge Y \wedge Z) \circ (E \wedge \bar\tau_{X\wedge Y,E} \wedge Z))_*(w \cdot z)\\ &= (((E \times f) \circ (\mu \wedge X) \circ (E \wedge \bar\tau_{X,E})) \wedge Y \wedge Z)_*((E \wedge X \wedge \bar\tau_{Y,E} \wedge Z)_*(w \cdot z))\\ &= ((\mu \circ ((E \times f) \wedge E)) \wedge Y \wedge Z)_*((E \wedge X \wedge \bar\tau_{Y,E} \wedge Z)_*(w \cdot z))\\ &= ((\mu \wedge Y \wedge Z) \circ ((E \times f) \wedge E \wedge Y \wedge Z))_*((E \wedge X \wedge \bar\tau_{Y,E} \wedge Z)_*(w \cdot z))\\ &= ((\mu \wedge Y \wedge Z) \circ (E \wedge \bar\tau_{Y,E} \wedge Z) \circ ((E \times f) \wedge Y \wedge E \wedge Z))_*(w \cdot z)\\ &= ((\mu \wedge Y \wedge Z) \circ (E \wedge \bar\tau_{Y,E} \wedge Z))_*(((E \times f) \wedge Y)_*(w) \cdot z)\\ &= ((\mu \wedge Y \wedge Z) \circ (E \wedge \bar\tau_{Y,E} \wedge Z))_*(\langle f, w\rangle \cdot z)\\ &= \langle f, w\rangle \times z\end{aligned}$$

(vi) We have

$$\begin{aligned} x \times \langle g, w\rangle &\overset{(6.9)}{=} (-1)^{lm}\cdot(\bar\tau_{Z,X})_*(\langle g,w\rangle \times x) \\ &\overset{(v)}{=} (-1)^{lm}\cdot(\bar\tau_{Z,X})_*(\langle g,\, w\times x\rangle) \\ &\overset{(ii)}{=} (-1)^{lm}\cdot\langle g,\, (Y\wedge\bar\tau_{Z,X})_*(w\times x)\rangle \\ &= (-1)^{lm}\cdot\langle g,\, ((\bar\tau_{X,Y}\wedge Z)\circ\bar\tau_{Y\wedge Z,X})_*(w\times x)\rangle \\ &\overset{(6.9)}{=} (-1)^{kl}\cdot\langle g,\, (\bar\tau_{X,Y}\wedge Z)_*(x\times w)\rangle \end{aligned}$$ □

For $Y=\mathbb{S}$ we have $E_m(\mathbb{S})=\pi_m E$ and the Kronecker pairing (6.14) specializes to pairing

$$\langle -,-\rangle \;:\; E^k(X)\;\otimes\; E_{k+m}(X)\;\longrightarrow\;\pi_m E\ . \tag{6.16}$$

Explicitly, for a cohomology class $f\in E^k(X)$ a homology class $x\in E_{k+m}(X)=\pi_{k+m}(E\wedge^L X)$ we have

$$\langle f,x\rangle \;=\; (E\times f)_*(x)\;\in\;\pi_m(E)\ .$$

The relations of Proposition 6.15 specialise to the following relations.

Proposition 6.17. *The Kronecker pairing* (6.16) *has the following properties.*

(i) *For a morphism $\varphi: X\longrightarrow X'$ in $\mathcal{SHC}$, $f'\in E^k(X')$ and $x\in E_{k+m}(X)$ the relation*

$$\langle\varphi^*(f'),\,x\rangle \;=\; \langle f'\circ\varphi,\,x\rangle \;=\; \langle f',\,\varphi_*(x)\rangle$$

holds in $\pi_m E$.

(ii) *For $f\in E^k(X)$ and $x\in E_{k+m}(X)$ the relation*

$$\langle\Sigma f,\,\Sigma x\rangle \;=\; \langle f,\,x\rangle$$

holds in $\pi_m E$.

(iii) *For all E-cohomology classes $f\in E^k(X)$, $g\in E^l(Y)$ and E-homology classes $x\in E_{k+m}(X)$, $y\in E_{l+n}(Y)$ such that f is central, the relation*

$$\langle f\times g,\,x\times y\rangle \;=\; (-1)^{(k+m)l}\cdot\langle f,x\rangle\cdot\langle g,y\rangle \tag{6.18}$$

holds in the group $\pi_{m+n}E$.

Proof. Parts (i) and (ii) are special cases of (i) and (ii) of Proposition 6.15. For part (iii) we combine parts (iv), (v) and (vi) of Proposition 6.15 and obtain

$$\langle f\times g,\,x\times y\rangle \;=\; (-1)^{kl}\cdot\langle g,\,\langle f,\,x\times y\rangle\rangle \;=\; (-1)^{kl}\cdot\langle g,\,\langle f,\,x\rangle\times y\rangle \;=\; (-1)^{kl+lm}\cdot\langle f,x\rangle\cdot\langle g,y\rangle\ .$$ □

In the special case $X=Y=\mathbb{S}$ of the sphere spectrum, the Kronecker pairing with the unit $1\in\pi_0E=E_0(\mathbb{S})$ is a group homomorphism

$$\langle -,1\rangle \;:\; E^k(\mathbb{S})\;\longrightarrow\;\pi_{-k}E\ .$$

Relation (6.18) specializes to [commutative or x central...]

$$\langle f,1\rangle\cdot\langle g,1\rangle \;=\; \langle f\times g,1\rangle\ ,$$

so the maps $\langle -,1\rangle$ form an isomorphism of graded rings from $E^*(\mathbb{S})$, under exterior product with reversed grading, to π_*E.

In the special case $Y=\mathbb{S}$ the unit map $\iota:\mathbb{S}\longrightarrow E$ is central as an element in $E^0(\mathbb{S})$. In this case we have $\langle\iota,y\rangle=y$ and $f\times\iota=f$, and the relation (6.18) says that for all $f\in E^k(X)$, $x\in E_{k+m}(X)$ and $y\in\pi_nE$ the relation

$$\langle f,\,x\times y\rangle \;=\; \langle f,x\rangle\times y$$

holds in the group $\pi_{m+n}E$.

We can adjoin the Kronecker pairing to a map

$$E^0(X)\;\longrightarrow\;\mathrm{Hom}_{\pi_*E}(E_*(X),\pi_*E)\ ,\quad f\;\longmapsto\;\langle f,-\rangle\ . \tag{6.19}$$

Here Hom_{π_*E} refers to the group of homomorphisms of graded right π_*E-modules, where π_*E acts on $E_*(X)$ by exterior product as in (6.12). The above relation precisely says that for fixed f the map $\langle f,-\rangle : E_*(X) \longrightarrow \pi_*E$ is indeed right π_*E-linear.

Proposition 6.20. *Let E be a homotopy ring spectrum and X be symmetric spectrum.*

(i) *If the E-homology $E_*(X)$ is projective as a graded left π_*E-module, then for every symmetric spectrum Y the map*

$$\mathcal{SHC}(X, E \wedge^L Y) \ \longrightarrow\ \mathrm{Hom}_{\pi_*E}(E_*(X), E_*(Y)) \ , \quad f \ \longmapsto\ (E \cdot f)_*$$

is an isomorphism. In particular, the adjoint Kronecker pairing (6.19) *is an isomorphism from the E-cohomology of X to the π_*E-dual of the E-homology of X.*

(ii) *If $E_*(X)$ is flat as a graded right π_*E-module or if $E_*(Y)$ is flat as a graded left π_*E-module then the map*

$$E_*(X) \otimes_{\pi_*E} E_*(Y) \ \longrightarrow\ E_*(X \wedge^L Y)$$

induced by the extorior product is an isomorphism.

Proof. (i) We prove this claim by comparing both sides of the map (6.19) to the set of morphisms of homotopy E-modules (compare Section 4.1) from $E \wedge^L X$ to $E \wedge^L Y$. A *morphism* between two homotopy E-modules is a morphism $\varphi : M \longrightarrow N$ in the stable homotopy category such that the following square commutes

$$\begin{tikzcd} M \wedge^L E \arrow[r, "\varphi\wedge E"] \arrow[d, "a"'] & N \wedge^L E \arrow[d, "a"] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd}$$

If a morphism φ is an isomorphism in the underlying stable homotopy category, then the inverse φ^{-1} is also a morphism of E-modules.

For every symmetric spectrum X the derived smash product $E \wedge^L X$ becomes a homotopy E-module via the action map

$$E \wedge^L X \wedge^L E \ \xrightarrow{X\wedge\mu}\ E \wedge^L X \ .$$

We claim that the functor which sends a symmetric spectrum X to the E-module $E \wedge^L X$ is left adjoint to the forgetful functor. More precisely, for every homotopy E-module N the map

$$\mathrm{Hom}_{\text{mod-}E}(E \wedge^L X, N) \ \longrightarrow\ \mathcal{SHC}(X, N) \ , \quad f \ \longmapsto\ f \circ (X \wedge \iota)$$

is bijective, where $\iota : \mathbb{S} \longrightarrow E$ is the unit of E. So we call $E \wedge^L X$ the *free E-module generated by X*.

As a special case of this adjunction we obtain that the group $\mathcal{SHC}(X, E \wedge^L Y)$ is naturally isomorphic to the group of E-module homomorphisms from $E \wedge^L X$ to $E \wedge^L Y$. The map in question factors as the composite

$$\mathcal{SHC}(X, E \wedge^L Y) \longrightarrow \mathrm{Hom}_{\text{mod-}E}(E \wedge^L X, E \wedge^L Y) \ \xrightarrow{\pi_*}\ \mathrm{Hom}_{\pi_*E}(E_*(X), E_*(Y))$$

where the first map is the adjunction bijection and the second map takes the effect of a morphism on homotopy groups. So it remains to show that the second map is bijective whenever $E_*(X)$ is projective as a π_*E-module. We prove more generally that for any pair of homotopy E-modules M and N such that π_*M is projective as a graded right π_*E-module the map

$$\pi_* \ : \ \mathrm{Hom}_{\text{mod-}E}(M, N) \ \longrightarrow\ \mathrm{Hom}_{\pi_*E}(\pi_*M, \pi_*N)$$

is bijective. The special case $M = E \wedge^L X$ and $N = E \wedge^L Y$ finishes the proof.

We start with the special case where π_*M is free, and not just projective, as a graded π_*E-module. [better: free of rank one; stable under suspension; wedges] We choose a π_*E-basis $\mathcal{B} = \{x_i\}_{i\in I}$ of homogeneous elements in π_*M. Every element x_i is a homotopy class, so is represented by a morphism $\bar{x}_i : \mathbb{S}^{|x_i|} \longrightarrow M$

in the stable homotopy category. We adjoint this to a morphism of E-modules $\tilde{x}_i : \mathbb{S}^{|x_i|} \wedge E \longrightarrow M$ and form the map

$$\kappa \ : \ \bigvee_{i\in I} \mathbb{S}^{|x_i|} \wedge E \ \longrightarrow \ M$$

whose i-th summand is $\tilde{x}_i$. On homotopy groups, the map κ sends the preferred π_*E-basis of the source to the chosen basis $\mathcal{B}$ on the right, so it is a π_*-isomorphism, hence stable equivalence. In other words, M is isomorphic, as an E-module to the wedge of free E-modules $\bigvee_{i\in I} \mathbb{S}^{|x_i|} \wedge E$. By naturality we can assume that M equals this wedge of free modules. In that case, we have have a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{E\text{-mod}}(\bigvee_{i\in I} \mathbb{S}^{|x_i|} \wedge E, N) \arrow[r, "\pi_*"] \arrow[d] & \mathrm{Hom}_{\pi_*E}(\pi_*(\bigvee_{i\in I} \mathbb{S}^{|x_i|} \wedge E), \pi_*N) \arrow[d, "\mathrm{ev}_{\mathcal{B}}"] \\
\prod_{i\in I} \mathcal{SHC}(\mathbb{S}^{|x_i|}, N) \arrow[r] & \prod_{i\in I} \pi_{|x_i|}N
\end{tikzcd}$$

The left vertical map is bijective by the universal property of the wedge and the free-forgetful adjunction. The right vertical map is evaluation at the preferred π_*E-basis, hence bijective. The lower horizontal map is evaluation at the fundamental classes, hence bijective.

Now we treat the general case, i.e., we suppose that π_*M is projective as a graded right module over π_*E. So there is a graded free π_*E-module and and π_*E-linear maps

$$\pi_*M \ \xrightarrow{\ i\ } \ F \ \xrightarrow{\ q\ } \ \pi_*M$$

such that qi is the identity of π_*M. As in the first part we can realize F by a homotopy E-module G, so we can assume without loss of generality that $F = \pi_*G$. Then by the first part there is a morphism $p : G \longrightarrow M$ such that $\pi_*p = q : \pi_*G \longrightarrow \pi_*M$.

Claim: there is a morphism $s : M \longrightarrow G$ of homotopy E-modules such that ps is the identity of M.

By the claim, M is a retract of the homotopy E-module G. The class of E-modules for which the map [...] is bijective is stable under retracts, and contains G by the above. So the map [...] is bijective for M

It remains to prove the claim. By the first part, the idempotent π_*E-linear self map $iq : \pi_*G \longrightarrow \pi_*G$ is realizable by a homomorphism of E-modules $\epsilon : G \longrightarrow G$.

Claim: The map

$$\mathrm{Hom}(p, N) \ : \ \mathrm{Hom}_{\text{mod-}E}(M, N) \ \longrightarrow \ \mathrm{Hom}_{\text{mod-}E}(G, N)$$

is injective and its image is equal to the image of the endomorphism $\mathrm{Hom}(\epsilon, N)$ of $\mathrm{Hom}_{\text{mod-}E}(G, N)$.

When X is fixed, both sides of the map are homological functors in Y; for the right hand side this uses that $E_*(X)$ is projective, so that the functor $\mathrm{Hom}_{\pi_*E}(E_*(X), -)$ an exact functor on graded left π_*E-modules. The transformation is an isomorphism for $Y = \mathbb{S}$ the sphere spectrum, by part (i). The map is thus an isomorphism for all symmetric spectra Y by [...]

(ii) We consider the case where $E_*(X)$ is flat, the other case is analogous. The functor $E_k(X \wedge^L -)$ is homological as the composite of an exact functor $X \wedge^L -$ and a homological functor $E_k(-)$. The functor $(E_*(X) \otimes_{\pi_*E} E_*(-))_k$ takes wedges to sums, and it takes triangles to exact sequences because Since $E_*(X)$ is flat and hence $E_*(X) \otimes_{\pi_*E} -$ an exact functor on graded left π_*E-modules. In the case $Y = \mathbb{S}$ both sides of the map reduce to $E_*(X)$. So the map in question is a natural transformation between exact functors on $\mathcal{SHC}$ that is an isomorphism for the sphere spectrum. The map is thus an isomorphism for all symmetric spectra Y by [...] □

6.2. Ordinary homology and cohomology. The 'ordinary' homology and cohomology groups of a symmetric spectrum are the homology and cohomology groups with respect to the Eilenberg-Mac Lane spectrum HA of an abelian group A. In this section we look more closely at this special case.

Definition 6.21. Let A be an abelian group, k an integer and X a symmetric spectrum. The k-th *homology group of X with coefficients in A* is defined as

$$H_k(X, A) \ = \ (HA)_k(X) \ = \ \pi_k(HA \wedge X) \ .$$

The k-th *cohomology group* of the symmetric spectrum X with coefficients in A is defined as

$$H^k(X,A) \;=\; HA^k(X) \;=\; \mathcal{SHC}(\mathbb{S}^{-k}\wedge X, HA)\ ,$$

the group of morphisms of degree $-k$ from X to the Eilenberg-Mac Lane spectrum of A in the stable homotopy category.

Since the Eilenberg-Mac Lane spectrum HA is flat [ref], deriving the smash product is not necessary (compare the remark above); so we have taken the liberty to use the pointset level smash product, and not the derived smash product, in the definition of ordinary homology groups.

In Exercise E.II.13 we show that for semistable symmetric spectra X the (co-)homology of X with coefficients in an abelian groups can be calculate from a chain complex associated to the symmetric spectrum.

If the symmetric spectrum X is semistable, then its A-(co-)homology groups can be calculated from the (co-)homology groups of the simplicial sets X_n. Indeed, if we specialize Proposition 6.5 to $E = HA$ and combine with the isomorphism $H_*(\Sigma^\infty K, A)\cong \tilde{H}_*(K,A)$ we obtain that the natural map

$$\operatorname{colim}_n \tilde{H}_{k+n}(X_n,A) \;\longrightarrow\; H_k(X,A)$$

is an isomorphism, where the colimit is formed over the system of maps

$$\tilde{H}_{k+n}(X_n,A) \xrightarrow{-\wedge\iota} \tilde{H}_{k+n+1}(X_n\wedge S^1,A) \xrightarrow{(\sigma_n)_*} \tilde{H}_{k+n+1}(X_{n+1},A)$$

with the first map being the suspension isomorphism.

Now we identify the cohomology of suspension spectra with the reduced cohomology of simplicial sets. We let K be a based simplicial set, A and abelian group and $k\geq 0$. The simplicial set $(HA)_k = A[S^k]$ is an Eilenberg-Mac Lane space of type (A,k), and as such represents cohomology. So every reduced cohomology class in $\tilde{H}^k(K,A)$ is of the form $f^*(\iota_{A,k})$ for a based map $f : K\longrightarrow A[S^k]$, unique up to based homotopy, where $\iota_{A,k}$ is the fundamental class in $\tilde{H}^k(A[S^k],A)$. We let

$$\hat{f} \;:\; \mathbb{S}^{-k}\wedge\Sigma^\infty K \;\longrightarrow\; HA$$

be the morphism of symmetric spectra that is freely generated by $f : K\longrightarrow A[S^k]$ (using that $\mathbb{S}^{-k}\wedge\Sigma^\infty K = F_k\wedge\Sigma^\infty K$ is isomorphic to the free symmetric spectrum F_kK). Then the homotopy class of $\hat{f}$ only depends on the cohomology class $f^*(\iota_{A,k})$ so we can define a map

$$\tilde{H}^k(K,A) \;\longrightarrow\; \mathcal{SHC}(\mathbb{S}^{-k}\wedge\Sigma^\infty K, HA) \;=\; H^k(\Sigma^\infty K,A) \qquad\text{by}\qquad f^*(\iota_{A,n}) \;\longmapsto\; \gamma(\hat{f})\ . \tag{6.22}$$

Proposition 6.23. *For every based simplicial set K, every abelian group A and all $k\geq 0$ the map (6.22) from the reduced cohomology of K to the cohomology of the suspension spectrum $\Sigma^\infty K$ is an isomorphism. Moreover, the cohomology group $H^k(\Sigma^\infty K,A)$ is trivial for negative k.*

[also for homology]

Proof. The cohomology group $H^k(\Sigma^\infty K,A) = \mathcal{SHC}(\mathbb{S}^{-k}\wedge\Sigma^\infty K, HA)$ is naturally isomorphic to the group $\mathcal{SHC}(\Sigma^\infty K, \mathrm{sh}^k(HA))$ in such a way that the map

$$[K, A[S^k]] \;\longrightarrow\; \mathcal{SHC}(\mathbb{S}^{-k}\wedge\Sigma^\infty K, HA)\ ,\quad [f]\longmapsto\gamma(\hat{f})$$

corresponds to the map

$$[K, A[S^k]] \;\longrightarrow\; \mathcal{SHC}(\Sigma^\infty K, \mathrm{sh}^k(HA))\ ,\quad [f]\longmapsto\gamma(\hat{f})$$

that we recognized as bijective in Example 1.18. So the first map, and hence also (6.22), is bijective. $\square$

Again for semistable symmetric spectra X the cohomology groups can be related to the cohomology groups of the simplicial sets which make up X. Indeed, as a special case of Proposition 6.5 (ii) for the Eilenberg-Mac Lane spectrum HA we obtain a natural short exact sequence

$$0\longrightarrow \lim^1_n \tilde{H}^{k+n-1}(X_n,A)\longrightarrow H^k(X,A)\longrightarrow \lim_n \tilde{H}^{k+n}(X_n,A)\longrightarrow 0$$

where the limit is taken over the inverse system of reduced cohomology groups

$$\tilde{H}^{k+n+1}(X_{n+1}, A) \xrightarrow{(\sigma_n)^*} \tilde{H}^{k+n+1}(X_n \wedge S^1, A) \cong \tilde{H}^{k+n}(X_n, A)$$

and the derived limit is taken of the analogous sequence with dimensions shifted by 1.

We consider a short exact sequence of abelian groups

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0 .$$

Then the triangle

$$HA \xrightarrow{Hi} HB \xrightarrow{Hj} HC \xrightarrow{\delta_{i,j}} \Sigma(HA)$$

is distinguished, where $\delta_{i,j}$ is the associated Bockstein morphism (see Proposition 5.28 (i)). By Proposition 3.19 the triangle

$$HA \wedge X \xrightarrow{Hi \wedge X} HB \wedge X \xrightarrow{Hj \wedge X} HC \wedge X \xrightarrow{\kappa_{HA,X}(\delta_{i,j} \wedge X)} \Sigma(HA \wedge X)$$

is then again distinguished. Passing to homotopy groups gives a long exact sequence of homology groups [ref: gradings]

$$\cdots \longrightarrow H_k(X, A) \xrightarrow{H_k(X,i)} H_k(X, B) \xrightarrow{H_k(X,j)} H_k(X, C) \xrightarrow{\delta} H_{k-1}(X, A) \longrightarrow \cdots .$$

By mapping X into the distinguished triangle (compare Proposition 2.10 (i)) we obtain a long exact sequence of cohomology groups

$$\cdots \longrightarrow H^k(X, A) \xrightarrow{H^k(X,i)} H^k(X, B) \xrightarrow{H^k(X,j)} H^k(X, C) \xrightarrow{\delta} H^{k+1}(X, A) \longrightarrow \cdots .$$

The 'ordinary' spectrum homology $H_k(X, A)$ admits another interpretation as the homotopy groups of the 'linearization' of X.

We let A be an abelian group and K and L based simplicial sets. The *assembly map*

$$A[K] \wedge L \longrightarrow A[K \wedge L]$$

is given by 'reparenthesising', i.e., by sending $(\sum a_i k_i) \wedge l$ to $\sum a_i(k_i \wedge l)$.

Definition 6.24. For a symmetric spectrum of simplicial sets X and an abelian group A the *A-linearization* $A[X]$ is obtained by applying the A-linearization functor to X levelwise and dimensionswise. More precisely, $A[X]$ is the symmetric spectrum given in level n by $A[X]_n = A[X_n]$, the dimensionwise reduced A-linearization of X_n. The Σ_n-action is induced by the action on X_n and the structure map is the composite

$$A[X_n] \wedge S^1 \xrightarrow{\text{assembly}} A[X_n \wedge S^1] \xrightarrow{A[\sigma_n]} A[X_{n+1}] .$$

If B is a ring and A is a B-module, then the linearization $A[X]$ is naturally an HB-module spectrum [...].

As an example of linearization which we have already seen, the Eilenberg-Mac Lane spectrum HA of Example I.1.14 equals (the underlying symmetric spectrum of) the linearization $A[\mathbb{S}]$ of the sphere spectrum. The assembly maps assemble into an assembly map of symmetric spectra. Indeed, for $n, m \geq 0$ the maps

$$(HA)_n \wedge X_m = A[S^n] \wedge X_m \xrightarrow{\text{assembly}} A[S^n \wedge X_m] \longrightarrow A[X_{n+m}]$$

form a bimorphism from the pair (HA, X) to the linearization $A[X]$. So the universal property of the smash product provides a corresponding assembly morphism of symmetric spectra

$$HA \wedge X \longrightarrow A[X] .$$

This assembly map is a $\hat{\pi}_*$-isomorphism by Proposition I.7.14 (iii), thus a stable equivalence. The A-homology of X can then be calculated as the true homotopy groups of the linearization $A[X]$, i.e., the assembly map induces an isomorphism

$$H_k(X, A) = \pi_k(HA \wedge X) \xrightarrow{\cong} \pi_k A[X] .$$

The composite

$$X = \mathbb{S} \wedge X \xrightarrow{\iota \wedge X} H\mathbb{Z} \wedge X \xrightarrow{\text{assembly}} \mathbb{Z}[X]$$

is the morphism given by the inclusion of generators. So under the assembly isomorphism between $H_k(X, \mathbb{Z})$ and $\pi_k \mathbb{Z}[X]$ the Hurewicz homomorphism $\pi_* X \longrightarrow H_*(X, \mathbb{Z})$ becomes the effect on homotopy groups of the 'inclusion of generators'.

Let us specialize the results of the previous Section 6.1 on pairings in generalized (co)homology theories to ordinary (co)homology, i.e., for $E = HA$, the symmetric Eilenberg-Mac Lane ring spectrum of an ordinary ring A. In Construction 6.6 we defined exterior products in generalized E-homology and E-cohomology. If we specialize to $E = HA$, the Eilenberg-Mac Lane spectrum of a ring A, we obtain *exterior products* in ordinary homology and cohomology:

$$\begin{aligned} \times \ &: \ H_k(X, A) \otimes H_l(Y, A) \ \longrightarrow \ H_{k+l}(X \wedge^L Y, A) \\ \times \ &: \ H^k(X, A) \otimes H^l(Y, A) \ \longrightarrow \ H^{k+l}(X \wedge^L Y, A) \ . \end{aligned}$$

Because of the associativity property of the exterior products, both pairings factor over the tensor products over the ring A. For $E = HA$, part (ii) of Proposition 6.20 then specializes to:

Proposition 6.25 (Künneth theorem)**.** *Let A be a commutative ring and let X and Y be symmetric spectra. Suppose that for each $k \in \mathbb{Z}$ the homology $H_k(X, A)$ is flat as an A-module. Then for every $n \in \mathbb{Z}$ the map*

$$\bigoplus_{k+l=n} H_k(X, A) \ \otimes_A \ H_l(Y, A) \ \longrightarrow \ H_n(X \wedge^L Y, A) \tag{6.26}$$

given by exterior product is an isomorphism.

For an Eilenberg-Mac Lane spectrum the Kronecker pairing (6.16) then specializes to a map

$$\langle -, - \rangle \ : \ H^k(X, A) \ \otimes \ H_k(X, A) \longrightarrow \ A \ ,$$

where we set $m = 0$ and used the preferred isomorphism between $\pi_0 HA$ and A. (Since the homotopy groups of HA are concentrated in dimension zero, the cap product is uninteresting for $m \neq 0$.) The relation (6.18) between Kronecker pairing and exterior product becomes

$$a \cdot \langle f, x \rangle \ = \ \langle f, \, a \cdot x \rangle \quad \text{and} \quad \langle f, x \rangle \cdot a \ = \ \langle f, \, x \cdot a \rangle$$

in the ring A for $f \in H^k(X, A)$, $x \in H_k(X, A)$ and $a \in A$. So the adjoint of the Kronecker pairing is a right A-linear map

$$H^k(X, A) \ \longrightarrow \ \mathrm{Hom}_{\text{mod-}A}(H_k(X, A), A) \ , \quad f \ \longmapsto \ \langle f, - \rangle \ . \tag{6.27}$$

[Exercise: for (co)homology of spaces, pair becomes evaluation of a cocycle on a cycle]

As a special case of Proposition 6.20 (i) we obtain the following proposition. Note that the projectivity hypothesis on $H_k(X, A)$ is automatically satisfied when every right A-module is projective, for example, when A is a field. When $A = \mathbb{Z}$ is the ring of integers, then following proposition also follows from the universal coefficient theorem for cohomology [ref].

Proposition 6.28. *Let A be a ring and X symmetric spectrum such that all homology modules $H_k(X, A)$ of X with coefficients in A are projective as right A-modules. Then the adjoint Kronecker pairing* (6.27) *is an isomorphism from the cohomology $H^k(X, A)$ to the A-dual of $H_k(X, A)$.*

In (6.18) we established a compatibility between the Kronecker pairing and the exterior product in generalized (co)homology. When we specialize this to ordinary homology and cohomology, this becomes the relation

$$\langle f \times g, \, x \times y \rangle \ = \ (-1)^{kl} \cdot \langle f, x \rangle \cdot \langle g, y \rangle \tag{6.29}$$

in A, for symmetric spectra X and Y and all cohomology classes $f \in H^k(X, A)$, $g \in H^l(Y, A)$ and homology classes $x \in H_k(X, A)$, $y \in H_l(Y, A)$.

There are various relations between homotopy and homology groups of spectra. The Hurewicz and Whitehead theorems for singular homology of spaces immediately imply Hurewicz and Whitehead theorems for spectrum homology. The only caveat is that the Whitehead theorem only applies to spectra whose homotopy groups are bounded below. The unit morphism $\mathbb{S} \longrightarrow H\mathbb{Z}$ can be smashed with a symmetric spectrum to yield a morphism $X = \mathbb{S} \wedge X \longrightarrow H\mathbb{Z} \wedge X$. The *Hurewicz homomorphism* is the natural morphism of abelian groups

$$h \; : \; \pi_k X \; \longrightarrow \; H_k(X, \mathbb{Z})$$

induced by this morphism on homotopy groups. This is closely related to the classical Hurewicz homomorphism for topological spaces: if X is semistable, then the source of the Hurewicz homomorphism is isomorphic to the colimit of the homotopy groups $\pi_{k+n} X_n$, whereas the target group is isomorphic to the colimit of the reduced homology groups $H_{k+n}(X_n, \mathbb{Z})$ [...]. Exercise E.II.15 shows the Hurewicz homomorphism which we just defined corresponds to the map induced by the classical Hurewicz homomorphisms for the spaces X_n by suitable passage to colimits.

Proposition 6.30. (i) (Stable Hurewicz theorem) *Let X be a $(k-1)$-connected symmetric spectrum for some integer k. Then the homology groups of X are trivial below dimension k and the Hurewicz homomorphism $\pi_k X \longrightarrow H_k(X, \mathbb{Z})$ is an isomorphism.*

(ii) (Stable Whitehead theorem) *Let $f : X \longrightarrow Y$ be a morphism between symmetric spectrum whose homotopy groups are bounded below. Then f is a stable equivalence if and only if it induces isomorphisms on all integral homology groups.*

(iii) *Let A be a uniquely divisible abelian group (i.e., a $\mathbb{Q}$-vector space). Then the smash product pairing*

$$A \otimes \pi_k X = \pi_0(HA) \otimes \pi_k X \;\stackrel{\cdot}{\longrightarrow}\; \pi_k(HA \wedge X) = H_k(X, A)$$

is an isomorphism for every integer k. In particular, the Hurewicz homomorphism induces an isomorphisms

$$\mathbb{Q} \otimes \pi_k X \;\cong\; \mathbb{Q} \otimes H_k(X, \mathbb{Z}) \;\cong\; H_k(X, \mathbb{Q})$$

for all symmetric spectra X and integers k.

Proof. (i) This is a special case of Proposition 5.22, or rather its corollary Remark 5.23. Since the Eilenberg-Mac Lane spectrum $H\mathbb{Z}$ is flat and (-1)-connected, the natural map

$$\cdot \; : \; \pi_0 H\mathbb{Z} \; \otimes \; \pi_k X \; \longrightarrow \; \pi_k(H\mathbb{Z} \wedge X)$$

is an isomorphism of abelian groups. The group $\pi_0(H\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and the Hurewicz map is given by $1 \cdot - : \pi_k X \longrightarrow \pi_k(H\mathbb{Z} \wedge X)$, so it is an isomorphism.

(ii) One direction does not need the hypothesis that X and Y are bounded below: if $f : X \longrightarrow Y$ is a stable equivalence, then so is $H\mathbb{Z} \wedge f : H\mathbb{Z} \wedge X \longrightarrow H\mathbb{Z} \wedge Y$ since the Eilenberg-Mac Lane spectrum is flat [ref].

Suppose conversely that f is an integral homology isomorphism. Then [level cofibrant] the mapping cone $C(f)$ has trivial integral homology by the long exact sequence of homology groups [ref]. Since the homotopy groups of X and Y are bounded below, so are the homotopy groups of the cone, by the long exact sequence of homotopy groups [ref]. So if $C(f)$ had a non-trivial homotopy group, there would be a minimal one, and by part (i) there would also be a non-trivial homology group in that minimal dimension. This would contract what we concluded before, so all homotopy groups of the mapping cone $C(f)$ vanish, and so f is a stable equivalence, one more time by the long exact sequence of homotopy groups.

(iii) We let $\mathcal{X}$ be the smallest class of symmetric spectra X for which that map $A \otimes \pi_k X \longrightarrow H_k(X, A)$ is an isomorphism for every integer k. This class is closed under stable equivalences, thus under isomorphism in the stable homotopy category. The sphere spectrum $\mathbb{S}$ belongs to $\mathcal{X}$ by Serre's calculation of homotopy groups of spheres modulo torsion. Both sides of the map commute with sums, so the class $\mathcal{X}$ is closed under sums in $\mathcal{SHC}$. Finally, both sides of the map take distinguished triangles in the stable homotopy category to long exact sequences, so the class $\mathcal{X}$ is closed under extensions by the 5-lemma. In other words, $\mathcal{X}$ is a localizing subcategory of the stable homotopy category which contains the sphere spectrum, so it contains all symmetric spectra by Proposition 5.16 □

Example 6.31. The hypothesis in the Whitehead theorem (Proposition 6.30 (ii)) that homotopy groups are bounded below is essential. In general, $H_*(-;\mathbb{Z})$-isomorphisms need not be stable equivalences. As an example we let $\mathbb{S}/p$ denote the mod-p Moore spectrum, for a prime number p. This spectrum is a mapping of the degree p map of the sphere spectrum, i.e., can be defined as the third term in a distinguished triangle

$$\mathbb{S} \xrightarrow{\cdot p} \mathbb{S} \xrightarrow{j} \mathbb{S}/p \xrightarrow{\delta} \mathbb{S}^1 .$$

Alternatively, $\mathbb{S}/p$ could be defined as $\Sigma^{-1}\Sigma^\infty M(p)$, the desuspension of a mod-p Moore space $M(p) = S^1 \cup_p D^2$. The mod-p Moore spectrum is characterized up to stable equivalence by the property that it is connective, its integral spectrum homology is concentrated in dimension 0, and $H_0(\mathbb{S}/p, \mathbb{Z})$ is cyclic of order p.

Since $\mathbb{S}/p$ is the mapping cone of the degree p map on the sphere spectrum, every generalized homology theory E_* gives rise to a long exact sequence of homology groups

$$\cdots \longrightarrow E_k(\mathbb{S}) \xrightarrow{\times p} E_k(\mathbb{S}) \xrightarrow{j_*} E_k(\mathbb{S}/p) \xrightarrow{\delta} E_{k-1}(\mathbb{S}) \xrightarrow{\times p} E_{k-1}(\mathbb{S}) \xrightarrow{j_*} \cdots$$

which splits up into short exact sequences

$$0 \longrightarrow \mathbb{Z}/p \otimes E_k(\mathbb{S}) \longrightarrow E_k(\mathbb{S}/p) \longrightarrow {}_pE_{k-1}(\mathbb{S}) \longrightarrow 0 \tag{6.32}$$

(${}_pA$ denotes the subgroup of an abelian group A consisting of those elements a which satisfy $pa = 0$).

In particular, this applies to periodic complex topological K-theory KU_* (compare Example I.1.20). The coefficients of complex K-theory are 2-periodic and $KU_k(\mathbb{S}) = \pi_k KU$ is a free abelian group of rank one when k is even and trivial when k is odd. So for KU-theory, the short exact sequence (6.32) shows that

$$KU_k(\mathbb{S}/p) \cong \begin{cases} \mathbb{Z}/p & \text{if } k \text{ is even, and} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Adams showed that for every odd prime p there exists a morphism

$$v_1 \ : \ \Sigma^{2p-2}\mathbb{S}/p \longrightarrow \mathbb{S}/p$$

in the stable homotopy category which induces an isomorphism in KU-homology. This implies that every iterated composition of (suspended copies of) the map v_1 induces an isomorphism in KU-homology, hence every such composite is stably essential. A map with this property is called a *periodic* self-map. The Moore space at the prime 2 also admits a periodic self-map, but the period and some other details are different from the situation at odd primes.

We define a spectrum $v_1^{-1}\mathbb{S}/p$ by 'inverting the self-map v_1', i.e., as the homotopy colimit of the sequence of symmetric spectra

$$\mathbb{S}/p \xrightarrow{\Sigma^{-2p+2}v_1} \mathbb{S}^{-2p+2}/p \xrightarrow{\Sigma^{-4p+4}v_1} \mathbb{S}^{-4p+4}/p \xrightarrow{\Sigma^{-6p+6}v_1} \cdots .$$

Every spectrum in this sequence is essentially a suspension spectrum, shifted down finitely many dimensions, but the telescope is no longer a shift of any suspension spectrum. The homology of a homotopy colimit is the colimit of the homologies, i.e., the natural map

$$\operatorname{colim}_n E_k(\mathbb{S}^{(-2p+2)n}/p) \longrightarrow E_k(v_1^{-1}\mathbb{S}/p)$$

is an isomorphism for every generalized homology theory E. The Adams map v_1 and any desuspension of it induces the trivial map in spectrum homology simply because the homology of source and target are concentrated in different dimensions; so the integral spectrum homology of the mapping telescope $v_1^{-1}\mathbb{S}/p$ is trivial. On the other hand, the Adams map induces an isomorphicm in KU-homology, so the map $\mathbb{S}/p \longrightarrow v_1^{-1}\mathbb{S}/p$ from the initial term to the homotopy colimit induces an isomorphism

$$KU_*(\mathbb{S}/p) \longrightarrow KU_*(v_1^{-1}\mathbb{S}/p) .$$

In particular, the spectrum $v_1^{-1}\mathbb{S}/p$ has non-trivial KU-homology.

To sum up, $v_1^{-1}\mathbb{S}/p$ has trivial spectrum homology, but it is not stably contractible. We conclude that $v_1^{-1}\mathbb{S}/p$ must have non-trivial homotopy groups in arbitrarily low dimensions. The homotopy groups of this mapping telescope were determined by Miller [**59**, Cor. 4.12], who showed that

$$\pi_n\left(v_1^{-1}\mathbb{S}/p\right) = \begin{cases} \mathbb{Z}/p & \text{for } n \equiv -1, 0 \text{ (modulo } 2p-2), \\ 0 & \text{else.} \end{cases}$$

This calculation requires much more sophisticated tools than we have available.

6.3. Moore spectra.

Definition 6.33. A *Moore spectrum* is a connective symmetric spectrum X for which the homology groups $H_k(X, \mathbb{Z})$ are trivial for $k \neq 0$. Given an abelian group A, a *Moore spectrum for* A is a Moore spectrum X endowed with an isomorphism between $H_0(X, \mathbb{Z})$ and A.

When no confusion can arise we suppress the isomorphism and abuse notation by writing $H_0(X, \mathbb{Z}) = A$ (instead of using the isomorphism which is part of the data of a Moore spectrum). We often write $\mathbb{S}A$ for a Moore spectrum for A; however, we emphasize that $\mathbb{S}A$ is *not* functorial in the abelian groups. In general, the construction of Moore spectra is not even functorial in the stable homotopy category, although it is when we stay away from 2-torsion. We discuss the functoriality (or rather the lack thereof) of Moore spectra in Theorem 6.43 below.

Remark 6.34. The spectrum homology group $H_k(X, \mathbb{Z})$ was defined as $\pi_k(H\mathbb{Z} \wedge X)$, so if X is a Moore spectrum then the spectrum $H\mathbb{Z} \wedge X$ has trivial homotopy groups in all nonzero dimensions. By Theorem 5.25 (ii) we deduce that $H\mathbb{Z} \wedge X$ is then stably equivalent to the Eilenberg-Mac Lane spectrum $H(\pi_0 X)$. We conclude that X is a Moore spectrum if and only if it is connective and $H\mathbb{Z} \wedge X$ and the stably equivalent linearization $\mathbb{Z}[X]$ are stably equivalent to $H(\pi_0 X)$.

If the homotopy groups of X are bounded below, then by the Hurewicz theorem (Proposition 6.30) the first non-trivial homotopy group of X is isomorphic to the first non-trivial integral homology group. So the condition that a Moore spectrum is connective can be weakened to the requirement that the homotopy groups are bounded below. However, some condition of this kind is necessary if we want Moore spectra to be determined by the 0th homology groups. In fact, there exist spectra X with non-trivial homotopy groups in arbitrary low dimensions such that the homology group $H_k(X, \mathbb{Z})$ is trivial for all integers k. An example is *mod-n topological* K*-theory* for any $n \geq 2$, which is the symmetric spectrum $\mathbb{S}\mathbb{Z}/n \wedge KU$ where $\mathbb{S}\mathbb{Z}/n$ is a flat mod-n Moore spectrum and KU is the symmetric ring spectrum representing complex topological K-theory from Example I.**??**. Compare Example 6.31.

Example 6.35. The sphere spectrum $\mathbb{S}$ is a Moore spectrum for the group $\mathbb{Z}$ of integers. The 'sphere spectrum with m inverted' $\mathbb{S}[1/m]$ of Example I.3.47 is a Moore spectrum for the group $\mathbb{Z}[1/m]$ of integers with m inverted. The symmetric spectrum underlying the ring spectrum $\mathbb{S}[1/2, i]$ of Example I.3.49 is a Moore spectrum for the abelian group underlying the Gaussian integers with 2 inverted.

Example 6.36. If A is any uniquely divisible abelian group (i.e., a vector space over the rational numbers), then the Eilenberg-Mac Lane spectrum HA is also a Moore spectrum for the group A. Indeed, HA is connective, so $H_0(HA, \mathbb{Z}) \cong \pi_0 HA = A$ by the Hurewicz Theorem (Proposition 6.30 (i)). The group $H_k(HA, \mathbb{Z})$ is isomorphic to $H_k(H\mathbb{Z}, A)$ (induced by the symmetry isomorphism $\tau : HA \wedge H\mathbb{Z} \cong H\mathbb{Z} \wedge HA$) which by Proposition 6.30 (iii) is isomorphic to $A \otimes \pi_k H\mathbb{Z}$ and thus trivial for $k \neq 0$.

Example 6.37. Suppose K is a *Moore space* for the abelian group A of dimension n, i.e., the reduced integral homology of K is concentrated in dimension n where we have $\tilde{H}_n(K, \mathbb{Z}) \cong A$. Then the spectrum $\Omega^n(\Sigma^\infty K)$ is a Moore spectrum for the group A. Indeed, suspension spectra are connective, and looping shifts homotopy groups, so $\Omega^n(\Sigma^\infty K)$ is connective. A chain of isomorphisms

$$H_k(\Omega^n(\Sigma^\infty K), \mathbb{Z}) \cong H_{k+n}(S^n \wedge \Omega^n(\Sigma^\infty K), \mathbb{Z}) \cong H_{k+n}(\Sigma^\infty K, \mathbb{Z}) \cong \tilde{H}_{k+n}(K, \mathbb{Z})$$

is given by the suspension isomorphism for homology, the fact that the adjunction counit $S^n \wedge \Omega^n(\Sigma^\infty K)$ is a stable equivalence [ref] and the isomorphism [...]. So the spectrum homology of $\Omega^n(\Sigma^\infty K)$ is indeed concentrated in dimension 0, where it is isomorphic to A.

Example 6.38. Let A be a subring of the ring $\mathbb{Q}$ of rational numbers. Then we can write down a commutative symmetric ring spectrum which is a Moore spectrum for A. First we introduce a more general construction based on a pair (K, f) consisting of a base space (or simplicial set) K and a based map $f : S^1 \longrightarrow K$. From this data we define a commutative symmetric ring spectrum $S(K, f)$ with levels

$$S(K, f)_n \;=\; K^{\wedge n}$$

with Σ_n permuting the smash factors. The multiplication map $\mu_{n,m} : K^{\wedge n} \wedge K^{\wedge m} \longrightarrow K^{\wedge(n+m)}$ is the canonical isomorphism. The unit map $\iota_0 : S^0 \longrightarrow K^{\wedge 0}$ is the identity and the unit map $\iota_1 : S^1 \longrightarrow K$ is the given map f. We have already seen special cases of this construction: $S(S^1, \mathrm{Id}) = \mathbb{S}$ is the sphere spectrum, and if $\varphi_m : S^1 \longrightarrow S^1$ is a map of degree m, then $S(S^1, \varphi_m) = \mathbb{S}[1/m]$ is the 'sphere spectrum with m inverted' as defined in Example I.3.47. The functor S from the category of based space under S^1 to commutative symmetric ring spectra is left adjoint to the 'evaluation' or forgetful functor which sends a commutative symmetric ring spectrum R to the pair (R_1, ι_1).

We return to the situation of a subring A of $\mathbb{Q}$. We choose a Moore space M for A, i.e., a CW-complex (respectively simplicial set) whose reduced integral homology is concentrated in dimension 1, where it is isomorphic to A. We also choose a based map $\iota : S^1 \longrightarrow M$ which sends the fundamental homology class of the circle to the class in $\tilde{H}_1(M; \mathbb{Z}) \cong A$ which corresponds to the unit element of the ring A. We claim that then the commutative symmetric ring spectrum $S(M, \iota)$ is a Moore spectrum for the ring A.

For $n \geq 1$ the structure map

$$S(M, \iota)_n \wedge S^1 = M^{\wedge n} \wedge S^1 \;\xrightarrow{\;\sigma_n\;}\; M^{\wedge n+1} = S(M, \iota)_{n+1}$$

is a homology isomorphism by the Künneth theorem for space level singular homology and since $A \otimes A$ is isomorphic to A. Since source and target of σ_n are simply connected CW-complexes, the structure map is a weak equivalence. As a consequence, the shifted spectrum $\mathrm{sh}(S(M, \iota))$ is level equivalent to the suspension spectrum of M. So $S(M, \iota)$ is stably equivalent to $\Omega(\Sigma^\infty M)$, and thus a Moore spectrum by Example 6.37. This also shows that the symmetric spectrum $S(M, \iota)$ is semistable, which is not generally the case for $S(X, f)$.

Construction 6.39. Now we give a construction of a Moore spectrum for a given group A in terms of the triangulated structure of the stable homotopy category. This construction also allows us to calculate the homotopy groups of Moore spectra and the groups of maps out of Moore spectra.

We choose a free presentation of A, i.e., a short exact sequence

$$0 \;\longrightarrow\; \mathbb{Z}[I] \;\xrightarrow{\;d\;}\; \mathbb{Z}[J] \;\xrightarrow{\;e\;}\; A \;\longrightarrow\; 0$$

where I and J are indexing sets. We have

$$\pi_0(\bigoplus_I \mathbb{S}) \;\cong\; \bigoplus_I \pi_0\mathbb{S} \;\cong\; \mathbb{Z}[I]$$

and similarly for the sum of sphere spectra indexed by the set J. Since $\mathbb{S}$ represents the functor π_0 we can realize the map $d : \mathbb{Z}[I] \longrightarrow \mathbb{Z}[J]$ by a morphism $\bar{d} : \bigoplus_I \mathbb{S} \longrightarrow \bigoplus_J \mathbb{S}$, i.e., $\pi_0(\bar{d})$ equals d under the isomorphisms. Now we choose a distinguished triangle

$$\bigoplus_I \mathbb{S} \;\xrightarrow{\;\bar{d}\;}\; \bigoplus_J \mathbb{S} \;\longrightarrow\; \mathbb{S}A \;\longrightarrow\; \bigoplus_I \Sigma\mathbb{S}\;. \tag{6.40}$$

We claim that $\mathbb{S}A$ is a Moore spectrum for the group A.

We first describe the homotopy groups of $\mathbb{S}A$ in terms of A and the stable homotopy groups of spheres. The long exact homotopy sequence of this triangle contains the exact sequence

$$\pi_n(\bigoplus_I \mathbb{S}) \;\xrightarrow{\;\pi_n\bar{d}\;}\; \pi_n(\bigoplus_J \mathbb{S}) \;\longrightarrow\; \pi_n\mathbb{S}A \;\longrightarrow\; \pi_{n-1}(\bigoplus_I \mathbb{S}) \;\longrightarrow\; \pi_{n-1}(\bigoplus_J \mathbb{S})\;.$$

Using that homotopy groups preserve sums we can rewrite this as an exact sequence

$$\mathbb{Z}[I] \otimes \pi_n\mathbb{S} \xrightarrow{d\otimes\pi_n\mathbb{S}} \mathbb{Z}[J] \otimes \pi_n\mathbb{S} \longrightarrow \pi_n\mathbb{S}A \longrightarrow \mathbb{Z}[I] \otimes \pi_{n-1}\mathbb{S} \xrightarrow{d\otimes\pi_{n-1}\mathbb{S}} \mathbb{Z}[J] \otimes \pi_{n-1}\mathbb{S}\;.$$

Since we started from a free resolution of A, cokernel respectively kernel of the morphism $d \otimes \pi_n \mathbb{S}$ are the groups $A \otimes \pi_n \mathbb{S}$ respectively $\mathrm{Tor}(A, \pi_n \mathbb{S})$. So the long exact homotopy sequence decomposes into short exact sequences

$$0 \longrightarrow A \otimes \pi_n^{\mathrm{s}} \longrightarrow \pi_n \mathbb{S}A \longrightarrow \mathrm{Tor}(A, \pi_{n-1}^{\mathrm{s}}) \longrightarrow 0 . \tag{6.41}$$

This shows in particular that $\mathbb{S}A$ is connective and gives an isomorphism between $\pi_0 \mathbb{S}A$ and A. The long exact homology sequence of the triangle (6.40) and the vanishing of the homology of $\mathbb{S}$ in positive dimensions show that $H_k(\mathbb{S}A, \mathbb{Z})$ is trivial for $k \geq 1$ (for $k = 1$ this also uses that $H_0(\bar{d}, \mathbb{Z})$ is injective). So $\mathbb{S}A$ is indeed a Moore spectrum for the group A.

With a similar argument we can calculate the morphisms from $\mathbb{S}A$ to any other object X of the stable homotopy category. We apply $[-, X]$ to the distinguished triangle (6.40) and use the isomorphisms

$$[\bigoplus_I \mathbb{S}, X] \cong \prod\nolimits_I \pi_0 X \cong \mathrm{Hom}(\mathbb{Z}[I], \pi_0 X)$$

and similarly for J instead of I and for morphisms of degree 1. We obtain an exact sequence

$$\mathrm{Hom}(\mathbb{Z}[I], \pi_0 X) \xleftarrow{\mathrm{Hom}(d, \pi_0 X)} \mathrm{Hom}(\mathbb{Z}[J], \pi_0 X) \leftarrow [\mathbb{S}A, X] \leftarrow \mathrm{Hom}(\mathbb{Z}[I], \pi_1 X) \xleftarrow{\mathrm{Hom}(d, \pi_1 X)} \mathrm{Hom}(\mathbb{Z}[J], \pi_1 X) .$$

Since kernel respectively cokernel of the morphism $\mathrm{Hom}(d, \pi_n X)$ are the groups $\mathrm{Hom}(A, \pi_0 X)$ respectively $\mathrm{Ext}(A, \pi_0 X)$, the long exact homotopy sequence decomposes into short exact sequences

$$0 \longrightarrow \mathrm{Ext}(A, \pi_1 X) \longrightarrow \mathcal{SHC}(\mathbb{S}A, X) \xrightarrow{\pi_0} \mathrm{Hom}(A, \pi_0 X) \longrightarrow 0 . \tag{6.42}$$

Theorem 6.43. *Let X and Y be two Moore spectra. Then every homomorphism $f : H_0(Y, \mathbb{Z}) \longrightarrow H_0(X, \mathbb{Z})$ can be realized by a morphism $\bar{f} : Y \longrightarrow X$ in the stable homotopy category. If f is an isomorphism, then so is $\bar{f}$. In particular, Moore spectra for a given group are unique up to isomorphism in the stable homotopy category.*

Moreover, when restricted to Moore spectra for 2-divisible groups, the functor π_0 becomes an equivalence of categories. In particular, Moore spectra for 2-divisible groups can be chosen functorially in the stable homotopy category.

Proof. In a first step we let $Y = \mathbb{S}A$ be a Moore spectrum of the special kind constructed in 6.39. Then the realizability of f is simply the surjectivity of the exact sequence (6.42), using that the zeroth homotopy and homology groups of a Moore spectrum are naturally isomorphic.

If f is an isomorphism and $\bar{f} : Y \longrightarrow X$ realizes f, then it induces an isomorphism on $H_0(-, \mathbb{Z})$, and thus on all integral homology groups (since X and Y are Moore spectra). Since X and Y are connective, $\bar{f}$ is an isomorphism in the stable homotopy category by the Whitehead theorem (Proposition 6.30).

The first step applied to $Y = \mathbb{S}(\pi_0 X)$ shows in particular that every Moore spectrum X is isomorphic in the stable homotopy category to $\mathbb{S}(\pi_0 X)$. We may thus assume without loss of generality that $X = \mathbb{S}A$ and $Y = \mathbb{S}B$ are both of the form constructed in 6.39.

The stable 0-stem π_0^{s} is free abelian, hence flat, and $\pi_1^{\mathrm{s}} \cong \mathbb{Z}/2$ generated by the Hopf map η. So for $n = 1$ the short exact sequence (6.41) reduces to an isomorphism $B \otimes \mathbb{Z}/2 \cong \pi_1(\mathbb{S}B)$. The exact sequence (6.42) then becomes a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(A, B \otimes \mathbb{Z}/2) \longrightarrow \mathcal{SHC}(\mathbb{S}A, \mathbb{S}B) \xrightarrow{\pi_0} \mathrm{Hom}(A, B) \longrightarrow 0 .$$

So if B is 2-divisible, then $\pi_0 : \mathcal{SHC}(\mathbb{S}A, \mathbb{S}B) \longrightarrow \mathrm{Hom}(A, B)$ is bijective, i.e., π_0 is fully faithful on this class of Moore spectra. □

Remark 6.44 (Limited functoriality of Moore spectra)**.** There are certain similarities between Eilenberg-Mac Lane and Moore spectra; for example, both are defined by the property that a certain homology theory (homotopy respectively integral homology) is concentrated in dimension zero, and both exist for every abelian group and are unique up to isomorphism in the stable homotopy category. We want to emphasize however, that the functoriality properties of these two kinds of spectra are very different.

Eilenberg-Mac Lane spectra have models which are functorial in the group on the point set level, i.e., as functors to the category of symmetric spectra. We have given such a construction in Example I.1.14, and that model also takes rings to symmetric ring spectra. One can also deduce this from the fact the Eilenberg-Mac Lane functor H with values in symmetric spectra is a symmetric monoidal functor with respect to smash product of spectra and tensor product of abelian groups (compare Example I.5.28).

For Moore spectra the situation is different. While they exist for every abelian group and we have given several different constructions, there are no pointset level models which are functorial in the group A. Even worse, we will see in Example 6.46 below that in general, Moore spectra are not even functorial in the homotopy category. Theorem 6.43 say that after inverting the prime 2, Moore spectra can be chosen functorially in the stable homotopy category, but even then, the construction is not compatible with tensor product respectively smash product. After inverting 2 and 3, Moore spectra can be made into a symmetric monoidal functor to the stable homotopy category. This implies that away from 6 the Moore spectrum associated to any ring can be made into a homotopy ring spectrum inducing the given multiplication on π_0. However, Moore spectra of rings can in general *not* be realized as a symmetric ring spectrum. Specifically, for no $n \geq 2$ can the mod-n Moore spectrum be realized as symmetric ring spectra. (prove this later) If we invert all primes, i.e., for uniquely divisible abelian groups, all these problems go away since rationally, Moore and Eilenberg-Mac Lane spectra coincide, see Example 6.36.

Let us simplify the notation a little by writing $\mathbb{S}/p$ for the *mod-p Moore spectrum* $\mathbb{S}(\mathbb{Z}/p\mathbb{Z})$. We calculate the mod-$p$ cohomology of $\mathbb{S}/p$. Let us denote by $\iota : \mathbb{S} \longrightarrow H\mathbb{Z}/p$ the unit morphism of the ring spectrum structure, which we can view as a cohomology class in the group $H^0(\mathbb{S}, \mathbb{F}_p)$. Both rows in the diagram

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\cdot p"] \arrow[d, "\iota"'] & \mathbb{S} \arrow[r, "j"] \arrow[d, "\tilde{\iota}"'] & \mathbb{S}/p \arrow[r, "\delta"] \arrow[d, dotted, "e_0"] & \Sigma\mathbb{S} \arrow[d, "\Sigma\iota"] \\
H\mathbb{Z}/p \arrow[r, "\cdot p"'] & H\mathbb{Z}/p^2 \arrow[r] & H\mathbb{Z}/p \arrow[r, "\beta"'] & \Sigma H\mathbb{Z}/_p
\end{tikzcd}$$

are distinguished triangles in the stable homotopy category, and the left square commutes. Here β is the mod-p Bockstein morphism, compare (5.31). So we can choose a morphism $e_0 : \mathbb{S}/p \longrightarrow H\mathbb{Z}/p$ which makes the entire diagram commute. The long exact sequence in mod-p cohomology associated to the upper distinguished triangle shows that the mod-p cohomology of $\mathbb{S}/p$ is concentrated in dimensions 0 and 1, where they are 1-dimensional generated by $e_0 \in H^0(\mathbb{S}/p, \mathbb{Z}/p)$ respectively $e_1 = \beta(e_0) \in H^1(\mathbb{S}/p, \mathbb{Z}/p)$.

Example 6.45. The mod-p Moore spectra (for p a prime) behave quite differently when $p = 2$ or p is odd. Let us first discuss the case of odd primes p. Since $\pi_1^{\mathrm{s}} \cong \mathbb{Z}/2$, the exact sequence (6.41) shows that $\pi_1(\mathbb{S}/p)$ is trivial. So the map $\pi_0 : [\mathbb{S}/p, \mathbb{S}/p] \longrightarrow \mathrm{Hom}(\mathbb{Z}/p, \mathbb{Z}/p)$ is an isomorphism. Thus p times the identity of $\mathbb{S}/p$ is trivial in the stable homotopy category, and all groups of the form $[X, \mathbb{S}/p]$ or $[\mathbb{S}/p, X]$ for X in the stable homotopy category are $\mathbb{F}_p$-vector spaces. In particular this holds for the homotopy groups, and so the exact sequence (6.41)

$$0 \longrightarrow \mathbb{Z}/p \otimes \pi_n^{\mathrm{s}} \to \pi_n(\mathbb{S}/p) \to {}_p\{\pi_{n-1}^{\mathrm{s}}\} \longrightarrow 0$$

splits. Here we write ${}_pA$ for $\{x \in A \mid px = 0\}$ which is isomorphic to the group $\mathrm{Tor}(\mathbb{Z}/p, A)$.

Example 6.46. Now we discuss the mod-2 Moore spectrum $\mathbb{S}/2$ and what is different compared to mod-p Moore spectra for odd primes p. Since the stable stems π_1^{s} and π_2^{s} are both cyclic of order 2 with generators η respectively η^2, the exact sequence (6.41) specializes to a short exact sequence

$$0 \longrightarrow \pi_2^{\mathrm{s}} \to \pi_2(\mathbb{S}/2) \to \pi_1^{\mathrm{s}} \longrightarrow 0 . \tag{6.47}$$

Proposition 6.48. *The short exact sequence* (6.47) *does not split, and so the group* $\pi_2(\mathbb{S}/2)$ *is cyclic of order four, generated by any of the two preimages of* η.

Proof. We let $\bar{\eta} : \mathbb{S}^2 \longrightarrow \mathbb{S}/2$ be a morphism in the stable homotopy category whose composite with the connecting morphism $\delta : \mathbb{S}/2 \longrightarrow \mathbb{S}^1$ is η and suppose that $2\bar{\eta} = 0$. Then we could choose an extension $\tilde{\eta} : \mathbb{S}^2 \wedge \mathbb{S}/2 \longrightarrow \mathbb{S}/2$ of $\bar{\eta}$ to the mod-2 Moore spectrum. Let $C(2, \eta, 2)$ be a mapping cone of this extension, i.e., a spectrum which is part of a distinguished triangle

$$\mathbb{S}^2 \wedge \mathbb{S}/2 \xrightarrow{\tilde{\eta}} \mathbb{S}/2 \xrightarrow{j} C(2, \eta, 2) \xrightarrow{\delta} \mathbb{S}^3 \wedge \mathbb{S}/2 .$$

Since the mod-2 cohomology of $\mathbb{S}/2$ respectively its double suspension $\mathbb{S}^2 \wedge \mathbb{S}/2$ are concentrated in dimensions 0 and 1 respectively 2 and 3, the morphism $\tilde{\eta}$ must induce the zero map on mod-2 cohomology. The long exact cohomology sequence of the defining triangle for $C(2,\eta,2)$ thus show that $H^*(C(2,\eta,2),\mathbb{F}_2)$ is one-dimensional in dimensions 0, 1, 3 and 4 and trivial in all other dimensions. Since $j : \mathbb{S}/2 \longrightarrow C(2,\eta,2)$ (respectively $\delta : C(2,\eta,2) \longrightarrow \Sigma^2\mathbb{S}/2$) are surjective (respectively injective) in mod-2 cohomology, we deduce that the Bockstein operation $\beta = \mathrm{Sq}^1$ is an isomorphism from dimension 0 to 1 and from dimension 3 to 4. [draw picture] Since η is detected by the Steenrod operation Sq^2 (compare Example 10.11), the composite operation $\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1 : H^0(C(2,\eta,2);\mathbb{F}_2) \longrightarrow H^4(C(2,\eta,2);\mathbb{F}_2)$ would be a non-trivial isomorphism. By the Adem relations we have $\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1 = \mathrm{Sq}^2\,\mathrm{Sq}^2$ which factors through the trivial group $H^2(C(2,\eta,2);\mathbb{F}_2)$. We have obtained a contradiction, and so the class $2\bar{\eta}$ must be nonzero, i.e., $\bar{\eta} \in \pi_2(\mathbb{S}/2)$ generates a cyclic group of order 4. $\square$

Now we calculate the ring of self maps of $\mathbb{S}/2$ in the stable homotopy category. Since the stable 0-stem π_0^{s} is infinite cyclic and thus torsion free, the exact sequence (6.41) shows that $\pi_1(\mathbb{S}/2) \cong \mathbb{Z}/2$, generated by the image of η. The sequence (6.42) thus specializes to a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z}/2,\mathbb{Z}/2) \to \mathcal{SHC}(\mathbb{S}/2,\mathbb{S}/2) \xrightarrow{\pi_0} \mathrm{Hom}(\mathbb{Z}/2,\mathbb{Z}/2) \longrightarrow 0 \ . \tag{6.49}$$

The sequence does not split, because otherwise we would have $2 \cdot \mathrm{Id}_{\mathbb{S}/2} = 0$ and $\pi_n(\mathbb{S}/2)$ would be an $\mathbb{F}_2$-vector space for all n, contradicting the calculation $\pi_2(\mathbb{S}/2) \cong \mathbb{Z}/4$ of the previous proposition. Thus the endomorphism ring $[S\mathbb{Z}/2, S\mathbb{Z}/2]$ is isomorphic to $\mathbb{Z}/4$. The exact sequence (6.49) show that $2\,\mathrm{Id}_{S\mathbb{Z}/2}$ equals the image of the generator of $\mathrm{Ext}(\mathbb{Z}/2,\mathbb{Z}/2)$ which proves the relation

$$2 \cdot \mathrm{Id}_{\mathbb{S}/2} \ = \ j\eta\delta \tag{6.50}$$

in the group $[\mathbb{S}/2,\mathbb{S}/2]$, where $j : \mathbb{S} \longrightarrow \mathbb{S}/2$ and $\delta : \mathbb{S}/2 \longrightarrow \mathbb{S}^1$ and the two morphisms from the defining triangle for the mod-2 Moore spectrum.

In contrast to the case of odd primes, the exact sequence

$$0 \ \longrightarrow \ \mathbb{Z}/2 \otimes \pi_n^{\mathrm{s}} \ \xrightarrow{j\cdot} \ \pi_n(\mathbb{S}/2) \ \xrightarrow{\delta\cdot} \ {}_2\{\pi_{n-1}^{\mathrm{s}}\} \ \longrightarrow \ 0$$

does not generally split (as we already saw for $n = 2$). The relation (6.50) implies that η-multiplication completely determines the class of this extension: if $\bar{x} \in \pi_n(\mathbb{S}/2)$ is a preimage of a 2-torsion element $x \in \pi_{n-1}^{\mathrm{s}}$, then $2\bar{x}$ is the image of $\eta x \in \pi_n^{\mathrm{s}}$ (because $2\bar{x} = j\eta\delta\bar{x} = j\eta x$). More generally, for any spectrum X the sequence

$$0 \ \longrightarrow \ \mathbb{Z}/2 \otimes \pi_n X \ \xrightarrow{j\cdot} \ \pi_n(X;\mathbb{Z}/2) \ \xrightarrow{\delta\cdot} \ {}_2\{\pi_{n-1}X\} \ \longrightarrow \ 0$$

is short exact but need not split. The extension is determined by the action of $\eta \in \pi_1^{\mathrm{s}}$ on the homotopy groups of X in the same way as for $X = \mathbb{S}$ above.

Remark 6.51 (Generalized (co)homology with coefficients)**.** Moore spectra can be used to introduce coefficients into generalized homology and cohomology theories. For this we let E be any symmetric spectrum, A and abelian group and $\mathbb{S}A$ a Moore spectrum for A. For any other symmetric spectrum X and an integer k, we define the k-th *E-homology group of X with coeffiencts in A* as

$$E_k(X,A) \ = \ (E \wedge^L \mathbb{S}A)_k(X) \ = \ \pi_k(E \wedge^L \mathbb{S}A \wedge^L X) \ .$$

The k-th *E-cohomology group with coeffiencts in A* of X is defined as

$$E^k(X,A) \ = \ (E \wedge^L \mathbb{S}A)^k(X) \ = \ [X, E \wedge^L \mathbb{S}A]_{-k} \ .$$

The E-homology and E-cohomology with coeffienects in A has the same functoriality in E and X as the theory without coefficients has. Functoriality in the group A is more subtle because the asignment $A \mapsto \mathbb{S}A$ is not in general functorial, even as a functor to the stable homotopy category. Strictly speaking, the definition above is not even as well-defined as one usually likes: while the Moore spectrum $\mathbb{S}A$ is determined by A up to isomorphism in the stable homotopy category, there is in general no preferred isomorphism (however, this is only an issue when A has 2-torsion). So hence $E_*(X, A)$ and $E^*(X, A)$ are determined by A up to isomorphism, but the isomorphism is generally non-canonical. We refer to Remark 6.44 for a detailed discusssion of this functoriality (or the lack thereof).

In special cases we get back known theories. For example, when $A = \mathbb{Z}$ is the group of integers, then the sphere spectrum $\mathbb{S}$ serves as a Moore spectrum; since the sphere spectrum is a strict unit for the derived smash product, we have

$$E_k(X, \mathbb{Z}) \;=\; \pi_k(E \wedge^L \mathbb{S} \wedge^L X) \;=\; \pi_k(E \wedge^L X) \;=\; E_k(X) \ .$$

When $E = H\mathbb{Z}$ is the integral Eilenberg-Mac Lane spectrum, then $E \wedge^L \mathbb{S}A = H\mathbb{Z} \wedge^L \mathbb{S}A$ has its homotopy groups concentrated in dimension 0, so in the stable homotopy category there is a preferred isomorphism between $H\mathbb{Z} \wedge^L \mathbb{S}A$ and the Eilenberg-Mac Lane spectrum HA [ref]. Hence the homology group $(H\mathbb{Z})_k(X, A) = \pi_k(H\mathbb{Z} \wedge^L \mathbb{S}A \wedge^L X)$ is naturally isomorphic to $H_k(X, A) = \pi_k(HA \wedge^L X)$, the ordinary homology with coefficients in A.

The distinguished triangle

$$\mathbb{S} \xrightarrow{\cdot n} \mathbb{S} \xrightarrow{j} \mathbb{S}/n \xrightarrow{\delta} \Sigma\mathbb{S}$$

involving the mod-n Moore spectrum gives rise to a long exact sequence

$$\cdots \longrightarrow E_k(X) \xrightarrow{\cdot n} E_k(X) \longrightarrow E_k(X, \mathbb{Z}/n) \longrightarrow E_{k-1}(X) \longrightarrow \cdots$$

and similarly for E-cohomology.

Whenever A is a subring of the ring of rational numbers, the effect of introducing coefficients is easy to describe: the map [...]

$$A \otimes E_k(X) \longrightarrow E_k(X, A)$$

is an isomorphism.

7. Finite spectra

7.1. The Spanier-Whitehead category. In Proposition 5.2 we saw that the symmetric sphere spectrum $\mathbb{S}$ is a compact object in the stable homotopy category as well as a weak generator. We will now identify the full subcategory of all compact objects in the stable homotopy category with a more 'concrete' category defined from the homotopy category of finite CW-complexes by 'inverting the suspension functor'. This category is known as the *Spanier-Whitehead category*, and historically it predates the stable homotopy category.

The Freudenthal suspension theorem asserts that for every n-connected pointed space Y and every pointed CW-complex X whose dimension is less than $2n$ (numbers right ?), the suspension map

$$\Sigma \;:\; [X, Y] \longrightarrow [\Sigma X, \Sigma Y]$$

is bijective. So when defining the stable homotopy classes of maps

$$\{X, Y\} \;=\; \mathrm{colim}_n\, [\Sigma^n X, \Sigma^n Y] \ ,$$

the colimit system actually stabilizes whenever X is a finite-dimensional CW-complex. A natural idea is thus to define a category in which these stable values live, and this is the so-called Spanier-Whitehead category.

Definition 7.1. The *Spanier-Whitehead category* $\mathcal{SW}$ has a objects the pairs (K, n) where K is a pointed space which admits the structure of a finite CW-complex and $n \in \mathbb{Z}$ is an integer. Morphisms in $\mathcal{SW}$ are defined by

$$\mathcal{SW}((K, n), (L, m)) \;=\; \mathrm{colim}_k\, [K \wedge S^{n+k}, L \wedge S^{m+k}] \ .$$

The colimit is taken over the suspension maps and ranges over large enough values of k for which both $n+k$ and $m+k$ are non-negative. Composition is defined by composition of representatives, suitably suspended so that composition is possible.

[is this historically correct, or was SW without the formal dimension?]

By Freudenthal's suspension theorem, the colimit is attained at a finite stage. We often identify a finite CW-complex X with the object $(X,0)$ of the Spanier-Whitehead category. With this convention the morphism set

$$\mathcal{SW}(S^n,K) \ = \ \mathrm{colim}_k\ [S^{n+k},K\wedge S^k]_* \ = \ \mathrm{colim}_k\ \pi_{n+k}(K\wedge S^k)$$

agrees with the n-th homotopy group of the suspension spectrum $\Sigma^\infty K$, i.e., the n-th stable homotopy group $\pi_n^{\mathrm{s}} K$ of K.

Tautologically, the identity map of $K\wedge S^{n+m}$ represents an isomorphism between $(K\wedge S^n,m)$ and $(K,n+m)$ in $\mathcal{SW}$, so suspension becomes invertible in $\mathcal{SW}$. In fact, $\mathcal{SW}$ is in a certain precise sense the universal example of this [...].

The Spanier-Whitehead category is naturally endowed with the structure of a triangulated category as follows. The shift functor is simply given by reindexing, i.e., $\Sigma(K,n)=(K,1+n)$. The Spanier-Whitehead category is additive because in it every objects is isomorphic to a double suspension, so the morphism-sets in $\mathcal{SW}$ are all abelian groups.

The distinguished triangles arise from homotopy cofiber sequences: for every based map $f:K\longrightarrow L$ between finite CW-complexes and every integer n the diagram

$$(K,n) \xrightarrow{\ f\ } (L,n) \xrightarrow{\ i\ } (C(f),n) \xrightarrow{\ p\ } (\Sigma K,n)\cong(K,1+n)$$

is a distinguished triangle (where $C(f)$ is the mapping cone), and a general triangle is distinguished if and only if it is isomorphic to one of these.

We will not verify the axioms of a triangulated category for $\mathcal{SW}$, but this can be found in Chapter 1, § 2 of [**54**]. Alternatively, one could check that distinguished triangle in $\mathcal{SW}$ go to distinguished triangle in $\mathcal{SHC}$ and then use the fact that Σ^∞ is fully faithful to deduce the axioms of a triangulated category for $\mathcal{SW}$ from those for $\mathcal{SHC}$.

The following theorem says that the stable homotopy category contains the Spanier-Whitehead category as a full subcategory. For every pointed space K we have a suspension spectrum $\Sigma^\infty K$ as in Example 1.13. We apply the singular complex functor $\mathcal{S}$ levelwise (compare Section I.1) to obtain a symmetric spectrum of simplicial sets and then use the localization functor $\gamma:\mathcal{S}p\longrightarrow\mathcal{SHC}$ to get into the stable homotopy category. We also have to compensate the formal dimension attached to an object in the Spanier-Whitehead category by shifting in the triangulated structure of the stable homotopy category, so altogether we define a functor

$$\underline{\Sigma}^\infty \ : \ \mathcal{SW} \ \longrightarrow \ \mathcal{SHC}$$

on objects by

$$\underline{\Sigma}^\infty(K,n) \ = \ \gamma(\mathcal{S}(\Sigma^\infty K))\wedge^L \mathbb{S}^n \ .$$

To define this functor on morphisms we define an isomorphism in $\mathcal{SHC}$ between $\underline{\Sigma}^\infty(K\wedge S^1,n)$ and $\Sigma\underline{\Sigma}^\infty(K,n)$.

$$\gamma(\mathcal{S}(\Sigma^\infty(K\wedge S^1)))\wedge^L\mathbb{S}^n \ \cong \ (S^1\wedge\gamma(\mathcal{S}(\Sigma^\infty K)))\wedge^L\mathbb{S}^n \xrightarrow{\mathrm{Id}\wedge^L\alpha_{1,n}} S^1\wedge\big(\gamma(\mathcal{S}(\Sigma^\infty K))\wedge^L\mathbb{S}^n\big)$$

[define the remaining iso] Using these isomorphisms we get natural maps

$$\begin{aligned}[K\wedge S^{n+k},L\wedge S^{m+k}] \ &\longrightarrow \ \mathcal{SHC}(\gamma(\mathcal{S}(\Sigma^\infty K))\wedge^L\mathbb{S}^{n+k},\gamma(\mathcal{S}(\Sigma^\infty L))\wedge^L\mathbb{S}^{m+k})\\ &\xrightarrow{-\wedge^L\mathbb{S}^{-k}} \ \mathcal{SHC}(\gamma(\mathcal{S}(\Sigma^\infty K))\wedge^L\mathbb{S}^{n},\gamma(\mathcal{S}(\Sigma^\infty L))\wedge^L\mathbb{S}^{m})\end{aligned}$$

where we have implicitly used the derived associativity isomorphisms and the isomorphism $\alpha_{n+k,-k}$: $\mathbb{S}^{n+k}\wedge^L\mathbb{S}^{-k}\cong\mathbb{S}^n$. These maps are compatible as k increases [associativity of α's ?], so the induce a well defined map

$$\underline{\Sigma}^\infty \ : \ \mathcal{SW}((K,n),(L,m)) \ \longrightarrow \ \mathcal{SHC}(\underline{\Sigma}^\infty(K,n),\underline{\Sigma}^\infty(L,m)) \ ,$$

which we take as the effect of the functor $\underline{\Sigma}^\infty$ on morphisms. The following theorem and the characterization of compact objects in Theorem 7.4 says that the Spanier-Whitehead category 'is' (up to equivalence of triangulated categories which preserves the smash products) the full subcategory of compact objects in $\mathcal{SHC}$.

Theorem 7.2. *The functor*

$$\underline{\Sigma}^\infty \ : \ \mathcal{SW} \ \longrightarrow \ \mathcal{SHC}$$

is fully faithful and exact.

PROOF. [specify the isomorphism $\underline{\Sigma}^\infty(K, 1+n) \cong S^1 \wedge \underline{\Sigma}^\infty(K,n)$; prove exactness].

Essentially by construction of $\underline{\Sigma}^\infty$ there are natural isomorphisms $\underline{\Sigma}^\infty(K, n+m) \cong \underline{\Sigma}^\infty(K,n) \wedge^L \mathbb{S}^m$. For showing that the map on morphism sets

$$\underline{\Sigma}^\infty \ : \ \mathcal{SW}((K,n),(L,m)) \ \longrightarrow \ \mathcal{SHC}(\underline{\Sigma}^\infty(K,n), \underline{\Sigma}^\infty(L,m))$$

is bijective we can thus assume $n = m = 0$.[...] Every finite CW-complex is homotopy equivalent, thus isomorphic in $\mathcal{SW}$, to the geometric realization of a finite simplicial set, so we can assume that $K = |K'|$ and $L = |L'|$ for finite pointed simplicial sets K' and L'. Then the spectrum $\underline{\Sigma}^\infty(|K'|, 0) = \mathcal{S}(\Sigma^\infty|K'|)$ is level equivalent to the suspension spectrum $\Sigma^\infty K'$, and similarly for L'. So in the special case the claim boils down to the statement that the map

$$\{|K'|, |L'|\} = \mathrm{colim}_k\, [|K'| \wedge S^k, |L'| \wedge S^k] \ \longrightarrow \ \mathcal{SHC}(\Sigma^\infty K', \Sigma^\infty L')$$

[which map?] is an isomorphism. This, however, was already shown in Example 1.20. □

The Spanier-Whitehead category has a symmetric monoidal smash product which is defined by $(K,n) \wedge (L,m) = (K \wedge L, n+m)$ on objects, and with unit object $(S^0, 0)$. Moreover, the embedding $\underline{\Sigma}^\infty$ of the Spanier-Whitehead category into $\mathcal{SHC}$ is compatible with smash products, i.e., it can be made into a strong symmetric monoidal functor. We leave the details as Exercise E.II.12.

Remark 7.3. There is an important conceptual difference in the construction of the Spanier-Whitehead and the stable homotopy category. The difference is in which order the processes of 'inverting suspension' and 'passage to homotopy classes' are taken. [...] There is no known construction of the stable homotopy category where the two processes are taken in the other order.

Now we can explain why we restrict to *finite* CW-complexes when defining the Spanier-Whitehead category. The definition of morphisms in $\mathcal{SW}$ make sense for arbitrary pointed spaces, but the natural map

$$\{X, Y\} = \mathrm{colim}_k [\Sigma^k X, \Sigma^k Y] \ \longrightarrow \ \mathcal{SHC}(\Sigma^\infty X, \Sigma^\infty Y)$$

is not generally a bijection. For example, the identity map of QS^0 is adjoint to a morphism $\Sigma^\infty QS^0 \longrightarrow \Sigma^\infty S^0$ in the stable homotopy category which is not in the image of $\{QS^0, S^0\}$. An injective Ω-spectrum X is isomorphic to $\Sigma^\infty(K,n)$ for some finite pointed simplicial set K and integer n if and only if it is *compact* as an object of the triangulated category $\mathcal{SHC}$.

We recall that an object X of a triangulated category with infinite sums is called *compact* (sometimes called *small* or *finite*) if for every family $\{Y_i\}_{i \in I}$ of objects the natural map

$$\bigoplus_{i \in I} [X, Y_i] \ \longrightarrow \ [X, \bigoplus_{i \in I} Y_i]$$

is an isomorphism. We saw in Proposition 5.2 that the sphere spectrum is compact as an object of the stable homotopy category. We will now characterize the compact objects of the stable homotopy category, which are often referred to as 'finite spectra'.

One of the characterizations below refers to the contravariant *Spanier-Whitehead dual* defined by $DX = F(X, \mathbb{S})$, the derived function spectrum with sphere spectrum in the second variable. For every spectrum X there is an evaluation morphism $\epsilon_{X,\mathbb{S}} : DX \wedge^L X = F(X, \mathbb{S}) \wedge^L X \longrightarrow \mathbb{S}$, adjoint to the identity, see (3.9). If Y is another spectrum, there is a natural morphism $Y \wedge^L DX \longrightarrow F(X,Y)$ which is adjoint to the morphism $Y \wedge \epsilon_{X,\mathbb{S}} : Y \wedge^L DX \wedge^L X \longrightarrow Y \wedge^L \mathbb{S} = Y$.

A symmetric spectrum X is *bounded below* if there is an integer k such that all homotopy groups below dimension k are trivial.

Theorem 7.4. *For a symmetric spectrum X the following five conditions are equivalent. Spectra which satisfy these equivalent conditions are called* finite spectra*.*

(i) *X is isomorphic to $(\Sigma^\infty K) \wedge^L \mathbb{S}^n$ for a finite pointed simplicial set K and an integer n;*
(ii) *X is* strongly dualizable*, i.e., for every object Y of $\mathcal{SHC}$ the morphism $Y \wedge^L DX \longrightarrow F(X,Y)$ is an isomorphism;*
(iii) *X is a compact object of the triangulated category $\mathcal{SHC}$;*
(iv) *X is bounded below and its integral homology $H_*(X,\mathbb{Z})$ is totally finitely generated;*
(v) *X belongs to the thick subcategory generated by the sphere spectrum.*

[can we add: $X \longrightarrow DDX$ is an isomorphism?]

Proof. (i)$\Longrightarrow$(ii) induction on number of non-degenerate simplices of K

(ii)$\Longrightarrow$(iii) For every symmetric spectrum X and every family $\{Y_i\}_{i\in I}$ of spectra we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{i\in I}(Y_i \wedge^L DX) & \xrightarrow{\cong} & (\bigoplus_{i\in I} Y_i) \wedge^L DX \\ \downarrow & & \downarrow \\ \bigoplus_{i\in I} F(X,Y_i) & \longrightarrow & F(X, \bigoplus_{i\in I} Y_i) \end{array}$$

in $\mathcal{SHC}$ in which the upper horizontal map is an isomorphism since the derived smash product is a left adjoint. If X is strongly dualizable, then the two vertical morphism are isomorphism, hence so is the lower horizontal map. If we take the 0th homotopy group of the lower morphism, exploit that π_0 commutes with sums, and use $\pi_0 F(X,Z) \cong \mathcal{SHC}(X,Z)$, we see that X is compact.

(iii)$\Longrightarrow$(iv) The canonical morphism $\bigoplus_{n\in\mathbb{Z}} H\mathbb{Z}[n] \longrightarrow \prod_{n\in\mathbb{Z}} H\mathbb{Z}[n]$ is a π_*-isomorphism, thus an isomorphism in $\mathcal{SHC}$. If X is compact, the composite map

$$\bigoplus_{n\in\mathbb{Z}} [X, H\mathbb{Z}[n]] \ \longrightarrow\ [X, \bigoplus_{n\in\mathbb{Z}} H\mathbb{Z}[n]] \ \xrightarrow{\cong}\ [X, \prod_{n\in\mathbb{Z}} H\mathbb{Z}[n]] \ \xrightarrow{\cong}\ \prod_{n\in\mathbb{Z}} [X, H\mathbb{Z}[n]]$$

is thus an isomorphism. This means that the group $[X, H\mathbb{Z}[n]] \cong H^n(X,\mathbb{Z})$ is trivial for almost all integers n, i.e., the integral cohomology of X is concentrated in finitely many dimensions. By the universal coefficient theorems, the integral homology is then also concentrated in finitely many dimensions.

We show next that $H_n(X,\mathbb{Z})$ is finitely generated for every integer n. We consider a family $\{A_i\}_{i\in I}$ of abelian groups and form the sum of the associated Eilenberg-Mac Lane spectra, which is stably equivalent (even isomorphic as a symmetric spectrum) to the Eilenberg-Mac Lane spectrum of the sum. Since X is compact, the map

$$\bigoplus_{i\in I} H^n(X,A_i) \ \cong\ \bigoplus_{i\in I} [X, HA_i]^n \ \longrightarrow\ [X, \bigoplus_{i\in I} HA_i]^n \ \xrightarrow{\cong}\ [X, H(\bigoplus_{i\in I} A_i)]^n \ \xrightarrow{\cong}\ H^n(X, \bigoplus_{i\in I} A_i)$$

is an isomorphism. We have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{i\in I} H^n(X,A_i) & \longrightarrow & H^n(X, \bigoplus_{i\in I} A_i) \\ \downarrow & & \downarrow \\ \bigoplus_{i\in I} \mathrm{Hom}(H_n(X,\mathbb{Z}), A_i) & \longrightarrow & \mathrm{Hom}(H_n(X,\mathbb{Z}), \bigoplus_{i\in I} A_i) \end{array}$$

in which the lower map is the canonical one. The upper map is an isomorphism and the two horizontal maps are surjective by the universal coefficient theorem. So the lower map is also surjective. Since this holds for all families $\{A_i\}_{i\in I}$ of abelian groups the homology group $H_n(X,\mathbb{Z})$ must be finitely generated.

The last thing we have to verify is that X is bounded below. By Theorem 8.1 below X is a homotopy colimit of its connective covers, i.e., there is a distinguished triangle

$$\bigoplus_{n\geq 0} X\langle -n\rangle \xrightarrow{1-\text{shift}} \bigoplus_{n\geq 0} X\langle -n\rangle \xrightarrow{\oplus q_{-n}} X \longrightarrow \Sigma\left(\bigoplus_{n\geq 0} X\langle -n\rangle\right)$$

Since X is compact, Lemma 5.6 (ii) shows that the group $[X, X]$ a colimit of the sequence of groups $[X, X\langle -n\rangle]$ which implies that the identity morphism of X in the stable homotopy category factors through the spectrum $X\langle -n\rangle$ for some $n \geq 0$. So X is a retract of a bounded below spectrum, hence it is itself bounded below.

(iv)$\Longrightarrow$(v) Build X from spheres. Use induction on width of homology.

(v)$\Longrightarrow$(i) We exploit that the functor $\underline{\Sigma}^\infty : \mathcal{SW} \longrightarrow \mathcal{SHC}$ is fully faithful, compare Theorem 7.2, and deduce from this that the essential image of the functor $\underline{\Sigma}^\infty$ is closed under extensions in the stable homotopy category. This is special case of the following more general fact, which we state as Proposition 7.5 below. The essential image of the functor $\underline{\Sigma}^\infty$ is thus a full triangulated subcategory of the stable homotopy category. Moreover, this essential image containes the sphere spectrum since that is isomorphic to the suspension spectrum of S^0. [closed under retracts] So the essential image of $\underline{\Sigma}^\infty$ contains the thick subcategory generated by the sphere spectrum. $\square$

The *essential image* of a functor is the class of all objects which are isomorphic to an object in the image of the functor.

Proposition 7.5. *Let $F : \mathcal{S} \longrightarrow \mathcal{T}$ be a fully faithful and exact functor between triangulated category. Then the essential image of F is closed under extensions in $\mathcal{T}$.*

Proof. Suppose that

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

is a distinguished triangle in $\mathcal{T}$ with A and B in the essential image of F. So there exist objects A' and B' of $\mathcal{S}$ and isomorphisms $\alpha : FA' \longrightarrow A$ and $\beta : FB' \longrightarrow B$ in $\mathcal{T}$. Since F is full, there exists a morphism $f' : A' \longrightarrow B'$ in $\mathcal{S}$ such that $F(f') = \beta^{-1} f\alpha$. We choose a distinguished triangle

$$A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$$

in $\mathcal{S}$. The image of this triangle is distinguished in $\mathcal{T}$, so the axiom (T3) of a triangulated category provides a morphism $\gamma : FC' \longrightarrow C$ which makes the diagram

$$\begin{tikzcd} FA' \arrow[r, "F(f')"] \arrow[d, "\alpha"'] & FB' \arrow[r, "F(g')"] \arrow[d, "\beta"'] & FC' \arrow[r, "\tau\circ F(h')"] \arrow[d, "\gamma"] & \Sigma(FA') \arrow[d, "\Sigma(\alpha)"] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma A \end{tikzcd}$$

commute, and γ is then an isomorphism since α and β are. Since F commutes with suspensions and desuspensions (up to natural isomorphism), the essential image of F is closed under suspensions and desuspensions. Since triangles can be rotated, this also shows that if A and C are in the essential image of F, then so is B, and if B and C are in the essential image of F, then so is A. $\square$

Remark 7.6. Let E be any symmetric spectrum. We can now explain how the E-homology of the dual X tries to be the E-cohomology of X. We have a morphism

$$DX \wedge^L E \wedge^L X \xrightarrow{DX\wedge\bar{\tau}_{E,X}} DX \wedge^L X \wedge^L E \xrightarrow{\epsilon_{X,\mathbb{S}}\wedge E} \mathbb{S} \wedge^L E \;=\; E$$

where $\epsilon_{X,\mathbb{S}} : DX \wedge^L X = F(X, \mathbb{S}) \wedge^L X \longrightarrow \mathbb{S}$ is the evaluation morphism (3.9) adjoint to the identity. Adjoining we obtain

$$DX \wedge^L E \;=\; F(X, \mathbb{S}) \wedge^L E \;\longrightarrow\; F(X, E)\ ;$$

taking k-th homotopy group and applying the isomorphism $\pi_k F(X,E) \cong \mathcal{SHC}(X, \mathbb{S}^k \wedge^L E)$ we obtain a natural map

$$E_k(DX) = \pi_k(E \wedge^L DX) \longrightarrow \pi_k F(X,E) \cong E^k(X) \ . \tag{7.7}$$

By Theorem 7.4 (ii), the morphism $DX \wedge^L E \longrightarrow F(X,E)$ is an isomorphism in $\mathcal{SHC}$ for *finite* spectra X. Hence (7.7) is an isomorphism for finite spectra X, compare Remark 7.8 below.

Remark 7.8 (Spanier-Whitehead duality)**.** The contravariant duality functor $D = F(-, \mathbb{S})$ preserves compact objects and restricts to a contravariant self-equivalence of $\mathcal{SHC}^c$. Since the Spanier-Whitehead category is equivalent to the category $\mathcal{SHC}^c$, the duality should be describable only in terms of finite CW-complexes. This is in fact possible as follows. [...] This duality on $\mathcal{SW}$ is called *Spanier-Whitehead duality*; historically, it was one of the origins for the stable homotopy theory. [classical definition for finite CW-complexes; prove that SW-duality is the restriction of 'S-duality' $D = F(-, \mathbb{S})$ to compact objects; S-dual for manifolds via Thom space of normal bundle]

We draw some consequences of the characterization of compact objects.

Proposition 7.9. *The full subcategory $\mathcal{SHC}^c$ of finite spectra is closed in the stable homotopy category under derived smash product, derived function spectra and duality. The duality functor restricts to a contravariant self-equivalence of $\mathcal{SHC}^c$.*

Proof. To show that the restriction of the duality to $\mathcal{SHC}^c$ is an self-equivalence it remains to show that every finite spectrum X is *dualizable*, i.e., the double duality morphism $X \longrightarrow DDX$, adjoint to the identity of DX, is an isomorphism. However, the class of spectra X for which this morphism is an isomorphism is closed under extensions and retract, and it contains the sphere spectrum $\mathbb{S}$. Since the thick subcategory generated by $\mathbb{S}$ coincides with the class of compact spectra, $X \longrightarrow DDX$ is an isomorphism for all compact X. □

An object X in a symmetric monoidal category is called *invertible* of there exists another object Y such that $X \wedge Y$ is isomorphic to the unit object. In the stable homotopy category we have $\mathbb{S}^n \wedge^L \mathbb{S}^{-n} \cong \mathbb{S}$, so $\mathbb{S}^n$ is invertible for all integers n. We can now show that $\mathcal{SHC}$ has no other invertible objects.

Proposition 7.10. *Every invertible object in the stable homotopy category is isomorphic to a sphere spectrum $\mathbb{S}^n$ for some integer n.*

Proof. Suppose X is invertible. Then $X \wedge^L -$ is an autoequivalence of the stable homotopy category, and thus it preserves all categorical properties. Since the sphere spectrum $\mathbb{S}$ is compact, so is its image $X \cong X \wedge^L \mathbb{S}$. By Theorem 7.4, the spectrum X is bounded below and has totally finitely generated homology. Since $X \wedge^L Y \cong \mathbb{S}$ the Künneth and universal coefficient theorems imply that the integral homology of X is concentrated in a single dimension n, where it is free abelian of rank one. Since X is also bounded below, X is isomorphic to $\mathbb{S}^n$ in the stable homotopy category. □

Remark 7.11. The *Freyd generating hypothesis* is a prominent open problem about the stable homotopy category. The question is whether the sphere spectrum is a (strong) categorical generator of the homotopy category of finite spectra. This means the following: given a morphism $f : X \longrightarrow Y$ between *finite* spectra such that the induced map $\pi_* f$ on homotopy groups is trivial, is f then necessarily the trivial morphism ?

This notion of generator which asks whether the sphere spectrum detects morphisms should be contrasted with the fact that the sphere spectrum is a weak generator, i.e., detects isomorphisms in the stable homotopy category (not necessarily finite), see Proposition 5.2.

The restriction to *finite* spectra is clearly necessary in the generating hypothesis: the mod-p Bockstein morphism $\beta : H\mathbb{Z}/p \longrightarrow \Sigma(H\mathbb{Z}/p)$ (see (5.31)) between mod-p Eilenberg-Mac Lane spectra is non-trivial, but induces the trivial map on stable homotopy groups for dimensional reasons.

Remark 7.12. The nilpotence theorem gives a criterion for when a (shifted) selfmap $f : X \wedge^L \mathbb{S}^n \longrightarrow X$ is nilpotent, i.e., some iterate of f becomes trivial. [expand...]

8. Connective covers and Postnikov sections

We denote by $\mathcal{SHC}_{\geq n}$ the full subcategory of the stable homotopy category with objects the $(n-1)$-connected spectra. We recall from Proposition 5.21 that the $(n-1)$-connected spectra coincide with $\langle \mathbb{S}^n \rangle_+$, the smallest class of symmetric spectra which contains the n-dimensional sphere spectrum and is closed under sums (possibly infinite) and 'extensions to the right'.

The next theorem uses homotopy colimits, which were defined in Definition 5.3 for general triangulated categories, and we gave a more explicit construction in the stable homotopy category in Proposition 5.11.

Theorem 8.1. *Let n be any integer. The inclusion of the full subcategory of $(n-1)$-connected spectra into $\mathcal{SHC}$ has a right adjoint $\langle n \rangle : \mathcal{SHC} \longrightarrow \mathcal{SHC}_{\geq n}$. The adjunction counit $q_n : X\langle n \rangle \longrightarrow X$ is called the* $(n-1)$-connected cover *of the symmetric spectrum X. There is a unique natural transformation $i_n : X\langle n \rangle \longrightarrow X\langle n-1 \rangle$ satisfying $q_{n-1} \circ i_n = q_n$. Moreover, the morphisms $q_n : X\langle n \rangle \longrightarrow X$ express every symmetric spectrum X as the homotopy colimit of the sequence of morphisms i_n as n goes to $-\infty$.*

Before we prove the theorem we spell out explicitly the main properties of this cover:

- the spectrum $X\langle n \rangle$ is $(n-1)$-connected and for every $(n-1)$-connected symmetric spectrum A the map $[A, q_n] : [A, X\langle n \rangle] \longrightarrow [A, X]$ is an isomorphism.
- by taking $A = \mathbb{S}^k$ for $k \geq n$ and using that $\mathbb{S}^k$ represents the homotopy group functor π_k we see that q_n induces isomorphisms of homotopy groups in dimensions n and above.
- The $(n-1)$-connected cover $q_n : X\langle n \rangle \longrightarrow X$ is a natural transformation,

The above conditions which refer to homotopy groups also characterizes the $(n-1)$-connected cover. Indeed, if $f : A \longrightarrow X$ is any morphism of symmetric spectra which induces isomorphisms of homotopy groups in dimensions n and above and such that A is $(n-1)$-connected, then by the above there is a unique morphism $\tilde{f} : A \longrightarrow X\langle n \rangle$ in the stable homotopy category satisfying $q_n \tilde{f} = f$, and $\tilde{f}$ is necessarily a π_*-isomorphism, thus an isomorphism in $\mathcal{SHC}$.

Proof of Theorem 8.1. We let $C = \{\mathbb{S}^k\}_{k \geq n}$ be the set of sphere spectra of dimensions at least n. For a given symmetric spectrum X we apply Proposition 5.14 to the representable functor $[-, X]$. We obtain a spectrum $X\langle n \rangle$ belonging to $\langle C \rangle_+$ and a morphism $q_n : X\langle n \rangle \longrightarrow X$ which induces isomorphisms on $[\mathbb{S}^k, -]$ for all $k \geq n$, i.e., isomorphisms on homotopy groups in dimensions n and above. Since $\langle C \rangle_+ = \langle \mathbb{S}^n \rangle_+$ equals the class of $(n-1)$-connected spectra, $X\langle n \rangle$ is $(n-1)$-connected.

Now we claim that for every $(n-1)$-connected spectrum A the map $[A, q_n] : [A, X\langle n \rangle] \longrightarrow [A, X]$ is an isomorphism. We let Z be any symmetric spectrum such that the homotopy groups $\pi_k Z$ are trivial for all $k \geq n$. We consider the class $\mathcal{X}$ of symmetric spectra A with the property that the groups $\mathcal{SHC}(\Sigma^k A, Z)$ are trivial for all $k \geq n$. The class $\mathcal{X}$ is closed under sums and contains the n-dimensional sphere spectrum $\mathbb{S}^n$. For a distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

such that A and B belong to $\mathcal{X}$ we apply $[\Sigma^k(-), Z]$ and get an exact sequence

$$[\Sigma^{k+1} A, Z] \longrightarrow [\Sigma^k C, Z] \longrightarrow [\Sigma^k B, Z] \longrightarrow [\Sigma^k A, Z] \ .$$

For $k \geq n$ the first and third group are trivial, hence so is the second, This shows that $\mathcal{X}$ is also closed under extensions to the right, so altogether we have $\langle \mathbb{S}^n \rangle_+ \subset \mathcal{X}$. By Proposition 5.21 we then know that $[\Sigma^k A, Z] = 0$ for every connective spectrum A and $k \geq n$.

We choose a distinguished triangle

$$X\langle n \rangle \ \xrightarrow{q_n} \ X \ \longrightarrow \ Z \ \longrightarrow \ \Sigma(X\langle n \rangle) \ .$$

Since $\pi_k(p)$ is bijective for $k \geq n$ and $\pi_{n-1}(X\langle n \rangle) = 0$, the long exact sequence of true homotopy groups shows that $\pi_k Z = 0$ for all $k \geq n$. By the above we thus have $[\Sigma A, Z] = 0 = [A, Z]$ for every $(n-1)$-connected spectrum A. The exact sequence

$$0 = [\Sigma A, Z] \ \longrightarrow \ [A, X\langle n \rangle] \ \xrightarrow{[A, q_n]} \ [A, X] \ \longrightarrow \ [A, Z] = 0$$

then shows that $[A, p]$ is an isomorphism.

So we now know that the restriction of the functor $[-, X]$ to the category $\mathcal{SHC}_{\geq 0}$ of connective spectra is representable, namely by the pair $(X\langle n\rangle, q_n)$ constructed above. It is then a formal consequence that these choices of connective covers $q_n : X\langle n\rangle \longrightarrow X$ can be made into a right adjoint to the inclusions, in a unique way such that q_n becomes the adjunction counit. We only show how to define $\langle n\rangle$ on morphisms and omit the remaining verifications. If $f : X \longrightarrow Y$ is a morphism in $\mathcal{SHC}$, then $[-, f] : [-, X] \longrightarrow [-, Y]$ is a natural transformation of functors, which we can restrict to the full subcategory $\mathcal{SHC}_{\geq n}$. On this subcategory, the two functors are represented by $X\langle n\rangle$ respectively $Y\langle n\rangle$, so by the Yoneda lemma there is a unique morphism $Q_n f : X\langle n\rangle \longrightarrow Y\langle n\rangle$ which represents the natural transformation.

Since $X\langle n+1\rangle$ is n-connected, thus also $(n-1)$-connected, the morphism $q_{n+1} : X\langle n+1\rangle \longrightarrow X$ is adjoint to a morphism $i_{n+1} : X\langle n+1\rangle \longrightarrow X\langle n\rangle$ which is in then a natural transformation satisfying $q_n \circ i_{n+1} = q_{n+1}$.

It remains to show that every object X of the stable homotopy category is the homotopy colimit of its n-connective covers as n goes to $-\infty$. We let $(X_{-\infty}, \{\varphi_n : X\langle n\rangle \longrightarrow X_{-\infty}\}_{n\leq 0})$ be a homotopy colimit, in the sense of Definition 5.3, of the sequence $X\langle 0\rangle \xrightarrow{i_0} X\langle -1\rangle \xrightarrow{i_{-1}} \cdots$. Since the morphism $q_n : X\langle n\rangle \longrightarrow X$ are compatible with the sequence, there is a morphism $q : X_{-\infty} \longrightarrow X$, not necessarily uniquely determined, satisfying $q\varphi_n = q_n$ for all $n \leq 0$. For every integer k we have a commutative triangle

$$\begin{tikzcd}
\operatorname{colim}_{n\longrightarrow -\infty}\ \pi_k X\langle n\rangle \arrow[rr, "\pi_k(\varphi_n)_n"] \arrow[dr, "\pi_k(q_n)_n"'] & & \pi_k(X_{-\infty}) \arrow[dl, "\pi_k(q)"] \\
& \pi_k X &
\end{tikzcd}$$

Since the sphere spectrum is compact, the horizontal map is an isomorphism by Lemma 5.6 (ii). The left diagonal map is also an isomorphism since $\pi_k(q_n)$ is bijective for $n \leq k$. Thus the map $\pi_k(q)$ is an isomorphism for all $k \in \mathbb{Z}$, which proves that $q : X_{-\infty} \longrightarrow X$ is an isomorphism in the stable homotopy category. $\square$

[rk: connective cover via $\mathbf{\Gamma}$-spaces $(\Lambda X)(\mathbb{S}) \longrightarrow X$]

Remark 8.2. In Section III.7 we will return to the topic of connected covers from a different angle, and in a much more general context. There we consider a connective operad $\mathcal{O}$ of symmetric spectra and construct connective (i.e., (-1)-connected) covers for algebras over the operad. When $\mathcal{O}$ is the initial operad, then its algebras are just symmetric spectra with no extra structure. In that case the general theory gives a refinement of the functor $X \mapsto X\langle 0\rangle$ as constructed here, namely a connective cover functor on the level of symmetric spectra which descends to $\langle 0\rangle$ on the level of the stable homotopy category. For more details we refer to Example III.7.10.

Instead of 'killing' all the homotopy groups of a spectrum below a certain dimension, we can also 'kill' the homotopy groups above a certain point. This leads to the following notion of 'Postnikov section' which is somewhat dual to that of a connective cover.

Theorem 8.3. *Let n be any integer. The inclusion of the full subcategory of $(n+1)$-coconnected spectra into $\mathcal{SHC}$ has a left adjoint $P_n : \mathcal{SHC} \longrightarrow \mathcal{SHC}_{\leq n}$. The adjunction unit $p_n : X \longrightarrow P_n X$ is called the n-th* Postnikov section *of the symmetric spectrum X. There is a unique natural transformation $j_n : P_n X \longrightarrow P_{n-1} X$ satisfying $j_n \circ p_n = p_{n-1}$.*

There is a unique morphism $\delta : P_n X \longrightarrow \Sigma(X\langle n+1\rangle)$ such that the diagram

$$X\langle n+1\rangle \xrightarrow{q_{n+1}} X \xrightarrow{p_n} P_n X \xrightarrow{\delta} \Sigma(X\langle n+1\rangle) \tag{8.4}$$

is a distinguished triangle in the stable homotopy category.

Moreover, the morphisms $p_n : X \longrightarrow P_n X$ express every symmetric spectrum X as the homotopy limit of the tower of morphisms j_n as n goes to ∞.

We take the time to spell out the properties of Postnikov sections 'dual' to certain properties of connective covers:

- the spectrum P_nX is $(n+1)$-coconnected and for every $(n+1)$-coconnected symmetric spectrum Y the map $[p_n, Y] : [P_nX, Y] \longrightarrow [X, Y]$ is an isomorphism.
- by the long exact homotopy group sequence of the triangle (8.4) the Postnikov section p_n induces isomorphisms of homotopy groups in dimensions n and below.
- The n-th Postnikov section $p_n : X \longrightarrow P_nX$ is a natural transformation,

The above conditions which refer to homotopy groups also characterizes the n-th Postnikov section. Indeed, if $f : X \longrightarrow Y$ is any morphism of symmetric spectra which induces isomorphisms of homotopy groups in dimensions n and below and such that Y is $(n+1)$-connected, then by the above there is a unique morphism $\tilde{f} : P_nX \longrightarrow Y$ in the stable homotopy category satisfying $\tilde{f}p_n = f$, and $\tilde{f}$ is necessarily a π_*-isomorphism, thus an isomorphism in $\mathcal{SHC}$.

Proof of Theorem 8.3. We define the spectrum P_nX and the Postnikov section $p_n : X \longrightarrow P_nX$ by choosing a distinguished triangle

$$X\langle n+1\rangle \xrightarrow{q_{n+1}} X \xrightarrow{p_n} P_nX \xrightarrow{\delta} \Sigma(X\langle n+1\rangle) .$$

For every $(n+1)$-coconnected spectrum Y the groups $[X\langle n+1\rangle, Y]$ and $[\Sigma(X\langle n+1\rangle), Y]$ vanish by is trivial in $\mathcal{SHC}$ by Proposition 5.24 (or rather its $(n+1)$-fold shifted version). The exact sequence

$$[\Sigma(X\langle n+1\rangle), Y] \xrightarrow{[\delta,Y]} [P_nX, Y] \xrightarrow{[p_n,Y]} [X, Y] \xrightarrow{[q_{n+1},Y]} [X\langle n+1\rangle, Y]$$

then shows that the map $[p_n, Y] : [P_nX, Y] \longrightarrow [X, Y]$ is an isomorphism. The remaining arguments to extend P_n to a functor which is left adjoint to the inclusions and such that p_n becomes the adjunction unit are formal, and 'dual' to the corresponding arguments for the connective covers in the proof of Theorem 8.1. The same goes for the construction of the natural transformation $j_n : P_n \longrightarrow P_{n-1}$.

The triangle (8.4) came with the construction of the Postnikov section. Lemma 5.27, applied to the composable morphisms $q_{n+1} : X\langle n+1\rangle \longrightarrow X$ and $p_n : X \longrightarrow P_nX$, shows that the connecting morphism δ is uniquely determined.

For the last claim we let $(X_\infty, \{\varphi_n : X_\infty \longrightarrow P_nX\}_{n\leq 0})$ be a homotopy limit, in the sense of [...] of the tower $\cdots \xrightarrow{j_2} P_1X \xrightarrow{j_1} P_0X$. Since the morphisms $p_n : X \longrightarrow P_nX$ are compatible with the sequence, there is a morphism $p : X \longrightarrow X_\infty$, not necessarily uniquely determined, satisfying $\varphi_n p = p_n$ for all $n \leq 0$. For every integer k, the system of homotopy groups π_kP_nX eventually stabilizes to π_kX, so in the short exact sequence [ref]

$$0 \longrightarrow \lim^1_n \pi_k(\Sigma P_nX) \longrightarrow \pi_k(\mathrm{holim}_n P_nX) \longrightarrow \lim_n \pi_k(P_nX) \longrightarrow 0 .$$

the $\lim^1$-term is trivial and the inverse limit is isomorphic to π_kX, via the Postnikov sections $p_n : X \longrightarrow P_nX$. Thus the morphism $p : X \longrightarrow X_\infty$ induces isomorphisms of all homotopy groups, so it is an isomorphism in the stable homotopy category. □

Remark 8.5. For $n \geq m$ we have

$$(X\langle m\rangle)\langle n\rangle \cong X\langle n\rangle \cong (X\langle n\rangle)\langle m\rangle \quad \text{and} \quad P_n(P_mX) \cong P_mX \cong P_m(P_nX)$$

via instances of the maps p_k respectively q_k. Moreover

$$P_n(X\langle m\rangle) \cong (P_nX)\langle m\rangle$$

and this spectrum has its homotopy groups concentrated in dimensions m through n.

[Exercise: $[HA, HB]^n = 0$ for $n < 0$. Thus X is coconnective if and only if it belongs to $\langle HA\rangle_-$ [check]]
[Exercise:

$$X\langle n+k\rangle \;\cong\; \mathbb{S}^n \wedge ((\mathbb{S}^{-n} \wedge X)\langle k\rangle) \quad \text{and} \quad P_{n+k}X \;\cong\; \mathbb{S}^n \wedge P_k(\mathbb{S}^{-n} \wedge X).$$

]

By Proposition 5.24 (or rather its n-fold suspension) there is a unique morphism $i : \mathbb{S}^n \wedge H(\pi_n X) \longrightarrow P_n X$ such that the composite

$$\pi_n X = \pi_0 H(\pi_n X) \xrightarrow{S^n \wedge -} \pi_n(\mathbb{S}^n \wedge H(\pi_n X)) \xrightarrow{\pi_n i} \pi_n(P_n X)$$

coincides with the isomorphism induced by the Postnikov section $p_n : X \longrightarrow P_n X$. Lemma 5.27 provides a unique morphism $\mathbf{k}_n : P_{n-1}X \longrightarrow \Sigma^{n+1} H(\pi_n X)$ such that the triangle

$$\mathbb{S}^n \wedge H(\pi_n X) \xrightarrow{i} P_n X \xrightarrow{j_n} P_{n-1} X \xrightarrow{\mathbf{k}_n} \mathbb{S}^{n+1} \wedge H(\pi_n X) \tag{8.6}$$

is distinguished. The morphism $\mathbf{k}_n$ is an element of the cohomology group $H^{n+1}(P_{n-1}X, \pi_n X)$ and is called the n-th *k-invariant* of the spectrum X. We will see below [...] that for *rational* spectra X (i.e., all homotopy groups of X are uniquely divisible) all k-invariants are trivial since X decomposes as a product of suspended Eilenberg-Mac Lane spectra.

It is tempting to say that 'the homotopy type of a spectrum is determined by the sequence of homotopy groups and k-invariants'. However, this statement has to be treated with care. The caveat is that the cohomology group in which the n-th k-invariant lies depends on the $(n-1)$th Postnikov section which in turn depends on all the previous k-invariants. So for a fixed sequence of groups, there is no universally defined group which houses the k-invariant of all spectra which have the given groups as homotopy groups. [inverse limit problem?]

[exercise: the k-invariant can also be defined as the composite

$$P_{n-1}X \cong P_{n-1}(P_n X) \xrightarrow{\delta} \Sigma((P_n X)\langle n+1\rangle) \cong \Sigma^{n+1} H(\pi_n X)$$

where δ is the connecting morphism of the distinguished triangle (8.4) for the spectrum $P_n X$ and n replaced by $n-1$.]

Remark 8.7. For spectra with additional structure one can typically refine the k-invariants and lift them from the spectrum cohomology group $H^{n+1}(P_{n-1}X, \pi_n X)$ to a 'refined' cohomology group which depends on the type of structure under consideration. We indicate a first example of this phenomenon in the case of homotopy ring spectra. We intend to return to this later, treating in particular the cases of symmetric ring spectra (where the appropriate home for the k-invariant are *topological derivation groups*, closely related to *topological Hochschild cohomology*) and commutative ring spectra (where the appropriate theory is *topological André-Quillen cohomology*).

Suppose R is a homotopy ring spectrum and M a homotopy R-bimodule. By a *derivation* of R with coefficients in M we mean a morphism $d : R \longrightarrow M$ in the stable homotopy category which satisfies

$$d\mu \;=\; d \wedge^L R \;+\; R \wedge^L d$$

as morphisms from $R \wedge^L R$ to M in $\mathcal{SHC}$.

Proposition 8.8. *Let R be a connective homotopy ring spectrum and $n \geq 0$ such that the homotopy groups of R are trivial above dimension n. Then the spectrum $P_n R$ inherits a unique structure of homotopy ring spectrum such that the Postnikov section $p_n : R \longrightarrow P_n R$ is a morphism of homotopy ring spectra. Moreover, the Eilenberg-Mac Lane spectrum $H(\pi_{n+1}R)$ inherits a natural structure of homotopy R-bimodule over $P_n R$ and the k-invariant $\mathbf{k}_{n+1} : P_n R \longrightarrow \Sigma^{n+2}(H(\pi_{n+1}R))$ is a derivation of $P_n R$ with coefficients in $\Sigma^{n+2}(H(\pi_{n+1}R))$. [is connectivity necessary?]*

Proof. To simplify notation we write P for the Postnikv section $P_n R$, we write A for the n-th homotopy group $\pi_n R$ and we write $\wedge$ instead of $\wedge^L$ during the course of this proof. [...]

It remains to establish the derivation property of the k-invariant. We do this by comparing the defining distinguished triangle with another distinguished triangle involving $P \wedge P$.

We choose a distinguished triangle

$$\Sigma^n HA \wedge \Sigma^n HA \xrightarrow{q_n \wedge q_n} R \wedge R \xrightarrow{g} C \longrightarrow \Sigma(\Sigma^n HA \wedge \Sigma^n HA)\ .$$

The spectrum $\Sigma^n HA \wedge \Sigma^n HA$ is $(2n-1)$-connected and R has trivial homotopy groups above dimenion n, the composite of $q_n \wedge q_n$ with the multiplication map $\mu : R \wedge R \longrightarrow R$ is trivial. So there is an extension [unique?] $\bar{\mu} : C \longrightarrow R$ such that $\bar{\mu}g = \mu$. Since the composite $p_{n-1}q_n : \Sigma^n HA \longrightarrow P$ is trivial, so is the map

$$(p_{n-1}q_n) \wedge (p_{n-1}q_n) = (p_{n-1} \wedge q_{n-1})(q_n \wedge q_n) \ : \ \Sigma^n HA \wedge \Sigma^n HA \ \longrightarrow \ P \wedge P \ ,$$

so there is a morphism $\lambda : C \longrightarrow P \wedge P$ such that $\lambda g = p_{n-1} \wedge p_{n-1}$.

We claim that $\mu_P \lambda = p_{n-1}\bar{\mu}$ as maps from C to P in $\mathcal{SHC}$. Indeed, this relation holds after composition with $g : R \wedge R$, and the defining distinguished triangle for C gives an exact sequence

$$\mathcal{SHC}(\Sigma(\Sigma^n HA \wedge \Sigma^n HA), P) \ \longrightarrow \ \mathcal{SHC}(C, P) \ \longrightarrow \ \mathcal{SHC}(R \wedge R, P) \ .$$

The first group is zero [...]

Now we consider the diagram

$$\begin{tikzcd}
(P \wedge \Sigma^n HA) \vee (\Sigma^n HA \wedge P) \arrow[r] \arrow[d, "\mathrm{act}"'] & C \arrow[r, "\lambda"] \arrow[d, "\bar{\mu}"'] & P \wedge P \arrow[r, "(P\wedge \mathbf{k}, \mathbf{k}\wedge P)"] \arrow[d, "\mu"] & \Sigma((P \wedge \Sigma^n HA) \vee (\Sigma^n HA \wedge P)) \arrow[d, "\Sigma(\mathrm{act})"] \\
\Sigma^n HA \arrow[r, "q_n"'] & R \arrow[r, "p_{n-1}"'] & P \arrow[r, "\mathbf{k}"'] & \Sigma^{n+1} HA
\end{tikzcd}$$

We claim that: (a) this diagram commutes and (b) the upper triangle is distinguished. [do we need (b)?]

[use (a) to define the action map? would need to show associativity and unitality...] □

Example 8.9. The first k-invariant of the sphere spectrum $\mathbb{S}$ is a morphism $\mathbf{k}_1(\mathbb{S}) \in H^2(H\mathbb{Z}, \mathbb{Z}/2)$ (where we identify $P_0\mathbb{S}$ with $H\mathbb{Z}$ and the first stable stem $\pi_1^s \cong \pi_1\mathbb{S}$ with the group $\mathbb{Z}/2$). This k-invariant is non-zero [justify] and is in fact the pullback of the Steenrod operation $\mathrm{Sq}^2 \in H^2(H\mathbb{Z}/2, \mathbb{Z}/2)$ along the projection morphism $H\mathbb{Z} \longrightarrow H\mathbb{Z}/2$.

The second (and first non-trivial) k-invariant of the connective complex K-theory spectrum ku is a morphism $\mathbf{k}_2(ku) \in H^2(H\mathbb{Z}, \mathbb{Z})$ (where we identify $P_0\mathbb{S}$ with $H\mathbb{Z}$ and $\pi_2 ku$ with $\mathbb{Z}$). This k-invariant is non-zero [justify] and is in fact equal to $\beta \circ \mathbf{k}_1(\mathbb{S})$ where $\beta : H\mathbb{Z}/2 \longrightarrow \Sigma(H\mathbb{Z})$ is the Bockstein operator (see Proposition 5.28) associated to the extension $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2$. The equation $\mathbf{k}_2(ku) = \beta \circ \mathbf{k}_1(\mathbb{S})$ is a way of rephrasing the Toda bracket relation $u \in \langle 1, \eta, 2 \rangle$ (modulo $2u$) in $\pi_2(ku)$ (compare I.2.5).

For an odd prime p the first non-trivial homotopy group of the localized sphere spectrum $\mathbb{S}_{(p)}$ in positive dimension is a copy of $\mathbb{Z}/p$ generated by the class α_1 in dimension $2p-3$. So the first potentially non-trivial k-invariant of $\mathbb{S}_{(p)}$ is a morphism $\mathbf{k}_{2p-3}(\mathbb{S}_{(p)}) \in H^{2p-2}(H\mathbb{Z}_{(p)}, \mathbb{Z}/p)$. As in the case $p = 2$, this k-invariant is non-zero [justify] and is in fact the pullback of the Steenrod operation $P^1 \in H^{2p-2}(H\mathbb{Z}/p, \mathbb{Z}/p)$ along the projection morphism $H\mathbb{Z}_{(p)} \longrightarrow H\mathbb{Z}/p$.

The k-invariants $\mathbf{k}_1(\mathbb{S})$, $\mathbf{k}_{2p-3}(\mathbb{S}_{(p)})$ and $\mathbf{k}_2(ku)$ arise from symmetric ring spectra, so they are in fact derivations by Proposition 8.8. The only derivations (up to units) in the mod-p Steenrod algebra $\mathcal{A}_p$ are the Milnor elements $Q_n \in H^{2p^n-1}(H\mathbb{F}_p, \mathbb{F}_p)$. We have $Q_0 = \beta$, the mod-p Bockstein, and the other classes are inductively defined as commutators $Q_{n+1} = [Q_n, P^{p^n}]$. These derivations are all realized as k-invariants of the suitable symmetric ring spectra, namely the connective Morava K-theory spectra $k(n)$ (see Example I.6.63), i.e., we have $Q_n = \mathbf{k}_{2p^n-2}(k(n))$. [check this]

9. Localization and completion

9.1. Localization at a set of primes. The localization of an abelian group at a set of prime numbers has an analogue in stable homotopy theory, also called 'localization'. With this tool, and the completion that we discuss below, one can often study a problem 'one prime at a time'.

To fix notation and language, we quickly review the localization of abelian groups. For every set S of prime numbers we define a subring $\mathbb{Z}_S$ of the ring of natural numbers by

$$\mathbb{Z}_S = \left\{ \frac{a}{b} \in \mathbb{Q} \mid b \text{ has only prime factors in } S \right\} \ .$$

For example, we have

$$\mathbb{Z}_\emptyset = \mathbb{Z} \ , \quad \mathbb{Z}_{\text{all primes}} = \mathbb{Q} \quad \text{and} \quad \mathbb{Z}_{\text{all}\setminus\{p\}} = \mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid \ p \text{ does not divide } b \right\} \ .$$

Every subring of $\mathbb{Q}$ is of the form $\mathbb{Z}_S$ for a unique set of primes S: given a subring $R \subset \mathbb{Q}$, then $R = \mathbb{Z}_S$ for the set S defined by $S = \{p \mid \frac{1}{p} \in R\}$.

For a subring $R \subset \mathbb{Q}$, the functor of abelian groups $A \mapsto R \otimes A$ is exact. Since the multiplication map $R \otimes R \longrightarrow R$ is an isomorphism, the functor is idempotent with respect to the natural map

$$A \longrightarrow R \otimes A \ , \quad a \longmapsto 1 \otimes a \ .$$

This construction is called *R-localization*. An abelian group A is called *R-local* if the following equivalent conditions hold:

(i) The group A has the structure of an R-module (necessarily unique).
(ii) The map $A \longrightarrow R \otimes A$ is bijective.
(iii) For every prime p with $\frac{1}{p} \in R$, multiplication by p on A is bijective.

Example 9.1.
- Every abelian group is $\mathbb{Z}$-local.
- A is $\mathbb{Z}_{(p)}$-local ('p-local') if and only if multiplication by q is bijective on A for all primes $q \neq p$.
- A is $\mathbb{Q}$-local ('rational') if and only if A is uniquely divisible, i.e., for all $a \in A$ and $n \in \mathbb{N}_+$ there is a unique $b \in A$ such that $a = n \cdot b$.
- Any finitely generated abelian group is a finite direct sum copies of $\mathbb{Z}$ and $\mathbb{Z}/q^m$ for various primes q and $m \geq 1$. Such a sum is p-local if and only if only $\mathbb{Z}/p^m$'s occur. The p-localization functor turns every copy of $\mathbb{Z}$ into a copy of $\mathbb{Z}_{(p)}$, it leaves all summands of the form $\mathbb{Z}/p^m$ untouched, and it kills summands of the form $\mathbb{Z}/q^m$ for primes $q \neq p$.

Moore spectra were introduced and discussed in Section 6.3. We recall that a Moore spectrum for an abelian group A is a connective symmetric spectrum $\mathbb{S}A$ equipped with an isomorphism $H_0(\mathbb{S}A, \mathbb{Z}) \cong A$ such that the integral homology groups of $\mathbb{S}A$ in dimensions different from 0 are trivial. The isomorphism is part of the data, but usually not mentioned explicitly.

Theorem 9.2. *Let R be a subring of $\mathbb{Q}$ and $\mathbb{S}R$ a Moore spectrum for R.*

(i) *For every symmetric spectrum X and integer k the natural map*

$$R \otimes \pi_k X \ \longrightarrow \ \pi_k(\mathbb{S}R \wedge^L X)$$

induced by the homotopy group pairing is an isomorphism. Hence a morphism $f : X \longrightarrow Y$ is an $\mathbb{S}R$-equivalence if and only if the map $R \otimes \pi_k f$ is an isomorphism for all integers k.

(ii) *For a symmetric spectrum X the following are equivalent.*
  (a) *All homotopy groups of X are R-local.*
  (b) *For every morphism $f : A \longrightarrow B$ of symmetric spectra such that*

$$R \otimes \pi_*(f) \ : \ R \otimes \pi_* A \longrightarrow R \otimes \pi_* B$$

  is an isomorphism, the induced map $\mathcal{SHC}(f, X) : \mathcal{SHC}(B, X) \longrightarrow \mathcal{SHC}(A, X)$ of morphisms in the stable homotopy category is a bijection.
  (c) *For every symmetric spectrum C such that $R \otimes \pi_* C$ is trivial the morphism group $\mathcal{SHC}(C, X)$ is trivial.*
  (d) *The endomorphism ring $\mathcal{SHC}(X, X)$ in the stable homotopy category is an R-algebra.*

If the homotopy groups of X are bounded below, then conditions (a)-(d) *above are also equivalent to the condition that the integral spectrum homology groups $H_*(X, \mathbb{Z})$ are R-local.*

As we shall see in Proposition 9.19 below, R-localization is a special case of E-localization (or Bousfield localization), namely when we take E as a Moore-spectrum $\mathbb{S}R$. More precisely, part (i) of the Theorem above says that the Hurewicz map $X \longrightarrow \mathbb{S}R \wedge^L X$ is an $\mathbb{S}R$-localization. So we could add the condition'X is $\mathbb{S}R$-local' to the above list of equivalent conditions.

A morphism $f : A \longrightarrow B$ such that $R \otimes \pi_* f : R \otimes \pi_* A \longrightarrow R \otimes \pi_* B$ is an isomorphism is called an *R-equivalence*. A spectrum C such that $R \otimes \pi_* C = 0$, or (equivalently) such that $\mathbb{S}R \wedge^L X$ is stably contractible is called *R-acyclic*. Instead of '$\mathbb{S}R$-localization', we simply say 'R-localization' and write X_R for $\mathbb{S}R \wedge^L X$. For a prime p we say 'p-local' instead of $\mathbb{Z}_{(p)}$-local and write $X_{(p)}$ for the p-localization

$\mathbb{S}\mathbb{Z}_{(p)} \wedge^L X$. Finally, we use the term 'rational' as synonymous for $\mathbb{Q}$-local and denote the rationalization by $X_{\mathbb{Q}}$.

Remark 9.3. The class of R-local spectra has various closure properties. Direct sums and products of R-local abelian groups are again R-local. Since homotopy groups take sums to sums and products to product, any sum or product of R-local spectra is again R-local.

A spectrum X is R-local if and only if the suspension ΣX is R-local. The class of R-local abelian groups is closed under subgroups, quotient groups and extensions. So if two out of three groups in a long exact sequence are R-local, so are the remaining groups. Hence in any distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

if two of the spectra A, B and C are R-local, so is the third.

If X is R-local, then for any other spectrum Y, the smash product $X \wedge^L Y$ and the function spectra $F(X,Y)$ and $F(Y,X)$ are also R-local. Indeed, if q is any prime which is invertible in R, then $q \cdot \mathrm{Id}_X$ is an automorphism of X. Now the degree q maps of $X \wedge^L Y$, $F(X,Y)$ and $F(Y,X)$ are obtained from $q \cdot \mathrm{Id}_X$ by smashing with Y or taking functions into respectively out of Y. So q acts invertibly on the spectra $X \wedge^L Y$, $F(X,Y)$ and $F(Y,X)$, and so all three are R-local.

Example 9.4. • Every symmetric spectrum is $\mathbb{Z}$-local. (This is not to be confused with the class of $H\mathbb{Z}$-local spectra, compare Remark 9.24).

• If A is an R-local abelian group, then the Eilenberg-Mac Lane spectrum HA is R-local.

• If X is a spectrum and p a prime, we denote by X/p the mapping cone of an endomorphism of X which represents $p \cdot \mathrm{Id}_X$ in $\mathcal{SHC}(X,X)$. Equivalently, we could define X/p as $\mathbb{S}/p \wedge^L X$, the derived smash product of a mod-p Moore spectum with X. The homotopy groups of X/p sit in a long exact sequence

$$\cdots \longrightarrow \pi_m X \xrightarrow{\times p} \pi_m X \longrightarrow \pi_m(X/p) \xrightarrow{\delta} \pi_{m-1} X \xrightarrow{\times p} \pi_{m-1} X \longrightarrow \cdots$$

which breaks up into short exact sequences

$$0 \longrightarrow \mathbb{Z}/p \otimes \pi_m X \longrightarrow \pi_m(X/p) \xrightarrow{\delta} {}_p\{\pi_{m-1}X\} \longrightarrow 0 \ .$$

The groups $\mathbb{Z}/p \otimes \pi_m X$ and ${}_p\{\pi_{m-1}X\}$ are $\mathbb{F}_p$-vector spaces, so the homotopy group $\pi_m(X/p)$ is killed by multiplication by p^2; hence the spectrum X/p is p-local. As a special case of this, the Moore spectrum $\mathbb{S}/p$ is p-local.

• The connectivity assumption in part (c) of the above theorem is important. Indeed, for the mapping telescope of the Adams map $v : \mathbb{S}/p \longrightarrow \mathbb{S}/p[-2p+2]$ (p an odd prime) we have

$$H_*(v_1^{-1}\mathbb{S}/p, \mathbb{Z}) \ = \ 0 \ ,$$

as in Example 6.31. As trivial groups, the integral spectrum homology groups of $v_1^{-1}\mathbb{S}/p$ are in particular rational. But the homotopy $\pi_*(v_1^{-1}\mathbb{S}/p)$ is a nontrivial graded $\mathbb{F}_p$-vector space, so it is not rational.

9.2. Rational stable homotopy theory. We shall now see that the *rational stable homotopy category*, i.e., the full subcategory of $\mathcal{SHC}$ consisting of rational spectra, is very easy to describe. As above, an abelian group is *rational* if it admits the structure of a $\mathbb{Q}$-vector space (necessarily unique). Equivalently, A is rational if and only it is uniquely divisible, i.e., for all $a \in A$ and $n > 0$ there is a unique $b \in A$ such that $a = n \cdot b$. We call a symmetric spectrum *rational* if all its homotopy groups are rational. By Theorem 9.2, a spectrum X is rational if and only if it is $\mathbb{S}\mathbb{Q}$-local, where $\mathbb{S}\mathbb{Q}$ is a Moore spectrum for the group $\mathbb{Q}$. We recall from Example 6.36 that the Eilenberg-Mac Lane spectrum $H\mathbb{Q}$ is a possible choice for the rational Moore spectrum $\mathbb{S}\mathbb{Q}$. So the rationalization of X, denoted $X_{\mathbb{Q}}$, is given by $H\mathbb{Q} \wedge^L X$, or by the $\mathbb{Q}$-linearization $\mathbb{Q}[X]$.

Theorem 9.6 below shows that a rational spectrum is completely determined by its homotopy groups: the homotopy group functor is an equivalence from rational stable homotopy category to graded vector spaces over $\mathbb{Q}$. In particular, two rational spectra are stably equivalent if and only if the homotopy groups are abstractly isomorphic. The key ingredient for all this is Serre's calculation of homotopy groups of spheres modulo torsion (Theorem 1.9 of Chapter I). The integral analogue of this is not at all true, i.e., the homotopy group functor

$$\pi_* \ : \ \mathcal{SHC} \ \longrightarrow \ \text{(graded abelian groups)}$$

is very far from being an equivalence for general spectra.

Let $V_* = \{V_n\}_{n\in\mathbb{Z}}$ be a $\mathbb{Z}$-graded abelian group. We defined the *generalized Eilenberg-Mac Lane spectrum* associated to V_* as the product of suspended Eilenberg-Mac Lane spectra for the groups V_n,

$$HV_* \ = \ \prod_{n\in\mathbb{Z}} \Sigma^n(HV_n) \ .$$

Here the suspension and the product are taken in the stable homotopy category; this particular product is isomorphic in $\mathcal{SHC}$ to the sum of the suspended Eilenberg-Mac Lane spectra. Then there is a natural isomorphism

$$\pi_k(HV_*) \ \cong \ \prod_{n\in\mathbb{Z}} \pi_k(\Sigma^n(HV_n)) \ \cong \ V_k \tag{9.5}$$

so that the generalized Eilenberg-Mac Lane spectrum HV_* realizes the graded abelian group V_* on homotopy.

Now we can show the main result of this section:

Theorem 9.6. (i) *Let V_* be a $\mathbb{Z}$-graded $\mathbb{Q}$-vector space and A a symmetric spectrum. Then the map*

$$\begin{aligned} \pi_* \ : \ [A, HV_*] \quad &\longrightarrow \quad \mathrm{Hom}_{\mathrm{gr}.\mathcal{A}b}(\pi_* A, V_*) \\ (f : A \longrightarrow HV_*) \quad &\longmapsto \quad (\pi_* f : \pi_* A \longrightarrow \pi_*(HV_*) \cong V_*) \end{aligned}$$

is an isomorphism of abelian groups.

(ii) *Every rational spectrum is a generalized Eilenberg-Mac Lane spectrum. More precisely, if A is a symmetric spectrum with rational homotopy groups, then there exists a unique homotopy class of morphism $A \longrightarrow H(\pi_* A)$ which realizes the isomorphism $\pi_* A \cong \pi_* H(\pi_* A)$ of* (9.5) *on homotopy groups.*

(iii) *The homotopy group functor*

$$\pi_* \ : \ \mathcal{SHC}_{\mathbb{Q}} \ \longrightarrow \ \text{(graded } \mathbb{Q}\text{-vector spaces)}$$

is an equivalence from the rational stable homotopy category to the category of graded $\mathbb{Q}$-vector spaces. An inverse functor is given by the generalized Eilenberg-Mac Lane spectra.

Proof. (i) We start with the special case $A = \mathbb{S}$ of the sphere spectrum. Since the sphere spectrum represents π_0 (Example 1.15), the group $[\mathbb{S}, HV_*]$ is isomorphic to $\pi_0(HV_*)$. Thus $[\mathbb{S}, HV_*]$ is trivial for $n \neq 0$ and isomorphic to V_0 for $n = 0$. The right hand side

$$\mathrm{Hom}_{\mathrm{gr}.\mathcal{A}b}(\pi_*\mathbb{S}, V_*) \ = \ \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}b}(\pi_n\mathbb{S}, V_n)$$

is isomorphic to $\mathrm{Hom}_{\mathcal{A}b}(\pi_0\mathbb{S}, V_0)$ since $\pi_n\mathbb{S}$ is a torsion group for $n \neq 0$ (by Serre's theorem I.1.9) whereas V_n is rational. Since $\pi_0\mathbb{S}$ is free abelian generated by the fundamental class, the group $\mathrm{Hom}_{\mathrm{gr}.\mathcal{A}b}(\pi_*\mathbb{S}, V_*)$ is altogether isomorphic to V_0, via evaluation at the fundamental class of $\mathbb{S}$. So the claim is true for the sphere spectrum.

We let $\mathcal{X}$ denote the class of symmetric spectra A with the property that the map $\pi_* : [A, HV_*] \longrightarrow \mathrm{Hom}_{\mathrm{gr}.\mathcal{A}b}(\pi_* A, V_*)$ is an isomorphism. By the above the sphere spectrum $\mathbb{S}$ belongs to $\mathcal{X}$. Moreover, the class $\mathcal{X}$ is closed under sums and extensions in distinguished triangles. The latter needs that every rational abelian group is injective, i.e., the functor $\mathrm{Hom}_{\mathrm{gr}.\mathcal{A}b}(-, V_*)$ is exact. Since the spectrum $\mathbb{S}$ generates the stable homotopy category (see Proposition 5.16), the class $\mathcal{X}$ contains all spectra, which proves (i).

Part (ii) follows by applying part (i) to $V_* = \pi_* A$. (iii) We have already seen that every graded $\mathbb{Q}$-vector space is isomorphic to an object in the image of the functor π_* (namely the generalized Eilenberg-Mac Lane spectrum). So it remains to show that π_* is fully faithful, i.e., that the map

$$\pi_* \ : \ \mathcal{SHC}_{\mathbb{Q}}(X, Y) \ \longrightarrow \ \mathrm{Hom}_{\text{gr-}\mathbb{Q}\text{-vs}}(\pi_* X, \pi_* Y)$$

is an isomorphism for all rational spectra X and Y. By (ii) we can assume that Y is a generalized Eilenberg-Mac Lane spectrum, i.e., $Y = HV_*$ for a graded $\mathbb{Q}$-vector space V_*. But then part (i) give the fully-faithfulness. So the functor π_* is an equivalence of categories. □

Part (ii) of the previous theorem is very particular for rational spectra. A general spectrum is *not* a generalized Eilenberg-Mac Lane spectrum. For example, the sphere spectrum $\mathbb{S}$, the mod-p Moore spectrum $S\mathbb{Z}/p$ or the real and complex K-theory spectra KO and KU are not stably equivalent to any generalized Eilenberg-Mac Lane spectrum.

Remark 9.7. By Proposition 6.30 (iii) the stable Hurewicz homomorphism

$$\pi_n X \longrightarrow H_n(X, \mathbb{Z})$$

is a *rational isomorphism* for every spectrum X and every integer n, i.e., it becomes an isomorphism after tensoring both sides with the group $\mathbb{Q}$ of rational numbers.

Now suppose that X is a spectrum whose integral spectrum homology groups $H_*(X, \mathbb{Z})$ are trivial. If X is not connective, then this need not imply that X is stably trivial, but it does have consequences for the homotopy groups of X. Since the stable Hurewicz morphism is a rational isomorphism, the rationalized stable homotopy groups $\mathbb{Q} \otimes \pi_* X$ are trivial. This is equivalent to the property that every homotopy element is *torsion*, i.e., annihilated by multiplication by some positive natural number.

In Example 6.31 we have seen this phenomenon happen; the mapping telescope $v^{-1}\mathbb{S}/p$ of the Adams map on the mod-p Moore spectrum has trivial spectrum homology, but it is not stably contractible. In that example, all homotopy groups are $\mathbb{F}_p$-vector spaces, so in particular annihilated by multiplication by p. [move this earlier ?]

9.3. Completion. We let $\mathbb{S}\mathbb{Q}/\mathbb{Z}$ denote the mapping cone of the unit map $\mathbb{S} \longrightarrow H\mathbb{Q}$; by [...] this construction comes with a distinguished triangle

$$\mathbb{S} \xrightarrow{\iota} H\mathbb{Q} \xrightarrow{q} \mathbb{S}\mathbb{Q}/\mathbb{Z} \xrightarrow{\delta} \mathbb{S}^1 .$$

Since $\mathbb{S}$ and $H\mathbb{Q}$ are Moore spectra for the groups $\mathbb{Z}$ respectively $\mathbb{Q}$ and the unit map $\iota : \mathbb{S} \longrightarrow H\mathbb{Q}$ induces the monomorphism on homology, the mapping cone $S\mathbb{Q}/\mathbb{Z}$ is a Moore spectrum for the group $\mathbb{Q}/\mathbb{Z}$. We define the *profinite completion* functor

$$(-)^\wedge \; : \; \mathcal{SHC} \longrightarrow \mathcal{SHC}$$

as the derived function spectrum

$$X^\wedge \; = \; F(\mathbb{S}\mathbb{Q}/\mathbb{Z}, \Sigma X) \; .$$

A natural morphism $X \longrightarrow X^\wedge$ in the stable homotopy category is obtained as the adjoint of the morphism

$$X \wedge^L \mathbb{S}\mathbb{Q}/\mathbb{Z} \xrightarrow{\mathrm{Id} \wedge \delta} X \wedge^L \mathbb{S}^1 \cong \Sigma X$$

where δ is the connecting morphism.

For a fixed prime p we similarly define the *p-completion* of X as

$$X^\wedge_p \; = \; F(\mathbb{S}/p^\infty, \Sigma X) \; .$$

where $\mathbb{S}/p^\infty$ is a Moore spectrum for the group $\mathbb{Z}/p^\infty$, which can either be defined as the colimit of the groups $\mathbb{Z}/p^n$ under multiplication by p maps or, equivalently, as the factor group $\mathbb{Z}[1/p]/\mathbb{Z}$ or, equivalently, as the p-power torsion subgroup of $\mathbb{Q}/\mathbb{Z}$. The p-adic completion comes with a natural map $X \longrightarrow X^\wedge_p$ defined similarly as for the profinite completion.

We add a remark about the uniqueness of the completions constructions. The group $\mathbb{Q}/\mathbb{Z}$ and the groups $\mathbb{Z}/p^\infty$ for any prime p are 2-divisible. So by Theorem 6.43 the corresponding Moore spectra $\mathbb{S}\mathbb{Q}/\mathbb{Z}$ and $\mathbb{S}/p^\infty$ are unique up to preferred isomorphism in the stable homotopy category.

Remark 9.8. The p-completion $X^\wedge_p$ of every spectrum is p-local. Indeed, the Moore spectrum $\mathbb{S}/p^\infty$ is connective and has p-local integral homology, so it is p-local. Thus the function spectrum $X^\wedge_p = F(\mathbb{S}/p^\infty, \Sigma X)$ is p-local, compare Remark 9.3.

Recall that we denote by $X \longrightarrow X_{\mathbb{Q}}$ the rationalization of a spectrum, where $X_{\mathbb{Q}} = H\mathbb{Q} \wedge X$.

Theorem 9.9. *Let X be a symmetric spectrum.*

(i) *The square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X^\wedge \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & (X^\wedge)_{\mathbb{Q}} \end{tikzcd}$$

is homotopy cartesian, where the vertical maps are rationalizations and the horizontal maps are (obtained from) completions.

(ii) *The map*

$$X^\wedge \longrightarrow \prod_{p\,\mathrm{prime}} X_p^\wedge$$

whose p-component is obtained from the morphism $\mathbb{S}\mathbb{Z}/p^\infty \longrightarrow \mathbb{S}\mathbb{Q}/\mathbb{Z}$ realizing the inclusion $\mathbb{Z}/p^\infty \longrightarrow \mathbb{Q}/\mathbb{Z}$ on homology by applying $F(-,\Sigma X)$ is an isomorphism in $\mathcal{SHC}$.

(iii) *The p-completion $X_p^\wedge$ is a homotopy limit in $\mathcal{SHC}$ of the tower*

$$\cdots \xrightarrow{F(\psi_3,\Sigma X)} F(\mathbb{S}/p^3,\Sigma X) \xrightarrow{F(\psi_2,\Sigma X)} F(\mathbb{S}/p^2,\Sigma X) \xrightarrow{F(\psi_1,\Sigma X)} F(\mathbb{S}/p,\Sigma X)$$

where $\psi_n : \mathbb{S}/p^n \longrightarrow \mathbb{S}/p^{n+1}$ realizes the map $p\cdot - : \mathbb{Z}/p^n \longrightarrow \mathbb{Z}/p^{n+1}$ on integral homology.

(iv) *There is a natural short exact sequence of abelian groups*

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z}/p^\infty,\pi_k X) \longrightarrow \pi_k(X_p^\wedge) \longrightarrow \mathrm{Hom}(\mathbb{Z}/p^\infty,\pi_{k-1}X) \longrightarrow 0\,. \tag{9.10}$$

If we combine parts (i) and (ii) of the previous theorem we obtain a homotopy cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_{p\,\mathrm{prime}} X_p^\wedge \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \left(\prod_{p\,\mathrm{prime}} X_p^\wedge\right)_{\mathbb{Q}} \end{tikzcd}$$

which is called the *arithmetic square*. This square encodes the way in which a spectrum can be assembled from rational information and profinite information at each prime. For example on homotopy groups the square gives rise to a long exact sequence:

$$\cdots \longrightarrow \pi_k X \longrightarrow \mathbb{Q}\otimes\pi_k X \,\oplus\, \prod_p \pi_k(X_p^\wedge) \longrightarrow \mathbb{Q}\otimes\left(\prod_p \pi_k(X_p^\wedge)\right) \longrightarrow \pi_{k-1}X \longrightarrow \cdots$$

Proof of Theorem 9.9. (i) The triangle

$$H\mathbb{Q} \xrightarrow{q} \mathbb{S}\mathbb{Q}/\mathbb{Z} \xrightarrow{\delta} \mathbb{S}^1 \xrightarrow{-\Sigma\iota} \Sigma H\mathbb{Q}$$

is a rotation of the defining triangle for $\mathbb{S}\mathbb{Q}/\mathbb{Z}$, and thus distinguished. The functor $F(-,X)$ and rationalization are exact, so we have a commutative diagram of distinguished triangles

$$\begin{tikzcd} F(\Sigma H\mathbb{Q},\Sigma X) \arrow[r] \arrow[d, "\cong"'] & F(\mathbb{S}^1,\Sigma X) \arrow[r, "F(\delta,\Sigma X)"] \arrow[d] & F(S\mathbb{Q}/\mathbb{Z},\Sigma X) \arrow[r] \arrow[d] & \Sigma F(H\mathbb{Q},X) \arrow[d, "\cong"] \\ F(\Sigma H\mathbb{Q},\Sigma X)_{\mathbb{Q}} \arrow[r] & F(\mathbb{S}^1,\Sigma X)_{\mathbb{Q}} \arrow[r, "F(\delta,\Sigma X)_{\mathbb{Q}}"'] & F(S\mathbb{Q}/\mathbb{Z},\Sigma X)_{\mathbb{Q}} \arrow[r] & \Sigma F(\Sigma H\mathbb{Q},\Sigma X)_{\mathbb{Q}} \end{tikzcd}$$

The derived function spectrum $F(\Sigma H\mathbb{Q},\Sigma X)$ is rational since $\Sigma H\mathbb{Q}$ is (compare Remark 9.3), so it is already rational and so the left and right vertical morphisms are stable equivalences. Thus the square in the middle is homotopy cartesian. Via the isomorphism $X \cong F(\mathbb{S}^1, \Sigma X)$ adjoint to $X \wedge \mathbb{S}^1 \cong \Sigma X$, the middle square becomes the square in question.

(ii) We start from the algebraic fact that the sum of the inclusions $\mathbb{Z}/p^\infty \longrightarrow \mathbb{Q}/\mathbb{Z}$ is an isomorphism

$$\bigoplus_{p \text{ prime}} \mathbb{Z}/p^\infty \longrightarrow \mathbb{Q}/\mathbb{Z} .$$

So if $\phi_p : \mathbb{S}/p^\infty \longrightarrow \mathbb{S}\mathbb{Q}/\mathbb{Z}$ is a morphism that realizes the inclusion on $H_0(-, \mathbb{Z})$, then the map

$$\sum \phi_p \ : \ \bigvee_{p \text{ prime}} \mathbb{S}/p^\infty \longrightarrow \mathbb{S}\mathbb{Q}/\mathbb{Z}$$

is a stable equivalence. Hence the induced morphism of derived function spectra

$$X^\wedge = \mathrm{Hom}(\mathbb{S}\mathbb{Q}/\mathbb{Z}, \Sigma X) \longrightarrow \mathrm{Hom}(\bigvee_{p \text{ prime}} \mathbb{S}/p^\infty, \Sigma X) \cong \prod_{p \text{ prime}} X_p^\wedge$$

is a stable equivalence.

(iii) We let $\psi_n : \mathbb{S}/p^n \longrightarrow \mathbb{S}/p^{n+1}$ be a morphism in the stable homotopy category that realizes multiplication by $p \cdot - : \mathbb{Z}/p^n \longrightarrow \mathbb{Z}/p^{n-1}$ on 0-th homology. Then any homotopy colimit of the sequence

$$\mathbb{S}/p \xrightarrow{\psi_1} \mathbb{S}/p^2 \xrightarrow{\psi_2} \mathbb{S}/p^3 \xrightarrow{\psi_3} \cdots$$

is a Moore spectrum for the group $\mathbb{Z}/p^\infty$ (because that group is a colimit of the groups $\mathbb{Z}/p^n$ under multiplication by p maps). So the spectrum $X_p^\wedge = F(\mathbb{S}/p^\infty, \Sigma X)$ is a homotopy inverse limits [ref] of the tower of the spectra

$$\cdots \xrightarrow{F(\psi_1, \Sigma X)} F(\mathbb{S}/p^3, \Sigma X) \xrightarrow{F(\psi_2, \Sigma X)} F(\mathbb{S}/p^2, \Sigma X) \xrightarrow{F(\psi_1, \Sigma X)} F(\mathbb{S}/p, \Sigma X) .$$

(iv) In (6.42) we derived a short exact sequence for morphisms in $\mathcal{SHC}$ out of a Moore spectrum for an arbitrary abelian group. For the abelian group $A = \mathbb{Z}/p^\infty$ and the spectrum $\mathbb{S}^{1-k} \wedge X$ this becomes a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_1(\mathbb{S}^{1-k} \wedge X)) \longrightarrow [\mathbb{S}/p^\infty, \mathbb{S}^{1-k} \wedge X] \xrightarrow{\pi_0} \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_0(\mathbb{S}^{1-k} \wedge X)) \longrightarrow 0 .$$

Modulo the identifications $\pi_n(\mathbb{S}^{1-k} \wedge X) \cong \pi_{n-k+1} X$ and $[\mathbb{S}/p^\infty, \mathbb{S}^{1-k} \wedge X] \cong \pi_k F(\mathbb{S}/p^\infty, \Sigma X) = \pi_k(X_p^\wedge)$, this is the desired exact sequence. $\square$

Remark 9.11. Beause the mod-p^n Moore spectra are 'self-dual', part (iii) of the previous Theorem 9.9 can be rephrased in terms of the smash products $\mathbb{S}/p^n \wedge X$ instead of the function spectra $F(\mathbb{S}/p^n, \Sigma X)$. Indeed, because of the isomorphism (7.7)

$$H_k(DX, \mathbb{Z}) \longrightarrow H^k(X, \mathbb{Z}) ,$$

the suspension $\Sigma(D\mathbb{S}/p^n)$ of the Spanier-Whitehead dual of the mod-p^n Moore spectrum is also a mod-p^n Moore spectrum. More precisely, we can choose isomorphism

$$j_n \ : \ \mathbb{S}/p^n \longrightarrow \Sigma(D\mathbb{S}/p^n)$$

in $\mathcal{SHC}$ such that the composite

$$\rho_n \ : \ \mathbb{S}/p^{n+1} \xrightarrow{j_{n+1}} \Sigma(D\mathbb{S}/p^{n+1}) \xrightarrow{\Sigma D\psi_n} \Sigma(D\mathbb{S}/p^n) \xrightarrow{j_n^{-1}} \mathbb{S}/p^n$$

realizes the reduction $\mathbb{Z}/p^{n+1} \longrightarrow \mathbb{Z}/p^n$ on homology. Then for every symmetric spectrum X the composite

$$\mathbb{S}/p^n \wedge X \longrightarrow \Sigma F(\mathbb{S}/p^n, \mathbb{S}) \wedge X \xrightarrow{\psi_{\mathbb{S}/p^n, X}} F(\mathbb{S}/p^n, \Sigma X)$$

is an isomorphism such that the squares

$$\begin{tikzcd} \mathbb{S}/p^{n+1} \wedge X \arrow[r, "\rho_n \wedge X"] \arrow[d, "\cong"'] & \mathbb{S}/p^n \wedge X \arrow[d, "\cong"] \\ F(\mathbb{S}/p^{n+1}, \Sigma X) \arrow[r, "F(\psi_n, \Sigma X)"'] & F(\mathbb{S}/p^n, \Sigma X) \end{tikzcd}$$

commute in $\mathcal{SHC}$. So the p-completion $X^\wedge_p$ is also a homotopy limit of the tower

$$\cdots \xrightarrow{\rho_n \wedge X} \mathbb{S}/p^n \wedge^L X \xrightarrow{\rho_{n-1} \wedge X} \cdots \xrightarrow{\rho_2 \wedge X} \mathbb{S}/p^2 \wedge^L X \xrightarrow{\rho_1 \wedge X} \mathbb{S}/p \wedge X \ .$$

In this form the result very much resembles the algebraic p-completion, where $\mathbb{S}/p^n$, smash product and homotopy limit are replaced, respectively, by $\mathbb{Z}/p^n$, tensor product and category limit.

Remark 9.12. For an abelian B, the group $\mathrm{Ext}(\mathbb{Z}/p^\infty, B)$ that occurs in the short exact sequence (9.10) is sometimes called the *Ext p-completion* of B. In general, there is a natural short exact sequence

$$0 \longrightarrow \lim{}^1 \mathrm{Hom}(\mathbb{Z}/p^n, B) \longrightarrow \mathrm{Ext}(\mathbb{Z}/p^\infty, B) \longrightarrow B^\wedge_p \longrightarrow 0 \ , \tag{9.13}$$

so in particular, the Ext p-completion surjects onto the algebraic p-completion. Indeed, the short exact sequence of abelian groups

$$0 \longrightarrow \bigoplus_{n \geq 1} \mathbb{Z}/p^n \xrightarrow{1-\mathrm{sh}} \bigoplus_{n \geq 1} \mathbb{Z}/p^n \longrightarrow \mathbb{Z}/p^\infty \longrightarrow 0$$

induces a long exact sequence of Ext groups

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/p^\infty, B) \longrightarrow \prod_{n\geq 1} \mathrm{Hom}(\mathbb{Z}/p^n, B) &\xrightarrow{1-\mathrm{sh}} \prod_{n \geq 1} \mathrm{Hom}(\mathbb{Z}/p^n, B) \\ \longrightarrow \mathrm{Ext}(\mathbb{Z}/p^\infty, B) \longrightarrow \prod_{n \geq 1} \mathrm{Ext}(\mathbb{Z}/p^n, B) &\xrightarrow{1-\mathrm{sh}} \prod_{n\geq 1} \mathrm{Ext}(\mathbb{Z}/p^n, B) \longrightarrow 0 \ , \end{aligned}$$

where we used that $\mathrm{Hom}(-, B)$ and $\mathrm{Ext}(-, B)$ take sums to products. Kernel and cokernel of the '$1 - \mathrm{sh}$' map are the limit respectively derived limit, and we deduce a short exact sequence

$$0 \longrightarrow \lim{}^1 \mathrm{Hom}(\mathbb{Z}/p^n, B) \longrightarrow \mathrm{Ext}(\mathbb{Z}/p^\infty, B) \longrightarrow \lim \mathrm{Ext}(\mathbb{Z}/p^n, B) \longrightarrow 0 \ .$$

The group $\mathrm{Ext}(\mathbb{Z}/p^n, B)$ is canonically isomorphic to $B/p^n B$, so the inverse limit on the right hand side is isomorphic to the p-completion $B^\wedge_p$.

If the homotopy groups of a symmetric spectrum X are finitely generated, then the terms in the short exact sequence (9.10) for the homotopy groups of the p-completion $X^\wedge_p$ simplify as follows. Indeed, if an abelian group B is finitely generated, then the group $\mathrm{Hom}(\mathbb{Z}/p^n, B)$ is trivial for all sufficiently large n, so the inverse limit and the derived limit over this sequence both vanish. Hence the map $\mathrm{Ext}(\mathbb{Z}/p^\infty, B) \longrightarrow B^\wedge_p$ from the Ext p-completion to the p-completion is an isomorphism and the short exact sequences (9.10) and (9.13) reduces to an isomorphism

$$\pi_k(X^\wedge_p) \ \cong \ (\pi_k X)^\wedge_p$$

between the homotopy theoretic and the algebraic completions.

9.4. Bousfield localization. In this section we discuss a more general kind of localization, namely localization at a generalized homology theory, also known as 'Bousfield localization'. Bousfield localization is a device to isolate the properties of a spectrum which 'are seen by' a fixed spectrum E.

Bousfield localization at a homology theory is a special case of localization in triangulated categories. So we discuss this more general localization first. We let $\mathcal{T}$ be a triangulated category with infinite sums. We recall that a *localizing subcategory* of $\mathcal{T}$ is a triangulated subcategory $\mathcal{C}$ that is closed under arbitrary sums.

Definition 9.14. Let $\mathcal{T}$ be a triangulated category with infinite sums and $\mathcal{C}$ a localizing subcategory of $\mathcal{T}$.

- An object L of $\mathcal{T}$ is $\mathcal{C}$-*local* if the group $\mathcal{T}(A, L)$ is trivial for all A in $\mathcal{C}$.
- A morphism $f : A \longrightarrow B$ in $\mathcal{T}$ is a $\mathcal{C}$-*equivalence* if the some (hence any) cone of f belongs to $\mathcal{C}$.
- A morphism $a : A \longrightarrow L$ in $\mathcal{T}$ is a $\mathcal{C}$-*localization* if a is a $\mathcal{C}$-equivalence whose target L is $\mathcal{C}$-local.

In this generality, localizations need not always exist.

Proposition 9.15. *Let $\mathcal{T}$ be a triangulated category with infinite sums and $\mathcal{C}$ a localizing subcategory of $\mathcal{T}$.*

(i) *An object X of $\mathcal{T}$ is $\mathcal{C}$-local if and only if for every $\mathcal{C}$-equivalence $g : B \longrightarrow C$ the induced map*

$$\mathcal{T}(g, X) \ : \ \mathcal{T}(C, X) \ \longrightarrow \ \mathcal{T}(B, X)$$

is bijective.

(ii) *An object X of $\mathcal{T}$ is $\mathcal{C}$-local if and only if the suspension ΣX is $\mathcal{C}$-local.*

(iii) *Let*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A \ .$$

be a distinguished triangle in $\mathcal{T}$. If two of the objects A, B and C are $\mathcal{T}$-local, so is the third.

(iv) *Every product of $\mathcal{C}$-local objects is again $\mathcal{C}$-local.*

(v) *Every homotopy limit of a tower of $\mathcal{C}$-local objects is $\mathcal{C}$-local.*

(vi) *Every $\mathcal{C}$-equivalence between $\mathcal{C}$-local objects is an isomorphism.*

(vii) *Every $\mathcal{C}$-localization $a : X \longrightarrow L$ is initial among morphisms in $\mathcal{T}$ from X to $\mathcal{C}$-local object.*

(viii) *Every $\mathcal{C}$-localization $a : X \longrightarrow L$ is terminal among $\mathcal{C}$-equivalences out of X.*

(ix) *If $a : X \longrightarrow L$ and $a' : X \longrightarrow L'$ are $\mathcal{C}$-localizations, then there is a unique morphism $f : L \longrightarrow L'$ such that $fa = a'$, and f is an isomorphism.*

Proof. (i) Let X be an object of $\mathcal{T}$ such that for every $\mathcal{C}$-equivalence g the map $\mathcal{T}(g, X)$ is bijective. If C is any object of $\mathcal{C}$, then C is a cone of the zero morphism $z : 0 \longrightarrow C$, so this zero morphism is a $\mathcal{C}$-equivalence and the induced map

$$\mathcal{T}(z, X) \ : \ \mathcal{T}(C, X) \ \longrightarrow \ \mathcal{T}(0, X)$$

is bijective. Hence the group $\mathcal{T}(C, X)$ is trivial, and so X is $\mathcal{C}$-local.

Conversely, suppose that X is $\mathcal{C}$-local. Given a $\mathcal{C}$-equivalence g, we choose a distinguished triangle

$$A \ \xrightarrow{f} \ B \ \xrightarrow{g} \ C \ \xrightarrow{h} \ \Sigma A \ .$$

Then ΣA is a cone of g, so ΣA and A, belong to $\mathcal{C}$ (by definition of $\mathcal{C}$-equivalence). By Proposition 2.10 (i) the sequence of abelian groups

$$\mathcal{T}(\Sigma A, X) \ \xrightarrow{\mathcal{T}(h,X)} \ \mathcal{T}(C, X) \ \xrightarrow{\mathcal{T}(g,X)} \ \mathcal{T}(B, X) \ \xrightarrow{\mathcal{T}(f,X)} \ \mathcal{T}(A, X)$$

is exact. If X is $\mathcal{C}$-local, then the groups $\mathcal{T}(\Sigma A, X)$ and $\mathcal{T}(A, X)$ vanish, so the map $\mathcal{T}(g, X)$ is bijective.

(ii) Since $\mathcal{C}$ is a triangulated subcategory of $\mathcal{T}$, and object X belongs to $\mathcal{C}$ if and only if ΣX does. If ΣL is $\mathcal{C}$-local, then for every X in $\mathcal{C}$ we have

$$\mathcal{T}(X, L) \ \cong \ \mathcal{T}(\Sigma X, \Sigma L) \ = \ 0 \ .$$

So L is $\mathcal{C}$-local. If conversely L is $\mathcal{C}$-local, then for every X in $\mathcal{C}$ we have

$$\mathcal{T}(X, \Sigma L) \ \cong \ \mathcal{T}(\Sigma^{-1} X, L) \ = \ 0 \ .$$

So ΣL is $\mathcal{C}$-local.

(iii) We start with the case where A and B are $\mathcal{C}$-local. For every object X of $\mathcal{T}$ we have an exact sequence

$$\mathcal{T}(X, \Sigma A) \ \longrightarrow \ \mathcal{T}(X, C) \ \longrightarrow \ \mathcal{T}(X, B) \ .$$

If X belongs to $\mathcal{C}$, then since B and ΣA (by (ii)) are $\mathcal{C}$-local, the two outer groups are trivial, hence so is the group $\mathcal{T}(X, C)$. So C is $\mathcal{C}$-local. The other two cases can be proved similarly, or can be reduced to the previous case by shifting the triangle and using (ii).

(iv) Given a family $\{L_i\}_{i\in I}$ of $\mathcal{C}$-local objects and any X in $\mathcal{C}$ we consider the canonical map

$$\mathcal{T}(X, \prod_{i\in I} L_i) \longrightarrow \prod_{i\in I} \mathcal{T}(X, L_i) .$$

This map is bijective by the universal property of a product, and the target is trivial by assumption. So the source is trivial, which means that the product $\prod_{i\in I} L_i$ is $\mathcal{C}$-local.

(v) We consider a tower $\{f_i : X_i \longrightarrow X_{i-1}\}_{i\geq 1}$ of morphisms in $\mathcal{T}$ such that all X_i are $\mathcal{C}$-local. By definition the homotopy limit of the tower participates in a distinguished triangle

$$\mathrm{holim}_n X_n \longrightarrow \prod_{n\geq 0} X_n \xrightarrow{1-f} \prod_{n\geq 0} X_n \longrightarrow \Sigma(\mathrm{holim}_n X_n) .$$

By part (iv) the product of the X_n is $\mathcal{C}$-local, so the homotopy limit is $\mathcal{C}$-local by part (iii).

(vi) Let $f : L \longrightarrow L'$ be a $\mathcal{C}$-equivalence between $\mathcal{C}$-local objects. By (ii) the two maps

$$\mathcal{T}(f, L) : \mathcal{T}(L', L) \longrightarrow \mathcal{T}(L, L) \qquad \text{and} \qquad \mathcal{T}(f, L') : \mathcal{T}(L', L') \longrightarrow \mathcal{T}(L, L')$$

are bijective. So there is a unique morphism $g : L' \longrightarrow L$ such that $gf = \mathrm{Id}_L$. The two endomorphisms fg and $\mathrm{Id}_{L'}$ then have the same image under $\mathcal{T}(f, L')$, so $fg = \mathrm{Id}_{L'}$. So g is inverse to f.

(vii) If L' is any $\mathcal{C}$-local object, then $\mathcal{T}(a, L') : \mathcal{T}(L, L') \longrightarrow \mathcal{T}(X, L')$ is bijective. So for any morphism $f : X \longrightarrow L'$ in $\mathcal{T}$ there is a unique morphism $b : L \longrightarrow L'$ such that $ba = f$.

(viii) If $f : X \longrightarrow Y$ is a $\mathcal{C}$-equivalence, then $\mathcal{T}(f, L) : \mathcal{T}(Y, L) \longrightarrow \mathcal{T}(X, L)$ is bijective. So there is a unique morphism $b : Y \longrightarrow L$ such that $bf = a$.

(ix) By (v) (or by (vi)) there is a unique morphism $f : L \longrightarrow L'$ satisfying $fa = a'$. Reversing the roles of a and a' gives a unique mrphism $g : L' \longrightarrow L$ such that $ga' = a$.

Since L is $\mathcal{C}$-local and a a $\mathcal{C}$-equivalence, the map $\mathcal{T}(a, L) : \mathcal{T}(L, L) \longrightarrow \mathcal{T}(X, L)$ is bijective. But gf and the identity are two endomorphisms of L satisfying $(gf)a = ga' = a = \mathrm{Id}_L a$, so $gf = \mathrm{Id}_L$. Reversing the roles of a and a' gives $fg = \mathrm{Id}_{L'}$. □

We recall from Definition 6.1 that a symmetric spectrum E defines a generalized homology theory E_* via

$$E_k(X) = \pi_k(E \wedge^L X) ,$$

where X is a symmetric spectrum and k an integer.

Definition 9.16. Let E be a symmetric spectrum. A symmetric spectrum X is *E-acyclic* if $E \wedge^L X$ is trivial, or equivalently, if the E-homology groups $E_k(X)$ are trivial for all integers k.

The class of E-acyclic symmetric spectra is a localizing subcategory of the stable homotopy category [...], so we can talk about localization, in the sense of Definition 9.14, with respect to the class of E-acyclic spectra.

Definition 9.17. Let E be a symmetric spectrum. A symmetric spectrum L is *E-local* if for every E-acyclic symmetric spectrum A the group $\mathcal{SHC}(A, L)$ is trivial. A morphism $f : A \longrightarrow B$ is an *E-equivalence* if the induced map $E_k(f) : E_k(A) \longrightarrow E_k(B)$ on E-homology groups is an isomorphism for every integer k. A morphism $a : A \longrightarrow L$ in the stable homotopy category is an *E-localization* if a is an E-equivalence whose target L is E-local.

By the very definition of E-homology, $f : X \longrightarrow Y$ is an E-equivalence if and only if the morphism $E \wedge^L f : E \wedge^L X \longrightarrow E \wedge^L Y$ induces isomorphisms on all true homotopy groups, i.e., if and only if $E \wedge^L f$ is a stable equivalence. Derived smash product with E commutes with taking mapping cones, so $f : X \longrightarrow Y$ is an E-equivalence if and only if the mapping cone $C(f)$ is E-acyclic. So the E-equivalences in the sense of the previous definition are precisely the equivalences, in the sense of Definition 9.14 with respect to the class of E-acyclics.

E-localizations always exist, but we shall defer a construction to a later chapter [ref], where we treat it as part of an 'E-local model structure' for symmetric spectra. In this section we investigate properties of E-localization and reinterpret it for certain specific E. The E-localization is often referred to as *Bousfield* *localization* with respect to E because Bousfield was the first to construct E-localizations for arbitrary spectra.

We will eventually discuss two generalizations of the E-localization above. In [...] we discuss Bousfield localizations in general triangulated categories, and in [...] we 'lift' E-localizations to categories of structured symmetric spectra, such as ring and module spectra.

Proposition 9.15 applied to the stable homotopy category and the class of E-acyclic spectra specializes as follows.

Proposition 9.18. *Let E be a symmetric spectrum.*

(i) *A symmetric spectrum X is E-local if and only if for every E-equivalence $g : B \longrightarrow C$ the induced map*
$$\mathcal{SHC}(g, X) \ : \ \mathcal{SHC}(C, X) \ \longrightarrow \ \mathcal{SHC}(B, X)$$
is bijective.

(ii) *A symmetric spectrum X is E-local if and only if the suspension ΣX is E-local.*

(iii) *Let*
$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A \ .$$
be a distinguished triangle in the stable homotopy category. If two of the spectra A, B and C are E-local, so is the third.

(iv) *Every product of E-local spectra in the stable homotopy category is again E-local.*

(v) *Every homotopy limit of a tower of E-local spectra is E-local.*

(vi) *Every E-equivalence between E-local spectra is a stable equivalence.*

(vii) *Every E-localization $a : X \longrightarrow L$ is initial among morphisms in the stable homotopy category from X to E-local spectra.*

(viii) *Every E-localization $a : X \longrightarrow L$ is terminal among E-equivalences out of X.*

(ix) *If $a : X \longrightarrow L$ and $a' : X \longrightarrow L'$ are E-localizations, then there is a unique morphism $f : L \longrightarrow L'$ such that $fa = a'$, and f is a stable equivalence.*

As we now explain, the localization at a subring of the rationals, and the completion at a prime, are special cases of E-localizations.

Proposition 9.19. (i) *Let R be a subring of the ring of rational numbers. Then the R-localization $X \longrightarrow X_R = \mathbb{S}R \wedge^L X$ is a Bousfield localization with respect to the Moore spectrum $\mathbb{S}R$.*

(ii) *Let p be a prime. The p-completion map $X \longrightarrow X^\wedge_p$ is a Bousfield localization with respect to the mod-p Moore spectrum $\mathbb{S}/p$.*

Proof. (i)

(ii) We show first that the p-adic completion $X^\wedge_p$ is $\mathbb{S}/p$-local. For every $\mathbb{S}/p$-acyclic spectrum A the distinguished triangles
$$\mathbb{S}/p \wedge^L A \ \xrightarrow{(\cdot p)\wedge A} \ \mathbb{S}/p^{n+1} \wedge^L A \ \xrightarrow{(\cdot p)\wedge A} \ \mathbb{S}/p^n \wedge^L A \ \xrightarrow{(\cdot p)\wedge A} \ \Sigma(\wedge \mathbb{S}/p \wedge^L A)$$
allow an induction which shows that $\mathbb{S}/p^n \wedge A$ is trivial in the stable homotopy category for all $n \geq 0$. Since $\mathbb{S}/p^\infty$ is a homotopy colimit of the spectra $\mathbb{S}/p^n$, the spectrum $A \wedge \mathbb{S}/p^\infty$ is also trivial. Thus the group $[A, X^\wedge_p] = [A, F(\mathbb{S}\mathbb{Z}/p^\infty, \Sigma X)] \cong [A \wedge \mathbb{S}\mathbb{Z}/p^\infty, \Sigma X]$ is trivial, so $X^\wedge_p$ is indeed $\mathbb{S}/p$-local.

It remains to show that the completion map $X \longrightarrow X^\wedge_p$ is is a mod-p equivalence. We can take the distinguished triangle
$$\mathbb{S}_{(p)} \ \xrightarrow{\iota} \ H\mathbb{Q} \ \xrightarrow{p} \ \mathbb{S}\mathbb{Z}/p^\infty \ \xrightarrow{\delta} \ \mathbb{S}^1_{(p)}$$
apply the exact functor $\mathbb{S}/p \wedge F(-, \Sigma X)$ and obtain a distinguished triangle
$$\mathbb{S}/p \wedge F(H\mathbb{Q}, X) \ \longrightarrow \ \mathbb{S}/p \wedge X \ \longrightarrow \ \mathbb{S}/p \wedge X^\wedge_p \ \longrightarrow \ \Sigma(\mathbb{S}/p \wedge F(H\mathbb{Q}, X)) \ .$$
The derived function spectrum $F(H\mathbb{Q}, X)$ is rational, so multiplication by p on it is a self-equivalence, and thus the smash product $\mathbb{S}/p \wedge F(H\mathbb{Q}, X)$ is trivial on $\mathcal{SHC}$. So the map $\mathbb{S}/p \wedge X \longrightarrow \mathbb{S}/p \wedge X^\wedge_p$ is a stable equivalence, as claimed. □

Proposition 9.20. *Let E be a symmetric spectrum and suppose that every spectrum X has an E-localization $a_X : X \longrightarrow LX$.*

(i) *Then there is a unique way to extend the assignment $X \mapsto LX$ to an endofunctor L of the stable homotopy category in such a way that the collection of morphisms a forms a natural transformation from the identity functor to L.*

(ii) *For every symmetric spectrum X there is a unique morphism $\tau_X : L(\Sigma X) \longrightarrow \Sigma(LX)$ such that $\tau_X a_{\Sigma X} = \Sigma(a_X)$. Moreover, τ is a natural isomorphism and makes $L : \mathcal{SHC} \longrightarrow \mathcal{SHC}$ into an exact functor.*

Proof. (i) We need to define L on morphisms $f : X \longrightarrow Y$. If we want the morphisms $a_X : X \longrightarrow LX$ to form a natural transformation, then the relation $LF \circ a_X = a_Y \circ f$ must hold for the value of the functor on f. Since a_X is an E-equivalence and LY is E-local, the map

$$\mathcal{SHC}(a_X, LY) \; : \; \mathcal{SHC}(LX, LY) \; \longrightarrow \; \mathcal{SHC}(X, Y)$$

is bijective, so there is a unique morphism $Lf : LX \longrightarrow LY$ satisfying $Lf \circ a_X = a_Y \circ f$. So there is only one way to define Lf, and it remains to check that we obtain a functor.

If $f = \mathrm{Id}_X$ is an identity morphism, then $L\,\mathrm{Id}_X \circ a_X = a_X$, so $L\,\mathrm{Id}_X = \mathrm{Id}_{LX}$ by the uniqueness clause. If $g : Y \longrightarrow Z$ is another morphism, then

$$Lg \circ Lf \circ a_X \; = \; Lg \circ a_Y \circ f \; = \; a_Z \circ g \circ f = L(gf) \circ a_X \; .$$

By uniqueness, again, this forces $Lg \circ Lf = L(gf)$, so we have ideed constructed a functor. Moreover, the morphisms $a_X : X \longrightarrow LX$ form a natural transformation by construction.

(ii) This is a special case of the fact that adjoints of exact functors are again exact, compare Proposition 3.16. Indeed, by Proposition 9.18 the full subcategory of E-local spectra is a triangulated subcategory of the stable homotopy category, and so the inclusion of E-local spectra into $\mathcal{SHC}$ is an exact functor, with respect to the identity transformation of the suspension functor Σ. The localization functor L is left adjoint to the inclusion, with $a_X : X \longrightarrow LX$ the unit of the adjunction. The counit of the adjunction is the identity, so the transformation $\tau : L \circ \Sigma \longrightarrow \Sigma \circ L$ that corresponds to the identity transformation of Σ under the recipe of Proposition 3.16 (i) is precisely the adjoint of $\Sigma a_X : \Sigma X \longrightarrow \Sigma(LX)$. This adjoint $\tau_X : L(\Sigma X) \longrightarrow \Sigma(LX)$ in turn is the unique morphism such that $\tau_X a_{\Sigma X} = \Sigma(a_X)$. Part (ii) of Proposition 3.16 then shows that τ is natural and part (iii) of the same proposition shows that the pair (L, τ) is exact. □

Example 9.21. Localization preserves homotopy ring and module structures. More precisely, if S is a homotopy ring spectrum and $a : S \longrightarrow LS$ an E-localization, then LS has a unique structure of homotopy ring spectrum such that a is a morphism of homotopy ring spectra. The multiplication of LS is commutative if the original multiplication of S is. Similarly, if M is a homotopy S-module and $b : M \longrightarrow LM$ an S-localization, then LM has a unique structure of homotopy LS-module such that b is a morphism of homotopy modules over S (where S acts on LM by restriction of scalars along $a : S \longrightarrow LS$.) We show this for the ring case; the module case is similar.

Since $a : S \longrightarrow LS$ is an E-equivalence, so is $a \wedge a : S \wedge^L S \longrightarrow LS \wedge^L LS$. Since LS is E-local, there is a unique morphism $\bar{\mu} : LS \wedge^L LS \longrightarrow LS$ satisfying $\bar{\mu}(a \wedge a) = a\mu$, where $\mu : S \wedge^L S \longrightarrow S$ is the multiplication of S. When composed with the E-equivalence $a \wedge a \wedge a : LS \wedge^L LS \wedge^L LS \longrightarrow S \wedge^L S \wedge^L S$, the two morphisms $\bar{\mu}(\bar{\mu} \wedge LS), \bar{\mu}(LS \wedge \bar{\mu}) : LS \wedge^L LS \wedge^L LS \longrightarrow LS$ restrict to $a\mu(\mu \wedge S)$ respectively $a\mu(S \wedge \mu)$. Since S is homotopy associative, these two composite coincide. Since L is local, we must have $\bar{\mu}(\bar{\mu} \wedge LS) = \bar{\mu}(LS \wedge \bar{\mu})$, i.e., the multiplication of LS is homotopy associative. Left and right unitality are proved in a similar way. The localization map a is compatible with the multiplications by construction, and with unit morphisms [...] So a is indeed a homomorphism of homotopy ring spectra.

If R is a symmetric ring spectrum, then as an object of the stable homotopy category, R is naturally a homotopy ring spectrum. By the above, any E-localization of R is again a homotopy ring spectrum, but it is not clear at this point whether the E-localization can be 'rigidified' to a symmetric ring spectrum. This is true, and will be shown (much more generally) in Chapter III below, using model category techniques.

Example 9.22. Let S be a homotopy ring spectrum. We claim that then every left or right homotopy S-module M is S-local as a spectrum. Indeed, suppose that A is S-acyclic and let $f : A \longrightarrow M$ be any morphism in the stable homotopy category. Then f equals the composite

$$A = \mathbb{S} \wedge^L A \xrightarrow{\eta\wedge A} S \wedge^L A \xrightarrow{S\wedge f} S \wedge^L M \xrightarrow{\alpha} M ,$$

where $\eta : \mathbb{S} \longrightarrow S$ is the unit morphism and $\alpha : S \wedge^L M \longrightarrow M$ is the left action of S on M. Since A is acyclic the spectrum $S \wedge^L A$ is trivial, and so f is trivial. Consequently, the group $\mathcal{SHC}(A, M)$ is trivial and M is S-local, as claimed.

While every S-module is S-local, there are typically also S-local spectra that do not admit the structure of a homotopy S-module [example].

There is a situation, though, when the S-local spectra coincide with the homotopy S-modules, namely when S is 'smash idempotent', i.e., when the multiplication map $\mu : S \wedge^L S \longrightarrow S$ is a stable equivalence. More precisely, in this situation every S-local spectrum admits a unique structure of homotopy S-module. Indeed, if S is homotopy idempotent, then also the morphism $S \wedge \eta : S \longrightarrow S \wedge^L S$ is a stable equivalence, hence for every spectrum X the morphism $\eta \wedge^L X : X \longrightarrow S \wedge^L X$ is an S-equivalence. Since the target $S \wedge^L X$ is a (free) S-module, it is S-local by the above, and so $\eta \wedge^L X : X \longrightarrow S \wedge^L X$ is an S-localization for every spectrum X. So if X is already S-local, there is a unique morphism $\alpha : S \wedge^L X \longrightarrow X$ such that $\alpha(\eta \wedge X) = \mathrm{Id}_X$. [associativity]

We make another observation: if S is homotopy idempotent, then the unit morphism $\eta : \mathbb{S} \longrightarrow S$ is an S-equivalence with S-local target, hence S is necessarily the localization of the sphere spectrum. A localization is called *smashing* if the localized sphere spectrum (which is always a homotopy ring spectrum as in Example 9.21) is homotopy idempotent. Examples of smashing localizations are localizations at a set of primes (see Section 9.1) and localizations at the Johnson-Wilson spectra $E(n)$ (for a prime which is fixed implicitly and some $n \geq 1$).

Example 9.23. Suppose the spectrum E is such that the class of E-acyclics is generated by a set $\mathcal{G}$ of *finite* spectra. Then we can construct E-localizations with the methods of Section 5.

Indeed, let C be a set of finite, E-acyclic spectra such that the localizing subcategory generated by C coincides with the class of all E-acyclics. By adding suspensions and desuspensions, if necessary, we can assume that C is closed (up to isomorphism) under suspensions and desuspensions. Given any symmetric spectrum X we can apply Proposition 5.14 to the stable homotopy category and the represented cohomological functor $E = \mathcal{SHC}(-, X)$. The Proposition provides a spectrum R in the class $\langle C\rangle_+$ and a morphism $u \in \mathcal{SHC}(R, X)$ such that for every object G of C the map

$$\mathcal{SHC}(G, u) \ : \ \mathcal{SHC}(G, R) \longrightarrow \mathcal{SHC}(G, X)$$

is bijective. We choose a distinguished triangle

$$R \xrightarrow{u} X \xrightarrow{a} L \longrightarrow \Sigma R .$$

Then the group $\mathcal{SHC}(G, L)$ is trivial for all $G \in C$. Since C generates the class of E-acyclics, the group $\mathcal{SHC}(A, L)$ is trivial for all E-acyclic spectra A. In other words, the spectrum L is E-local.

On the other hand, the spectrum R is in the class $\langle C\rangle_+$ of objects obtained from C by taking sums and extensions to the right. So R is E-acyclic, and hence $a : X \longrightarrow L$ is an E-equivalence. Altogether constructs an E-localization for any given spectrum X.

Unfortunately, the class of acyclic spectra with respect to a given generalized homology theory is not generally generated by a set of finite spectra. So we cannot always construct E-localizations as in Example 9.23. [telescope conjecture]

Remark 9.24. By the Whitehead theorem 6.30 (ii), every *connective* spectrum which is $H\mathbb{Z}$-acyclic is already stably contractible. However, there are non-trivial spectra (necessarily non-connective) which are $H\mathbb{Z}$-acyclic, for example $v_1^{-1}\mathbb{S}/p$, the mapping telesope of the Adams map of the mod-p Moore spectrum, (compare Example 6.31). Since the mapping telescope is $H\mathbb{Z}$-acyclic but not stably contractible, it is not

$H\mathbb{Z}$-local. So the Bousfield class of $H\mathbb{Z}$ is strictly smaller than the maximal Bousfield class of the sphere spectrum, $\langle H\mathbb{Z}\rangle < \langle \mathbb{S}\rangle$.

On the other hand we claim that every connective spectrum is $H\mathbb{Z}$-local. We argue in various steps: for every abelian group A, the Eilenberg-Mac Lane spectrum HA is a module spectrum over $H\mathbb{Z}$, hence $H\mathbb{Z}$-local by Example 9.22. Local spectra are stable under suspensions, so every spectrum of the form $\Sigma^n HA$ is $H\mathbb{Z}$-local.

Suppose X is k-connected for some integer k. Then the Postnikov section $P_k X$ is trivial, hence $H\mathbb{Z}$-local. For $n > k$ there is a distinguished triangle (8.6)

$$\mathbb{S}^n \wedge H(\pi_n X) \xrightarrow{i} P_n X \xrightarrow{j_n} P_{n-1} X \xrightarrow{\mathbf{k}_n} \mathbb{S}^{n+1} \wedge H(\pi_n X)$$

hence all Postnikov sections $P_n X$ are $H\mathbb{Z}$-local by induction. Since X is the homotopy limit of the Postnikov sections $P_n X$ (see Theorem 8.3), X is $H\mathbb{Z}$-local by Proposition 9.18 (iv).

Remark 9.25 (Bousfield class)**.** Let E and F be two symmetric spectra. We say that E has *Bousfield class* smaller or equal to F if every F-acyclic spectrum is also E-acyclic. In this situation, we also say that the Bousfield class of F is greater or equal to that of E, and we use the notation $\langle E\rangle \leq \langle F\rangle$ or $\langle F\rangle \geq \langle E\rangle$. So we may interpret $\langle E\rangle$ as the class of all E-acyclic spectra, and then $\leq$ is the reverse inclusion. More acyclics implies fewer local spectra, so if $\langle E\rangle \leq \langle F\rangle$, then every E-local spectrum is also F-local.

If the Bousfield class of E is smaller or equal to that of F, and the Bousfield class of F is smaller or equal to that of E, we say that E and F have the *same Bousfield class* and write $\langle E\rangle = \langle F\rangle$. Of course, this simply means that E and F have the same acyclics, and also the same local spectra. Thus any E-localization is also an F-localization and conversely.

Some properties of Bousfield classes which follow straight from the definition are the following. The trivial spectrum has the smallest Bousfield class and the sphere spectrum has the largest Bousfield class, in symbols

$$\langle * \rangle \ \leq\ \langle E\rangle \ \leq\ \langle \mathbb{S}\rangle$$

for all spectra E. Moreover, $\langle E\rangle = \langle \Sigma E\rangle$, i.e., the Bousfield class is invariant under suspension. A spectrum E has Bousfield class smaller or equal (respectively greater or equal) to the sum (respectively the smash product) of E with any other spectrum:

$$\langle E \wedge^L F\rangle \ \leq\ \langle E\rangle \ \leq\ \langle E \vee F\rangle \ .$$

Given a distinguished triangle

$$E \xrightarrow{f} F \xrightarrow{g} G \xrightarrow{h} \Sigma E$$

in the stable homotopy category, then any of the three spectra E, F and G has smaller Bousfield class than the sum of the other two:

$$\langle E\rangle \leq \langle F \vee G\rangle \ , \quad \langle F\rangle \leq \langle G \vee E\rangle \quad \text{and} \quad \langle G\rangle \leq \langle E \vee F\rangle \ .$$

[the collection of Bousfield classes forms a set]

A little less obvious is the following property of Bousfield classes.

Proposition 9.26. *Let $f : \Sigma^n E \longrightarrow E$ be a nilpotent degree n endomorphism of a spectrum E. Then E and any mapping cone $C(f)$ of f have the same Bousfield class.*

Proof. We need to show that every E-acyclic spectrum if $C(f)$-acyclic and conversely. Suppose first that a spectrum X is E-acyvlic. The distinguished triangle

$$\Sigma^n E \xrightarrow{f} E \longrightarrow C(f) \longrightarrow \Sigma^{n+1} E$$

implies the relation

$$\langle C(f)\rangle \ \leq\ \langle E \vee \Sigma^n E\rangle \ =\ \langle E\rangle$$

among Bousfield classes.

Suppose conversely that X is acyclic for the mapping cone $C(f)$. We show by induction on m that then X is acyclic for the mapping cone $C(f^m)$ of the m-th power of f. For $m = 1$ this was assumed, so we can take $m \geq 2$. The octahedral axiom (TR3) applied to the composable morphisms $\Sigma^n(f^{m-1})$ and f yields a distinguished triangle

$$\Sigma^n C(f^{m-1}) \longrightarrow C(f^m) \longrightarrow C(f) \longrightarrow \Sigma^{n+1}C(f^{m-1}) .$$

By induction, the smash products $C(f^{m-1}) \wedge^L X$ and $C(f) \wedge^L X$ are stably contractible, hence so is $C(f^m) \wedge^L X$.

By assumption the endomorphism f is nilpotent, i.e., $f^m = 0$ for some $m \geq 1$. Then the cone $C(f^m)$ is isomorphic in the stable homotopy category to the sum $E \oplus \Sigma^{mn+1}E$. By the previous paragraph, $C(f^m) \wedge^L X$ is stably contractible, hence so is its direct summand $E \wedge^L X$. In other words, the spectrum X is E-acyclic. □

9.5. Localization with respect to topological K-theory. We cannot refrain from giving some idea of what Bousfield localization with respect to a non-connective spectrum can look like. So we recall some result about localization with respect to topological K-theory. However, we will need some facts whose proofs are beyond the scope of this book, so this section is much less self-contained than the rest of the book.

Since we have isomorphisms $KU \cong KO \wedge^L C(\eta)$ and $KT \cong KO \wedge C(\eta^2)$ and the stable homotopy class of the Hopf map η satisfies $\eta^4 = 0$, a special case of Proposition 9.26 is that the complex, self-conjugate and real topological K-theory spectra KU, KT respectively KO have the same Bousfield class. We call a spectrum X K-*local* if it is local with respect to KU (or equivalently local with respect to KT or KO).

In Example 6.31 we discussed the Adams maps, certain graded selfmaps $v_1 : \Sigma^q\mathbb{S}/p \longrightarrow \mathbb{S}/p$ of the mod-p Moore spectra, where $q = 8$ for $p = 2$ and $q = 2p - 2$ for odd primes p. [for $p = 2$ this should better be called v_1^4] These selfmaps induce an isomorphism in complex topological K-theory (and hence also in KO and KT-theory). In particular, the map v_1 is *periodic* in the sense that all iterates v_1^n are non-trivial in the stable homotopy category. We recall that the mod-p homotopy groups of a spectrum X are defined by

$$\pi_k(X, \mathbb{Z}/p) \;=\; \pi_k(\mathbb{S}/p \wedge^L X) .$$

Smashing with the Adams map and taking homotopy group gives an operator on mod-p homotopy groups

$$v_1 \;:\; \pi_k(X, \mathbb{Z}/p) \cong \pi_{k+q}(\Sigma^q\mathbb{S}/p \wedge^L X) \xrightarrow{\pi_{k+q}(v_1 \wedge X)} \pi_{k+q}(\mathbb{S}/p \wedge^L X) = \pi_{k+q}(X, \mathbb{Z}/p)$$

which is called v_1-*multiplication*. The mod-p homotopy groups are called v_1-*periodic* if v_1-multiplication is an isomorphism for all integers k.

Theorem 9.27. *A spectrum X is K-local if and only if for every prime p the mod-p homotopy groups $\pi_k(X, \mathbb{Z}/p)$ are v_1-periodic. The K-localization functor is smashing in the sense that the morphism $X \longrightarrow L_K\mathbb{S}_{(p)} \wedge^L X$ is a K-localization for every p-local spectrum.*

So to complete the picture of K-localization we should describe the localized sphere spectrum in a more explicit way. [...]

10. The Steenrod algebra

In this section we identify the graded maps between Eilenberg-Mac Lane spectra with stable operations in singular cohomology. Especially important are the operations in mod-p cohomology, which give rise to the mod-p Steenrod algebra.

For us, the *reduced cohomology* $\tilde{H}^n(X, A)$ of a based simplicial set X with coefficients in an abelian group A is the relative cohomology $H^n(X, \{x\}, A)$, relative to the simplicial subset consisting only of the basepoint x and its degeneracies. Of course, the difference between reduced and unreduced cohomology is very minor: the restriction map

$$\tilde{H}^n(X, A) = H^n(X, \{x\}, A) \longrightarrow H^n(X, \emptyset, A) = H^n(X, A)$$

is an isomorphism for $n \neq 0$; for $n = 0$, this map is a split monomorphism whose complementary summand is a copy of the coefficient group A.

Definition 10.1. Let A and B be abelian groups and n, m natural numbers. A *reduced cohomology operation* of type (A, n, B, m) is a natural transformation

$$\tau : \tilde{H}^n(-, A) \longrightarrow \tilde{H}^m(-, B)$$

of set valued functors on the category of based simplicial sets.

Note that we do not demand that the individual maps $\tau_X : \tilde{H}^n(X, A) \longrightarrow \tilde{H}^m(X, B)$ are additive. However, in Exercise E.II.19 below we show that reduced cohomology operations are automatically additive *on suspensions*. In any case, two reduced cohomology operations of the same type can be added pointwise, so the set of all reduced cohomology operations of a fixed type forms an abelian group, which we denote $\mathrm{Oper}(A, n, B, m)$. We could just as well replace simplicial sets by topological spaces and would arrive at an equivalent definition of reduced cohomology operations, provided we restrict to CW-complexes.

Remark 10.2. *Unreduced* cohomology operations are natural transformations

$$\tau : H^n(-, A) \longrightarrow H^m(-, B)$$

of unreduced cohomology functors. There is only a minor difference between reduced and (non-reduced) cohomology operations. Since the one-point space has trivial cohomology in positive dimensions, this is only a condition for $n = 0$, and for $n \geq 1$ every cohomology operation is reduced. The set of reduced cohomology operations of a fixed type forms a subgroup of the group of all cohomology operations.

We recall from Example I.1.14 that the A-linearization $A[S^n]$ of the n-sphere is an Eilenberg-Mac Lane space of type (A, n), and as such it represents cohomology with coefficients in A. More precisely, there is a *fundamental class* $\iota_{n,A} \in \tilde{H}^n(A[S^n], A)$ such that for every based simplicial set X the evaluation map

$$[X, A[S^n]] \longrightarrow \tilde{H}^n(X, A) , \quad [f] \longmapsto f^*(\iota_{n,A})$$

is bijective, where the left hand side denotes based homotopy classes of based morphisms (we recall here that $A[S^n]$ is a Kan complex, so these homotopy classes are the 'right thing' to consider). Since the reduced cohomology functor $\tilde{H}^n(-, A)$ is representable, the following is an instance of the Yoneda lemma:

Lemma 10.3. *The map*

$$\mathrm{Oper}(A, n, B, m) \longrightarrow \tilde{H}^m(A[S^n], B)$$

which takes a reduced cohomology operation $\tau : \tilde{H}^n(-, A) \longrightarrow \tilde{H}^m(-, B)$ to the image of the fundamental class $\tau(\iota_{n,A}) \in \tilde{H}^m(A[S^n], B)$ is an isomorphism from the group of reduced cohomology operations of type (A, n, B, m) and the m-th rduced cohomology group of $A[S^n]$ with coefficients in B.

Example 10.4. *(i)* The simplicial set $A[S^n]$ is $(n-1)$-connected, so the group $\tilde{H}^m(A[S^n], B)$ is trivial for $m < n$. Hence there are no non-trivial reduced cohomology operations of type (A, n, B, m) for $m < n$.

(ii) A homomorphism of coefficient groups $f : A \longrightarrow B$ induces a reduced cohomology operation of type (A, n, B, n) for every n. Since a $A[S^n]$ is $(n-1)$-connected and $H_n(A[S^n], \mathbb{Z}) \cong \pi_n(A, *) \cong A$, the universal coefficient theorem for cohomology yields an isomorphism

$$H^n(A[S^n], B) \cong \mathrm{Hom}(A, B) ,$$

which shows that the cohomology operations of type (A, n, B, n) all arise from coefficient homomorphisms.

(iii) The Bockstein homomorphism $\delta : \tilde{H}^n(X; A) \longrightarrow \tilde{H}^{n+1}(X; B)$ associated to a short exact sequence of abelian groups

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0 .$$

is a reduced cohomology operation of type $(A, n, B, n+1)$ for every n which only depends on the Yoneda class of the extension. This gives a map

$$\mathrm{Ext}(A, B) \longrightarrow \mathrm{Oper}(A, n, B, n+1) .$$

The universal coefficient theorem for cohomology yields a short exact sequence

$$0 \longrightarrow \operatorname{Ext}(A,B) \longrightarrow \tilde{H}^{n+1}(A[S^n],B) \longrightarrow \operatorname{Hom}(\tilde{H}_{n+1}(A[S^n],\mathbb{Z}),B) \longrightarrow 0 \ ,$$

so this map is injective. Moreover, for $n \geq 2$, the homology group $\tilde{H}_{n+1}(A[S^n],\mathbb{Z})$ is trivial (see e.g. [**25**, Thm. 20.5]), so in that case every cohomology operation of type $(A,n,B,n+1)$ is the Bockstein homomorphism of an abelian group extension.

The group $\tilde{H}_2(A[S^1],\mathbb{Z})$ is not generally trivial, so there are reduced cohomology operations of type $(A,1,B,2)$ which do not come from short exact sequences of abelian groups. Indeed, $\tilde{H}^2(A[S^1],B)$ classifies equivalence classes of *central group extension* of A by B, i.e., short exact sequences of groups

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 1$$

such that A is contained in the center of E (which need not be abelian). The image of $\operatorname{Ext}(A,B)$ in $H^2(A[S^1],B)$ corresponds to those extensions (necessarily central) for which E is abelian. Exercise E.II.11 explains how to construct a non-abelian Bockstein operation from such a central extension. A proof of the correspondence between $H^2(A[S^1];B)$ and classes of central extensions can be found in [**49**, IV Thm. 6.2] (in the special case of trivial coefficient modules).

(iv) Let R be any ring and $k \geq 0$. Then the cup product power operation

$$H^n(X,R) \longrightarrow H^{kn}(X,R) \ , \quad x \longmapsto x^k$$

is a cohomology operation of type (R,n,R,kn). In some cases the cup powers give all operations of a certain type. For example, the group $H^n(\mathbb{F}_2[S^1],\mathbb{F}_2)$ is cyclic of order 2, generated by the n-th power of the fundamental class. So by the representability of cohomology the n-th cup-power operation is the only non-trivial cohomology operation of type $(\mathbb{F}_2,1,\mathbb{F}_2,n)$. Similarly, since the integral cohomology algebra of $\mathbb{Z}[S^2]$ is polynomial on the fundamental class, there is only the trivial operation of type $(\mathbb{Z},2,\mathbb{Z},n)$ for odd n, and all cohomology operations of type $(\mathbb{Z},2,\mathbb{Z},2k)$ are multiples of the k-th cup power operation. Rationally, there are no other cohomology operations whatsoever, besides multiples of cup powers. Indeed, the cohomology algebra $H^*(\mathbb{Q}[S^n],\mathbb{Q})$ is polynomial on the fundamental class for even n, and is an exterior algebra on the fundamental class for odd n.

Now we get to the concept of a *stable* cohomology operation, which is really a family of compatible cohomology operations.

Definition 10.5. Let A and B be abelian groups and i a natural number. A *stable cohomology operation* of type (A,B) and degree i is a family $\{\tau_n\}_{n\geq 0}$ of reduced cohomology operations of type $(A,n,B,i+n)$ which are compatible with suspension isomorphisms in the following sense. For every based simplicial set X and every $i \geq 0$ the square

$$\begin{array}{ccc} \widetilde{H}^n(X,A) & \xrightarrow{-\wedge\iota} & \widetilde{H}^{n+1}(X\wedge S^1,A) \\ \downarrow{\scriptstyle\tau_n} & & \downarrow{\scriptstyle\tau_{n+1}} \\ \widetilde{H}^{i+n}(X,B) & \xrightarrow[-\wedge\iota]{} & \widetilde{H}^{i+n+1}(X\wedge S^1,B) \end{array}$$

commutes, where the horizontal maps are given by exterior product with the generator $\iota \in \tilde{H}^1(S^1,\mathbb{Z})$. We denote by $\operatorname{StOp}(A,B,i)$ the abelian group of stable cohomology operations of type (A,B) and degree i.

Our next task is to identify stable cohomology operations with morphisms in the stable homotopy category between Eilenberg-Mac Lane spectra. For this purpose we define a map

$$\operatorname{ev} \ : \ [HA,HB]^i \longrightarrow \operatorname{StOp}(A,B,i)$$

from the latter to the former as follows. Let $f : HA \longrightarrow \mathbb{S}^i \wedge^L HB$ be a morphism of degree i. For every $n \geq 0$ we consider the diagram

$$\begin{tikzcd} \tilde{H}^n(X,A) \arrow[r, "\cong"] \arrow[d, "\mathrm{ev}(f)_n"'] & H^n(\Sigma^\infty X, A) = [\Sigma^\infty X, HA]^n \arrow[d, "f\circ"] \\ \tilde{H}^{i+n}(X,B) \arrow[r, "\cong"'] & H^{i+n}(\Sigma^\infty X, B) = [\Sigma^\infty X, HB]^{i+n} \end{tikzcd}$$

of cohomology groups. Here the horizontal maps are the isomorphisms of [...] and the right vertical map is the graded composition with the morphism f. There is thus a unique map $\mathrm{ev}(f)_n$ making the square commute. For fixed f, the map $\mathrm{ev}(f)_n$ is clearly natural in the space X, hence a cohomology operation. We claim that as n varies, the sequence $\{\mathrm{ev}(f)_n\}_{n\geq 0}$ forms a *stable* cohomology operation [...]

Theorem 10.6. *For all abelian groups A and B the map*

$$\mathrm{ev} \ : \ [HA, HB]^i \ \longrightarrow \ \mathrm{StOp}(A,B,i)$$

is an isomorphism of groups.

Proof. The group $\mathcal{SHC}(HA, HB)^i$ is, by definition, the cohomology group $H^i(HA,B)$ of the Eilenberg-Mac Lane spectrum HA. Since the Eilenberg-Mac Lane spectrum HB is semistable, [...] Proposition 6.5 (ii) provides a short exact sequence

$$0 \longrightarrow \lim^1_n \tilde{H}^{i+n-1}(A[S^n], B) \longrightarrow H^i(HA,B) \longrightarrow \lim_n \tilde{H}^{i+n}(A[S^n], B) \longrightarrow 0$$

Since the simplicial set $A[S^n]$ is $(n-1)$-connected, so the suspension homomorphism $-\wedge S^1 : \pi_k(A[S^n]) \longrightarrow \pi_{k+1}(A[S^n]\wedge S^1)$ is an isomorphism for $k \leq 2n-2$, by Freudenthal's suspension theorem. Hence the structure map $\sigma_n : A[S^n] \wedge S^1 \longrightarrow A[S^{n+1}]$ is $(2n-1)$-connected, so it induces an isomorphism of all cohomology groups below dimension $2n-1$. For a fixed integer i, almost all of the maps in the inverse system of groups $H^{i+n-1}(A[S^n], B)$ are thus isomorphisms, so the derived limit above vanishes.

On the other hand, the map

$$\mathrm{StOp}(A,B,i) \ \longrightarrow \ \lim_n \tilde{H}^{i+n}(A[S^n], B) \ , \quad \tau \longmapsto \tau_n(\iota_{A,n})$$

is bijective. Indeed, by Lemma 10.3 a family $\tau = \{\tau_n\}_{n\geq 0}$ of unreduced cohomology operations corresponds bijectively to the family $\tau_n(\iota_{n,A}) \in \tilde{H}^{i+n}(A[S^i], B)$ of unreduced cohomology classes. Moreover, the stability condition for the family $\{\tau_n\}_{n\geq 0}$ corresponds to the condition to be an element in the inverse limit. □

If $\tau = \{\tau_n\}_{n\geq 0}$ is a stable cohomology operation of type (A,B,i) and $\lambda = \{\lambda_n\}_{n\geq 0}$ is a stable cohomology operation of type (B,C,j), then they compose to yield a stable cohomology operation

$$\lambda \circ \tau \ = \ \{\lambda_{i+n} \circ \tau_n\}_{n\geq 0}$$

of type (A,C) and degree $j+i$. On the other hand, graded morphisms between Eilenberg-Mac Lane spectra can be composed as in (4.6), specializing to a map

$$[HB, HC]^j \ \otimes \ [HA, HB]^i \ \longrightarrow \ [HA, HC]^{j+i} \ .$$

The two kinds of composition coincide, i.e., we have

$$\mathrm{ev}(g \circ f) \ = \ \mathrm{ev}(g) \circ \mathrm{ev}(f) \ .$$

[proof?]

Example 10.7. (a) By Example 10.4 (i) there are no stable cohomology operations of negative degree. If $f : A \longrightarrow B$ is a homomorphism of coefficient groups, then the associated cohomology operations of type (A,m,B,m) for every $m \geq 0$ form a stable cohomology operation, and so the group all stable cohomology operations of type (A,B) of degree 0 is naturally isomorphic to $\mathrm{Hom}(A,B)$,

$$\mathrm{StOp}(A,B,0) \ \cong \ \mathrm{Hom}(A,B) \ .$$

This is consistent with Theorem 10.6 since the morphism from HA to HB in the stable homotopy category also biject with the group of homomorphisms from A to B (by Theorem 5.25 (i)).

(b) The Bockstein homomorphisms $\delta : \tilde{H}^n(X, A) \longrightarrow \tilde{H}^{n+1}(X, B)$ in reduced cohomology associated to a short exact sequence of abelian groups

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$$

form a stable cohomology operation of type (A, B) and degree 1 (for varying $n \geq 0$). For $n \geq 2$, the homology group $H_{n+1}(A[S^n], \mathbb{Z})$ is trivial (see e.g. [**25**, Thm. 20.5]), so the universal coefficient theorem implies that this construction gives all stable operations of type (A, B) and degree 1,

$$\mathrm{StOp}(A, B, 1) \;\cong\; \mathrm{Ext}(A, B) \;.$$

Again, this is consistent with Theorem 10.6, since the morphism from HA to $\Sigma(HB)$ in the stable homotopy category also biject with the Ext group (by Proposition 5.28 (iii)).

(c) If R is a ring, then the cup product power operation $x \longmapsto x^k$ is usually not additive, and whenever it fails to be so, then as an operation of type (R, n, R, kn) it does not extend to a stable operation of degree $(k-1)n$ (by part (b) of Exercise E.II.19). However, if p is a prime number and R is an $\mathbb{F}_p$-algebra, then the p-th power operation $x \longmapsto x^p$ is additive. And indeed, as we recall in 10.8 and 10.10 below, the p-th cup power

$$H^i(X, \mathbb{F}_p) \;\longrightarrow\; H^{pi}(X, \mathbb{F}_p) \;,\; x \longmapsto x^p$$

extends to a stable mod-p cohomology operation of degree $2i(p-1)$ whenever $p = 2$ or i is even. For $p = 2$ this stable operation is denoted Sq^i and is called the i-th *Steenrod divided square operation*. For odd p this stable operation is called the i-th divided power operation and is denoted P^i.

Now we specialize to the most useful kind of stable cohomology operations: For a prime p, the *mod-p Steenrod algebra* $\mathcal{A}_p$ is the graded $\mathbb{F}_p$-algebra of mod-p stable cohomology operations; in other words, in degree n we have

$$(\mathcal{A}_p)^n \;=\; \mathrm{StOp}(\mathbb{F}_p, \mathbb{F}_p, n)$$

and the product structure is by composition of operations.

The mod-p Steenrod algebra has an explicit description in terms of generators and relations, which we now review. There are various sources available where the construction of the mod-p cohomology operations and the relations between them are discussed in detail, and we will not reprove these facts. The explicit results take a slightly different form for the prime 2 and for odd primes, and read as follows.

10.8. The 2-primary Steenrod algebra. The *i-th Steenrod square*, for $i \geq 0$, is a stable mod-2 cohomology operation Sq^i of degree i with the following properties:

(i) The operation Sq^0 is the identity and Sq^1 coincides with the mod-2 Bockstein operation.

(ii) (Unstability condition) For every simplicial set X and cohomology class $x \in H^n(X, \mathbb{F}_2)$ we have $\mathrm{Sq}^i(x) = x \cup x$ if $i = n$ and $\mathrm{Sq}^i(x) = 0$ for $i > n$.

(iii) (Cartan formula) For $x, y \in H^*(X, \mathbb{F}_2)$ and $i \geq 0$ we have

$$\mathrm{Sq}^i(x \cup y) \;=\; \sum_{a=0}^{i} \mathrm{Sq}^a x \cup \mathrm{Sq}^{i-a} y \;.$$

(iv) (Adem relations) The Steenrod squaring operations satisfy the following relations

$$\mathrm{Sq}^a \mathrm{Sq}^b \;=\; \sum_{j=0}^{[a/2]} \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j$$

for all $a < 2b$.

The stable operation Sq^i is in fact already uniquely determined by the property that $\mathrm{Sq}^i(x) = x \cup x$ for all classes x of degree i.

With respect to binomial coefficients $\binom{n}{m}$ for integers n and m, possibly negative, we recall that

$$\binom{n}{m} = \begin{cases} \frac{n\cdot(n-1)\cdots(n-m+1)}{m\cdot(m-1)\cdots 1} & \text{if } m\geq 1, \\ 1 & \text{if } m=0, \\ 0 & \text{if } m<0. \end{cases}$$

With these conventions, the formula

$$(1+t)^n = \sum_{i=0}^{\infty} \binom{n}{i} \cdot t^i$$

holds for all integers n, positive or negative, in the power series ring $\mathbb{Z}[[t]]$.

Example 10.9. The first Adem relations are:

$$\begin{array}{rclcrcl} \mathrm{Sq}^1\,\mathrm{Sq}^1 &=& 0 & \qquad & \mathrm{Sq}^2\,\mathrm{Sq}^2 &=& \mathrm{Sq}^3\,\mathrm{Sq}^1 \\ \mathrm{Sq}^1\,\mathrm{Sq}^2 &=& \mathrm{Sq}^3 & & \mathrm{Sq}^2\,\mathrm{Sq}^3 &=& \mathrm{Sq}^5 + \mathrm{Sq}^4\,\mathrm{Sq}^1 \\ \mathrm{Sq}^1\,\mathrm{Sq}^3 &=& 0 & & \mathrm{Sq}^3\,\mathrm{Sq}^2 &=& 0 \end{array}$$

One can see that some of the relations are redundant. For example, the vanishing of $\mathrm{Sq}^3\,\mathrm{Sq}^2$ can be deduced from the previous relations. For all integers n we have

$$\mathrm{Sq}^1\,\mathrm{Sq}^n = \binom{n-1}{1}\mathrm{Sq}^{n+1} = \begin{cases} \mathrm{Sq}^{n+1} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

For all integers $n\geq 2$ we have

$$\begin{aligned} \mathrm{Sq}^2\,\mathrm{Sq}^n &= \binom{n-1}{2}\mathrm{Sq}^{n+2} + \binom{n-2}{0}\mathrm{Sq}^{n+1}\,\mathrm{Sq}^1 \\ &= \begin{cases} \mathrm{Sq}^{n+1}\,\mathrm{Sq}^1 & \text{if } n\equiv 1,2 \mod 4, \\ \mathrm{Sq}^{n+2} + \mathrm{Sq}^{n+1}\,\mathrm{Sq}^1 & \text{if } n\equiv 0,3 \mod 4. \end{cases} \end{aligned}$$

10.10. The odd-primary Steenrod algebra Let p be an odd prime. There is a unique sequence of stable mod-p cohomology operations P^i of degree $2i(p-1)$, for $i\geq 0$, such that the following properties (i)-(iii) below hold. The operation P^i is called the *i-th Steenrod power operation.*

(i) The operation P^0 is the identity operation.

(ii) (Unstability condition) For every simplicial set X and cohomology class $x\in H^n(X,\mathbb{F}_p)$ we have $P^i(x)=x^p$ if $n=2i$ and $P^i(x)=0$ if $2i>n$.

(iii) (Cartan formula) Let X be a simplicial set and $x,y\in H^*(X;\mathbb{F}_p)$ cohomology classes. Then we have

$$P^i(x) = \sum_{a+b=i} P^a(x)\cup P^b(x)\ .$$

Moreover, these operations enjoy the following properties:

(iv) (Adem relations) The power operations P^i satisfy the relations

$$P^aP^b = \sum_{j=0}^{[a/p]} (-1)^{a+j}\binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j}P^j$$

for all $0<a<pb$. Moreover, the power operations and the mod-p Bockstein β satisfy the relations

$$\begin{aligned} P^a\beta P^b &= \sum_{j=0}^{[a/p]} (-1)^{a+j}\binom{(p-1)(b-j)}{a-pj}\beta P^{a+b-j}P^j \\ &+ \sum_{j=0}^{[(a-1)/p]} (-1)^{a+j-1}\binom{(p-1)(b-j)-1}{a-pj-1} P^{a+b-j}\beta P^j \end{aligned}$$

for all $0<a\leq pb$.

The Steenrod operations and Adem relations provide a complete description of the mod-p Steenrod algebra. Indeed, the Steenrod operations Sq^i's generate the mod-2 Steenrod algebra, the reduced power opertations P^i and the Bockstein operation β generate the mod-p Steenrod algebra for odd p, and in each case the Adem relations form a complete set of relations between these operations. More precisely, we can consider the free graded $\mathbb{F}_2$-algebra generated by symbols Sq^i of degree i for $i \geq 0$, subject to the relation $\mathrm{Sq}^0 = 1$ and the Adem relations. This algebra maps to the Steenrod algebra of stable mod-2 cohomology operations by sending the symbol Sq^i to the operation Sq^i. The resulting map

$$\mathbb{F}_2\langle \mathrm{Sq}^i \rangle / (\mathrm{Sq}^0 = 1, \text{ Adem relations}) \longrightarrow \mathcal{A}_2$$

is an isomorphism of graded $\mathbb{F}_2$-algebras, and similarly for odd primes p.

We want to emphasize that the unstability conditions of the Steenrod squaring operations Sq^i and the reduced power operations P^i are special for the cohomology *of simplicial sets* (or spaces), and they do not generally hold for the cohomology *of spectra*. The relations between $\mathrm{Sq}^i(x)$ and the cup square respectively $P^i(x)$ and the p-th cup power don't even make sense in the context of spectra, since there is no natural product structure on the mod-p cohomology groups of spectra. The unstability relations would make sense for spectra, but the class $\mathrm{Sq}^i(x)$ respectively $P^i(x)$ can be non-trivial for spectrum cohomology classes of arbitrily large i.

Here is a (tautological) example: we let $\iota \in H^0(H\mathbb{F}_p, \mathbb{F}_p) = \mathcal{SHC}(H\mathbb{F}_p, H\mathbb{F}_p)$ denote the fundamental class, i.e., the identity morphism of the Eilenberg-Mac Lane spectrum $H\mathbb{F}_p$. Then

$$\mathrm{Sq}^i(\iota) \in H^i(H\mathbb{F}_p, \mathbb{F}_p) \quad \text{respectively} \quad P^i(\iota) \in H^{2i(p-1)}(H\mathbb{F}_p, \mathbb{F}_p)$$

equal the non-zero operations Sq^i respectively P^i, hence they are non-zero as cohomology classes of the spectrum $H\mathbb{F}_p$.

10.1. Examples and applications. An important problem in homotopy theory is to find ways of telling when a continuous map $f : X \longrightarrow Y$ is null-homotopic. A map which is not null-homotopic is called *essential*.

Sometimes a map can be shown to be essential by checking that it induces a non-trivial map on cohomology with suitable coefficients. If this does not help, then one can use the mapping cone: Suppose that a map $f : X \longrightarrow Y$ between topological spaces or simplicial sets is trivial in cohomology with coefficients in an abelian group A. Then the long exact cohomology sequence yields an epimorphism

$$H^*(C(f), A) \xrightarrow{i^*} H^*(Y, A) ,$$

where $C(f) = * \cup_{X \times 0} X \times [0,1] \cup_{X \times 1} Y$ is the unreduced mapping cone of f and $i : Y \longrightarrow C(f)$ is the inclusion.

If f is null-homotopic, then a choice of null-homotopy provides a section $\sigma : C(f) \longrightarrow Y$ to the map i. On cohomology, this induces a map of graded abelian groups $\sigma^* : H^*(Y, A) \longrightarrow H^*(C(f), A)$ which is a section to the map i^*. Since the section σ^* is induced by a map σ, it also respects all additional structure which is natural for maps of space or simplicial sets. For example, if A is a ring, then σ^* is compatible with the cup-product. In some cases, the original map f can be seen to be essential because there is no section to i^* which is multiplicative with respect to the cup-product. The prime example of this kind for reasoning is the following proof that the Hopf maps $\eta : S^3 \longrightarrow S^2$, $\nu : S^7 \longrightarrow S^4$ and $\sigma : S^{15} \longrightarrow S^7$ are essential. The mapping cones of the three Hopf maps are isomorphic to the projective planes $\mathbb{C}\mathrm{P}^2$, $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$ over the complex numbers, the quaternions and the Cayley octaves respectively. The integral cohomology rings of these spaces are all of the from $\mathbb{Z}[x]/x^3$ where the dimension of the generator is 2, 4 or 8 respectively. Hence if $i : \mathbb{C}\mathrm{P}^1 \longrightarrow \mathbb{C}\mathrm{P}^2$ is the inclusion, then there is no multiplicative section to the map

$$i^* \ : \ H^*(\mathbb{C}\mathrm{P}^2; \mathbb{Z}) \longrightarrow H^*(\mathbb{C}\mathrm{P}^1; \mathbb{Z}) ,$$

and so the Hopf map η is essential. The same argument with $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$ shows that the Hopf maps ν and σ are essential.

However, this book is mainly concerned with *stable* homotopy theory, and the cup-product is useless for telling whether a map is *stably essential*, i.e., whether or not it becomes null-homotopic after some number of suspensions. This is because the cup-product is trivial on the reduced cohomology of any suspension. So if $f : X \longrightarrow Y$ is a map of spaces which is trivial in reduced cohomology with coefficients in a ring A, then we have $C(\Sigma f) \cong \Sigma C(f)$, and the map

$$H^*(\Sigma C(f), A) \;\cong H^*(C(\Sigma f), A) \;\xrightarrow{i^*}\; H^*(\Sigma Y, A)$$

always has a multiplicative section.

In general, the more highly structured and calculable homotopy functors we find, the better chances we have to show that such a section cannot exist. For detecting stably essential maps, stable cohomology operations are a good tool, since they don't change after suspension. In more detail, let $f : X \longrightarrow Y$ be a based map of spaces (or simplicial sets) which becomes null-homotopic after n suspensions. Then the mapping cone $C(\Sigma^n f)$ is homotopy equivalent to the wedge of $\Sigma^{n+1}X$ and $\Sigma^n Y$. Taking cones and suspension commute, so the mapping cone $C(\Sigma^n f)$ is homeomorphic to $\Sigma^n C(f)$ and we conclude that the cohomology $\tilde{H}^*(\Sigma^n C(f), \mathbb{F}_p)$ is the direct sum, as a module over the mod-p Steenrod algebra $\mathcal{A}_p$, of the mod-p cohomology groups of $\Sigma^{n+1}X$ and $\Sigma^n Y$. Since the Steenrod algebra consists of stable operations, suspension amounts to reindexing the cohomology of a space, including the $\mathcal{A}_p$-action. In other words, if a map $f : X \longrightarrow Y$ becomes null-homotopic after some number of suspensions, then not only is f trivial on reduced mod-p cohomology, in addition the map

$$i^* \;:\; \tilde{H}^*(C(f), \mathbb{F}_p) \;\xrightarrow{i^*}\; \tilde{H}^*(Y, \mathbb{F}_p)$$

has a section which is $\mathcal{A}_p$-linear. We apply this strategy to the Hopf maps.

Example 10.11 (The Hopf maps are stably essential)**.** The mapping cones of the Hopf maps η, ν and σ are isomorphic to the projective planes $\mathbb{C}\mathrm{P}^2, \mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$ over the complex numbers, the quaternions and the Cayley octaves respectively. The mod-2 cohomology algebras of these spaces are all of the from $\mathbb{F}_2[x]/x^3$ where the dimension of the generator is 2, 4 or 8 respectively. Hence we have the relation

$$\mathrm{Sq}^2(x_2) \;=\; x_2^2 \;\neq\; 0 \;\in\; H^4(\mathbb{C}\mathrm{P}^2, \mathbb{F}_2)\ ,$$

and similarly the classes $\mathrm{Sq}^4(x_4) \in H^8(\mathbb{H}\mathrm{P}^2, \mathbb{F}_2)$ and $\mathrm{Sq}^8(x_8) \in H^{16}(\mathbb{O}\mathrm{P}^2, \mathbb{F}_2)$ are non-zero. So the mod-2 cohomology groups of the mapping cones of η, ν and σ do not split as modules over the mod-2 Steenrod-algebra. Hence by the previous paragraph, these maps are stably essential. This, finally, finishes the proof proof that the first stable stem π_1^s is cyclic of order two, generated by the class of the Hopf map η (as was already claimed in Example I.1.8).

Example 10.12 (The degree 2 map of the mod-2 Moore space is stably essential)**.** Let p be a prime and let

$$M(p) \;=\; S^1 \cup_p D^2$$

denote the mod-p Moore space of dimension 2, obtained by attaching a 2-cell to the circle along the degree p map $S^1 \longrightarrow S^1$. Denote by $\times p : \Sigma M(p) \longrightarrow \Sigma M(p)$ the smash product of $M(p)$ with the degree p map of the circle. The degree p map induces multiplication by p in cohomology with any kind of coefficients, but the cohomology of $M(p)$, with any kind of coefficients, is annihilated by p. So $\times p$ induces the trivial map in cohomology, and we may ask whether this map is null-homotopic. The answer is different for the prime 2 and the odd primes. [p odd...]

In contrast to this, for the prime 2 the degree 2 map of $\Sigma M(2)$ is stably essential; note that another name for $M(2)$ is $\mathbb{R}\mathrm{P}^2$. Since the degree 2 map of $\Sigma M(2)$ is obtained by smashing the $M(2)$ with the degree 2 map of S^1, its mapping cone of $C(\times 2)$ is isomorphic to the smash product of two copies of the Moore space,

$$C(\times 2) \;\cong\; M(2) \wedge M(2)$$

in such a way that the inclusion $\Sigma M(2) \longrightarrow C(\times 2)$ corresponds to the smash product of the inclusion $i : S^1 \longrightarrow M(2)$ with $M(2)$. Now the mod-2 cohomology of $M(2)$ has an $\mathbb{F}_2$-basis given by a class $x \in \tilde{H}^1(M(2); \mathbb{F}_2)$ and $x^2 = \mathrm{Sq}^1(x) \in \tilde{H}^2(M(2); \mathbb{F}_2)$. By the Künneth theorem the cohomology of the smash product $M(2) \wedge M(2)$ is four dimensional with basis given by the classes $x \otimes x$ in dimension 2,

$\mathrm{Sq}^1(x)\otimes x$ and $\mathrm{Sq}^1(x)\otimes x$ in dimension 3, and $\mathrm{Sq}^1(x)\otimes\mathrm{Sq}^1(x)$ in dimension 4. Also by the Künneth theorem, the map

$$(i\wedge M(2))^* \;:\; \tilde{H}^*(M(2)\wedge M(2),\mathbb{F}_2) \;\longrightarrow\; \tilde{H}^1(S^1\wedge M(2),2\mathbb{F}_2)$$

is given by

$$(i\wedge M(2))^*(x\otimes x) \;=\; \Sigma x\ , \quad \text{and} \quad (i\wedge M(2))^*(x\otimes \mathrm{Sq}^1(x)) \;=\; \Sigma\,\mathrm{Sq}^1(x)\ ,$$

and it vanishes on the classes $\mathrm{Sq}^1(x)\otimes x$ and $\mathrm{Sq}^1(x)\otimes\mathrm{Sq}^1(x)$. All cup products are trivial in the reduced cohomology of $S^1\wedge M(2)$, but in the cohomology of $M(2)\wedge M(2)$, the cup-square of the two-dimensional class $x\otimes x$ is non-trivial. This shows that there is no section to $(i\wedge M(2))^*$ which is compatible with the cup-product, so the degree 2 map on $M(2)$ is essential.

However, after a single suspension, the cup products of both sides are trivial, so this argument does not give any hint as to whether the suspension of the degree 2 map on $M(2)$ is null-homotopic or not. However, we can calculate the action of the Steenrod-squares in the cohomology of $M(2)\wedge M(2)$. Note that the operation $\mathrm{Sq}^2(x)$ acts trivially on the cohomology of $S^1\wedge M(2)$ for dimensional reasons. On the other hand, the Cartan-formula gives

$$\mathrm{Sq}^2(x\otimes x) \;=\; \mathrm{Sq}^2(x)\otimes x+\mathrm{Sq}^1(x)\otimes\mathrm{Sq}^1(x)+x\otimes\mathrm{Sq}^2(x) \;=\; \mathrm{Sq}^1(x)\otimes\mathrm{Sq}^1(x) \;\neq\; 0$$

in $\tilde{H}^4(M(2)\wedge M(2),\mathbb{F}_2)$. So there does not exist a section to $(i\wedge M(2))^*$ which is compatible with the action of the Steenrod-algebra. Hence we conclude that the degree 2 map of the mod-2 Moore space is stably essential.

Construction 10.13. The Adem relations can be used to show that certain composites of Hopf maps are stably essential. We need the following observation: suppose that

$$\alpha \;:\; S^m \;\longrightarrow\; S^k \qquad \text{and} \qquad \beta \;:\; S^n \;\longrightarrow\; S^m$$

are continuous based maps between spheres. Suppose that the composite $\alpha\beta : S^n \longrightarrow S^k$ is null-homotopic, and let

$$H \;:\; [0,1]\wedge S^n \longrightarrow S^k$$

be a null-homotopy, i.e., a based map such that $H(1,-)=\alpha\beta$. The maps H and α glue to a map

$$H\cup\alpha \;:\; C(\beta)=([0,1]\wedge S^n)\cup_\beta CS^m \;\longrightarrow\; S^k$$

from the mapping cone of β. We let $C(\alpha,\beta,H)$ be the mapping cone of $H\cup\alpha : C(\beta)\longrightarrow S^k$. This space has a preferred CW-structure with 4 cells in dimensions 0, k, $m+1$ and $n+2$. Moreover, it contains the mapping cone of α as its $(m+1)$-skeleton, and the quotient of $C(\alpha,\beta,H)$ by its k-skeleton (which is the sphere S^k) is homoemorphic to the suspension of the mapping cone of β.

Example 10.14. Now we show how Construction 10.13 and the Adem relation $\mathrm{Sq}^2\,\mathrm{Sq}^2=\mathrm{Sq}^3\,\mathrm{Sq}^1$ can be used to show that the composite η^2 is stably essential. Suppose that for some n the composite

$$S^{n+2} \;\xrightarrow{\;\eta\;}\; S^{n+1} \;\xrightarrow{\;\eta\;}\; S^n$$

is null-homotopic. After choosing a null-homotopy H we can form the space $C(\eta,\eta,H)$ with cells in dimension 0, n, $n+2$ and $n+4$. The reduced mod-2 cohomology of this space is one-dimensional in dimensions n, $n+2$ and $n+4$, and trivial in all other dimensions. Since the $(n+2)$-cell is attached to the n-cell by η, the Steenrod operation Sq^2 is an isomorphism from $H^n(C(\eta,\eta,H);\mathbb{F}_2)$ to $H^{n+2}(C(\eta,\eta,H);\mathbb{F}_2)$, and similarly from there to $H^{n+4}(C(\eta,\eta,H);\mathbb{F}_2)$. But since the group $H^{n+1}(C(\eta,\eta,H);\mathbb{F}_2)$ vanishes, we get that

$$\mathrm{Sq}^2\,\mathrm{Sq}^2=\mathrm{Sq}^3\,\mathrm{Sq}^1 \;:\; H^n(C(\eta,\eta,H);\mathbb{F}_2) \;\longrightarrow\; H^{n+4}(C(\eta,\eta,H);\mathbb{F}_2)$$

is trivial, a contradiction. Hence no suspension of η^2 is ever null-homotopic.

The same kind of reasoning yields other non-triviality results for certain composites of Hopf maps, using that 2, η, ν and σ are detected in mod-2 cohomology by the Steenrod operations Sq^1, Sq^2, Sq^4 and Sq^8, respectively. In the following table we list some Adem relations along with the composite which are non-trivial by the above argument.

relation			stably essential product
$\mathrm{Sq}^1\,\mathrm{Sq}^4$	$=$	$\mathrm{Sq}^4\,\mathrm{Sq}^1+\mathrm{Sq}^2\,\mathrm{Sq}^3$	2ν
$\mathrm{Sq}^1\,\mathrm{Sq}^8$	$=$	$\mathrm{Sq}^8\,\mathrm{Sq}^1+\mathrm{Sq}^2\,\mathrm{Sq}^7$	2σ
$\mathrm{Sq}^2\,\mathrm{Sq}^2$	$=$	$\mathrm{Sq}^3\,\mathrm{Sq}^1$	$\eta\eta$
$\mathrm{Sq}^2\,\mathrm{Sq}^8$	$=$	$\mathrm{Sq}^9\,\mathrm{Sq}^1+\mathrm{Sq}^8\,\mathrm{Sq}^2+\mathrm{Sq}^4\,\mathrm{Sq}^6$	$\eta\sigma$
$\mathrm{Sq}^4\,\mathrm{Sq}^4$	$=$	$\mathrm{Sq}^7\,\mathrm{Sq}^1+\mathrm{Sq}^6\,\mathrm{Sq}^2$	$\nu\nu$
$\mathrm{Sq}^8\,\mathrm{Sq}^8$	$=$	$\mathrm{Sq}^{15}\,\mathrm{Sq}^1+\mathrm{Sq}^{14}\,\mathrm{Sq}^2+\mathrm{Sq}^{12}\,\mathrm{Sq}^4$	$\sigma\sigma$

The product of Hopf maps 2η, $\eta\nu$ and $\nu\sigma$ do not occur in the table, and in fact they are stably null-homotopic.

10.2. Hopf algebra structure. We will now define a map of graded $\mathbb{F}_p$-vector spaces

$$\Delta \;:\; \mathcal{A}_p \;\longrightarrow\; \mathcal{A}_p\otimes\mathcal{A}_p$$

which makes the mod-p Steenrod algebra into a Hopf algebra.

The multiplication map $\mu : H\mathbb{F}_p\wedge H\mathbb{F}_p \longrightarrow H\mathbb{F}_p$ induces a map

$$\mu^* \;:\; \mathcal{A}_p = H^*(H\mathbb{F}_p,\mathbb{F}_p) \;\longrightarrow\; H^*(H\mathbb{F}_p\wedge H\mathbb{F}_p,\mathbb{F}_p)$$

in mod-p cohomology. By the cohomological Künneth theorem [ref] the exterior product map

$$\times \;:\; \mathcal{A}_p\otimes\mathcal{A}_p = H^*(H\mathbb{F}_p,\mathbb{F}_p)\otimes H^*(H\mathbb{F}_p,\mathbb{F}_p) \;\longrightarrow\; H^*(H\mathbb{F}_p\wedge H\mathbb{F}_p,\mathbb{F}_p)$$

is an isomorphism since the Steenrod algebra has finite type. So we can define the diagonal $\Delta(f)$ as the unique class in the tensor product $\mathcal{A}_p\otimes\mathcal{A}_p$ which is taken to $\mu^*(f)$ by the exterior product map. An augmentation $\epsilon : \mathcal{A}_p^0 \longrightarrow \pi_0(H\mathbb{F}_p) = \mathbb{F}_p$ is given by evaluation at the unit $1\in\pi_0(H\mathbb{F}_p)$, i.e.,

$$\epsilon(f) \;=\; f_*(1) \;=\; \langle f, 1\rangle\ .$$

Now we show that the exterior product map (6.8) is $\mathcal{A}_p$-linear with respect to a certain action of the Steenrod algebra on the source through the diagonal map. In other words, for two graded $\mathcal{A}_p$-modules M^* and N^* we define define the action map

$$\circ \;:\; \mathcal{A}_p\otimes(M^*\otimes N^*) \;\longrightarrow\; M^*\otimes N^*$$

as the composite

$$\begin{aligned}\mathcal{A}_p\otimes(M^*\otimes N^*) &\xrightarrow{\Delta\otimes M^*\otimes N^*} \mathcal{A}_p\otimes\mathcal{A}_p\otimes M^*\otimes N^*\\ &\xrightarrow{\mathcal{A}_p\otimes\tau_{\mathcal{A}_p,M^*}\otimes N^*} \mathcal{A}_p\otimes M^*\otimes\mathcal{A}_p\otimes N^* \xrightarrow{\circ\otimes\circ} M^*\otimes N^*\end{aligned}$$

where the second map involves the symmetric isomorphism with the graded sign. More explicitly, if $\Delta(f) = \sum f_i'\otimes f_i''$, then

$$f\circ(x\otimes y) \;=\; \sum\,(-1)^{|f_i''||x|}\cdot(f_i'\circ x)\otimes(f_i''\circ y)\ . \tag{10.15}$$

We have not yet shown that the diagonal morphism of the Steenrod algebra is a homomorphism of graded rings. So we do not yet know that the action of $\mathcal{A}_p$ on $M^*\otimes N^*$ is associative. However, the homomorphism property of the diagonal will be shown in Theorem 10.17 below.

Proposition 10.16. *For all symmetric spectra X and Y the exterior product map* (6.8)

$$\times \;:\; H^*(X,\mathbb{F}_p)\otimes H^*(Y,\mathbb{F}_p) \;\longrightarrow\; H^*(X\wedge^L Y,\mathbb{F}_p)$$

is $\mathcal{A}_p$-linear, with respect to the action of the Steenrod algebra on the source through the diagonal map.

PROOF. We write the diagonal of a given cohomology class $f \in \mathcal{A}_p^n$ as $\Delta(f) = \sum f_i' \otimes f_i''$ for homogenous classes $f_i', f_i'' \in \mathcal{A}_p$. Then $f \circ \mu = \mu^*(f) = \sum f_i' \times f_i''$, by definition of the diagonal. For all cohomology classes $x \in H^k(X, \mathbb{F}_p)$ and $y \in H^l(Y, \mathbb{F}_p)$ we then have

$$\begin{aligned} f \circ (x \times y) \;&=\; f \circ \mu \circ (x \wedge y) \;=\; \sum (f_i' \times f_i'') \circ (x \wedge y) \\ &=\; \sum \; \mu \circ (f_i' \wedge f_i'') \circ (x \wedge y) \\ &=\; \sum \; (-1)^{|f_i''||x|} \cdot \mu \circ ((f_i' \circ x) \wedge (f_i'' \circ y)) \\ &=\; \sum \; (-1)^{|f_i''||x|} \cdot (f_i' \circ x) \times (f_i'' \circ y) \;=\; (- \times -)\big(f \circ (x \otimes y)\big) \; . \end{aligned}$$

in $H^{n+k+l}(X \wedge^L Y, \mathbb{F}_p)$, where the fourth equation is Proposition 4.8. This is the desired $\mathcal{A}_p$-linearity. $\square$

Theorem 10.17. *The diagonal map Δ makes the mod-p Steenrod algebra into a graded cocommutative Hopf algebra.*

PROOF. The map Δ is $\mathbb{F}_p$-linear since the action pairing and the Künneth map are $\mathbb{F}_p$-linear. The diagonal is co-associative and co-commutative because the multiplication of the ring spectrum $H\mathbb{F}_p$ is associative and commutative and because the exterior product map is associative and commutative. For homogeneous cohomology classes $f, g \in \mathcal{A}_p^*$ we have

$$(\epsilon \otimes \mathcal{A}_p)(f \otimes g) \;=\; (f \circ \iota) \times g \;=\; (\iota \wedge H\mathbb{F}_p)^*(f \times g)$$

in $H^*(H\mathbb{F}_p, \mathbb{F}_p)$, and hence

$$(\epsilon \otimes \mathcal{A}_p) \circ \Delta \;=\; (\iota \wedge H\mathbb{F}_p)^* \circ (- \times -) \circ \Delta \;=\; (\iota \wedge H\mathbb{F}_p)^* \circ \mu^* \;=\; (\mu \circ (\iota \wedge H\mathbb{F}_p))^* \;=\; \mathrm{Id} \; .$$

This shows that the diagonal is co-unital.

We now show that the diagonal Δ is a homomorphism of graded rings, where the multiplication on $\mathcal{A}_p \otimes \mathcal{A}_p$ is defined as the bilinear extension of

$$(f \otimes f') \circ (g \otimes g') \;=\; (-1)^{|f'||g|} \cdot (f \circ g) \otimes (f' \circ g') \; .$$

This means that for homogeneous classes $f, g \in \mathcal{A}_p$ we have $\Delta(f) \circ \Delta(g) = f \circ \Delta(g)$ in $\mathcal{A}_p^* \otimes \mathcal{A}_p^*$, where the right hand side is the action of $\mathcal{A}_p$ on the tensor product of two modules as defined in (10.15). For $X = Y = H\mathbb{F}_p$, Proposition 10.17 specializes to

$$\begin{aligned} (- \times -)(\Delta(f) \circ \Delta(g)) \;&=\; (- \times -)(f \circ \Delta(g)) \;=\; f \circ ((- \times -)(\Delta(g))) \\ &=\; f \circ \mu^*(g) \;=\; \mu^*(f \circ g) \;=\; (- \times -)(\Delta(f \circ g)) \; . \end{aligned}$$

Since the exterior product map is an isomorphism, this proves the desired relation. [antipode is automatic since $\mathcal{A}_p$ is graded connected] $\square$

Example 10.18 (Diagonal of Steenrod operations)**.** Theorem 10.6 provides an isomorphism between the mod-p Steenrod algebra $\mathcal{A}_p$ and the algebra of stable mod-p cohomology operations. We denote an element of $\mathcal{A}_p^k$ by the same symbol as its image in $\mathrm{StOp}(\mathbb{F}_p, \mathbb{F}_p, k)$. This way we view the Steenrod operations as classes

$$\mathrm{Sq}^i \;\in\; \mathcal{A}_2^i \qquad \text{respectively} \qquad P^i \;\in\; \mathcal{A}_p^{2i(p-1)} \; .$$

The Cartan formula for the action of the Steenrod operations then propagates to an *external Cartan formula* for mod-p cohomology of symmetric spectra. We claim that for all symmetric spectra X and Y, all mod-p cohomology classes $x \in H^n(X, \mathbb{F}_p)$ and $y \in H^m(Y, \mathbb{F}_p)$ and all $i \geq 0$ we have

$$\mathrm{Sq}^i(x \times y) \;=\; \sum_{a=0}^{i} \; \mathrm{Sq}^a \, x \times \mathrm{Sq}^b \, y$$

in $H^{n+m+i}(X \wedge^L Y, \mathbb{F}_2)$ if $p = 2$, respectively

$$P^i(x \times y) \ = \ \sum_{a=0}^{i} \ P^a(x) \times P^{i-a}(y)$$

in $H^{n+m+2i(p-1)}(X \wedge^L Y, \mathbb{F}_p)$ if p is odd. Indeed, if X and Y are suspension spectra of pointed simplicial sets, then the relation reduces to the Cartan formulas for the Steenrod operations. Since elements in $\mathcal{A}_p$ are determined by their actions on the cohomology of all suspension spectra, the relation holds in general.

Spectra, as opposed to spaces or simplicial sets, have no diagonal maps; so there is no cup product in spectrum cohomology, hence also no internal Cartan formula.

The Cartan formula for the Steenrod operations is equivalent to an identification of the Hopf algebra diagonal on the classes Sq^i and P^i.

Proposition 10.19. *The diagonals on the Steenrod operations are given by*

$$\Delta(\mathrm{Sq}^i) \ = \ \sum_{a=0}^{i} \ \mathrm{Sq}^a \otimes \mathrm{Sq}^{i-a} \qquad \textit{respectively} \qquad \Delta(P^i) \ = \ \sum_{a=0}^{i} \ P^a \otimes P^{i-a} \ .$$

For all primes, the diagonal on the mod-p Bockstein operation is given by

$$\Delta(\beta) \ = \ \beta \otimes 1 + 1 \otimes \beta \ .$$

Proof. We have $\mu = 1 \times 1$ in $H^0(H \wedge H, \mathbb{F}_p)$. For $p = 2$ this yields

$$\begin{aligned} (- \times -)(\Delta(\mathrm{Sq}^i)) \ &= \ \mathrm{Sq}^i \circ \mu \ = \ \mathrm{Sq}^i \circ (1 \times 1) \\ &= \ \sum_{a=0}^{i} \ \mathrm{Sq}^a(1) \times \mathrm{Sq}^{i-a}(1) \ = \ \sum_{a=0}^{i} \ \mathrm{Sq}^a \times \mathrm{Sq}^{i-a} \end{aligned}$$

Since the exterior product map $\times : \mathcal{A}_p \otimes \mathcal{A}_p \longrightarrow H^*(H \wedge H, \mathbb{F}_p)$ is an isomorphism, this proves the formula for $\Delta(\mathrm{Sq}^i)$. The argument for $\Delta(P^i)$ is analogous.

By [...] the Bockstein morphism $\beta : H\mathbb{F}_p \longrightarrow \Sigma(H\mathbb{F}_p)$ is a derivation. So we have

$$(- \times -)(\Delta(\beta)) \ = \ \beta \circ \mu \ = \ \mu \circ (\beta \wedge H\mathbb{F}_p) \ + \ \mu \circ (H\mathbb{F}_p \wedge \beta) \ = \ \beta \times 1 \ + \ 1 \times \beta$$

Again this calculates the diagonal on β because the exterior product map is an isomorphism. □

Suppose that H is a graded Hopf algebra over a field k which is of finite type, i.e., such that H^n is finite dimensional for every degree n. Then the *dual Hopf algebra* $H^\vee$ is given in degree n by the vector space dual of H^n,

$$(H^\vee)_n \ = \ \mathrm{Hom}_k(H^n, k) \ .$$

The multiplication (respectively diagonal) of $H^\vee$ is, by definition, the map dual to the diagonal (respectively multiplication) of the original Hopf algebra H, exploiting the canonical isomorphism of the linear dual of a tensor product with the tensor product of the linear duals. [...]

The mod-p Steenrod algebra is of finite type, so we can consider the *dual Steenrod algebra* $\mathcal{A}^p = (\mathcal{A}_p)^\vee$ in the sense of the previous paragraph. We now discuss the topological interpretation of this dual Steenrod algebra as the homotopy groups of the symmetric ring spectrum $H\mathbb{F}_p \wedge H\mathbb{F}_p$ and an explicit description of the multiplication and comultiplication in the dual Steenrod algebra $\mathcal{A}^p$.

The Eilenberg-Mac Lane spectrum $H\mathbb{F}_p$ is a symmetric ring spectrum, and so the smash product $H\mathbb{F}_p \wedge H\mathbb{F}_p$ of two copies is another symmetric ring spectrum. The homotopy groups of $H\mathbb{F}_p \wedge H\mathbb{F}_p$ thus form a graded commutative ring. The homotopy groups $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$ of this ring spectrum coincide, by definition, with the homology groups $H_*(H\mathbb{F}_p, \mathbb{F}_p)$. For every symmetric spectrum X the Kronecker pairing

$$\langle -, - \rangle \ : \ H^k(X, \mathbb{F}_p) \ \otimes \ H_k(X, \mathbb{F}_p) \ \longrightarrow \ \mathbb{F}_p$$

has an adjoint map

$$H^k(X, \mathbb{F}_p) \ \longrightarrow \ \mathrm{Hom}(H_k(X, \mathbb{F}_p), \mathbb{F}_p) = H_k(X, \mathbb{F}_p)^\vee \ , \quad f \longmapsto \langle f, - \rangle \ . \tag{10.20}$$

We know from Proposition 6.28 that this adjoint is an isomorphism of $\mathbb{F}_p$-vector spaces. In the special case $X = H\mathbb{F}_p$ this becomes a map

$$\mathcal{A}_p^k \longrightarrow (\pi_k(H\mathbb{F}_p \wedge H\mathbb{F}_p))^\vee , \quad f \longmapsto \langle f, - \rangle . \tag{10.21}$$

Now we compare the comultiplication of the Steenrod algebra with the multiplication in the homotopy ring of the ring spectrum $H\mathbb{F}_p \wedge H\mathbb{F}_p$.

Proposition 10.22. *Under the isomorphism* (10.21) *given by the Kronecker pairing, the comultiplication of the Steenrod algebra $\mathcal{A}_p^*$ coincides with the multiplication in the ring $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$.*

Proof. The claim is essentially a formal consequence of the compatibility (6.29) between Kronecker pairing and exterior products. The product of two classes $x, y \in \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p) = H_*(H\mathbb{F}_p, \mathbb{F}_p)$ is the image of the extorior product $x \times y \in H_*(H\mathbb{F}_p \wedge H\mathbb{F}_p, \mathbb{F}_p)$ under the map induced by the multiplication $\mu : H\mathbb{F}_p \wedge H\mathbb{F}_p \longrightarrow H\mathbb{F}_p$ on mod-p homology. The defining property of the diagonal $\Delta(f)$ of a cohomology class $f \in H_*(H\mathbb{F}_p, \mathbb{F}_p)$ is that the exterior product map takes it to $\mu^*(f) \in H^*(H\mathbb{F}_p \wedge H\mathbb{F}_p, \mathbb{F}_p)$. In other words, if $\Delta(f) = \sum_i f'_i \otimes f''_i$ then $\mu^*(f) = \sum_i f'_i \times f''_i$. We obtain the relations

$$\begin{aligned} \langle f,\, x \cdot y \rangle \;&=\; \langle f,\, \mu_*(x \times y) \rangle \;=\; \langle \mu^*(f),\, x \times y \rangle \;=\; \sum_i \langle f'_i \times f''_i,\, x \times y \rangle \\ &=\; \sum_i (-1)^{k|f''_i|} \cdot \langle f'_i, x \rangle \cdot \langle f''_i, y \rangle \;=\; \langle \sum_i f'_i \otimes f''_i,\, x \otimes y \rangle \;=\; \langle \Delta(f),\, x \otimes y \rangle \ , \end{aligned}$$

where the second equality is naturality of the Kronecker pairing and the fourth equality is the compatibility between the exterior product and Kronecker pairing. The fifth equation is the definition of the induced pairing between two tensor copies of $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$ and of $\mathcal{A}_p$. Altogether, this equation says that the product on $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$ and the diagonal on $\mathcal{A}_p$ are dual to each other. □

Example 10.23 (Dual Steenrod algebra)**.** As we just showed the product of $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$ is dual to the comultiplication in the Steenrod algebra $\mathcal{A}_p$. We can also *define* a comultiplication on $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$ by dualizing the composition product in Steenrod algebra $\mathcal{A}_p$. In other words, for $x \in \pi_k(H\mathbb{F}_p \wedge H\mathbb{F}_p) = H_k(H\mathbb{F}_p, \mathbb{F}_p)$ we define $\Delta(x) \in H_*(H\mathbb{F}_p, \mathbb{F}_p) \otimes H_*(H\mathbb{F}_p, \mathbb{F}_p)$ as the unique class such that the relation

$$\langle f \otimes g,\, \Delta(x) \rangle \;=\; \langle f \circ g,\, x \rangle$$

holds for all homogeneous cohomology classes $f, g \in \mathcal{A}_p^*$ whose degrees add up to k. Here the pairing on the left hand side is defined by [...] With this comultiplication, the graded $\mathbb{F}_p$-algebra $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$ then becomes a commutative Hopf algebra, isomorphic, via (10.21), to the Hopf algebra dual of the Steenrod algebra $\mathcal{A}_p$.

We describe the structure of the dual Steenrod algebra $H_*(H\mathbb{F}_p, \mathbb{F}_p) = \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$ explicitly. The first space $(H\mathbb{F}_p)_1 = \mathbb{F}_p[S^1]$ of the spectrum $H\mathbb{F}_p$ is an Eilenberg-Mac Lane space of type $(\mathbb{F}_p, 1)$, and its mod-p homology gives rise to a set of multiplicative generators of $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$, and thus of the dual Steenrod algebra. Indeed, the tautological morphism of symmetric spectra

$$i_1 \;:\; \mathbb{S}^{-1} \wedge \mathbb{F}_p[S^1] \;=\; F_1\mathbb{F}_p[S^1] \;\longrightarrow\; H\mathbb{F}_p$$

(freely generated by the identity of $\mathbb{F}_p[S^1] = (H\mathbb{F}_p)_1$) induces a map of homology groups

$$\tilde{H}_{k+1}(\mathbb{F}_p[S^1], \mathbb{F}_p) \cong H_k(\mathbb{S}^{-1} \wedge \mathbb{F}_p[S^1], \mathbb{F}_p) \xrightarrow{(i_1)_*} H_k(H\mathbb{F}_p, \mathbb{F}_p) \ . \tag{10.24}$$

The mod-p homology of $\mathbb{F}_p[S^1]$ is also the mod-p group homology of the cyclic group of order p, which is one-dimensional in every non-negative degree.

For $p = 2$ and $i \geq 0$ we define an element

$$\xi_i \;\in\; H_{2^i-1}(H\mathbb{F}_2, \mathbb{F}_2) = \pi_{2^i-1}(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$$

as the image of the non-trivial element of $H_{2^i}(\mathbb{F}_2[S^1], \mathbb{F}_2)$ under the map (10.24). For odd primes p we denote by $b = \beta(\iota)$ in $H^2(\mathbb{F}_p[S^1], \mathbb{F}_p)$ the Bockstein of the fundamental class. We let $x_{2n} \in H^{2n}(\mathbb{F}_p[S^1], \mathbb{F}_p)$

and $x_{2n+1} \in H^{2n+1}(\mathbb{F}_p[S^1], \mathbb{F}_p)$ the homology classes dual to the cohomology generators b^n respectively $\iota \cdot b^n$. For $i \geq 0$ we then define

$\xi_i \in H_{2p^i-2}(H\mathbb{F}_p, \mathbb{F}_p) = \pi_{2p^i-2}(H\mathbb{F}_p \wedge H\mathbb{F}_p)$ respectively $\tau_i \in H_{2p^i-1}(H\mathbb{F}_p, \mathbb{F}_p) = \pi_{2p^i-1}(H\mathbb{F}_p \wedge H\mathbb{F}_p)$

as the image of $x_{2p^i-1} \in H_{2p^i-1}(\mathbb{F}_p[S^1], \mathbb{F}_p)$ respectively the image of $x_{2p^i} \in H_{2p^i}(\mathbb{F}_p[S^1], \mathbb{F}_p)$ under the map (10.24). We have $\xi_0 = 1$, the unit of the multiplication in the graded ring $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$.

The following calculation of the dual Steenrod algebra in Theorem 10.25 is due to Milnor [**60**].

Theorem 10.25. *The dual mod-2 Steenrod is a polynomial algebra on the classes ξ_i for $i \geq 1$:*

$$\pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2) \;=\; \mathbb{F}_2[\xi_1, \xi_2, \dots] \;.$$

The diagonal is determined by

$$\Delta(\xi_i) \;=\; \sum_{j=0}^{i} \xi_{i-j}^{2^j} \otimes \xi_j \;,$$

where $\xi_0 = 1$. For odd primes p the dual mod-p Steenrod is a tensor product of an exterior algebra on the classes τ_i for $i \geq 0$ with a polynomial algebra on the classes ξ_i for $i \geq 1$:

$$\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p) \;=\; \Lambda(\tau_0, \tau_1, \dots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \dots] \;.$$

The diagonal is determined by

$$\Delta(\tau_i) \;=\; \tau_i \otimes 1 \;+\; \sum_{j=0}^{i} \xi_{i-j}^{p^j} \otimes \tau_j \qquad \textit{and} \qquad \Delta(\xi_i) \;=\; \sum_{j=0}^{i} \xi_{i-j}^{p^j} \otimes \xi_j \;,$$

where $\xi_0 = 1$.

As in any graded connected Hopf algebra, the antipode c is determined by the rest of the data. In the case of the dual Steenrod algebra, it is determined inductively on the Milnor generators by the formula

$$c(\xi_n) \;=\; \bar{\xi}_n \;=\; \sum_{i=0}^{n-1} \xi_{n-i}^{2^i} \cdot c(\xi_i) \;.$$

For example, we have

$$c(\xi_1) = \xi_1 \;, \quad c(\xi_2) \;=\; \xi_2 + \xi_1^3 \quad \text{and} \quad c(\xi_3) \;=\; \xi_3 + \xi_2^2 \cdot c(\xi_1) + \xi_1^4 \cdot c(\xi_2) \;=\; \xi_3 + \xi_1\xi_2^2 + \xi_1^4\xi_2 + \xi_1^7 \;.$$

Under the Hopf algebra isomorphism 10.21, every homotopy class of $H\mathbb{F}_p \wedge H\mathbb{F}_p$ corresponds to a linear form on the Steenrod algebra. For the generators ξ_i and τ_i, these forms are given as follows. For $p = 2$ we have

$$\langle \mathrm{Sq}^{2^{i-1}} \mathrm{Sq}^{2^{i-2}} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1, \xi_i \rangle \;=\; 1$$

and ξ_i pairs to 0 with all other admissible monomials of degree $2^i - 1$. Since the admissible monomials form a vector space basis of the Steenrod algebra, this determines the form $\langle -, \xi_i \rangle$. Conversely $\langle \mathrm{Sq}^i, \xi_1^i \rangle = 1$ and Sq^i pairs to 0 with all other monomials of degree i in the ξ_n's. We derive these and other formulas in Exercise E.II.23. [p odd]

Since the classes ξ_i, plus the classes τ_i in the odd-primary case, are multiplicative generators of the dual Steenrod algebra and the product of $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$ is dual to the coproduct of the Steenrod algebra, this determines the pairing in general by the formula

$$\langle f,\, x \cdot y \rangle \;=\; \langle \Delta(f),\, x \otimes y \rangle \;=\; \sum (-1)^{|y||f_i'|} \langle f_i', x \rangle \cdot \langle f_i'', y \rangle \;,$$

where $\Delta(f) = \sum_i f_i' \otimes f_i''$.

Milnor's description of the dual Steenrod algebra gives rise to another basis of the Steenrod algebra, different from the Serre-Cartan basis of admissible monomials. Indeed, the *Milnor basis* is simply the basis of $\mathcal{A}_p$ dual to the monomial basis of $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$ [spell out for odd p].

For $p = 2$ the elements of the Milnor basis dual to $\xi_1^{i_1} \cdots \xi_n^{i_n}$ is denoted $\mathrm{Sq}^{i_1, \dots, i_n}$ (which must be distinguished from the product $\mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_n}$). This notation is consistent since $\mathrm{Sq}^{i,0,\dots,0} = \mathrm{Sq}^i$. The first difference between the Serre-Cartan and the Milnor basis occurs as soon as there is room for it, i.e., in dimension 3, where

$$\mathrm{Sq}^3 = \mathrm{Sq}^{3,0} \qquad \text{and} \qquad \mathrm{Sq}^2\,\mathrm{Sq}^1 = \mathrm{Sq}^{3,0} + \mathrm{Sq}^{0,1} \ .$$

Remark 10.26. Milnor's explicit description of the dual Steenrod algebra can be used to calculate the homotopy rings of some related ring spectra as well: Exercise E.II.27 is devoted to showing that the reduction morphism $\pi : H\mathbb{Z} \longrightarrow H\mathbb{F}_p$ induces isomorphisms of graded $\mathbb{F}_2$-algebras

$$\pi_*(H\mathbb{Z} \wedge H\mathbb{F}_2) \ \cong\ \mathbb{F}_2[\xi_1^2, \xi_2, \xi_3, \dots] \quad \text{and} \quad \pi_*(H\mathbb{F}_2 \wedge H\mathbb{Z}) \ \cong\ \mathbb{F}_2[\bar{\xi}_1^2, \bar{\xi}_2, \bar{\xi}_3, \dots]$$

in the 2-primary case and

$$\pi_*(H\mathbb{Z} \wedge H\mathbb{F}_p) \ \cong\ \mathbb{F}_p[\xi_i \,|\, i \geq 1] \otimes \Lambda(\tau_i \,|\, i \geq 1) \quad \text{and} \quad \pi_*(H\mathbb{F}_p \wedge H\mathbb{Z}) \ \cong\ \mathbb{F}_p[\bar{\xi}_i \,|\, i \geq 1] \otimes \Lambda(\bar{\tau}_i \,|\, i \geq 1)$$

for odd primes, where $\bar{\xi}_i = c(\xi_i)$ is the antipode of the Milnor generator ξ_i.

Exercise E.II.28 explains that the 'reduction morphism' $\pi : ko \longrightarrow H\mathbb{F}_2$ induces isomorphisms of graded $\mathbb{F}_2$-algebras

$$\pi_*(ko \wedge H\mathbb{F}_2) \ \cong\ \mathbb{F}_2[\xi_1^4, \xi_2^2, \xi_3, \xi_4 \dots] \quad \text{and} \quad \pi_*(H\mathbb{F}_2 \wedge ko) \ \cong\ \mathbb{F}_2[\bar{\xi}_1^4, \bar{\xi}_2^2, \bar{\xi}_3, \bar{\xi}_2 \dots] \ .$$

Example 10.27. For odd primes, the classes τ_i in the homotopy ring $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$ satisfy the relation $\tau_i^2 = 0$, so Toda brackets of exterior generators are defined [ref]. For $p = 3$ we have

$$\langle \tau_0, \tau_0, \tau_0 \rangle \ = \ \{\xi_1\}$$

in $\mathcal{A}_4$. [show ! Do we have $\xi_{i+1} \in \langle \tau_i, \tau_i, \tau_i \rangle$?] For $p \geq 5$, the Toda bracket $\langle \tau_i, \tau_i, \tau_i \rangle$ consists only of 0 for dimensional reasons [check]. However, there are non-trivial higher Toda brackets (which we have not defined) such as $\xi_1 \in \langle \tau_0, \tau_0, \dots, \tau_0 \rangle$ (p-fold bracket).

Remark 10.28. We can consider a 'Steenrod algebra' $\mathcal{A}(A)$ for every abelian group A, generalizing the mod-p Steenrod algebra as $\mathcal{A}_p = \mathcal{A}(\mathbb{F}_p)$. We can define $\mathcal{A}(A)$ either as the graded ring of stable cohomology operations with coefficients in A, or equivalently as the graded endomorphism ring $\mathcal{SHC}(HA, HA)^*$ of the Eilenberg-Mac Lane spectrum in the stable homotopy category. If A is a ring, then sending an element $a \in A$ to the map $\lambda_a : A \longrightarrow A$ given by left multiplication by a gives a ring homomorphism

$$A \ \longrightarrow\ \mathrm{Hom}(A, A) \ \cong\ \mathcal{A}(A)^0 \ .$$

If A is commutative, then the image of λ is central in the graded ring $\mathcal{A}(A)^*$ so in this case $\mathcal{A}(A)^*$ becomes an A-algebra.

If A is a field, then cohomology with coefficients in A admits a Künneth isomorphism, so we can define a diagonal on $\mathcal{A}(A)^*$ just as in the special case $\mathbb{F}_p$ [finite type?], making it into a graded Hopf algebra. For fields of characteristic 0, this structure is rather boring: if A is any uniquely divisible abelian group, then there are no nontrivial morphisms from HA to itself of non-zero degree. So in that case $\mathcal{A}(A)^n$ is trivial for $n \neq 0$, and $\mathcal{A}(A)^0 = \mathrm{Hom}(A, A)$.

For fields k of positive characteristic p, the structure of the 'Steenrod algebra' $\mathcal{A}(k)$ is an extension of the mod-p Steenrod algebra. [spell out]

Since the components of a stable cohomology operation are always additive (part (b) of Exercise E.II.19), the reduced cohomology $\tilde{H}^*(X, A)$ of a pointed simplicial set X with coefficients in an abelian group A is tautologically a graded left module over the Steenrod-algebra $\mathcal{A}(A)^*$ via

$$\tau \cdot x \ = \ \tau_i(x) \ \in\ \widetilde{H}^{n+i}(X; A)$$

for $\tau = \{\tau_i\}_{i \geq 0} \in A(A)^n$ and $x \in \widetilde{H}^i(X; A)$. So cohomology with coefficients in A can be viewed as a functor

$$\widetilde{H}^*(-; A) \ : \ \mathrm{Ho}_*(\mathbf{sS}) \ \longrightarrow\ \mathcal{A}(A)^*\text{-mod} \ .$$

Moreover, the suspension isomorphism

$$\Sigma \ : \ \widetilde{H}^*(X; A)[1] \ \longrightarrow\ \widetilde{H}^*(\Sigma X; A)$$

is an isomorphism of graded $\mathcal{A}(A)^*$-modules, by the compatibility condition in the definition of a stable cohomology operation. Here the square brackets [1] denote the shift of a graded module.

The algebra $\mathcal{A}(A)$ also acts on the spectrum cohomology with coefficients in A by graded composition. For a based simplicial set, the suspension isomorphism

$$\widetilde{H}^*(X;A) \longrightarrow H^*(\Sigma^\infty X;A)$$

is an isomorphism of graded $\mathcal{A}(A)^*$-modules,

10.3. The Adams spectral sequence. We conclude this section with a brief discussion with the Adams spectral sequence based on mod-p homology. This spectral sequence is a very useful tool for the calculation of the homotopy groups of many spectra, as we indicate below.

Construction 10.29. Let p be a prime number and X a symmetric spectrum. For ease of notation we abbreviate the Eilenberg-Mac Mane spectrum $H\mathbb{F}_p$ to H for the course of this construction. Since $H = H\mathbb{F}_p$ is a symmetric ring spectrum, the functor $H \wedge -$ is a triple on the category of symmetric spectra. Like any triple, it gives rise to an augmented cosimplicial object $H^\bullet X$ for every symmetric spectrum X.

In more detail, we define the symmetric spectrum of n-cosimplices by

$$H^{(n)}X \;=\; \underbrace{H \wedge \ldots \wedge H}_{n+1} \wedge X \ .$$

For $i = 0, \ldots, n$ the coface morphism $d^i : H^{(n-1)}X \longrightarrow H^{(n)}X$ is given by inserting the unit morphism $\iota : \mathbb{S} \longrightarrow H$ into the ith slot (counting from $i = 0$). For $i = 0, \ldots, n$ the codegeneracy morphism s^i : $H^{(n+1)}X \longrightarrow H^{(n)}X$ is given by using multiplication $\mu : H \wedge H \longrightarrow H$ on the ith and $(i+1)$st factor.

The cosimplicial symmetric spectrum gives rise to a spectral sequence

$$E_1^{s,t} \;=\; \pi_{t-s}(H^{(s+1)}X) \implies \pi_{t-s}\,\mathrm{Tot}\,H^\bullet X \ .$$

which converges (conditionally ?) to the homotopy groups of the totalization of the cosimplicial spectrum [ref]. We will now identify the E^2-term of this spectral sequence as a homological invariant of the mod-p homology of X and investigate the relationship between the abutment and the homotopy groups of X.

Proposition 10.30. (i) *The E_2-term of the spectral sequence is naturally isomorphic to the* Ext*-groups of comodules over the dual Steenrod algebra, from $\mathbb{F}_p$ to the t-fold shift of the mod-p homology of X,*

$$E_2^{s,t} \;\cong\; \mathrm{Ext}^s_{\mathcal{A}^p\text{-comod}}(\mathbb{F}_p, H_*(X,\mathbb{F}_p)[t]) \ .$$

(ii) *Under some assumptions on the symmetric spectrum X, the natural map $X \longrightarrow \mathrm{Tot}(H^\bullet X)$ is a p-adic completion. In this case the Adams spectral sequence converges [how?] to the homotopy groups of the p-completion $X_p^\wedge$ of X.*

Proof. (i) By an iterated application of the Künnth theorem (Proposition 6.25) the group $\pi_{t-s}(H^{(s+1)}X) = H_{t-s}(H \wedge \ldots \wedge H \wedge X, \mathbb{F}_p)$ is naturally isomorphic to the tensor product

$$\mathcal{A}^p \otimes \cdots \otimes \mathcal{A}^p \otimes H_*(X,\mathbb{F}_p)$$

(there are s factors of H respectively of the dual Steenrod algebra), via the exterior product map.[generalizes m] So the bigraded abelian group $E_1^{s,t}$ is isomorphic to the cobar complex of the $\mathcal{A}$-comodule $H_*(X,\mathbb{F}_p)$, and the d^1-differential corresponds to the cobar differential. For every comodule M, the homology of the cobar complex $\mathcal{C}(\mathcal{A},M)$ calculates the Ext-groups $\mathrm{Ext}^s_{\mathcal{A}}(\mathbb{F}_p, M[t])$, which proves (i). □

Definition 10.31. Let p be prime and X a (connective, finite type ?) symmetric spectrum. By Proposition 10.30 the cosimplicial spectrum $H^\bullet X$ gives rise to a spectral sequence (how convergent?)

$$E_2^{s,t} \;=\; \mathrm{Ext}^{s,t}_{\mathcal{A}^p\text{-comod}}(\mathbb{F}_p, H_*(X,\mathbb{F}_p)) \implies \pi_{t-s}\left(X_p^\wedge\right)$$

which is called the *Adams spectral sequence* for the spectrum X.

[Describe the filtration on homotopy groups. If R is a semistable symmetric ring spectrum then the Adams spectral sequence is a spectral sequence of algebras]

Example 10.32 (Primitives in the dual Steenrod algebra)**.** For any Hopf algebra H (only coaugmented coalgebra?) over a field k and any comodule M over H, the first Ext-group $\mathrm{Ext}^1_{H\text{-comod}}(k,k)$ from the ground field to itself is isomorphic to the k-vector space

$$\mathrm{Prim}(H) \;=\; \{x \in H \,|\, \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

of primitive elements in H. This group is canonically isomorphic to the dual vector space of the indecomposables $Q(H^\vee)$ of the dual algebra $H^\vee$. Indeed, a natural k-linear map

$$\mathrm{Prim}(H^\vee) \;\cong\; Q(H)^\vee$$

is defined as follows. A linear form $\psi : H \longrightarrow k$, can be restricted to the augmentation ideal $I = \ker(\epsilon : H \longrightarrow k)$. If ψ is primitive as an element of the dual Hopf algebra $H^\vee$, then for all pairs of elements $x, y \in I$ in the augmentation ideal we have

$$\langle \psi, xy\rangle \;=\; \langle \Delta(\psi), x \otimes y\rangle \;=\; \langle \psi \otimes 1 + 1 \otimes \psi, x \otimes y\rangle \;=\; \langle \psi, x\rangle \cdot \langle 1, y\rangle \;+\; \langle 1, x\rangle \cdot \langle \psi, y\rangle \;=\; 0\;.$$

So ψ vanishes on I^2 and factors over a unique linear map $Q(H) = I/I^2 \longrightarrow k$. Conversely, every linear form on I/I^2 can be extended to a linear map on H by taking 1 to 0; the fact that the original form vanishes on I^2 then implies that the extension is a primitive element of H. In other words, x is primitive in the dual coalgebra if and only if its restriction to the square of the augmentation ideal of H vanishes. Hence the map

We conclude that to calculate the 1-line of the Adams spectral sequence, we should determine the primitive elements in the dual Steenrod algebra, or, equivalently, the vector space of indecomposables in the Steenrod algebra:

Proposition 10.33. *The graded vector space $Q\mathcal{A}_2$ of indecomposables in the mod-2 Steenrod algebra is trivial in dimensions not a power of 2, and $(Q\mathcal{A}_2)^{2^i}$ is 1-dimensional, generated by the image of Sq^{2^i} for all $i \geq 0$.*

The graded vector space $\mathrm{Prim}(\mathcal{A}^2)$ of primitive elements in the dual mod-2 Steenrod algebra is trivial in dimensions not a power of 2, and 1-dimensional in dimension 2^i, generated by $\xi_1^{2^i}$, for all $i \geq 0$.

For an odd prime p, the graded vector space $Q\mathcal{A}_p$ of indecomposables in the mod-p Steenrod algebra is trivial in dimensions different from 1 and $2p^i(p-1)$. In the remaining dimensions, the indecomposable are 1-dimensional generated by the images of β respectively P^{p^i} for $i \geq 0$.

The graded vector space $\mathrm{Prim}(\mathcal{A}^p)$ of primitive elements in the dual mod-p Steenrod algebra is trivial in dimensions different from 1 and $2p^i(p-1)$, and 1-dimensional in the other dimensions, generated by

$$\tau_0 \;\in\; \mathrm{Prim}(\mathcal{A}^p)_1 \quad \textit{and} \quad \xi_1^{p^i} \;\in\; \mathrm{Prim}(\mathcal{A}^p)_{2p^i(p-1)}\;.$$

Proof. Let us start to determine the indecomposables of the mod-p Steenrod algebra. Since every element in $\mathcal{A}_2$ (respectively $\mathcal{A}_p$ for p odd) is a sum of products of Steenrod operations (respectively of the Bockstein β and the operations P^i), we only have to find out which of the operations Sq^i (respectively β and P^i) are decomposable.

We start with the case $p = 2$. Let n be a positive integer which is not a power of 2. So we can write $n = 2^i(2k+1)$ with $i \geq 0$ and $k \geq 1$. We have the Adem relation

$$\mathrm{Sq}^{2^i}\,\mathrm{Sq}^{n-2^i} \;=\; \sum_{j=0}^{2^{i-1}} \binom{n-2^i-j-1}{2^i}\,\mathrm{Sq}^{n-j}\,\mathrm{Sq}^j\;.$$

The summand indexed by $j = 0$ contributes the term $\binom{n-2^i-1}{2^i}\mathrm{Sq}^n$. We claim that the binomial coefficient $\binom{n-2^i-1}{2^i} = \binom{2^{i+1}k-1}{2^i}$ is odd. This implies that

$$\mathrm{Sq}^n \;=\; \mathrm{Sq}^{2^i}\,\mathrm{Sq}^{n-2^i} \;+\; \sum_{j=1}^{2^{i-1}} \binom{n-2^i-j-1}{2^i}\,\mathrm{Sq}^{n-j}\,\mathrm{Sq}^j\;,$$

so we conclude that Sq^n is decomposable in the mod-2 Steenrod algebra if n is not a power of 2. To evaluate the binomial coefficient we use that $\binom{2^{i+1}k-1}{2^i}$ is the coefficient of t^{2^i} in the polynomial $(1+t)^{2^{i+1}k-1}$. In characteristic 2, that polynomial evaluates to

$$(1+t)^{2^{i+1}k-1} \;=\; ((1+t)^{2^{i+1}})^k \cdot (1+t)^{-1} \;=\; (1+t^{2^{i+1}})^k \cdot (1+t+t^2+\cdots) \;.$$

Since $(1+t^{2^{i+1}})^k$ is congruent to 1 modulo t^{2^i+1}, the coefficient of t^{2^i} in $(1+t)^{2^{i+1}k-1}$ is indeed congruent to 1 mod 2. Hence we have shown that the decomposables $Q\mathcal{A}_2$ are generated by the residue classes of the operations Sq^{2^i}; in particular, the indecompsables of $\mathcal{A}_2$ are trivial in all dimensions not a power of 2. Since the indecomposables of $\mathcal{A}_2$ are vector space dual to the primitives in $\mathcal{A}^2$, this also shows that the primitives vanish in dimensions which are not a power of 2.

[p odd]

We still need to show that the elements Sq^{2^i} respectively β and P^{p^i} are not decomposable. If any of these were decomposable, then the the indecomposable of $\mathcal{A}_p$ would be entirely trivial in the respective dimension, and hence the primitives in $\mathcal{A}^p$ would be trivial in the same dimension. So it suffices to find non-trivial primitive class in dimensions 2^i when $p=2$ and in dimensions 1 and $2p^i(p-1)$ when p is odd.

Evidently, the class $\xi_1 \in (\mathcal{A}^2)_1$ is primitive when $p=2$ and $\tau_0 \in (\mathcal{A}^p)_1$ and $\xi_1 \in (\mathcal{A}^p)_{2(p-1)}$ are primitive. Moreover, if x is primitive, then in characteristic p we have

$$\Delta(x^p) \;=\; \Delta(x)^p \;=\; (x\otimes 1 + 1\otimes x)^p \;=\; x^p\otimes 1 + 1\otimes x^p \;.$$

So in characteristic p, the p-th power of a primitive class is again primitive. So $\xi_1^{p^i}$ is primitive for all $i \geq 1$. So we have found all primitive elements, and thus also calculated the indecomposables in $\mathcal{A}_p$. $\square$

Remark 10.34 (Hopf invariant one problem)**.** The degree 2 map of S^1 and the Hopf maps $\eta : S^3 \longrightarrow S^2$, $\nu : S^7 \longrightarrow S^4$ and $\sigma : S^{15} \longrightarrow S^8$ have the property that the cohomology of their respective mapping cones is a truncated polynomial algebra, compare Example 10.11. Now we can show that if a map $f : S^{2n-1} \longrightarrow S^n$ has a mapping cone with truncated polynomial cohomology, then n must be a power of 2.

Indeed, let $Cf = S^n \cup_f D^{2n}$ denote the mapping cone of f. We may suppose $n \geq 2$, and then the integral cohomology of Cf is free abelian of rank 1 in dimensions 0, n and $2n$, and trivial in all other dimensions. We let $x_n \in H^n(Cf,\mathbb{Z})$ a generator of this infinite cyclic group and assume that $x^2 \in H^{2n}(Cf,\mathbb{Z})$ a also a generator. By the universal coefficient theorem, the mod-2 cohomology is 1-dimensional in dimensions 0, n and $2n$, and trivial in all other dimensions. Moreover, the reduction of x generates $H^n(Cf,\mathbb{F}_2)$ and its square generates $H^{2n}(Cf,\mathbb{F}_2)$. So we have

$$\mathrm{Sq}^n(x) \;=\; x^2 \;\neq 0 \;.$$

However, if n is not a power of 2, then the operation Sq^n is decomposable. Since the mod-2 cohomology groups of $C(x)$ are all trivial in dimensions $n+1$ through $2n-1$, we must have $\mathrm{Sq}^n(x)=0$; we reached a contradiction, so n must be power of 2.

The question for which i there is a map $f : S^{2^{i+1}-1} \longrightarrow S^{2^i}$ whose mapping cone Cf has truncated polynomial mod-2 cohomology was known as the *Hopf invariant one problem*. The answer, first obtained by Adams [...], is that $i = 0,1,2$ and 3 are the only possible values, and the the odd multiples of the maps 2 and the three Hopf maps are the only maps which qualify.

Now we have determined the primitive elements in the dual mod-p Steenrod algebra, and hence we can describe the 1-line of the Adams spectral sequence for the sphere spectrum. For $p=2$ we denote by

$$h_i = [\xi_1^{2^i}] \in \mathrm{Ext}^{1,2^i}(\mathbb{F}_2,\mathbb{F}_2)$$

the Ext class corresponding to the primitive element $\xi_1^{2^i}$. For odd primes p we denote by

$$a = [\tau_0] \in \mathrm{Ext}^{1,1} \quad \text{and} \quad h_{i+1} = [\xi_1^{p^i}] \in \mathrm{Ext}^{1,p^iq} \;.$$

the Ext class corresponding to the primitive elements τ_0 and $\xi_1^{p^i}$.

$$\mathrm{Ext}^{0,0}_{\mathcal{A}}(\mathbb{F}_p,\mathbb{F}_p) \;=\; \mathbb{F}_p$$

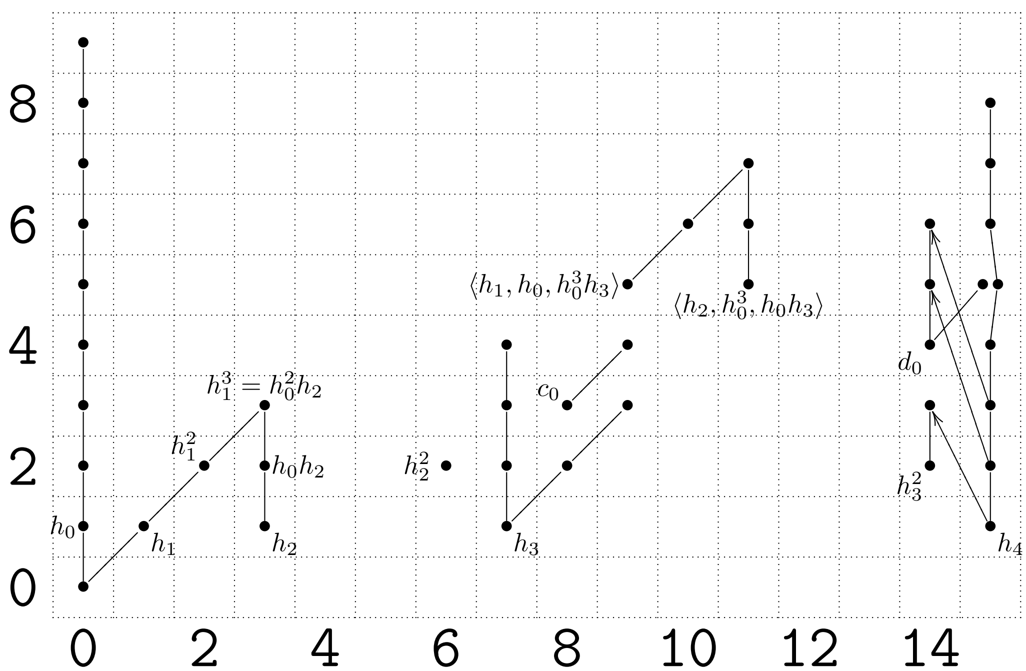
and $\mathrm{Ext}^{0,t}_{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p) = 0$ for $t \neq 0$.

$$\mathrm{Ext}^{1,*}_{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p) \;=\; \mathrm{Prim}(\mathcal{A}^p) \;=\; \begin{cases} \mathbb{F}_2\{\xi_1^{2^i} \mid i \geq 1\} & p = 2 \\ \mathbb{F}_p\{\tau_0, \xi_1^{p^i} \mid i \geq 1\} & p > 2. \end{cases}$$

Example 10.35. We discuss the Adams spectral sequence for the sphere spectrum

$$E_2^{s,t} \;=\; \mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p) \;\Longrightarrow\; (\pi_{t-s}\mathbb{S})^\wedge_p$$

and justify the table given in Example 1.11 of Chapter I for the homotopy groups spheres up to dimension 8. We start with the prime $p = 2$, where the following chart displays the relevant portion of the spectral sequence.



We explain how to interpret this chart (and similar pictures later on) and how to derived conclusions about the stable homotopy groups of spheres from it.

It is customary to draw Adams spectral sequences in *Adams indexing* which means that the homological degree, usually denoted s, is drawn vertically, and the difference $t - s$ of the degrees is drawn vertically. The number $t - s$ is referred to as the *topological degree.* So the slot with coordinated (p, q) represents the group $\mathrm{Ext}^{p,p+q}$. This way of drawing an Adams spectral sequence makes it easy to visualize which groups contribute to the nth stable stem, since these all lie on the vertical line of topological degree $t - s = n$. We have chosen to display this particular portion of the spectral sequence because two phenomena occur for the first time in topological degree $t - s = 15$: in bidegree $(15, 5)$ sits the first Ext group whose dimension is larger than 1 and the first non-zero differentials originate in topological degree $t - s = 15$, namely $d_2(h_4) = h_0 h_3^2$, compare the discussion below.

We have calculated the Ext groups in homological degrees 0 and 1. Clearly $E_2^{0,*} = \mathrm{Ext}^{0,*}(\mathbb{F}_2, \mathbb{F}_2)$ is one-dimensional with basis $1 \in E_2^{0,0}$. The graded group $E_2^{1,*} = \mathrm{Ext}^{1,*}(\mathbb{F}_2, \mathbb{F}_2)$ is isomorphic to the primitives in the dual Steenrod algebra, hence one-dimensional in dimensions a power of 2 and trivial otherwise (see Proposition 10.33). The standard notation is

$$h_i = [\xi_1^{2^i}] \in \mathrm{Ext}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$$

for the Ext class corresponding to the primitive element $\xi_1^{2^i}$.

Moreover $E_2^{s,s}$ is one-dimensional, generated by h_0^s, for all $s \geq 0$, which means that the vertical line with $t-s=0$ continues indefinitely upwards [justify]; we only labeled a few of these classes in the chart. The vertical lines represent multiplication by h_0 and the diagonal lines represent multiplication by h_1. So for example, the generator in bidegree $(t-s,s)=(3,3)$ is $h_1^3=h_0^2h_2$ and the generator in bidegree $(7,4)$ is $h_0^3h_3$. There are two non-trivial h_2-multiplications in this range (namely on 1 and h_2) which are not represented graphically. We have labeled all generators in the lower left part of the chart, and in the rest the multiplicative structure yields basis elements.

Remark 10.36. We collect some more facts about the mod-2 Adams spectral sequence for the sphere spectrum, mostly without proof. In the 2-line of the Adams spectral sequence the relation $h_{i+1}h_i=0$ holds for all $i \geq 0$: the class $h_{i+1}h_i$ is represented in the cobar complex by $[\xi_1^{2^{i+1}}|\xi_1^{2^i}]$, which is the coboundary of $[\xi_2^{2^i}]$. This relation and the commutativity is all that happens on the 2-line, so the classes

$$h_ih_j \qquad \text{for } i=j \text{ or } i \geq j+2$$

form a basis of $\mathrm{Ext}^{2,*}$.

The stable classes of the Hopf maps 2ι, η, ν and σ are all trivial in mod-2 homology, so they have Adams filtration 1. As we discussed in Example 10.11, these classes are detected by the Steenrod operations $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4$ respectively Sq^8; dually this means that their images under the map

$$F^1\pi_n^s \longrightarrow \mathrm{Ext}^{1,n+1}$$

are non-zero. In other words, the classes h_0, h_1, h_2 and h_3 are permanent cycles in the spectral sequence and detect 2ι, η, ν and σ respectively. The other classes on the 1-line support a non-trivial differential

$$d_2(h_{i+1}) = h_0h_i^2$$

often referred to as the *Adams differential*. (The formula holds for all $i \geq 1$, but both sides are zero for $i<3$.) This differential is a biproduct of Adams' non-existence theorem for maps of Hopf invariant one [**3**], which he solved by exhibiting an explicit decomposition of the Steenrod operation Sq^{2^i} in terms of secondary cohomology operations. The first of the non-zero Adams differentials can also be deduced from the fact that $2\sigma^2=0$ in the 14-stem (since the product of the stable homotopy groups is graded commutative), whereas the class $h_0h_3^2$ which detects $2\sigma^2$ is non-zero in $\mathrm{Ext}^{3,17}$. This can only happen if $h_0h_3^2$ is the image of some differential, but there is nothing except the class h_4 which could hit $h_0h_3^2$. So we must have $d_2(h_4)=h_0h_3^2$.

The classes h_i^2 are called *Kervaire invariant classes* because h_i^2 is a permanent cycle in the Adams spectral sequence if and only if there exists a framed (2^i-2)-manifold with Kervaire invariant 1. Then

$$\Theta_i \in \pi^s_{2^{i+1}-2}$$

denotes any stable homotopy element of filtration 2 which is detected by h_i^2. We have

$$\Theta_1=\eta^2 , \quad \Theta_2=\nu^2 , \quad \Theta_3=\sigma^2 , \quad \Theta_4=\langle 2,\Theta_3,2,\Theta_3\rangle ;$$

the class Θ_5 exists, but is more difficult to describe [**4**]. Whether the remaining classes Θ_i exist was an open question for a long time, known as the *Kervaire invariant problem*. For $i \geq 7$ this question was recently resolved in the negative by Hill, Hopkins and Ravenel [**34**], i.e., h_i^2 is not a permanent cycle for $i \geq 7$. The status of Θ_6 is still unknown at present.

The 3-line of the mod-2 Adams spectral sequence is also completely known. After dividing out by commutativity and the relations

$$h_ih_{i+1}=0 , \quad h_ih_{i+2}^2=0 , \quad \text{and} \quad h_i^2h_{i+2}=h_{i+1}^3$$

(see Exercise E.II.31), the triple products $h_ih_jh_k$ for $i,j,k \geq 0$ become linearly independent in the 3-line. The 3-line has another family of classes which are multiplicatively indecomposable. There are non-zero classes

$$c_i = \langle h_{i+1},h_i,h_{i+2}^2\rangle \in \mathrm{Ext}^{3,\,11\cdot 2^i}$$

for $i \geq 0$ which together with triple products of h_i's generate the entire 3-line. Bruner [**16**] showed that the classes $h_2h_j^2$ are permanent cycles for $j \geq 4$ [??]. The class c_0 is a permanent cycle and detects ε, the unique non-trivial element of filtration 3 in the 8-stem. The class c_1 (outside of our chart) is also a permanent cycle, but for $i \geq 2$, the class c_i supports a non-trivial d_2-differential. There is one more multiplicative generator on the chart which we have not yet introduced, namely

$$d_0 = \langle h_0, h_1, h_2, c_0 \rangle \in \mathrm{Ext}^{4,18}$$

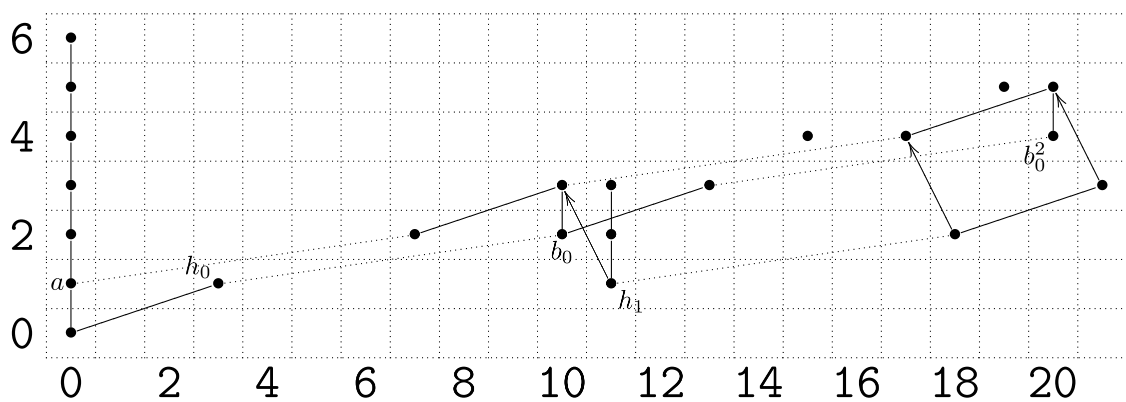
which is the first of an infinite family of classes d_i in the 4-line. This class d_0 is a permanent cycle and detects a homotopy class of filtration 4 in the 14-stem usually denoted κ.

Using 4-fold Toda brackets, the class κ can be decomposed as

$$\kappa \in \langle 2, \eta, \nu, \epsilon \rangle \cap \langle \nu, \epsilon, 2, \eta \rangle \cap \langle \nu, 2\nu, \nu, 2\nu \rangle .$$

Some more systematic information is available for the 4-line and above, but we'll stop our survey of the 2-primary Adams spectral sequence here. [vanishing line; Adams periodicity]

Now we take a brief look at the mod-p Adams spectral sequence for the stable stemes at an odd prime p. Here is a chart for $p = 3$ up to topological degree $t - s = 21$:



The graded group $E_2^{1,*} = \mathrm{Ext}^{1,*}(\mathbb{F}_p, \mathbb{F}_p)$ is isomorphic to the primitives in the dual Steenrod algebra, hence one-dimensional in dimensions 1 and $2p^i(p-1)$ and trivial otherwise (see Proposition 10.33). We write

$$a = [\tau_0] \in \mathrm{Ext}^{1,1} \quad \text{and} \quad h_i = [\xi_1^{p^i}] \in \mathrm{Ext}^{1,p^i q} .$$

for the Ext classes corresponding to the primitive elements τ_0 and $\xi_1^{p^i}$. The vertical lines represent multiplication by a and the diagonal lines of slope 1/2 represent multiplication by h_0. The dashed lines of slope 1/3 represent the Massey product operation $\langle h_0, h_0, - \rangle$.

Besides certain products [which are these, besides power of a?], the 2-line contains a non-trivial class $b_i \in \mathrm{Ext}^{2,qp^{i+1}}$ for each $i \geq 1$ which is represented in the cobar complex by

$$\sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} [\xi_1^{p^i k} | \xi_1^{p^i (p-k)}] .$$

As the picture for $p = 3$ indicates, the classes a, h_0 and b_0 are permanent cycles since there are no possible targets for differentials. They detect homotopy classes denoted $3\iota \in \pi_0^s$, $\alpha_1 \in \pi_{2p-3}^s$ and $\beta_1 \in \pi_{2p(p-1)-2}^s$ respectively. The classes α_1 and β_1 are the first in infinite families of classes in the p-local stable stems, called the *α-family* respectively the *β-family*.

The Ext class b_i admits a decomposition as a 'long' Massey product (which we have not defined), namely as the p-fold bracket

$$b_i \in \langle h_i, \ldots, h_i \rangle .$$

The homotopy class β_1 has a corresponding decomposition as a p-fold Toda bracket

$$\beta_1 = \langle \alpha_1, \ldots, \alpha_1 \rangle .$$

The first systematic families of differentials are the *Adams differential*

$$d_2(h_i) = h_0 b_{i-1}$$

for $i \geq 1$ and the *Toda differential*

$$d_{2p-1}(b_{i+1}) = h_1 b_i^p \ ,$$

which imply various other differentials by the derivation property. [vanishing line]

Exercises

Exercise E.II.1. Let us denote by $\gamma^{\mathrm{ss}} : \mathcal{S}p^{\mathrm{ss}} \longrightarrow \mathcal{SHC}$ the restriction of the localization functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ of Theorem 1.6 to the full subcategory of semistable symmetric spectra of simplicial sets. Show that γ^{ss} is a localization of the category of semistable symmetric spectra at the class of $\hat{\pi}_*$-isomorphism. Do the same for the pairs (Ω-spectra, level equivalences) and (injective Ω-spectra, homotopy equivalences).

Exercise E.II.2. Let K be a based simplicial set whose reduced integral homology is concentrated in a single dimension, where it is free abelian of rank 1. The *degree* of a based self map $\tau : K \longrightarrow K$ is the unique integer $\deg(\tau)$ such that τ induces multiplication by $\deg(\tau)$ on reduced integral homology.

Show that for every injective Ω-spectrum X the induced morphism $\tau^* : X^K \longrightarrow X^K$ equals $\deg(\tau) \cdot \mathrm{Id}$ in the group $[X^K, X^K]$. Show that for every symmetric spectrum A of simplicial sets, the image in the group $\mathcal{SHC}(K \wedge A, K \wedge A)$ of the self map $\tau \wedge \mathrm{Id}$ of $K \wedge A$ under the localization functor $\gamma : \mathcal{S}p \longrightarrow \mathcal{SHC}$ is multiplication by the degree of τ.

Exercise E.II.3. Let $a : X \longrightarrow Y$ be a morphism in the stable homotopy category. Construct a flat symmetric spectrum Z, a stable equivalence $f : Z \longrightarrow X$ and a morphism of symmetric spectra $g : Z \longrightarrow Y$ such that

$$\gamma(g) \circ \gamma(f)^{-1} \ = \ a \ .$$

Exercise E.II.4. Let $\mathcal{T}$ be a triangulated category. We call a triangle (f, g, h) in $\mathcal{T}$ *anti-distinguished* if the triangle $(-f, -g, -h)$ is distinguished in the original triangulation of $\mathcal{T}$. Show that the class of anti-distingusihed triangles is also a triangulation of $\mathcal{T}$ (with respect to the same suspension functor).

Exercise E.II.5. Let $\mathcal{T}$ be a triangulated category and $\Sigma^{-1} : \mathcal{T} \longrightarrow \mathcal{T}$ a quasi-inverse to the suspension functor, i.e., a functor endowed with a natural isomorphism $\psi_A : A \cong \Sigma(\Sigma^{-1}A)$. We call a triangle

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \xrightarrow{\ h\ } \Sigma^{-1}A$$

in the opposite category $\mathcal{T}^{\mathrm{op}}$ *op-distinguished* if the triangle

$$\Sigma^{-1}A \xrightarrow{\ h\ } C \xrightarrow{\ g\ } B \xrightarrow{\ \psi_A \circ f\ } \Sigma(\Sigma^{-1}A)$$

is distinguished in the original triangulation of $\mathcal{T}$. Show that the opposite category $\mathcal{T}^{\mathrm{op}}$ is a triangulated category with respect to the functor $\Sigma^{-1} : \mathcal{T}^{\mathrm{op}} \longrightarrow \mathcal{T}^{\mathrm{op}}$ as suspension functor and the class of op-distinguished triangles.

Exercise E.II.6. Let $\mathcal{T}$ be a triangulated category and $f_n : X_n \longrightarrow X_{n+1}$ a sequence of composable morphism for $n \geq 0$. Let $(\bar{X}, \varphi_n)$ and $(\bar{X}', \varphi'_n)$ be two homotopy colimits of the sequence (X_n, f_n). Construct an isomorphism $\psi : \bar{X} \longrightarrow \bar{X}'$ satisfying $\psi\varphi_n = \varphi'_n$ and commuting with the connecting morphisms to the suspension of $\bigoplus_{n \geq 0} X_n$. To what is extent it the isomorphism ψ unique?

Exercise E.II.7. Let $\mathcal{T}$ be a triangulated category with countable sums. Let X be any object of $\mathcal{T}$ and $e : X \longrightarrow X$ an idempotent endomorphism. Show that e splits in the following sense: there are objects eX and $(1-e)X$ and an isomorphism between X and the sum $eX \oplus (1-e)X$ under which $e : X \longrightarrow X$ corresponds to the endomorphism $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ of $eX \oplus (1-e)X$. (Hint: use that homotopy colimits exist in $\mathcal{T}$ and construct eX as the homotopy colimit of the sequence of e's).

Exercise E.II.8. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms in a triangulated category $\mathcal{T}$ such that the composite $gf : X \longrightarrow Z$ zero and the group $[\Sigma X, Z]$ is trivial. Show that there is at most one morphism $h : Z \longrightarrow \Sigma X$ such that (f, g, h) is a distinguished triangle.

Exercise E.II.9. In Proposition 3.5 we constructed the symmetric monoidal derived smash product on the stable homotopy category of flat symmetric spectra and proved some properties. In this exercise we show that these properties uniquely determine the derived smash product and the coherence isomorphisms on $\mathcal{SHC}^\flat$.

(i) Show that there is only one functor $\wedge^L : \mathcal{SHC}^\flat \times \mathcal{SHC}^\flat \longrightarrow \mathcal{SHC}^\flat$ which satisfies the equality $\wedge^L \circ (\gamma^\flat \times \gamma^\flat) = \gamma^\flat \circ \wedge$ as functors $\mathcal{S}p^\flat \times \mathcal{S}p^\flat \longrightarrow \mathcal{SHC}^\flat$.

(ii) Show that there is only one way to define unit, associativity and symmetry isomorphisms for $\wedge^L$ on $\mathcal{SHC}^\flat$ if we want the functor $\gamma^\flat : \mathcal{S}p^\flat \longrightarrow \mathcal{SHC}^\flat$ to be strong symmetric monoidal with respect to the identity transformation.

Exercise E.II.10. By Proposition 5.21 a symmetric spectrum X is connective if and only if X belongs to $\langle \mathbb{S} \rangle_+$, the smallest class of objects of the stable homotopy category which contains the sphere spectrum $\mathbb{S}$ and is closed under sums (possibly infinite) and extensions to the right. Show that another equivalent condition is that X stably equivalent to a symmetric spectrum of the form $A(\mathbb{S})$ for a $\mathbf{\Gamma}$-space A.

Exercise E.II.11. Given a central group extension

$$0 \longrightarrow B \longrightarrow E \longrightarrow G \longrightarrow 1$$

with G and B abelian, we define a cohomology operation

$$H^1(X; G) \longrightarrow H^2(X; B)$$

generalizing the Bockstein homomorphism for abelian extensions. Suppose $f : X_1 \longrightarrow G$ is a 1-cocycle, choose a lift $\bar{f} : X_1 \longrightarrow E$. Show that for every $x \in X_2$ the expression

$$(\delta \bar{f})(x) = \bar{f}(d_0 x) \cdot \bar{f}(d_1 x)^{-1} \cdot \bar{f}(d_2 x)$$

is contained in the subgroup B of E, and that it defines a 2-cocycle of X with values in B. Then show that the cohomology class of $\delta \bar{f}$ is independent of the choice of lift, and of the choice of cocycle f within its cohomology class.

Exercise E.II.12. Show that the Spanier-Whitehead category has a symmetric monoidal smash product which is defined by $(X, n) \wedge (Y, m) = (X \wedge Y, n + m)$ on objects and with unit object $(S^0, 0)$. Make the embedding $\underline{\Sigma}^\infty$ of the Spanier-Whitehead category into $\mathcal{SHC}$ compatible with smash products, i.e., make it into a strong symmetric monoidal functor.

Exercise E.II.13. For spaces, (co)homology is usually defined from the singular chain complex; in this exercise we show that also spectrum (co-)homology can be calculated from a functorial chain complex which is assembled from the singular chain complexes of the individual spaces in the spectrum.

The definition of spectrum homology is very analogous to the definition of singular homology for topological spaces. We recall that the definition of singular homology can be broken up as a composite of several functors:

$$\mathbf{T} \xrightarrow{\mathcal{S}} \mathbf{sS} \xrightarrow{\mathbb{Z}[-]} s\mathcal{A}b \xrightarrow{C} \text{(chain complexes)} \xrightarrow{H_k} \mathcal{A}b \ .$$

The first functor associates to a spaces its singular complex, a simplicial set. By taking free abelian groups in every simplicial dimension, the second step produces a simplicial abelian group. The third functor takes the alternative sum of face morphisms to turn a simplicial abelian group into a chain complex. The singular homology, finally, is the homology of this 'singular chain complex'.

In the context of symmetric spectra of simplicial sets we now a chain functor as the composite

$$\mathcal{S}p_{\mathbf{sS}} \xrightarrow{\mathbb{Z}[-]} \mathcal{S}p_{s\mathcal{A}b} \xrightarrow{C} \text{(tame } \mathcal{M}\text{-chain complexes)} \xrightarrow{\mathbb{Z} \otimes^L_{\mathcal{M}} -} \text{(chain complexes)} \ .$$

(For symmetric spectra of spaces we also precompose with the 'levelwise' singular complex functor $\mathcal{S} : \mathcal{S}p_{\mathbf{T}} \longrightarrow \mathcal{S}p_{\mathbf{sS}}$.) The first functor is the 'free abelian group spectrum', compare Definition 6.24, which takes reduced free abelian groups in every spectrum level and every simplicial dimension. It naturally lands in a category $\mathcal{S}p_{s\mathcal{A}b}$ of 'symmetric spectra of abelian groups', i.e., abelian group objects in the category of symmetric spectra. The second functor is a 'chain functor' that we define below, which takes values in chain complexes of tame modules over the injection monoid $\mathcal{M}$. In the third step we take derived tensor product over $\mathcal{M}$ with the trivial right $\mathcal{M}$-module $\mathbb{Z}$, i.e., we coequalize the $\mathcal{M}$-action in a homologically meaningful way.

There is one qualitative difference between the chain complex of a space or simplicial set and the chain complex of a symmetric spectrum: the former is concentrated in non-negative dimensions, whereas the chain complex of a symmetric spectrum is in general not bounded below.

Let A be a symmetric spectrum of abelian groups. We define the chain complex CA of A as

$$CA \;=\; \operatorname{colim}_n (NA_n)[-n] \ .$$

In more detail, NA_n is the normalized chain complex of the simplicial abelian group A_n. Then $(NA_n)[-n]$ is the shifted complex, which is now concentrated in dimensions $-n$ and above. Our convention for the shift of a complex C is that $C[n]_k = C_{k-n}$ and the differential $C[n]_k \longrightarrow C[n]_{k-1}$ is $(-1)^n d^C_{k-n}$. The above colimit is formed over the sequence of chain maps

$$\begin{aligned}(NA_n)[-n] \cong (NA_n)[-n-1] \otimes N\tilde{\mathbb{Z}}[S^1] \;&\xrightarrow{\nabla}\; (NA_n \otimes \tilde{\mathbb{Z}}[S^1])[-n-1] \\ &\xrightarrow{(N\sigma_n)[-n-1]}\; (NA_{n+1})[-(n+1)]\end{aligned}$$

[specify the iso] where ∇ is the (normalized) shuffle map [ref].

(i) Show that the complex CA is in fact the colimit, over inclusions, of a functorial, chain complex valued **I**-functor. So CA comes with a natural tame action by the injection monoid $\mathcal{M}$, and hence its homology groups are tame $\mathcal{M}$-modules.

(ii) Construct a natural isomorphism of $\mathcal{M}$-modules

$$H_k(CA) \;\cong\; \hat{\pi}_k A$$

for every integer k.

(iii) The *chain complex* CX of a symmetric spectrum of simplicial sets X is the chain complex $\mathbb{Z} \otimes^L_{\mathcal{M}} C(\mathbb{Z}[X])$, the derived tensor product, over the monoid ring of the injection monoid $\mathcal{M}$, of the chain complex of the linearization of X. Given an abelian groups A, construct isomorphisms

$$H_k(A \otimes CX) \;\cong\; H_k(X,A) \quad \text{and} \quad H^k(CX,A) \;\cong\; H^k(X,A) \ ,$$

for $k \in \mathbb{Z}$, which are natural in X and A.

Exercise E.II.14. Let A be an abelian group and X a symmetric spectrum. Prove universal coefficient theorems for the (co-)homology of X with coeffiecients in A: construct natural short exact sequences of abelian groups

$$0 \longrightarrow A \otimes H_k(X,\mathbb{Z}) \longrightarrow H_k(X,A) \longrightarrow \operatorname{Tor}(A, H_{k-1}(X,\mathbb{Z})) \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Ext}(H_{k-1}(X,\mathbb{Z}),A) \longrightarrow H^k(X,A) \longrightarrow \operatorname{Hom}(H_k(X,\mathbb{Z}),A) \longrightarrow 0$$

which split (non-naturally).

Exercise E.II.15. We recall that for a pointed space X the Hurewicz homomorphism $h : \pi_n X \longrightarrow \tilde{H}_n(X,\mathbb{Z})$ sends the homotopy class of a based map $f : S^n \longrightarrow X$ to the homology class $f_*(\iota_n)$ where $\iota_n \in \tilde{H}_n(S^n,\mathbb{Z})$ is the fundamental class, a chosen generator of the free abelian group $\tilde{H}_n(S^n,\mathbb{Z})$ [chose the right one to make it coincide on the nose with inclusion of generators]. Show that the 'unstable' Hurewicz homomorphisms for spaces converge to the 'stable' Hurewicz homomorphism for symmetric spectra in the following sense.

(i) For every symmetric spectrum of topological spaces X and every integer k and $n \geq 0$, the following square commutes

$$\begin{tikzcd} \pi_{k+n}X_n \arrow[r, "\iota"] \arrow[d, "h"'] & \pi_{k+n+1}X_{n+1} \arrow[d, "h"] \\ \tilde{H}_{k+n}(X_n,\mathbb{Z}) \arrow[r] & \tilde{H}_{k+n+1}(X_{n+1},\mathbb{Z}) \end{tikzcd}$$

where the lower horizontal map is the one from the colimit system [...].

(ii) Show that the composite map

$$\hat{\pi}_k X = \mathrm{colim}_n\, \pi_{k+n}X_n \xrightarrow{\mathrm{colim}_n\, h} \mathrm{colim}_n\, \tilde{H}_{k+n}(X_n,\mathbb{Z}) = \hat{H}_k(X,\mathbb{Z})$$

agrees with the effect of the morphisms $X \longrightarrow \mathbb{Z}[X]$, by inclusion of generators, on naive homotopy groups.

Exercise E.II.16. In Proposition 6.17 (iii) we established the relation

$$\langle f \times g,\, x \times y\rangle \;=\; (-1)^{(k+m)l} \cdot \langle f, x\rangle \cdot \langle g, y\rangle$$

between Kronecker pairing and exterior products for a central cohomology class f. Give an example to show that centrality is necessary here. (Hint: one could take a homotopy ring spectrum where the non-commutativity shows up in the graded homotopy ring, and take $X = Y = \mathbb{S}$.)

Exercise E.II.17. If E and X are homotopy ring spectra with multiplications $\mu_E : E \wedge^L E \longrightarrow E$ respectively $\mu_X : X \wedge^L X \longrightarrow X$. The E-homology $E_*(X)$ becomes a graded ring if we endow it with the product

$$E_k(X) \otimes E_l(X) \xrightarrow{\times} E_{k+l}(X \wedge^L X) \xrightarrow{E_{k+l}(\mu_X)} E_{k+l}(X)$$

where '$\times$' is the exterior product in E-homology (6.7).

On the other hand, the derived smash product $E \wedge^L X$ can be given a multiplication as the composite

$$(E \wedge^L X) \wedge^L (E \wedge^L X) \xrightarrow{E \wedge \tau_{X,E} \wedge X} E \wedge^L E \wedge^L X \wedge^L X \xrightarrow{\mu_E \wedge \mu_X} E \wedge^L X\ .$$

Hence the homotopy groups of $E \wedge^L X$ inherit a product structure. Show that these two products on $E_*(X) = \pi_*(E \wedge^L X)$ coincide.

Exercise E.II.18. Let A be an abelian group. The n-th level $A[S^n] = (HA)_n$ of the Eilenberg-Mac Lane spectrum HA is an Eilenberg-Mac Lane space of type (A, n) and as such it has a fundamental class

$$\iota_{A,n} \;\in\; H^n(A[S^n], A)$$

uniquely characterized by the property that the cap product map

$$H_n(A[S^n],\mathbb{Z}) \;\longrightarrow\; A\ , \quad x \;\longmapsto\; x \cap \iota_{A,n}$$

is inverse to the composite

$$A \xrightarrow{l} \pi_n(A[S^n], 0) \xrightarrow{\text{Hurewicz}} H_n(A[S^n],\mathbb{Z})$$

of the isomorphism l and the Hurewicz homomorphism. [here $l(a)$ is the 'left multiplication' map, i.e., the homotopy class of the map $S^n \longrightarrow A[S^n]$ sending x to ax. The composite sends a to $a \cdot \iota_{S^n}$] The purpose of this exercise is to exhibit the relations that these fundamendal classes satisfy as A and n vary.

(i) Let A and B be two abelian groups. In Example I.5.28 we defined natural maps

$$A[S^n] \wedge B[S^m] \longrightarrow (A \otimes B)[S^{n+m}]$$

that together constitute a bimorphism $m_{A,B} : (HA, HB) \longrightarrow H(A \otimes B)$ of symmetric spectra. We let

$$m_{A,n,B,m} \;:\; A[S^n] \times B[S^m] \longrightarrow (A \otimes B)[S^{n+m}]$$

denote the composite with the projection from the cartesian to the smash product. Show the relation

$$m^*_{A,n,B,m}(\iota_{A\otimes B,n+m}) \;=\; \iota_{A,n} \times \iota_{B,m} \quad \text{in} \quad H^{n+m}(A[S^n] \times B[S^m],\, A \otimes B)$$

where the product on the right hand side is the exterior product.

(ii) Let B be a ring and denote by

$$m_{B,n,m} \;:\; B[S^n] \times B[S^m] \longrightarrow B[S^{n+m}]$$

denote the composite of the projection from the cartesian to the smash product and the multiplication map $\mu_{n,m} : (HB)_n \wedge (HB)_m \longrightarrow (HB)_{n+m}$ of the Eilenberg-Mac Lane ring spectrum HB. Show the relation

$$m^*_{B,n,m}(\iota_{B,n+m}) \;=\; \iota_{B,n} \times \iota_{B,m} \quad \text{in} \quad H^{n+m}(B[S^n] \times B[S^m], B)$$

where the product on the right hand side is the exterior product, followed by the multiplication in the ring B.

(iii) Let K be a simplicial set. In Example 1.18 we constructed a natural isomorphism

$$\psi_{K,A,n} \;:\; H^n(\Sigma^\infty_+ K, A) \;\cong\; H^n(K, A)$$

between the cohomology of the unreduced suspension spectrum of K and the cohomology of K (where we are identifying the reduced cohomology of K_+ with the unreduced cohomology of K). Show that this isomorphism is multiplicative in the sense that the square

$$\begin{tikzcd} H^n(\Sigma^\infty_+ K, A) \times H^m(\Sigma^\infty_+ L, B) \arrow[r, "\psi_{K,A,n} \times \psi_{L,B,m}"] \arrow[d, "\times"'] & H^n(K, A) \times H^m(L, B) \arrow[d, "\times"] \\ H^{n+m}(\Sigma^\infty_+(K \times L),\, A \otimes B) \arrow[r, "\psi_{K \times L, A\otimes B, n+m}"'] & H^{n+m}(K \times L, A \otimes B) \end{tikzcd}$$

commutes, where the vertical maps are exterior products.

(iv) Let K and L be simplicial sets and B a ring. Show that the square

$$\begin{tikzcd} H^n(\Sigma^\infty_+ K, B) \times H^m(\Sigma^\infty_+ L, B) \arrow[r, "\psi_{K,B,n} \times \psi_{L,B,m}"] \arrow[d, "\times"'] & H^n(K, B) \times H^m(L, B) \arrow[d, "\times"] \\ H^{n+m}(\Sigma^\infty_+(K \times L),\, B) \arrow[r, "\psi_{K \times L, B, n+m}"'] & H^{n+m}(K \times L, B) \end{tikzcd}$$

commutes, where the vertical maps are exterior products.

Exercise E.II.19. (i) If τ is any reduced cohomology operation and X a pointed simplicial set, show that the value of τ at the suspension ΣX is an additive map.

(ii) Let $\tau = \{\tau_i\}_{i \geq 0}$ be a stable cohomology operation of degree n and type (A, B). Show that each individual cohomology operation $\tau_i : H^i(-, A) \longrightarrow H^{n+i}(-, B)$ is additive.

(iii) Show that composition of stable cohomology operations is bi-additive.

Exercise E.II.20. Let τ be a reduced cohomology operation of type (A, n, B, m) which extends to a stable cohomology operation of degee $m - n$. Show that the associated cohomology class $\tau(\iota_{A,n})$ in $\tilde{H}^m(A[S^n], B)$ is primitive in the sense that

$$\mu^*(\tau(\iota_{A,n})) = \tau(\iota_{A,n}) \times 1 \;+\; 1 \times \tau(\iota_{A,n})$$

in $\tilde{H}^m(A[S^n] \times A[S^n], B)$, where $\mu : A[S^n] \times A[S^n] \longrightarrow A[S^n]$ is the addition map of the simplicial abelian group $A[S^n]$.

Exercise E.II.21. Let R be a commutative ring. The chain level $\cup_1$-*product*

$$\cup_1 \;:\; C^n(X, R) \otimes C^m(X, R) \;\longrightarrow\; C^{n+m-1}(X, R)$$

in the cochain complex of a simplicial set X is defined by the formula

$$(f \cup_1 g)(x) \;=\; \sum_{i=0}^{n-1} (-1)^{(n-i)(m+1)}\, f(d_i^{\text{out}} x) \otimes g(d_i^{\text{inn}} x)$$

for $x \in X_{n+m-1}$. Here the *i-th outer face* $d_i^{\mathrm{out}} : X_{n+m-1} \longrightarrow X_n$ and the *i-th inner face* $d_i^{\mathrm{inn}} : X_{n+m-1} \longrightarrow X_m$ are induced by the monotone injective maps $\delta_i^{\mathrm{out}} : [n] \longrightarrow [n+m-1]$ and $\delta_i^{\mathrm{inn}} : [m] \longrightarrow [n+m-1]$ with respective images

$$\mathrm{Im}\left(\delta_i^{\mathrm{out}}\right) \;=\; \{0,\dots,i\} \cup \{i+m,\dots,n+m-1\} \quad \text{and} \quad \mathrm{Im}(\delta_i^{\mathrm{inn}}) \;=\; \{i,\dots,i+m\} \;.$$

Note that the images of d_i^{out} and d_i^{inn} intersect in exactly two points, namely i and $i+m$.

(i) Show that the $\cup_1$-product satisfies the coboundary formula

$$\delta(f \cup_1 g) \;=\; (\delta f) \cup_1 g \;+\; (-1)^n f \cup_1 (\delta g) \;-\; (-1)^{n+m} f \cup g \;-\; (-1)^{(n+1)(m+1)}(g \cup f) \;.$$

(ii) Show that if $f \in C^n(X,R)$ is a cocycle and n is even, then the $\cup_1$-square $f \cup_1 f$ is a cocycle whose cohomology class only depends on the class of f. If n is odd, then $f \cup_1 f$ is a mod-2 cocycle whose mod-2 cohomology class only depends on the class of f. In other words, the formula $\mathrm{Sq}_1[f] = [f \cup_1 f]$ defines cohomology operations

$$\begin{aligned} \mathrm{Sq}_1 \;&:\; H^n(X;R) \;\longrightarrow\; H^{2n-1}(X;R) && \text{if } n \text{ is even, and}\\ \mathrm{Sq}_1 \;&:\; H^n(X;R) \;\longrightarrow\; H^{2n-1}(X;R/2R) && \text{if } n \text{ is odd.} \end{aligned}$$

(iii) Show that $\mathrm{Sq}_1(x) = \mathrm{Sq}^{n-1}(x)$ for every mod-2 cohomology class x of dimension n, i.e., the $\cup_1$-product is a chain level construction of the Steenrod squaring operation Sq^{n-1}.

Remark: the first definition of the squaring operations Sq^i in the paper [**79**] by Steenrod was in fact by combinatorial formulas at the cochain level, generalizing the $\cup_1$-square above.

Exercise E.II.22. Show that for every 1-dimensional cohomology class x in the mod-2 cohomology of a space or simplicial set the following formula holds:

$$\mathrm{Sq}^i(x^n) \;=\; \binom{n}{i} x^{i+n} \;.$$

Show that for any sequence $i_1,\dots,i_n$ of positive integers, the product Steenrod operation acts as

$$\mathrm{Sq}^{i_1}\cdots\mathrm{Sq}^{i_n}(x) \;=\; \begin{cases} x^{2^n} & \text{if } (i_1,\dots,i_n) = (2^{n-1},\dots,2,1)\text{, and}\\ 0 & \text{else.}\end{cases}$$

Exercise E.II.23. In this exercise we determine the Kronecker pairing between the Steenrod algebra and its dual in terms of the generators $\mathrm{Sq}^i \in (\mathcal{A}_2)^i$ respectively $\xi_i \in \pi_{2^i-1}(H\mathbb{F}_2 \wedge H\mathbb{F}_2)$.

(i) Show the relation

$$\langle \mathrm{Sq}^{2^{n-1}}\,\mathrm{Sq}^{2^{n-2}}\cdots\mathrm{Sq}^2\,\mathrm{Sq}^1,\, \xi_n\rangle \;=\; 1$$

and show that ξ_n caps to 0 with all other products of Steenrod squares whose degrees sum up to 2^n-1. (Hint: recall the definition of the class ξ_n and the cohomology of $\mathbb{F}_2[S^1]$ including cup product; use naturality of the cap product.)

(ii) Show the relation

$$\langle \mathrm{Sq}^n, \xi_1^n\rangle \;=\; 1$$

and show that all other monomials of degree n in the ξ_i's caps to 0 with Sq^n.

(iii) Show the relation

$$\langle \mathrm{Sq}^{i_1}\cdots\mathrm{Sq}^{i_k},\, \xi_1^n\rangle \;=\; \frac{n!}{i_1!\cdots i_k!}$$

where $n = i_1 + \cdots i_k$.

Exercise E.II.24. Expand the Milnor basis elements $\mathrm{Sq}^{0,2}$ and $\mathrm{Sq}^{1,2}$ in the Serre-Cartan basis of the mod-2 Steenrod algebra. In other words, write $\mathrm{Sq}^{0,2}$ and $\mathrm{Sq}^{1,2}$ as a sum of admissible sequences of Steenrod squaring operations.

Exercise E.II.25. Let X be a symmetric spectrum.

(i) Show that the composite map

$$\text{(E.II.26)} \qquad \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p) \otimes \pi_*(H\mathbb{F}_p \wedge X) \xrightarrow{\ \cdot\ } \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge X) \xrightarrow{\pi_*(H\mathbb{F}_p \wedge \mu \wedge X)} \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge X)$$

is an isomorphism.

(ii) We define a map

$$\Delta \ : \ H_*(X, \mathbb{F}_p) = \pi_*(H\mathbb{F}_p \wedge X) \ \longrightarrow \ \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p) \otimes \pi_*(H\mathbb{F}_p \wedge X) = \mathcal{A}^p_* \otimes H_*(X, \mathbb{F}_p)$$

as the composite of the map induced on homotopy groups by the morphism

$$H\mathbb{F}_p \wedge \iota \wedge X \ : \ H\mathbb{F}_p \wedge X = H\mathbb{F}_p \wedge \mathbb{S} \wedge X \ \longrightarrow \ H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge X$$

and the inverse of the isomorphism (E.II.26). Show that for every element $f \in \mathcal{A}^*_p$, every cohomology class $g \in H^*(X, \mathbb{F}_p)$ and every homology class $x \in H_*(X, \mathbb{F}_p)$ of the same degree as $f \circ g$ the relation

$$\langle f \otimes g, \, \Delta(x) \rangle \ = \ \langle f \circ g, \, x \rangle$$

holds. In other words: under the Kronecker pairing, the morphism Δ is dual to the action of the Steenrod algebra $\mathcal{A}^*_p$ on the mod-p cohomology of X. In particular, for $X = H\mathbb{F}_p$ this gives an interpretation of the comultiplication in the dual Steenrod algebra.

(iii) Let $x \in H_{k+l}(X, \mathbb{F}_p)$ satisfy

$$\Delta(x) \ = \ \sum_i \, a_i \otimes x_i$$

for suitable $a_i \in \mathcal{A}^p_*$ and $x_i \in H_*(X, \mathbb{F}_p)$. Let $g \in H^l(X, \mathbb{F}_p) = [X, \mathbb{F}_p]_{-l}$ be a mod-p cohomology class. Show that then the map $g_* : H_{k+l}(X, \mathbb{F}_p) \longrightarrow H_k(H\mathbb{F}_p, \mathbb{F}_p)$ induced by g in mod-p homology satisfies

$$g_*(x) \ = \ \sum_i \, (-1)^{|a_i| l} \cdot a_i \cdot \langle g, x_i \rangle \ .$$

Exercise E.II.27. Let $\pi : H\mathbb{Z} \longrightarrow H\mathbb{F}_p$ denote the morphism of symmetric ring spectra which is induced by the reduction map $\mathbb{Z} \longrightarrow \mathbb{F}_p$. Show that π induces a surjection in mod-p cohomology with kernel the $\mathcal{A}_p$-submodule $\mathcal{A}_p \cdot \beta$. Show that π induces an injection in mod-p homology whose image is the subalgebra of the dual Steenrod algebra generated by $\bar{\xi}_1^2$ and $\bar{\xi}_i$ for $i \geq 2$ when $p = 2$, and the subalgebra of the dual Steenrod algebra generated by $\bar{\tau}_i$ and $\bar{\xi}_i$ for $i \geq 1$ when p is odd. Here $\bar{\xi}_i = c(\xi_i)$ and $\bar{\tau}_i = c(\tau_i)$ are the antipodes of the respective Milnor generators. Show that the morphism

$$\pi_*(\pi \wedge H\mathbb{F}_p) \ : \ \pi_*(H\mathbb{Z} \wedge H\mathbb{F}_p) \ \longrightarrow \ \pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p) = \mathcal{A}^p_*$$

is injective with image the subalgebra of the dual Steenrod algebra generated by ξ_1^2 and ξ_i for $i \geq 2$ when $p = 2$, and the subalgebra of the dual Steenrod algebra generated by τ_i and ξ_i for $i \geq 1$ when p is odd.

Exercise E.II.28. Let $\pi : ko \longrightarrow H\mathbb{F}_2$ denote the composite of the 'dimension' morphism $ko \longrightarrow H\mathbb{Z}$ and the 'reduction' morphism $\pi : H\mathbb{Z} \longrightarrow H\mathbb{F}_p$, so that it induces the reduction map $\pi_0(ko) \cong \mathbb{Z} \longrightarrow \mathbb{F}_2$ on the 0-th homotopy group. Show that π induces a surjection in mod-2 cohomology with kernel the $\mathcal{A}_2$-submodule $\mathcal{A}_2 \cdot (\mathrm{Sq}^1, \mathrm{Sq}^2)$ generated by Sq^1 and Sq^2. Show that π induces an injection in mod-2 homology whose image is the subalgebra of the dual Steenrod algebra generated by $\bar{\xi}_1^4$, $\bar{\xi}_2^2$ and $\bar{\xi}_i$ for $i \geq 3$ (where again $\bar{\xi}_i = c(\xi_i)$ is the antipodes of the Milnor generator). Show that the morphism

$$\pi_*(H\mathbb{F}_p \wedge \pi) \ : \ \pi_*(ko \wedge H\mathbb{F}_2) \ \longrightarrow \ \pi_*(H\mathbb{F}_2 \wedge H\mathbb{F}_2) = \mathcal{A}^2_*$$

is injective with image the subalgebra of the dual Steenrod algebra generated by ξ_1^4, ξ_2^2 and ξ_i for $i \geq 3$. [cohomology of ku]

Exercise E.II.29. Let $\mathbb{S}/2$ denote a mod-2 Moore spectrum. Show that the smash product $\mathbb{S}/2 \wedge^L \mathbb{S}/2$ is indecomposable in the stable homotopy category (Hint: translate Example 10.12 into the stable homotopy category.) Deduce that the mod-2 Moore spectrum does not admit the structure of a homotopy ring spectrum.

Exercise E.II.30. Show that the mod-3 Moore spectrum $\mathbb{S}/3$ is not a homotopy ring spectrum because the multiplication morphism $\mathbb{S}/3 \wedge^L \mathbb{S}/3 \longrightarrow \mathbb{S}/3$ is not associative in the stable homotopy category. This can be done in the following two steps.

(i) Show that the unique unit preserving morphism $\kappa : \mathbb{S}/3 \longrightarrow H\mathbb{Z}/3$ in $\mathcal{SHC}$ is compatible with the multiplications, i.e., the square

$$\begin{tikzcd} \mathbb{S}/3 \wedge^L \mathbb{S}/3 \arrow[r, "\kappa\wedge^L\kappa"] \arrow[d, "\mu"'] & H\mathbb{Z}/3 \wedge^L H\mathbb{Z}/3 \arrow[d, "\mu"] \\ \mathbb{S}/3 \arrow[r, "\kappa"'] & H\mathbb{Z}/3 \end{tikzcd}$$

commutes in the stable homotopy category.

(ii) Use the Toda bracket relation $\langle \tau_0, \tau_0, \tau_0\rangle = \{\xi_1\}$ in $H_4(H\mathbb{Z}/3, \mathbb{F}_3)$ to derive a contradiction from the assumption that the multiplication of $\mathbb{S}/3$ is homotopy associative.

Exercise E.II.31. Use the cobar complex of the dual Steenrod algebra at the prime 2 to deduce the following multiplicative and Massey product relations among the classes h_i and c_i in the Ext algebra $\mathrm{Ext}^{*,*}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$:

$$\begin{array}{rrcll} \text{(i)} & h_{i+1}^3 & = & h_i^2 h_{i+2} & \\ \text{(ii)} & h_{i+1}^4 & = & 0 & \\ \text{(iii)} & h_i h_{i+2}^2 & = & 0 & \\ \text{(iv)} & h_i^2 h_{i+3}^2 & = & 0 & \\ \text{(v)} & h_0^{2^i} h_i & = & 0 & \text{for } i \geq 1 \\ \text{(vi)} & h_0^{2^i} h_{i+2}^2 & = & 0 & \end{array} \qquad \begin{array}{rrcl} \text{(vii)} & \multicolumn{3}{l}{h_i c_i \;=\; h_{i+2} c_i \;=\; h_{i+3} c_i = 0} \\ \text{(viii)} & h_{i+1}^2 & = & \langle h_i, h_{i+1}, h_i\rangle \\ \text{(ix)} & h_i h_{i+2} & = & \langle h_{i+1}, h_i, h_{i+1}\rangle \\ \text{(x)} & h_{i-1}^3 h_{i+2} & = & \langle h_{i+1}, h_{i-1}^3, h_{i+1}\rangle \\ \text{(xi)} & \multicolumn{3}{l}{c_i \;=\; \langle h_{i+1}, h_i, h_{i+2}^2\rangle \;=\; \langle h_i, h_{i+2}^2, h_{i+1}\rangle} \\ \text{(xii)} & 0 & \in & \langle h_{i+1}, h_{i+2}, c_i\rangle \end{array}$$

(In (x), we supposedly already have $h_0^{2^i-1} h_i = 0$.) Hint: one can save some work by systematically exploiting the juggling formulas for Massey products.

Deduce that the products 2η, $\eta\nu$ and $\nu\sigma$ are trivial in the 2-primary homotopy groups of spheres and that the Toda bracket relations

$$\eta^2 \in \langle 2, \eta, 2\rangle \ , \quad 2\nu \in \langle \eta, 2, \eta\rangle \ , \quad \nu^2 \in \langle \eta, \nu, \eta\rangle \quad \text{and} \quad 8\sigma \in \langle \nu, 8, \nu\rangle$$

hold. What can you conclude about the brackets $\langle \nu, \eta, \nu\rangle$ and $\langle \nu, \sigma, \nu\rangle$? Can you express ν^3 in terms of $\eta^2\sigma$ and $\eta\varepsilon$?

Exercise E.II.32. It is an algebraic fact that the Ext group $\mathrm{Ext}^{3,2^{i+1}+1}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by the class $h_0 h_i^2$ and that the class $h_0 h_i^3$ is non-trivial in $\mathrm{Ext}^{4,3\cdot 2^i+1}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$ [ref]. Assuming this fact, use the first Adams differential $d_2(h_4) = h_0 h_3^2$ and the relations of Exercise E.II.31 to deduce the general Adams differential

$$d_2(h_i) = h_0 h_{i-1}^2$$

for $i \geq 4$ in the 2-primary Adams spectral sequence for the sphere spectrum.

History and credits

The stable homotopy category as we know it today is usually attributed to Boardman, who introduced it in his thesis [**5**] including the triangulated structure and the symmetric monoidal (derived !) smash product. Boardman's stable homotopy category is obtained from a category of *CW-spectra* by passing to homotopy classes of morphisms. Boardman's construction was widely circulated as mimeographed notes [**6**], but he never published these. Accounts of Boardman's construction appear in [**84**], [**87**], and [**2**, Part III]. Strictly speaking the 'correct' stable homotopy category had earlier been introduced by Kan [**41**] based on his notion of *semisimplicial spectra*. Kan and Whitehead [**42**] defined a smash product in the homotopy category of semisimplicial spectra and proved that it is homotopy commutative, but neither they, nor anyone else, ever addressed the associativity of that smash product. Before Kan and Boardman there had been various precursors of the stable homotopy category, and I recommend May's survey article [**57**] for a detailed discussion and an extensive list of references to these.

I am not aware of a complete published account that Boardman's category is really equivalent to the stable homotopy category as defined in Definition 1.1 using injective Ω-spectra. However, here is a short guide through the literature which outlines a comparison. In a first step, Boardman's stable homotopy category can be compared to Kan's homotopy category of semisimplicial spectra, which is done in Chapter IV of Boardman's unpublished notes [**6**]. An alternative source is Tierney's article [**84**] where he promotes the geometric realization functor to a functor from Boardman's category of CW-spectra to Kan's category of semisimplicial spectra. Tierney remarks that the singular complex functor from spaces to simplicial set does not lift to a pointset level functor in the other directions, but Section 3 of [**84**] then ends with the words "(...) it is more or less clear – combining various results of Boardman and Kan – that the singular functor exists at the level of homotopy and provides an inverse to the stable geometric realization, i.e. the two homotopy theories are equivalent. The equivalence of homotopy theories has also been announced by Boardman." I am not aware that the details have been carried out in the published literature.

Kan's semisimplicial spectra predate model categories, but Brown [**15**, Thm. 5] showed later that the π_*-isomorphisms used by Kan are part of a model structure on semisimplicial spectra. In the paper [**13**] Bousfield and Friedlander introduce a model structure on a category of 'sequential spectra' which are just like symmetric spectra, but without the symmetric group actions. In Section 2.5 of [**13**], Bousfield and Friedlander describe a chain of Quillen equivalences between semisimplicial and sequential spectra, which then in particular have equivalent homotopy categories. Hovey, Shipley and Smith show in [**36**, Thm. 4.2.5] that the forgetful functor is the right adjoint of a Quillen equivalence from symmetric spectra (with the stable absolute projective model structure in the sense of Chapter III) to the Bousfield-Friedlander stable model structure of sequential spectra. Since the weak equivalences used for symmetric spectra are the stable equivalences in the sense of Definition 4.11 we can conclude that altogether that Boardman's stable homotopy category is equivalent to the localization of the category of symmetric spectra at the class of stable equivalences, which coincides with the stable homotopy category in our sense by Theorem 1.6.

A word of warning: the comparison which I just summarized passes through the intermediate homotopy category of sequential spectra for which no intrinsic way to define a derived smash product has been studied. As a consequence, it is not clear to me if the combined equivalence takes Boardman's derived smash product to the derived smash product as discussed in Section 3. However, I would be surprised if the composite equivalence were not strongly symmetric monoidal.

CHAPTER III

Model structures

Symmetric spectra support many useful model structures and we will now develop several of these. We will mainly be interested in two kinds, namely *level model structures* (with weak equivalences the level equivalences) and *stable model structures* (with weak equivalences the stable equivalences). The level model structures are really an intermediate steps towards the more interesting stable model structures. We will develop the theory for symmetric spectra of simplicial sets first, and later say how to adapt things to symmetric spectra of topological spaces.

We have already seen pieces of some of the model structures at work. Our definition of the stable homotopy category in Section 1 of Chapter II is implicitly relying on the *absolute injective stable* model structure in which every object is cofibrant (as long as we use simplicial sets, not topological spaces) and the fibrant objects are the injective Ω-spectra. However, this model structure does not interact well with the smash product, so when we constructed the derived smash product in Section 3 of Chapter II we implicitly worked in the flat model structures. So it should already be clear that it can be useful to play different model structures off against each other.

Besides the injective and flat model structures there is another useful kind of cofibration/fibration pair which we will discuss, giving the *projective* model structures. Moreover, we will later need 'positive' versions of the model structures which discard all homotopical information contained in level 0 of a symmetric spectrum.

So each of the model structures which we discuss has four kinds of 'attributes':

- a kind of space (simplicial set or topological space)
- a kind of cofibration/fibration pair (injective, flat or projective)
- a type of equivalence (level or stable)
- which levels are used (absolute or positive)

Since all of these attributes can be combined, this already makes $2\times3\times2\times2 = 24$ different model structures on the two kinds of symmetric spectra. More variations are possible: one can also take $\hat{\pi}_*$-isomorphisms as weak equivalences, or even isomorphisms in some homology theory (giving model structures which realize Bousfield localizations), or one could study 'more positive' model structures which disregard even more than the level 0 information. And this is certainly not the end of the story...

1. Symmetric spectra in a simplicial category

The projective and flat level model structures on symmetric spectra of spaces or simplicial sets can be produced by one very general method that we develop in this section. For this purpose we consider a category $\mathcal{C}$ which is pointed, complete, cocomplete and simplicial, by which we mean enriched, tensored and cotensored over the category of pointed simplicial sets. Our main example will be $\mathcal{C} = \mathbf{T}$, the category of based spaces and $\mathcal{C} = \mathbf{sS}$, the category of based simplicial sets. In this situation, the category $G\mathcal{C}$ of G-objects in $\mathcal{C}$ is also complete, cocomplete and simplicial for every group G; indeed, limits, colimits, tensors and cotensors in $G\mathcal{C}$ are created in the underlying category $\mathcal{C}$. For consistency with the previous notions of symmetric spectra we use the smash symbol for the action of a simplicial set on an object of $\mathcal{C}$ or $G\mathcal{C}$, and let the based simplicial sets act on the right.

Definition 1.1. Let $\mathcal{C}$ be a pointed simplicial category. A *symmetric spectrum in* $\mathcal{C}$ consists of the following data:

- a Σ_n-object X_n in $\mathcal{C}$ for $n \geq 0$,
- $\mathcal{C}$-morphisms $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ for $n \geq 0$,

where $S^1 = \Delta[1]/\partial\Delta[1]$ is the 'small' simplicial circle. This data is subject to the following condition: for all $n, m \geq 0$, the composite

$$(1.2) \qquad X_n \wedge S^m \xrightarrow{\sigma_n \wedge \mathrm{Id}} X_{n+1} \wedge S^{m-1} \xrightarrow{\sigma_{n+1} \wedge \mathrm{Id}} \cdots \xrightarrow{\sigma_{n+m-2} \wedge \mathrm{Id}} X_{n+m-1} \wedge S^1 \xrightarrow{\sigma_{n+m-1}} X_{n+m}$$

is $\Sigma_n \times \Sigma_m$-equivariant, where $S^n = S^1 \wedge \ldots \wedge S^1$ is the n-fold smash product of copies of S^1.

A *morphism* $f : X \longrightarrow Y$ of symmetric spectra in $\mathcal{C}$ consists of Σ_n-equivariant morphisms $f_n : X_n \longrightarrow Y_n$ for $n \geq 0$, which satisfy $f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge S^1)$ for all $n \geq 0$. We denote the category of symmetric spectra in $\mathcal{C}$ by $\mathcal{S}p_{\mathcal{C}}$.

Of course, when $\mathcal{C} = \mathbf{T}$ is the category of based compactly generated weak Hausdorff spaces or $\mathcal{C} = \mathbf{sS}$ is the category of based simplicial sets, we recover the definitions of symmetric spectra of spaces respectively of simplicial sets of Section I.1. As in these special cases, we denote the composite map eqrefeq-general C symmetric axiom by σ^m and we refer to the object X_n as the *nth level* of the symmetric spectrum X. If we want a 'level' model structure on $\mathcal{S}p_{\mathcal{C}}$ we need to start with compatible model structure on the categories of Σ_n-objects in $\mathcal{C}$. We formalize what we mean by 'compatible model structures' in the following definition.

Many of the formal construction for symmetric spectra from Chapter I make sense in the more general context of symmetric spectra in $\mathcal{C}$, and many of the formal properties carry over. We will now quickly go through these generalizations but omit the proofs which are formally the same as in Chapter I.

Example 1.3 (Shift in $\mathcal{S}p_{\mathcal{C}}$)**.** For every symmetric spectrum X in $\mathcal{C}$ and every $m \geq 0$, the assignment $(\mathrm{sh}^m X)_n = X_{m+n}$ defines the nth level of a new symmetric spectrum $\mathrm{sh}^m X$, mth *shift* of X. The Σ_n-action on this object is obtained from the given Σ_{m+n}-action by restriction along $1 + - : \Sigma_n \longrightarrow \Sigma_{m+n}$ and the structure maps are reindexed structure maps of X. The symmetric group Σ_m acts on $\mathrm{sh}^m X$ by automorphisms through the 'shifted coordinates', i.e., by restriction along $- + 1 : \Sigma_m \longrightarrow \Sigma_{m+n}$. In other words: everything works in the same way as in Example I.3.9. We have $\mathrm{sh}^k(\mathrm{sh}^m X) = \mathrm{sh}^{m+k} X$ as symmetric spectra with $(\Sigma_m \times \Sigma_k)$-action.

Example 1.4 (Semifree and free symmetric spectra in $\mathcal{C}$)**.** As for symmetric spectra of spaces or simplicial sets, there is a class of semifree spectra in $\mathcal{C}$ with the same formal properties which are elementary building blocks for general symmetric spectra in $\mathcal{C}$. We let L be a Σ_m-object in $\mathcal{C}$ and define the *semifree symmetric spectrum* generated by L in level m in much the same way as in Example I.3.23. Below level m the spectrum $G_m L$ is trivial, i.e., consists of a zero object of $\mathcal{C}$. In the other levels we set

$$(G_m L)_{m+n} \ = \ \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge S^n \ .$$

The structure morphism $\sigma_{m+n} : (G_m K)_{m+n} \wedge S^1 \longrightarrow (G_m K)_{m+n+1}$ is defined by smashing the 'inclusion' $- + 1 : \Sigma_{m+n} \longrightarrow \Sigma_{m+n+1}$ with the identity of L and the preferred isomorphism $S^n \wedge S^1 \cong S^{n+1}$. As before, the semifree functor

$$G_m \ : \ \Sigma_m \mathcal{C} \ \longrightarrow \ \mathcal{S}p_{\mathcal{C}}$$

is left adjoint to the forgetful evaluation functor $\mathrm{ev}_m : \mathcal{S}p_{\mathcal{C}} \longrightarrow \Sigma_m \mathcal{C}$, by the essentially same adjunction.

Semifree spectra are again basic building blocks for symmetric spectra in $\mathcal{C}$ in the sense that the diagram (3.25) of Chapter I makes perfect sense in the present more general context and expresses an arbitrary symmetric spectrum X in $\mathcal{C}$ as a coequalizer of wedges of semifree spectra.

In the context of symmetric spectra of spaces and simplicial sets we defined free and semifree spectra seperately and then discussion how one kind can be obtained from each other. Now we turn this around and define free symmetric spectra as a special case of semifree ones. More precisely, we define the *free symmetric spectrum* generated by a $\mathcal{C}$-object K in level m as

$$F_m K \ = \ G_m(\Sigma^+_m \wedge K) \ .$$

Example 1.5 (Twisted smash products in $\mathcal{C}$)**.** For symmetric spectra in a based simplicial category $\mathcal{C}$ there are two kinds of twisted smash products. For every Σ_m-object L in $\mathcal{C}$ and symmetric spectrum of simplicial sets A we define the *twisted smash product* $L \triangleright_m A$ as the symmetric spectrum in $\mathcal{C}$ which is trivial below level m and is otherwise given by

$$(L \triangleright_m A)_{m+n} \ = \ \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge A_n \ .$$

The structure maps are defined in the same way as for symmetric spectra of spaces or simplicial sets (compare Example I.3.27). If X is a symmetric spectrum in $\mathcal{C}$ and L is a based Σ_m-simplicial set we define the 'right' twisted smash product $X \triangleleft_m L$ as the symmetric spectrum in $\mathcal{C}$ which is trivial below level m and is otherwise given by

$$(X \triangleleft_m L)_{n+m} \ = \ \Sigma^+_{n+m} \wedge_{\Sigma_n \times \Sigma_m} X_n \wedge L \ .$$

The structure maps are hopefully clear.

In the present context the asymmetry between category $\mathcal{C}$ and the category $\mathbf{sS}$ gives two different constructions based on different kind of input. In the context of spaces or simplicial sets, the 'left' and 'right' twisted smash products are isomorphic (which is why so far we have only discussed one of them). Indeed, if L is a based Σ_m-space (or simplicial set) and X a symmetric spectrum of spaces (or simplicial sets), then an isomorphism $L \triangleright_m X \cong X \triangleleft_m L$ is given in level $m+n$ by

$$\begin{array}{ccc} \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge X_n & \longrightarrow & \Sigma^+_{n+m} \wedge_{\Sigma_n \times \Sigma_m} X_n \wedge L \\ [\gamma \wedge l \wedge x] & \longmapsto & [\gamma \chi_{n,m} \wedge x \wedge l] \ . \end{array}$$

As the special case for $m = 1$ and $L = S^0$ we obtain an *induction* functor $\triangleright X = X \triangleleft_1 S^0$ which is left adjoint to the shift functor of Example 1.3. In the special case where $\mathcal{C} = \mathbf{T}$ or $\mathcal{C} = \mathbf{sS}$, this construction is different from, but naturally isomorphic to, the induction functor from Example I.3.17.

Now we define a 'action' of the category $\mathcal{S}p_{\mathbf{sS}}$ of symmetric spectra of simplicial sets on the category $\mathcal{S}p_{\mathcal{C}}$; the construction and various formal properties work in much the same way as the internal smash product for symmetric spectra in Section I.5. In fact, if we take $\mathcal{C} = \mathbf{sS}$ we recover the smash product of symmetric spectra of simplicial sets.

Let X and Z be symmetric spectra in $\mathcal{C}$ and let A be a symmetric spectrum of simplicial sets. A *bimorphism* $b : (X, A) \longrightarrow Z$ from the pair (X, A) to Z as a collection of $\Sigma_p \times \Sigma_q$-equivariant $\mathcal{C}$-morphisms or simplicial sets, depending on the context,

$$b_{p,q} \ : \ X_p \wedge A_q \ \longrightarrow \ Z_{p+q}$$

for $p, q \geq 0$, such that the 'bilinearity diagram'

(1.6)
$$\begin{tikzcd} & X_p \wedge A_q \wedge S^1 \arrow[r, "X_p \wedge \mathrm{twist}"] \arrow[dl, "X_p \wedge \sigma_q"'] \arrow[d, "b_{p,q} \wedge S^1"] & X_p \wedge S^1 \wedge A_q \arrow[d, "\sigma_p \wedge A_q"] \\ X_p \wedge A_{q+1} \arrow[dr, "b_{p,q+1}"'] & Z_{p+q} \wedge S^1 \arrow[d, "\sigma_{p+q}"] & X_{p+1} \wedge A_q \arrow[d, "b_{p+1,q}"] \\ & Z_{p+q+1} & Z_{p+1+q} \arrow[l, "1 \times \chi_{1,q}"] \end{tikzcd}$$

commutes in $\mathcal{C}$ for all $p, q \geq 0$. A smash product of X and A is a pair $(X \wedge A, i)$ consisting of a symmetric spectrum $X \wedge A$ in $\mathcal{C}$ and a universal bimorphism $i : (X, A) \longrightarrow X \wedge A$, i.e., a bimorphism such that for every symmetric spectrum Z in $\mathcal{C}$ the map

(1.7) $$\mathcal{S}p_{\mathcal{C}}(X \wedge A, Z) \ \longrightarrow \ \mathrm{Bimor}((X, A), Z) \ , \quad f \longmapsto fi = \{f_{p+q} \circ i_{p,q}\}_{p,q}$$

is bijective. Again it will be convenient to make the sphere spectrum $\mathbb{S}$ into a strict right unit for this mixed smash product. So we agree that for $A = \mathbb{S}$ we choose $X \wedge \mathbb{S} = X$ with universal bimorphism

$i : (X, \mathbb{S}) \longrightarrow X$ given by the iterated structure map,

$$i_{p,q} \;= \sigma^q \;:\; X_p \wedge S^q \;\longrightarrow\; X_{p+q} \;.$$

The existence of smash product $X \wedge A$ with X in $\mathcal{S}p_{\mathcal{C}}$ and A in $\mathcal{S}p_{\mathbf{sS}}$ is established by the same construction (C) as in Section I.5. For $n \geq 0$ we define the nth level $(X \wedge A)_n$ as the coequalizer, in the category $\Sigma_n\mathcal{C}$, of two maps

$$\alpha_X,\, \alpha_A \;:\; \bigvee_{p+1+q=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_1 \times \Sigma_q} X_p \wedge S^1 \wedge A_q \;\longrightarrow\; \bigvee_{p+q=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge A_q$$

defined exactly as before from the structure maps of X and A. The structure map $(X \wedge A)_n \wedge S^1 \longrightarrow (X \wedge A)_{n+1}$ is induced on coequalizers by the wedge of the maps

$$\Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge A_q \wedge S^1 \;\longrightarrow\; \Sigma_{n+1}^+ \wedge_{\Sigma_p \times \Sigma_{q+1}} X_p \wedge A_{q+1}$$

induced from $\mathrm{Id} \wedge \sigma_q^A : X_p \wedge A_q \wedge S^1 \longrightarrow X_p \wedge A_{q+1}$. One should check that this indeed passes to a well-defined map on coequalizers. Equivalently we could have defined the structure map by moving the circle past A_q, using the structure map of X (instead of that of A) and then shuffling back with the permutation $\chi_{1,q}$; the definition of $(X \wedge A)_{n+1}$ as a coequalizer precisely ensures that these two possible structure maps coincide. Morevoer the collection of maps

$$X_p \wedge A_q \xrightarrow{x \wedge y \mapsto 1 \wedge x \wedge y} \bigvee_{p+q=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge A_q \xrightarrow{\text{projection}} (X \wedge A)_{p+q}$$

forms a universal bimorphism.

The arguments of Section I.5 generalize in a straightforward way to show that the smash product $X \wedge A$ is a functor in two variables

$$\wedge \;:\; \mathcal{S}p_{\mathcal{C}} \times \mathcal{S}p_{\mathbf{sS}} \;\longrightarrow\; \mathcal{S}p_{\mathcal{C}} \;.$$

We have agreed above to make the sphere spectrum a strict right unit for this pairing, and the pairing is coherently associative as before. More precisely, for any symmetric spectrum X in $\mathcal{C}$ and all symmetric spectra of simplicial sets A and B the family

$$\left\{ X_p \wedge\; A_q \wedge B_r \xrightarrow{i_{p,q} \wedge B_r} (X \wedge A)_{p+q} \wedge B_r \xrightarrow{i_{p+q,r}} ((X \wedge A) \wedge B)_{p+q+r} \right\}_{p,q,r \geq 0}$$

and the family

$$\left\{ X_p \wedge\; A_q \wedge B_r \xrightarrow{X_p \wedge i_{q,r}} X_p \wedge (A \wedge B)_{q+r} \xrightarrow{i_{p,q+r}} (X \wedge (A \wedge B))_{p+q+r} \right\}_{p,q,r \geq 0}$$

both have the universal property of a *tri*morphism out of X, A and B. The uniqueness of representing objects gives a unique isomorphism of symmetric spectra in $\mathcal{C}$

$$\alpha_{X,A,B} \;:\; (X \wedge A) \wedge B \;\cong X \wedge (A \wedge B)$$

such that $(\alpha_{X,A,B})_{p,q,r} \circ i_{p+q,r} \circ (i_{p,q} \wedge B_r) = i_{p,q+r} \circ (X_p \wedge i_{q,r})$. The coherence property now takes the form of a commutative pentagon pentagon

$$\begin{tikzcd}[column sep=small]
 & ((X \wedge A) \wedge B) \wedge C \arrow[dl, "\alpha_{X,A,B} \wedge C"'] \arrow[dr, "\alpha_{X \wedge A,B,C}"] & \\
(X \wedge (A \wedge B)) \wedge C \arrow[d, "\alpha_{X,A \wedge B,C}"'] & & (X \wedge A) \wedge (B \wedge C) \arrow[d, "\alpha_{X,A,B \wedge C}"] \\
X \wedge ((A \wedge B) \wedge C) \arrow[rr, "X \wedge \alpha_{A,B,C}"'] & & X \wedge (A \wedge (B \wedge C))
\end{tikzcd}$$

where now X is in $\mathcal{S}p_{\mathcal{C}}$ and A, B and C are objects of $\mathcal{S}p_{\mathcal{C}}$. The associativity isomorphism

$$\begin{aligned} \alpha_{X,\mathbb{S},B} &: X \wedge B = (X \wedge \mathbb{S}) \wedge B \longrightarrow X \wedge (\mathbb{S} \wedge B) = X \wedge B \quad \text{and} \\ \alpha_{X,A,\mathbb{S}} &: X \wedge A = (X \wedge A) \wedge \mathbb{S} \longrightarrow X \wedge (A \wedge \mathbb{S}) = X \wedge A \end{aligned}$$

are the identity morphisms.

We define a bimorphism $j : (G_m L, A) \longrightarrow L \rhd_m A$ as follows. The component of j of bidegree (p, q) is trivial for $p < m$ and for $p = m + n$ the component $j_{m+n,q} : (G_m L)_{m+n} \wedge A_q \longrightarrow (L \rhd_m A)_{m+n+q}$ is the $(\Sigma_{m+n} \times \Sigma_q)$-equivariant extension of the $(\Sigma_m \times \Sigma_n \times \Sigma_q)$-equivariant composite

$$\begin{aligned} L \wedge S^n \wedge A_q &\xrightarrow{L\wedge \text{twist}} L \wedge A_q \wedge S^n \xrightarrow{L\wedge\sigma^q} L \wedge A_{q+n} \\ &\xrightarrow{L\wedge\chi_{q,n}} L \wedge A_{n+q} \xrightarrow{[1\wedge -]} \Sigma^+_{m+n+q} \wedge_{\Sigma_m\times\Sigma_{n+q}} L \wedge A_{n+q} \ . \end{aligned}$$

The same arguments as in the proof of Proposition I.5.5 show that $j : (G_m L, A) \longrightarrow L \rhd_m A$ is a universal bimorphism. Hence the pair $(L \rhd_m A, i)$ is a smash product of the semifree symmetric spectrum $G_m L$ and A.

The pairing between $\mathcal{S}p_{\mathcal{C}}$ and $\mathcal{S}p_{\mathbf{sS}}$ is not symmetric in the two smash factors, and there is another 'right' twisted smash product $X \lhd_m L$ where X is a symmetric spectrum in $\mathcal{C}$ and L is a based Σ_m-simplicial set. In much the same way as in the previous paragraph we can obtain a universal bimorphism $j : (X, G_m L) \longrightarrow X \lhd_m L$, so that the pair $(X \lhd_m L, j)$ is a smash product of X and the semifree symmetric spectrum $G_m L$.

Example 1.8 (Skeleta and latching objects in $\mathcal{S}p_{\mathcal{C}}$)**.** Latching objects $L_k X$ and the skeleta $F^k X$ of a symmetric spectrum X in $\mathcal{C}$ can be defined in much the same ways as in Section I.5.4. We start with $F^{-1}X = *$, the trivial spectrum consisting of the zero object of $\mathcal{C}$ in every level, and we let $i_{-1} : * \longrightarrow X$ denote the unique morphism. For $k \geq 0$ we define the latching object by

$$L_k X \ = \ (F^{k-1}X)_k \ , \tag{1.9}$$

the k-th level of the $(k-1)$-skeleton. The latching object $L_k X$ comes equipped with a Σ_k-action and an equivariant latching morphism $\nu_k : L_k X = (F^{k-1}X)_k \longrightarrow X_k$, namely the k-level of the previously constructed morphism $i_{k-1} : F^{k-1}X \longrightarrow X$. Then we define the k-skeleton $F^k X$ as the pushout

$$\begin{tikzcd} G_k L_k X \arrow[r, "G_k\nu_k"] \arrow[d] & G_k X_k \arrow[d] \\ F^{k-1}X \arrow[r, "j_k"'] & F^k X \end{tikzcd} \tag{1.10}$$

where the left vertical morphism is adjoint to the identity map of $L_k X = (F^{k-1}X)_k$. The morphism $\eta : G_k X_k \longrightarrow X$ (which is adjoint to the identity of X_k) and $i_{k-1} : F^{k-1}X \longrightarrow X$ restrict to the same morphism on $G_k L_k X$. So the universal property of the pushout provides a unique morphism $i_k : F^k X \longrightarrow X$ which satisfied $i_k j_k = i_{k-1}$ and whose restriction to $G_k X_k$ is η.

As in Construction I.5.29 we insist on some particular choices. Since the left vertical morphism $G_k L_k X \longrightarrow F^{k-1}X$ in the pushout (1.10) is an isomorphism, we can choose the kth level of $F^k X$ as

$$(F^k X)_k \ = \ X_k \ . \tag{1.11}$$

Since the morphism $G_k \nu_k$ is an isomorphism below level k (between zero objects), we can choose insist that $(F^k X)_n = (F^{k-1}X)_n$ for $n \leq k$. By induction, this implies that

$$(F^k X)_n \ = \ (F^{k-1}X)_n \qquad \text{for } n \leq k. \tag{1.12}$$

and that the morphisms $j_{k+1} : F^k X \longrightarrow F^{k+1}X$ and $i_k : F^k X \longrightarrow X$ are the identity in level k and below. Thus the structure maps of the symmetric spectrum $F^k X$ also coincide with those of X up to level k. Again the sequence of skeleta $F^k X$ stabilizes to X in the strong sense that in every given level, all maps are eventually identities. In particular, the spectrum X is a colimit, with respect to the morphisms i_k, of the sequence of morphisms j_k.

Given any morphism $f : X \longrightarrow Y$ in $\mathcal{S}p_{\mathcal{C}}$ we can define a relative skeleton filtration as follows. The *relative m-skeleton* of f is the pushout

$$F^m[f] \;=\; X \cup_{F^m X} F^m Y \tag{1.13}$$

where $F^m X$ is the m-skeleton of X as defined above. The relative m-skeleton comes with a unique morphism $i_m : F^m[f] \longrightarrow Y$ which restricts to $f : X \longrightarrow Y$ respectively to $i_m : F^m Y \longrightarrow Y$. Since $L_m X = (F^{m-1}X)_m$ we have

$$(F^{m-1}[f])_m \;=\; X_m \cup_{L_m X} L_m Y \ ,$$

the mth relative latching object. A morphism $j_m[f] : F^{m-1}[f] \longrightarrow F^m[f]$ is obtained from the commutative diagram

$$\begin{tikzcd}
X \arrow[d, equal] & F^{m-1}X \arrow[l] \arrow[r, "F^{m-1}f"] \arrow[d, "j^X_m"'] & F^{m-1}Y \arrow[d, "j^Y_m"] \\
X & F^m X \arrow[l] \arrow[r, "F^m f"'] & F^m Y
\end{tikzcd}$$

by taking pushouts. The square

$$\begin{tikzcd}
G_m(X_m \cup_{L_m X} L_m Y) \arrow[r, "G_m(\nu_m f)"] \arrow[d] & G_m Y_m \arrow[d] \\
F^{m-1}[f] \arrow[r, "j_m[f]"'] & F^m[f]
\end{tikzcd} \tag{1.14}$$

is a pushout [justify] and the original morphism $f : X \longrightarrow Y$ factors as the composite of the countable sequence

$$X = F^{-1}[f] \xrightarrow{j_0[f]} F^0[f] \xrightarrow{j_1[f]} F^1[f] \longrightarrow \cdots \xrightarrow{j_m[f]} F^m[f] \longrightarrow \cdots .$$

If we fix a level n, then the sequence stabilizes to the identity map of Y_n from $(F^n[f])_n$ on; in particular, the compatible maps $j_m : F^m[f] \longrightarrow Y$ exhibit Y as the colimit of the sequence.

[show that $L_m X \cong (X \wedge \bar{\mathbb{S}})_m$]

As in the special case of two symmetric spectra of spaces or simplicial sets we can express the shift of a smash product $X \wedge A$ as a pushout of the spectra $(\operatorname{sh} X) \wedge A$ and $X \wedge (\operatorname{sh} A)$ along $S^1 \wedge X \wedge A$, where now $X \in \mathcal{S}p_{\mathcal{C}}$ and $A \in \mathcal{S}p_{\mathbf{sS}}$. As in I.(5.17) the square

$$\begin{tikzcd}
(S^1 \wedge X) \wedge A \arrow[r, "(X\wedge\lambda_A)\circ\text{twist}"] \arrow[d, "\lambda_X \wedge A"'] & X \wedge (\operatorname{sh} A) \arrow[d, "\xi^{0,1}_{X,A}"] \\
(\operatorname{sh} X) \wedge A \arrow[r, "\xi^{1,0}_{X,A}"'] & \operatorname{sh}(X \wedge A)
\end{tikzcd} \tag{1.15}$$

of symmetric spectra in $\mathcal{C}$ commutes and the same proof as in Proposition I.5.18 shows that the square is a pushout. As an Example I.5.40 we can evaluate the special case $A = \bar{\mathbb{S}}$ of the truncated sphere spectrum at level m and obtain a pushout square in $\mathcal{C}$:

$$\begin{tikzcd}
L_m X \wedge S^1 \arrow[r, "\nu_m \wedge S^1"] \arrow[d, "L_m \lambda_X"'] & X_m \wedge S^1 \arrow[d, "\sigma_m"] \\
L_m(\operatorname{sh} X) \arrow[r] & L_{m+1} X
\end{tikzcd} \tag{1.16}$$

The following proposition is an immediate application of the relative skeleton filtration. It is the key ingredient to the lifting properties of the various level model structures on the category $\mathcal{S}p_{\mathcal{C}}$ that we will discuss in the next section. We recall that a pair $(i : A \longrightarrow B, f : X \longrightarrow Y)$ of morphisms in some category has the *lifting property* if for all morphism $\varphi : A \longrightarrow X$ and $\psi : B \longrightarrow Y$ such that $f\varphi = \psi i$ there exists a *lifting*, i.e., a morphism $\lambda : B \longrightarrow Y$ such that $\lambda i = \varphi$ and $f\lambda = \psi$. Instead of saying that the pair (i, f) has the lifting property we may equivalently say 'i has the left lifting property with respect to f' or 'f has the right lifting property with respect to i'.

Proposition 1.17. *Let $i : A \longrightarrow B$ and $f : X \longrightarrow Y$ be two morphisms of symmetric spectra in $\mathcal{C}$. If the pair $(\nu_m i : A_m \cup_{L_m A} L_m B \longrightarrow B,\ f_m : X_m \longrightarrow Y_m)$ has the lifting property in the category $\Sigma_m\mathcal{C}$ for every $m \geq 0$, then the pair (i, f) has the lifting property in $\mathcal{S}p_{\mathcal{C}}$.*

Proof. We consider the class f-cof of all morphisms in $\mathcal{S}p_{\mathcal{C}}$ that have the left lifting property with respect to f; this class is closed under cobase change and countable composition. Since the pair $(\nu_m i, f_m)$ has the lifting property in $\Sigma_m\mathcal{C}$, the semifree morphism $G_m(\nu_m i)$ belongs to the class f-cof by adjointness. The relative skeleton filtration (1.13) shows that i is a countable composite of cobase changes of the morphisms $\nu_m i$, so i belongs to the class f-cof. □

Example 1.18 (Mapping spaces in $\mathcal{S}p_{\mathcal{C}}$)**.** There is a whole simplicial set, and even a symmetric spectrum of simplicial sets worth of morphisms between two symmetric spectra X and Y in $\mathcal{C}$. We define the *mapping space* as the equalizer, in the category of based simplicial sets, of the diagram

$$\mathrm{map}_{\mathcal{C}}(X,Y) \longrightarrow \prod_{n\geq 0} \mathrm{map}(X_n, Y_n)^{\Sigma_n} \rightrightarrows \prod_{n\geq 0} \mathrm{map}(X_n \wedge S^1, Y_{n+1})^{\Sigma_n} .$$

For $\mathcal{C} = \mathbf{sS}$, i.e., symmetric spectra of simplicial sets, this precisely recovers the mapping space as defined in Example I.3.36; For $\mathcal{C} = \mathbf{T}$, i.e., symmetric spectra of spaces, the present mapping space is the singular complex of the topological mapping space of Example I.3.36.

For a based simplicial set K and symmetric spectra X and Y we have adjunction isomorphisms of simplicial sets

$$\mathrm{map}_{\mathbf{sS}}(K, \mathrm{map}_{\mathcal{C}}(X,Y)) \cong \mathrm{map}_{\mathcal{C}}(K \wedge X, Y) \cong \mathrm{map}_{\mathcal{C}}(X, Y^K) .$$

We can also define a symmetric 'function spectrum' $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, just as in Example I.3.38: in level n we set $\mathrm{Hom}_{\mathcal{C}}(X,Y)_n = \mathrm{map}_{\mathcal{C}}(X, \mathrm{sh}^n Y)$ with Σ_n-action induced by the action on $\mathrm{sh}^n Y$ as above. The structure map is defined as in Example I.3.38.

We have associative and unital composition maps

$$\circ \ : \ \mathrm{map}_{\mathcal{C}}(Y,Z) \wedge \mathrm{map}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{map}_{\mathcal{C}}(X,Z)$$

of simplicial sets and

$$\circ \ : \ \mathrm{Hom}_{\mathcal{C}}(Y,Z) \wedge \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

of symmetric spectra of simplicial sets. In the special case $X = Y = Z$ the unit map $\mathbb{S} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,X)$ and the composition morphism turn $\mathrm{Hom}_{\mathcal{C}}(X,X)$ into a symmetric ring spectrum.

2. Flat cofibrations

In this section we continue to study symmetric spectra in a based simplicial category $\mathcal{C}$, but we shift the emphasis to homotopical considerations. We assume that the simplicial category is also endowed with a model structure and introduce and study the class of flat cofibrations in the category of symmetric spectra in $\mathcal{C}$. These cofibrations will later be complemented by different types of equivalence and fibrations into various model structures.

Definition 2.1. Let $\mathcal{C}$ be a pointed simplicial model category and let $f : X \longrightarrow Y$ be a morphism of symmetric spectra in $\mathcal{C}$. We call f

- a *level equivalence* if for all $n \geq 0$ the morphism $f_n : X_n \longrightarrow Y_n$ is weak equivalence in the model category $\mathcal{C}$ after forgetting the group action,
- a *level cofibration* if for all $n \geq 0$ the morphism $f_n : X_n \longrightarrow Y_n$ is a cofibration in the model category $\mathcal{C}$ after forgetting the group action,
- a *flat cofibration* if for all $n \geq 0$ the latching morphism $\nu_n f : X_n \cup_{L_n X} L_n Y \longrightarrow Y_n$ is a cofibration in the model category $\mathcal{C}$ after forgetting the group action.

We will see in Corollary 3.12 below that flat cofibrations are level cofibrations.

Example 2.2. Let $f : K \longrightarrow L$ be a morphism of Σ_m-objects in $\mathcal{C}$. Then the morphism $G_m f : G_m K \longrightarrow G_m L$ of semifree symmetric spectra is a flat cofibration if and only if f is a cofibration in $\mathcal{C}$. Indeed, the same proof as in Example I.5.35 shows that in the present more general context the kth latching object of the semifree symmetric spectrum $G_m K$ is trivial for $k \leq m$ and for $k > m$ the map $\nu_k : L_k(G_m K) \longrightarrow (G_m K)_k$ is an isomorphism. This lets us identify the terms in the commutative square of Σ_k-objects:

$$\begin{tikzcd} L_k(G_m K) \arrow[r, "L_k(G_m f)"] \arrow[d, "\nu_k"'] & L_k(G_m L) \arrow[d, "\nu_k"] \\ (G_m K)_k \arrow[r, "(G_m f)_k"'] & (G_m L)_k \end{tikzcd}$$

For $k < m$ all four terms are zero objects. For $k = m$ the two upper objects are trivial and the lower vertical map is isomorphic to f. For $k > m$ both vertical maps are isomorphisms. So the latching morphism

$$\nu_k(G_m f) \ : \ (G_m K)_n \cup_{L_k(G_m K)} L_k(G_m L) \ \longrightarrow \ (G_m L)_k$$

is an isomorphism for $k \neq m$ and isomorphic to f for $k = m$. This proves the claims.

Let $\mathcal{Z}$ be a class of morphisms in a category. We say that $\mathcal{Z}$ is *closed under cobase change* if the following holds. For every pushout square

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "g'"'] & D \end{tikzcd} \tag{2.3}$$

such that g in in $\mathcal{Z}$, the morphism g' is also in $\mathcal{Z}$. Note that every commutative square (2.3) in which both vertical morphisms are isomorphism is a pushout. So $\mathcal{Z}$ is in particular closed under isomorphisms.

We say that $\mathcal{Z}$ is *closed under countable composition* if the following holds. [we don't really need countable compositions... in every level, finite compositions suffice...]

Example 2.4. If every model category the class of cofibrations is closed under cobase change and countable composition, and so is the class of acyclic cofibrations. If $\mathcal{F}$ is a class of morphisms in any category, then the class $\mathcal{F}$-cof of all morphisms that have the left lifting property with respect to $\mathcal{F}$ is closed under under cobase change and countable composition.

Proposition 2.5. *The class of flat cofibrations is the smallest class of morphisms in $\mathcal{S}p_{\mathcal{C}}$ that is closed under cobase change and countable composition and contains the semifree morphisms $G_n i$ for all $n \geq 0$ and all Σ_n-morphisms $i : L \longrightarrow L'$ that are cofibrations in $\mathcal{C}$.*

Proof. Let us denote by $[G_n i]$ the smallest class that is closed under cobase change and countable composition and contains the semifree morphisms $G_n i$ for all Σ_n-morphisms i that are cofibrations in $\mathcal{C}$. For every such equivariant cofibration $i : L \longrightarrow L'$ the semifree morphism $G_n i : G_n L \longrightarrow G_n L'$ is a flat cofibration by Example 2.2. The class of flat cofibrations is also closed under cobase change and countable composition [...] so theflat cofibrations contain the class $[G_n i]$.

For the reverse inclusion we consider a flat cofibration $f : X \longrightarrow Y$. We use the relative skeleton filtration $F^n[f]$ of the morphism f, see 1.13. Since f is a flat cofibration the latching morphism $\nu_n f$ is a cofibration in $\mathcal{C}$. The pushout square (1.14)

$$\begin{tikzcd} G_n(X_n \cup_{L_n X} L_n Y) \arrow[r, "G_n(\nu_n f)"] \arrow[d] & G_n Y_n \arrow[d] \\ F^{n-1}[f] \arrow[r, "j_n[f]"'] & F^n[f] \end{tikzcd}$$

implies that the skeleton 'inclusion' $j_n[f] : F^{n-1}[f] \longrightarrow F^n[f]$ belongs to $[G_n i]$. The symmetric spectrum Y is the colimit of the sequence

$$X = F^{-1}[f] \xrightarrow{j_0[f]} F^0[f] \xrightarrow{j_1[f]} F^1[f] \xrightarrow{j_2[f]} F^2[f] \cdots \xrightarrow{j_m[f]} F^m[f] \cdots ,$$

and f is the countable composition of the morphisms $j_n[f]$. So f belongs to $[G_n i]$. $\square$

Given morphisms $f : X \longrightarrow Y$ in $\mathcal{S}p_{\mathcal{C}}$ and $g : A \longrightarrow B$ in $\mathcal{S}p_{\mathbf{sS}}$ we denote by $f\Box g$ the *pushout product morphism* defined as

$$f\Box g = (Y \wedge g) \cup (f \wedge B) \ : \ Y \wedge A \cup_{X\wedge A} X \wedge B \ \longrightarrow \ Y \wedge B \ .$$

Proposition 2.6. (i) *Let $i : L \longrightarrow L'$ be a Σ_n-morphism that is a cofibration in $\mathcal{C}$ and let $g : A \longrightarrow B$ be a level cofibration of symmetric spectra of simplicial sets. Then the pushout product morphism $(G_n i)\Box g$ is a level cofibration of symmetric spectra in $\mathcal{C}$. If in addition i is a weak equivalence or if g is a level equivalence, then $(G_n i)\Box g$ is a level equivalence.*

(ii) *Let $f : X \longrightarrow Y$ be a level cofibration of symmetric spectra in $\mathcal{C}$ and let $j : A \longrightarrow A'$ be a monomorphism of based Σ_m-simplicial sets. Then the pushout product morphism $f\Box(G_m j)$ is a level cofibration of symmetric spectra in $\mathcal{C}$. If in addition f is a level equivalence or if j is a weak equivalence, then $f\Box(G_m j)$ is a level equivalence.*

Proof. We prove part (i); the proof of (ii) is similar. The smash product $(G_n L) \wedge A$ is naturally isomorphic to the twisted smash product $L \rhd_n A$. So in level $n+m$ the pushout product $(G_n i)\Box g$ is given by

$$\Sigma^+_{n+m} \wedge_{\Sigma_n\times\Sigma_m} (i\Box g_n) \ : \ \Sigma^+_{n+m} \wedge_{\Sigma_n\times\Sigma_m} (L \wedge B_n \cup_{L\wedge A_n} L' \wedge A_n) \ \longrightarrow \ \Sigma^+_{n+m} \wedge_{\Sigma_n\times\Sigma_m} (L' \wedge B_n) \ .$$

The morphism $i\Box g_n : L \wedge B_n \cup_{L\wedge A_n} L' \wedge A_n \longrightarrow L' \wedge B_n$ is a cofibration by the pushout product property in $\mathcal{C}$. If in addition i or g_m is a weak equivalence, then $i\Box g_n$ is a weak equivalence, again by the pushout product property in $\mathcal{C}$. Inducing from the group $\Sigma_n \times \Sigma_m$ to Σ_{n+m} amounts to taking an $\binom{n+m}{m}$-fold coproduct, which preserves acyclic cofibrations. So the morphism $(G_n i)\Box g$ is a level cofibration in $\mathcal{S}p_{\mathcal{C}}$, and it is a level equivalence if i is a weak equivalence or if g is a level equivalence. $\square$

Proposition 2.7. *Let $X, Y : \mathcal{C} \longrightarrow \mathcal{D}$ be two functors that preserves pushouts and countable composition and let $\psi : X \longrightarrow Y$ be a natural transformation. Let $\mathcal{Z}$ be a class of morphisms in $\mathcal{D}$ that is closed under cobase change and countable composition. Then the class*

$$\{g \in \mathcal{C} \mid \quad \psi B \cup Y g : XB \cup_{XA} YA \longrightarrow YB \ \in \mathcal{Z}\}$$

is closed under cobase change and countable composition.

Proof. Given any pushout square (2.3) in $\mathcal{C}$ the square

$$\begin{tikzcd} XB \cup_{XA} YA \arrow[r, "\psi B\cup Yg"] \arrow[d] & YB \arrow[d] \\ XD \cup_{XC} YC \arrow[r, "\psi D\cup Yg'"'] & YD \end{tikzcd}$$

is a pushout in $\mathcal{D}$ since the functors X and Y preserve pushouts [expand]. The class $\mathcal{Z}$ is closed under cobase change, so if $\psi B \cup Y g$ belongs to $\mathcal{Z}$, then so does $\psi D \cup Y g'$. Hence the class in question is closed under cobase change.

Now we consider a sequence of composable morphism

$$A_0 \xrightarrow{g_0} A_1 \xrightarrow{g_1} A_2 \xrightarrow{g_2} \cdots$$

and we let A_∞ be a colimit of the sequence and denote by $i_n : A_n \longrightarrow A_\infty$ the canonical map from A_n to the colimit. In particular, $i_0 : A_0 \longrightarrow A_\infty$ is the countable composition of the morphisms g_n.

In this situation the square

$$\begin{tikzcd} XA_{n+1}\cup_{XA_n} YA_n \arrow[r, "\psi A_{n+1}\cup Yg_n"] \arrow[d, "(Xi_{n+1})\cup\mathrm{Id}"'] & YA_{n+1} \arrow[d] \\ XA_\infty \cup_{XA_n} YA_n \arrow[r, "\mathrm{Id}\cup(Yg_n)"'] & XA_\infty\cup_{XA_{n+1}} YA_{n+1} \end{tikzcd}$$

is a pushout and the morphism $\psi A_\infty \cup Yi_0 : XA_\infty \cup_{XA_0} YA_0 \longrightarrow YA_\infty$ is isomorphic to the composite of the sequence

$$XA_\infty \cup_{XA_0} YA_0 \xrightarrow{\mathrm{Id}\cup(Yg_0)} XA_\infty \cup_{XA_1} YA_1 \xrightarrow{\mathrm{Id}\cup(Yg_1)} XA_\infty \cup_{XA_2} YA_2 \xrightarrow{\mathrm{Id}\cup(Yg_2)} \cdots .$$

So if $\psi A_{n+1}\cup Yg_n$ belongs to $\mathcal{Z}$ for all $n\geq 0$, then so do the morphisms $\mathrm{Id}\cup(Yg_n)$, and hence the morphism $\psi A_\infty \cup Yi_0$. Altogether this shows that the class of morphisms g with $\psi B\cup Yg\in\mathcal{Z}$ is closed under countable composition. □

Proposition 2.8. *Let f be a level cofibration of symmetric spectra in $\mathcal{C}$ and let g be a level cofibration of symmetric spectra of simplicial sets.*

(i) *If f or g is a flat cofibration then the pushout product morphism $f\Box g$ is a level cofibration in $\mathcal{S}p_{\mathcal{C}}$.*
(ii) *If f and g are flat cofibrations then the pushout product morphism $f\Box g$ is a flat cofibration in $\mathcal{S}p_{\mathcal{C}}$.*

Proof. We exploit that the classes of level cofibrations and flat cofibrations in $\mathcal{S}p_{\mathcal{C}}$ are closed under cobase change and countable composition. For flat cofibration this was shown in [...] Level cofibrations have these closure properties because colimits in $\mathcal{S}p_{\mathcal{C}}$ are created levelwise after forgetting the symmetric group actions, and because the cofibrations in the model category $\mathcal{C}$ are closed under cobase change and countable composition.

(i) We treat the case where f is a flat cofibration; the same arguments work with reversed roles when g is a flat cofibration. If we fix a level cofibration ψ in $\mathcal{S}p_{\mathbf{sS}}$, then the class

$$\{f\in\mathcal{S}p_{\mathcal{C}}\mid f\Box g \text{ is a level cofibration in } \mathcal{S}p_{\mathcal{C}}\}$$

is closed under cobase change and countable composition by Proposition 2.7 applied to the natural transformation $-\wedge g : -\wedge A \longrightarrow -\wedge B$. Proposition 2.6 shows that the semifree morphism $G_n i$ generated by a Σ_n-morphism $i : L\longrightarrow L'$ that is a $\mathcal{C}$-cofibration belongs to this class; since these semifree morphisms generate all flat cofibrations under cobase change and countable composition (Proposition 2.5 (i)),

(ii) Again we start with a special case, namely where $f = G_n i$ and $g = G_m j$ are semifree morphisms generated by a Σ_n-morphisms $i : L\longrightarrow L'$ that is a $\mathcal{C}$-cofibration respectively a monomorphism $j : A\longrightarrow A'$ of based Σ_m-simplicial sets. In this case $f\Box g = (G_n i)\Box(G_m j)$ is isomorphic to

$$G_{n+m}(\Sigma^+_{n+m}\wedge_{\Sigma_n\times\Sigma_m} i\Box j) .$$

The pushout product $i\Box j : L\wedge A'\cup_{L\wedge A} L'\wedge A \longrightarrow L\wedge A'$ is a $\mathcal{C}$-cofibration by the pushout product property in $\mathcal{C}$. The morphism $\Sigma^+_{n+m}\wedge_{\Sigma_n\times\Sigma_m} i\Box j$ is a coproduct of $\binom{n+m}{m}$ copies of $i\Box j$, hence a $\mathcal{C}$-cofibration. This shows that the morphism $G_{n+m}(\Sigma^+_{n+m}\wedge_{\Sigma_n\times\Sigma_m} i\Box j)$ is a flat cofibration, and so is $(G_n i)\Box(G_m j)$.

Now we fix a monomorphism j of based Σ_m-simplicial sets consider the class

$$\{f\in\mathcal{S}p_{\mathcal{C}}\mid f\Box(G_m j) \text{ is a flat cofibration in } \mathcal{S}p_{\mathcal{C}}\} .$$

This class is closed under cobase change and countable composition by Proposition 2.7 and contains all semifree morphism $G_n i$ with i a Σ_n-morphisms and $\mathcal{C}$-cofibration; since these semifree morphisms generate all flat cofibrations under cobase change and countable composition, $f\Box(G_m j)$ is a flat cofibration for all flat cofibrations f.

Finally, we fix a flat cofibration f in $\mathcal{S}p_{\mathcal{C}}$ and consider the class

$$\{g\in\mathcal{S}p_{\mathcal{C}}\mid f\Box g \text{ is a flat cofibration in } \mathcal{S}p_{\mathcal{C}}\} .$$

As before, this class is closed under cobase change and countable composition by Proposition 2.7 and contains all semifree morphism $G_m j$ with j a monomorphism of based Σ_m-simplicial sets; since these semifree morphisms generate all flat cofibrations under cobase change and countable composition, $f\Box g$ is a flat cofibration for all flat cofibrations g. $\Box$

Proposition 2.9. *Let $f : X \longrightarrow Y$ be a morphism of symmetric spectra in $\mathcal{C}$. If f is a flat cofibration, then so are the morphisms*

$$(\mathrm{sh}\, f) \cup \lambda_Y \ : \ \mathrm{sh}\, X \cup_{S^1\wedge X} (S^1 \wedge Y) \ \longrightarrow \ \mathrm{sh}\, Y \qquad \textit{and} \qquad \mathrm{sh}\, f \ : \ \mathrm{sh}\, X \ \longrightarrow \ \mathrm{sh}\, Y \ .$$

In particular, if Y is a flat symmetric spectrum in $\mathcal{C}$, then the morphism $\lambda_Y : S^1 \wedge Y \longrightarrow \mathrm{sh}\, Y$ is a flat cofibration and $\mathrm{sh}\, Y$ is again flat.

Proof. We start by analyzing the morphism $(\mathrm{sh}\, f) \cup \lambda_Y$ in the special case where $f = G_n i : G_n L \longrightarrow G_n L'$ is the semifree morphism generated by a morphism $i : L \longrightarrow L'$ of Σ_n-objects. The shift $\mathrm{sh}(G_n L)$ of a semifree symmetric spectra splits naturally as a wedge $G_{n-1}(\mathrm{sh}\, L) \vee (S^1 \wedge G_n L)$ and the morphism $\lambda_{G_n L} : S^1 \wedge G_n L \longrightarrow \mathrm{sh}(G_n L)$ is the inclusion of one of the wedge summands, compare [...]. Hence the pushout $\mathrm{sh}(G_n L) \cup_{S^1\wedge G_n L} (S^1 \wedge G_n L')$ is isomorphic to the wedge $G_{n-1}(\mathrm{sh}\, L) \vee (S^1 \wedge G_n L')$ and the morphism in question is isomorphic to

$$G_{n-1}(\mathrm{sh}\, i) \vee \mathrm{Id} \ : \ G_{n-1}(\mathrm{sh}\, L) \vee (S^1 \wedge G_n L') \ \longrightarrow \ G_{n-1}(\mathrm{sh}\, L') \vee (S^1 \wedge G_n L') \cong \mathrm{sh}(G_n L') \ .$$

If $i : L \longrightarrow L'$ is a $\mathcal{C}$-cofibration, then so is $\mathrm{sh}\, i : \mathrm{sh}\, L \longrightarrow \mathrm{sh}\, L'$. Hence the semifree morphism $G_{n-1}(\mathrm{sh}\, i)$, and thus also $G_{n-1}(\mathrm{sh}\, i) \vee \mathrm{Id}$, is a flat cofibration.

The suspension functor $S^1 \wedge -$ and the shift functor preserve colimits. By Proposition 2.7, applied to the natural transformation $\lambda_X : S^1 \wedge X \longrightarrow \mathrm{sh}\, X$, the class

$$\{f \in \mathcal{S}p_{\mathcal{C}} \mid \quad (\mathrm{sh}\, f) \cup \lambda_Y \text{ is a flat cofibration}\}$$

is closed under cobase change and countable composition. By the previous paragraph this class contains the semifree morphisms $G_n i$ for all Σ_n-morphisms $i : L \longrightarrow L'$ that are $\mathcal{C}$-cofibrations. Since these semifree morphisms generate all flat cofibrations under cobase change and countable composition, $(\mathrm{sh}\, f) \cup \lambda_Y$ is a flat cofibration for all flat cofibrations f.

If f is a flat cofibration, then so is its suspension $S^1 \wedge f : S^1 \wedge X \longrightarrow S^1 \wedge Y$. Hence the cobase change $\mathrm{sh}\, X \longrightarrow \mathrm{sh}\, X \wedge X \cup_{S^1\wedge X} S^1 \wedge Y$ is again a flat cofibration. Since $\mathrm{sh}\, f : \mathrm{sh}\, X \longrightarrow \mathrm{sh}\, Y$ is the composite of this cobase change and the morphism $(\mathrm{sh}\, f) \cup \lambda_Y$, the shift $\mathrm{sh}\, f : \mathrm{sh}\, X \longrightarrow \mathrm{sh}\, Y$ is also a flat cofibration.

The last sentence is the special case where $X = *$ is the trivial spectrum. $\Box$

3. Level model structures

In this section we continue to study symmetric spectra in a based simplicial category $\mathcal{C}$, but we shift the emphasis to homotopical considerations. We assume that the simplicial category is also endowed with a model structure. Then we can introduce certain notions of equivalences, fibrations and cofibrations in the category $G\mathcal{C}$ of G-objects in $\mathcal{C}$. These notions typically conspire into three model structures: the *weak*, *strong* and *mixed* equivariant model structure on $G\mathcal{C}$.

For the definitions we recall that the object X^G of *fixed points* of a G-object X is, by definition, the equalizer in $\mathcal{C}$ of the diagram

$$X^G \longrightarrow X \rightrightarrows \mathrm{map}(G, X)$$

where the two maps to be equalized are adjoint to the projection $G^+ \wedge X \longrightarrow X$ respectively the action morphism. The fixed point functor $G\mathcal{C} \longrightarrow \mathcal{C}$, $X \mapsto X^G$ is right adjoint to the free functor which sends a $\mathcal{C}$-object Y to $G^+ \wedge Y$.

The *homotopy fixed points* of a G-object X is the $\mathcal{C}$-object $X^{hG} = \mathrm{map}(EG, X)^G$ of G-equivariant maps from the free contractible G-space EG to X. The unique map of simplicial sets $EG \longrightarrow *$ is equivariant and induces a natural map $X^G = \mathrm{map}(*, X)^G \longrightarrow \mathrm{map}(EG, X)^G = X^{hG}$ from the fixed points to the homotopy fixed points.

Definition 3.1. Given a simplicial model category $\mathcal{C}$ and a group G, a morphism $f : X \longrightarrow Y$ in $G\mathcal{C}$ is called a

- a *weak G-equivalence* if the underlying morphism in $\mathcal{C}$ is weak equivalence;
- a *strong G-equivalence* if for every subgroup H of G the map of H-fixed points $f^H : X^H \longrightarrow Y^H$ is a weak equivalence in $\mathcal{C}$;
- a *weak G-fibration* if the underlying morphism in $\mathcal{C}$ is fibration;
- a *strong G-fibration* if for every subgroup H of G the map of H-fixed points $f^H : X^H \longrightarrow Y^H$ is a fibration in $\mathcal{C}$;
- a *strict G-fibration* if it is a strong fibration and for every subgroup H of G the square

$$\begin{tikzcd} X^H \arrow[r] \arrow[d, "f^H"'] & X^{hH} \arrow[d, "f^{hH}"] \\ Y^H \arrow[r] & Y^{hH} \end{tikzcd} \tag{3.2}$$

  is homotopy cartesian in the model category $\mathcal{C}$;
- a *G-cofibration* if it has the left lifting property for all strong G-acyclic fibrations;
- a *free G-cofibration* if it has the left lifting property for all weak G-acyclic fibrations.

For many simplicial model categories $\mathcal{C}$, the category of G-objects inherits the following equivariant model structures. The *weak equivariant model structure* on $G\mathcal{C}$ consists of the weak G-equivalences, weak G-fibrations and free G-cofibrations. The *strong equivariant model structure* on $G\mathcal{C}$ consists of the strong G-equivalences, strong G-fibrations and G-cofibrations. The *mixed equivariant model structure* on $G\mathcal{C}$ consists of the weak G-equivalences, strict G-fibrations and G-cofibrations. Often even more variations are possible, for example one can prescribe a family $\mathcal{F}$ of subgroups of G and define $\mathcal{F}$-weak equivalences and $\mathcal{F}$-fibrations by testing after taking H-fixed points for all $H \in \mathcal{F}$.

In [...] we have defined various classes of cofibrations. We recall that a morphism $f : X \longrightarrow Y$ of symmetric spectra in $\mathcal{C}$ is a level equivalence (respectively level cofibration) if f_n is a weak equivalence (respectively Σ_n-cofibration) for all $n \geq 0$. The morphisms f is a flat cofibration (respectively projective cofibration) if the latching morphism $\nu_n(f) : X_n \cup_{L_n X} L_n Y \longrightarrow Y_n$ is a Σ_n-cofibration (respectively free Σ_n-cofibration) for all $n \geq 0$.

Definition 3.3. Let $\mathcal{C}$ be a pointed simplicial model category and let $f : X \longrightarrow Y$ be a morphism of symmetric spectra in $\mathcal{C}$. We call f

- a *level fibration* if for all $n \geq 0$, the morphism $f_n : X_n \longrightarrow Y_n$ is a weak Σ_n-fibration, i.e., a fibration in the model category $\mathcal{C}$ after forgetting the group action.
- a *strict fibration* if for all $n \geq 0$, the morphism $f_n : X_n \longrightarrow Y_n$ is a strict Σ_n-fibration.

An *injective class* in a category $\mathcal{C}$ is a class F of morphisms with the following property: every morphism f in $\mathcal{C}$ can be factored as $f = qi$ such that q is in F and i has the left lifting property with respect to all morphisms in F. The obvious examples of injective classes are the fibrations and the acyclic fibrations in a closed model category.

Proposition 3.4. *Suppose we are given an injective class F_n in the category $\Sigma_n\mathcal{C}$ for every $n \geq 0$. Then every morphism f in $\mathcal{S}p_{\mathcal{C}}$ can be factored as $f = qi$ where q and i satisfy the following properties: for every $n \geq 0$ the morphism q_n is in F_n and the latching morphism $\nu_n i$ has the left lifting property with respect to all morphisms in F_n.*

Proof. Let $f : A \longrightarrow X$ be a morphism of symmetric spectra in $\mathcal{C}$. We construct a symmetric spectrum B and morphisms $i : A \longrightarrow B$ and $q : B \longrightarrow X$ by induction over the levels. In level 0 we choose a factorization

$$A_0 \xrightarrow{i_0} B_0 \xrightarrow{q_0} X_0$$

of f_0 such that q_0 belongs to F_0 and i_0 has the left lifting with respect to the class F_0.

Now suppose that the symmetric spectrum B and the morphisms i and q have already been constructed up to level $m-1$. Then we have all the data necessary to define the m-th latching object $L_m B$; moreover, the 'partial morphism' $q : B \longrightarrow X$ provides a Σ_m-morphism $L_m B \longrightarrow X_m$ such that the square

$$\begin{tikzcd} L_m A \arrow[r] \arrow[d, "\nu_m"'] & L_m B \arrow[d] \\ A_m \arrow[r, "f_m"'] & X_m \end{tikzcd}$$

commutes. We factor the resulting morphism $A_m \cup_{L_m A} L_m B \longrightarrow X_m$ in $\Sigma_m \mathcal{C}$ as

$$A_m \cup_{L_m A} L_m B \xrightarrow{\nu_m} B_m \xrightarrow{q_m} X_m$$

such that q_m belongs to F_m and ν_m has the left lifting with respect to the class F_m. The intermediate object B_m defines the m-th level of the symmetric spectrum B, and the second morphism q_m is the mth level of the morphism q. The structure morphism $\sigma_n : B_{m-1} \wedge S^1 \longrightarrow B_m$ is the composite

$$B_{m-1} \wedge S^1 \longrightarrow L_m B \longrightarrow A_m \cup_{L_m A} L_m B \xrightarrow{\nu_m} B_m$$

and the composite of ν_m with the canonical morphism $A_m \longrightarrow A_m \cup_{L_m A} L_m B$ is the m-th level of the morphism i.

At the end of the day we have indeed factored $f = qi$ in the category of symmetric spectra in $\mathcal{C}$ and q_m belongs to F_m for all $m \geq 0$. Morevoer, the mth latching morphism $\nu_m i$ comes out to be $\nu_m : A_m \cup_{L_m A} L_m B \longrightarrow B_m$ which has the left lifting property with respect to the class F_m by construction. □

Proposition 3.5. *Let $f : X \longrightarrow Y$ be a flat cofibration of symmetric spectra in $\mathcal{C}$ such that for every $m \geq 0$ the latching morphism $\nu_m f : X_m \cup_{L_m X} L_m Y \longrightarrow Y_m$ is a weak Σ_m-equivalence. Then the morphism f is a level equivalence.*

Proof. We use the relative skeleton filtration (1.13) of the morphism $f : X \longrightarrow Y$ by the intermediate spectra $F^m[f]$. Since the latching morphism $\nu_m f$ is a weak equivalence and cofibration in $\mathcal{C}$, the semifree morphism $G_m(\nu_m f) : G_m(X_m \cup_{L_m X} L_m Y) \longrightarrow G_m Y_m$ is a level cofibration and level equivalence [ref?]. The morphism $j_m[f] : F^{m-1}[f] \longrightarrow F^m[f]$ is a cobase change of $G_m(\nu_m f)$, compare the pushout square (1.14), so $j_m[f]$ is a level cofibration and level equivalence. In level m the skeleton filtration stabilizes after m steps, i.e., the map f_m is the composite

$$X_m = (F^{-1}[f])_m \xrightarrow{(j_0[f])_m} (F^0[f])_m \xrightarrow{(j_1[f])_m} \dots \xrightarrow{(j_{m-1}[f])_m} (F^{m-1}[f])_m \xrightarrow{(j_m[f])_m} (F^m[f])_m = Y_m \ .$$

Since each $(j_k[f])_m$ is a weak Σ_m-equivalence, so is f_m. □

Now we can easily establish two level model structures on the category $\mathcal{S}p_{\mathcal{C}}$ in which the weak equivalences are the level equivalences. We start with the projective level model structure.

Theorem 3.6 (Projective level model structure)**.** *Let $\mathcal{C}$ be a pointed simplicial category such that for every $n \geq 0$ the category $\Sigma_n \mathcal{C}$ admits the weak equivariant model structure. Then the level equivalences, level fibrations and projective cofibrations define the* projective level model structure *on the category $\mathcal{S}p_{\mathcal{C}}$ of symmetric spectra in $\mathcal{C}$. Moreover, the following properties holds.*

(i) *A morphism $i : A \longrightarrow B$ in $\mathcal{S}p_{\mathcal{C}}$ is simultaneously a projective cofibration and a level equivalence if and only if for all $n \geq 0$ the latching morphism $\nu_n i : A_n \cup_{L_n A} L_n B \longrightarrow B_n$ is a free Σ_n-cofibration and a weak equivalence after forgetting the group action.*

(ii) *If the model category $\mathcal{C}$ is right proper (respectively left proper), then the projective level model structure on $\mathcal{S}p_{\mathcal{C}}$ is right proper (respectively left proper).*

(iii) *If the fibrations in $\mathcal{C}$ are detected by a set of morphisms J, then the level fibrations in $\mathcal{S}p_{\mathcal{C}}$ are detected by the set of free morphisms*

$$\{F_n j \mid n \geq 0, j \in J\} \ .$$

(iv) *If the acyclic fibrations in $\mathcal{C}$ are detected by a set of morphisms I, then the level acyclic fibrations in $\mathcal{S}p_{\mathcal{C}}$ are detected by the set of free morphisms*

$$\{F_n i \mid n \geq 0, i \in I_n\} \ .$$

(v) *The projective level model structure is simplicial and monoidal over the projective level model structure for symmetric spectra of simplicial sets.*

Proof. Limits, colimits, tensors and cotensors are defined levelwise. The 2-out-of-3 property for level equivalences and the closure properties of the three distinguished classes under retracts are direct consequences of the corresponding properties in the projective model structures on $\Sigma_n\mathcal{C}$.

The factorization properties are obtained by applying by applying Proposition 3.4 to the two factorization systems (free Σ_n-cofibrations, weak Σ_n-acyclic fibrations) and (free Σ_n-cofibrations which are weak Σ_n-equivalences, weak Σ_n-fibrations). In the second case we need Proposition 3.5 to see that the resulting projective cofibration (which is in particular a flat cofibration) is also a level equivalence.

It remains to show the lifting axioms. Since free Σ_m-cofibrations have the left lifting property with respect to weak Σ_m-acyclic fibrations, projective cofibrations have the left lifting property with respect to acyclic level fibrations by Proposition 1.17.

We postpone the proof of the other lifting property and we pause to prove the claim (i) next. Suppose that $i : A \longrightarrow B$ is a projective cofibration and a level equivalence. The second factorization axiom proved above provides a factorization $i = pj$ where $j : A \longrightarrow D$ is a level equivalence such that each latching morphism $\nu_m j$ is an acyclic cofibration in the weak equivariant model structure on $\Sigma_m\mathcal{C}$, and $p : D \longrightarrow B$ is a level fibration. Since i and j are level equivalences, so is p. So the projective cofibration i has the left lifting property with respect to the level acyclic fibration p by what we already showed. In particular, a lift $\lambda : B \longrightarrow D$ in the square

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & D \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dotted, "\lambda"'] & B \end{tikzcd} \tag{3.7}$$

shows that the morphism i is a retract of the morphism j. In particular, the latching morphism $\nu_n i$ is a retract of the latching morphism $\nu_n j$, hence also an acyclic cofibration in the weak equivariant model structure on $\Sigma_m\mathcal{C}$. This proves one direction of claim (i); the other direction follows from Proposition 3.5 because every projective cofibration is in particular a flat cofibration.

Now we prove the remaining half of the lifting properties. We let $i : A \longrightarrow B$ be a projective cofibration that is also a level equivalence. By (i), which has just been shown, each latching morphism $\nu_m i$ is an acyclic cofibration in the weak equivariant model structure on $\Sigma_m\mathcal{C}$. So i has the left lifting property with respect to all level fibrations by Proposition 1.17.

(ii) Weak equivalences, fibrations and limits in $\mathcal{S}p_{\mathcal{C}}$ are all defined or detected levelwise after forgetting the group actions. Moreover, projective cofibrations are in particular level cofibrations by Corollary 3.12. So if the model category $\mathcal{C}$ is right (left) proper, then so is the projective level model structure.

Properties (iii) and (iv) are straightforward consequences of the fact that the free functor F_m is left adjoint to evaluation at level m and forgetting the Σ_m-action.

(v) We have to establish the pushout product property: Let f be a projective cofibration in $\mathcal{S}p\_Cc$ and let g be a projective cofibration in $\mathcal{S}p_{\mathbf{sS}}$. Then the pushout product $f\Box g$ is a projective cofibration in $\mathcal{S}p_{\mathcal{C}}$ by Proposition 2.8 (iii). If in addition f or g is a level equivalence, then so is the pushout product $f\Box g$ by Proposition 3.15. [simplicial is a special case] □

Another general kind of level model structure is the 'flat' level model structure which has the same weak equivalences as the projective model structure, but more cofibration (and hence fewer fibrations).

Theorem 3.8 (Flat level model structure)**.** *Let $\mathcal{C}$ be a pointed simplicial category such that for every $n \geq 0$ the category $\Sigma_n\mathcal{C}$ admits the mixed equivariant model structure. Then the level equivalences, strict fibrations and flat cofibrations define the* flat level model structure *on the category $\mathcal{S}p_{\mathcal{C}}$ of symmetric spectra in $\mathcal{C}$. Moreover, the following properties holds.*

(i) *A morphism $j : A \longrightarrow B$ in $\mathcal{S}p_{\mathcal{C}}$ is simultaneously a flat cofibration and a level equivalence if and only if for all $n \geq 0$ the latching morphism $\nu_n j : A_n \cup_{L_n A} L_n B \longrightarrow B_n$ is a Σ_n-cofibration and a weak equivalence after forgetting the group action.*

(ii) *If the model category $\mathcal{C}$ is right proper (respectively left proper), then the flat level model structure on $\mathcal{S}p_{\mathcal{C}}$ is right proper (respectively left proper).*

(iii) *Suppose that the fibrations in $\mathcal{C}$ are detected by a set of morphisms J and the acyclic fibrations in $\mathcal{C}$ are detected by a set of morphisms I. Then the level fibrations in $\mathcal{S}p_{\mathcal{C}}$ are detected by the set of semifree morphisms* **[check...]**

$$\{G_n(\Sigma_n/H^+ \wedge j) \mid n \geq 0,\, H \leq \Sigma_n,\, j \in J\}$$

together with the set of semifree morphisms **[check...]**

$$\{G_n(f \square i) \mid n \geq 0,\, i \in I\}$$

where $f : E\Sigma_n \longrightarrow C(E\Sigma_n)$ is the cone inclusion and for $i : A \longrightarrow B$ we denote by $f \square i$ is the pushout product

$$f \square i \ : \ C(E\Sigma_n)^+ \wedge A \cup_{E\Sigma_n^+ \wedge A} E\Sigma_n^+ \wedge B \ \longrightarrow \ C(E\Sigma_n)^+ \wedge B \ .$$

(iv) *If the acyclic fibrations in $\mathcal{C}$ are detected by a set of morphisms I, then the level equivalences in $\mathcal{S}p_{\mathcal{C}}$ which are also strict fibrations are detected by the set of semifree morphisms*

$$\{G_n(\Sigma_n/H^+ \wedge i) \mid n \geq 0,\, H \leq \Sigma_n,\, i \in I\} \ .$$

(v) *The flat level model structure is simplicial and monoidal over the flat level model structure for symmetric spectra of simplicial sets.*

Proof. Limits, colimits, tensors and cotensors are defined levelwise. The 2-out-of-3 property for level equivalences and the closure properties of the three distinguished classes under retracts are direct consequences of the corresponding properties in the mixed model structures on $\Sigma_n\mathcal{C}$.

The factorization properties are obtained by applying by applying Proposition 3.4 to the two factorization systems (Σ_n-cofibrations, mixed Σ_n-acyclic fibrations) and (Σ_n-cofibrations which are weak Σ_n-equivalences, strict Σ_n-fibrations). In the second case we need Proposition 3.5 to see that the resulting flat cofibration is also a level equivalence.

It remains to show the lifting axioms. Since Σ_n-cofibrations have the left lifting property with respect to mixed Σ_n-acyclic fibrations, flat cofibrations have the left lifting property with respect to level equivalences which are also strict fibrations by Proposition 1.17.

We postpone the proof of the other lifting property and we pause to prove the claim (i) next. Suppose that $i : A \longrightarrow B$ is a flat cofibration and a level equivalence. The second factorization axiom proved above provides a factorization $i = pj$ where $j : A \longrightarrow D$ is a level equivalence such that each latching morphism $\nu_m j$ is an acyclic cofibration in the mixed equivariant model structure on $\Sigma_m\mathcal{C}$, and $p : D \longrightarrow B$ is a strict fibration. Since i and j are level equivalences, so is p. So the flat cofibration i has the left lifting property with respect to the level equivalence and strict fibration p by what we already showed. In particular (compare the same step (3.7) in the proof of Proposition 3.6), the morphism i is a retract of the morphism j and so the latching morphism $\nu_n i$ is a retract of the latching morphism $\nu_n j$. Since ν_j is an acyclic cofibration in the weak equivariant model structure on $\Sigma_m\mathcal{C}$, so is its retract $\nu_n i$. This proves one direction of claim (i); the other direction is given by Proposition 3.5.

Now we prove the remaining half of the lifting properties. We let $i : A \longrightarrow B$ be a flat cofibration that is also a level equivalence. By (i), which has just been shown, each latching morphism $\nu_m i$ is an acyclic cofibration in the mixed equivariant model structure on $\Sigma_m\mathcal{C}$. So i has the left lifting property with respect to all strict fibrations by Proposition 1.17.

(ii) Weak equivalences, fibrations and limits in $\mathcal{S}p_{\mathcal{C}}$ are all defined or detected levelwise after forgetting the group actions. Moreover, flat cofibrations are in particular level cofibrations by Corollary 3.12 So if the model category $\mathcal{C}$ is right (left) proper, then so is the flat level model structure.

Properties (iii) and (iv) are straightforward consequences of the fact that the free functor F_m is left adjoint to evaluation at level m and forgetting the Σ_m-action.

(v) We have to establish the pushout product property: Let f be a flat cofibration in $\mathcal{S}p_{-}Cc$ and let g be a flat cofibration in $\mathcal{S}p_{\mathbf{sS}}$. Then the pushout product $f\Box g$ is a flat cofibration in $\mathcal{S}p_{\mathcal{C}}$ by Proposition 2.8 (ii). If in addition f or g is a level equivalence, then so is the pushout product $f\Box g$ by Proposition 3.15. [simplicial is a special case] □

Besides the projective and flat level model structures, another general class of Σ-model structure arise from the 'strong' equivariant model structures, see Exercise 4.1 for details.

Remark 3.9 (Functorial factorization)**.** If the projective (respectively mixed) model structure on the category $\Sigma_n\mathcal{C}$ has functorial factorizations for all $n \geq 0$, then the projective (respectively flat) level model structure on $\mathcal{S}p_{\mathcal{C}}$ has functorial factorizations. Indeed, the explicit construction on the factorization in Proposition 3.4 does not introduce any new choices.

Later we want to obtain stable model structures for certain categories of more structured symmetric spectra, for example for commutative symmetric ring spectra (or, more generally, algebras over operads). The projective and flat model structures cannot always be lifted to categories of more structured objects. However, a small modification solves this problem, namely replacing the 'absolute' level (and stable) model structure by 'positive' version as follows. The essence of the positive level model structure is that the objects in level 0 have no homotopical significance.

Definition 3.10. A morphism $f : X \longrightarrow Y$ of symmetric spectra in $\mathcal{C}$ is

- a *positive level equivalence* if $f_n : X_n \longrightarrow Y_n$ is weak equivalence in the model category $\mathcal{C}$ for all $n > 0$,
- a *positive level fibration* if $f_n : X_n \longrightarrow Y_n$ is fibration in the model category $\mathcal{C}$ for all $n > 0$,
- a *positive strict fibration* if $f_n : X_n \longrightarrow Y_n$ is mixed Σ_n-fibration for all $n > 0$,
- a *positive projective cofibration* it is a projective cofibration and the morphism $f_0 : X_0 \longrightarrow Y_0$ is an isomorphism.
- a *positive flat cofibration* it is a flat cofibration and the morphism $f_0 : X_0 \longrightarrow Y_0$ is an isomorphism.

A morphism in $\mathcal{S}p_{\mathcal{C}}$ is then simultaneously a positive level equivalence and positive cofibration if if the morphism $f_0 : X_0 \longrightarrow Y_0$ is an isomorphism and the latching morphism $\nu_n(f) : X_n \cup_{L_n X} L_n Y \longrightarrow Y_n$ is a cofibration and a weak equivalence in the model category $\Sigma_n\mathcal{C}$ for all $n > 0$.

Theorem 3.11 (Positive projective level model structure)**.** *Let $\mathcal{C}$ be a pointed simplicial category such that for every $n \geq 0$ the category $\Sigma_n\mathcal{C}$ admits the weak equivariant model structure. Then the positive level equivalences, positive level fibrations and positive projective cofibrations define the* positive projective level model structure *on the category $\mathcal{S}p_{\mathcal{C}}$ of symmetric spectra in $\mathcal{C}$. Moreover, the following properties holds.*

(i) *A morphism $j : A \longrightarrow B$ in $\mathcal{S}p_{\mathcal{C}}$ is simultaneously a positive projective cofibration and a positive level equivalence if and only $j_0 : A_0 \longrightarrow B_0$ is an isomorphism and for all $n > 0$ the latching morphism $\nu_n(j) : A_n \cup_{L_n A} L_n B \longrightarrow B_n$ is a free Σ_n-cofibration and a weak equivalence after forgetting the group action.*

(ii) *If the model category $\mathcal{C}$ is right proper (respectively left proper), then the projective level model structure on $\mathcal{S}p_{\mathcal{C}}$ is right proper (respectively left proper).*

(iii) *If the fibrations in $\mathcal{C}$ are detected by a set of morphisms J, then the level fibrations in $\mathcal{S}p_{\mathcal{C}}$ are detected by the set of free morphisms*

$$\{F_n j \mid n > 0, j \in J\} \ .$$

(iv) *If the acyclic fibrations in $\mathcal{C}$ are detected by a set of morphisms I, then the level acyclic fibrations in $\mathcal{S}p_{\mathcal{C}}$ are detected by the set of free morphisms*

$$\{F_n i \mid n > 0, i \in I_n\} \ .$$

(v) *The positive projective level model structure is simplicial and monoidal over the absolute projective level model structure for symmetric spectra of simplicial sets.*

Proof. Most properties follow from the absolute projective level model structure, or are straightforward, such as the 2-out-of-3 property for level equivalences and the closure properties under retracts.

Since free Σ_m-cofibrations have the left lifting property with respect to weak Σ_m-acyclic fibrations, projective cofibrations have the left lifting property with respect to acyclic level fibrations by Proposition 1.17.

By Proposition 3.13 every projective cofibration i which is also a level equivalence has the property that all latching morphisms $\nu_m i$ are acyclic cofibrations in the weak equivariant model structure on $\Sigma_m\mathcal{C}$, and these have the left lifting property with respect to weak Σ_m-fibrations. So projective cofibrations which are also level equivalences have the left lifting property with respect to level fibrations by Proposition 1.17.

The factorization properties are obtained by applying by applying Proposition 3.4 to the two factorization systems (free Σ_n-cofibrations, weak Σ_n-acyclic fibrations) and (free Σ_n-cofibrations which are weak Σ_n-equivalences, weak Σ_n-fibrations). In the second case we need Corollary **??** to see that the resulting projective cofibration is also a level equivalence.

(ii) Weak equivalences, fibrations and limits in $\mathcal{S}p_{\mathcal{C}}$ are all defined or detected levelwise after forgetting the group actions. Moreover, projective cofibrations are in particular level cofibrations by Corollary 3.12. So if the model category $\mathcal{C}$ is right (left) proper, then so is the projective level model structure.

Properties (iii) and (iv) are straightforward consequences of the fact that the free functor F_m is left adjoint to evaluation at level m and forgetting the Σ_m-action.

(v) The pushout product property for the positive projective level model structure is immediate from the pushout product property for the absolute projective level model structure (part (v) of Theorem3.6) and the additional observation that if one of the two morphisms f or g is an isomorphism in level zero, then the pushout product $f\Box g$ is an isomorphism in level zero. $\Box$

same for positive flat level

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[Stuff on projective cofibrations]

Thus we have the implications

$$\text{projective cofibration} \implies \text{flat cofibration} \implies \text{level cofibration}\ ,$$

and these containments are typically strict.

We remark that in the context of simplicial sets, a Σ_n-cofibration is just an equivariant morphism which is injective. Hence a morphism of symmetric spectra of simplicial sets is a level cofibration if and only if it is levelwise injective, i.e., a categorical monomorphism.

(1) The class of projective cofibrations is the smallest the class of morphisms in $\mathcal{S}p_{\mathcal{C}}$ that is closed under cobase change and countable composition by the free morphisms $F_n i$ for all $n \geq 0$ and all cofibrations $i : K \longrightarrow K'$ in $\mathcal{C}$.
(2) If f and g are projective cofibrations then the pushout product morphism $f\Box g$ is a projective cofibration in $\mathcal{S}p_{\mathcal{C}}$.

(iii) This is the same argument as in (ii) except that the initial special case is where $f = F_n i$ and $g = F_m j$ are free morphisms generated by a cofibration $i : L \longrightarrow L'$ in $\mathcal{C}$ respectively a cofibration $j : A \longrightarrow A'$ of based simplicial sets. In this case $f\Box g = (F_n i)\Box(F_m j)$ is isomorphic to $F_{n+m}(i\Box j)$ and morphism $i\Box j : L \wedge A' \cup_{L\wedge A} L' \wedge A \longrightarrow L \wedge A'$ is a cofibration by the pushout product property in the simplicial model category $\mathcal{C}$. Hence the morphism $F_{n+m}(i\Box j)$ and thus $(F_n i)\Box(F_m j)$ are projective cofibrations. The general case then follows as in (ii) using that the free morphisms $F_n i$ for cofibrations i generate all projective cofibrations under cobase change and countable composition.

[make exercise?]

Corollary 3.12. *A morphism f of symmetric spectra in $\mathcal{C}$ is a flat cofibration if and only if for every level cofibration $g : A \longrightarrow B$ of symmetric spectra of simplicial sets the pushout product map $f\Box g$ is a level cofibration in $\mathcal{S}p_{\mathcal{C}}$. In particular, every flat cofibration is a level cofibration.*

Proof. The 'only if' direction is part (i) of Proposition 2.8. The inclusion $\bar{\mathbb{S}} \longrightarrow \mathbb{S}$ of the truncated sphere spectrum is a level cofibration of symmetric spectra of simplicial sets and in level m the pushout product of f with this inclusion is the latching morphism $\nu_m f : X_m \cup_{L_m X} L_m Y \longrightarrow Y_m$. So the pushout product condition for all level cofibrations implies that $f : X \longrightarrow Y$ is a flat cofibration. The pushout product of f with the level cofibration $* \longrightarrow \mathbb{S}$ is isomorphic to f. So the pushout product condition for all level cofibrations implies that $f : X \longrightarrow Y$ is a level cofibration. $\square$

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If f is a projective cofibration, then so are the morphisms

$$(\mathrm{sh}\, f) \cup \lambda_Y \ : \ \mathrm{sh}\, X \cup_{S^1 \wedge X} (S^1 \wedge Y) \ \longrightarrow \ \mathrm{sh}\, Y \qquad \text{and} \qquad \mathrm{sh}\, f \ : \ \mathrm{sh}\, X \ \longrightarrow \ \mathrm{sh}\, Y \ .$$

In particular, if Y is a projective symmetric spectrum in $\mathcal{C}$, then the morphism $\lambda_Y : S^1 \wedge Y \longrightarrow \mathrm{sh}\, Y$ is a projective cofibration and $\mathrm{sh}\, Y$ is again projective.

The class of projective cofibrations that are also level equivalences is the smallest the class of morphisms in $\mathcal{S}p_{\mathcal{C}}$ that is closed under cobase change and countable composition and contains the free morphisms $F_n i$ for all $n \geq 0$ and all cofibrations $i : K \longrightarrow K'$ in $\mathcal{C}$ that are also weak equivalence.

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Proposition 3.13. *Let $f : X \longrightarrow Y$ be a flat cofibration of symmetric spectra in $\mathcal{C}$. Then the following are equivalent:*

(i) *the morphism f is a level equivalence;*
(ii) *for every $m \geq 0$ the latching morphism $\nu_m f : X_m \cup_{L_m X} L_m Y \longrightarrow Y_m$ is a weak Σ_m-equivalence.*

Proof. (i)$\Longrightarrow$(ii) We show by induction on m that $\nu_m f$ weak Σ_m-equivalence for all flat cofibrations $f : X \longrightarrow Y$ that are also weak equivalences. The induction starts with $m = -1$, where there is nothing to show because $L_{-1} X = *$. In the inductive step we exploit the square (1.16) that describes $L_{m+1} X$ as a pushout of the objects $L_m(\mathrm{sh}\, X)$, $L_m X \wedge S^1$ and $X_m \wedge S^1$. The morphism $L_{m+1} f : L_{m+1} X \longrightarrow L_{m+1} Y$ is thus induced on horizontal pushouts by the commutative diagram:

$$\begin{tikzcd}
L_m(\mathrm{sh}\, X) \arrow[d, "L_m(\mathrm{sh}\, f)"'] & L_m X \wedge S^1 \arrow[l, "L_m \lambda_X"'] \arrow[r, "\nu_m \wedge S^1"] \arrow[d, "L_m f \wedge S^1"'] & X_m \wedge S^1 \arrow[d, "f_m \wedge S^1"] \\
L_m(\mathrm{sh}\, Y) & L_m Y \wedge S^1 \arrow[l, "L_m \lambda_Y"] \arrow[r, "\nu_m \wedge S^1"'] & Y_m \wedge S^1
\end{tikzcd}$$

The morphism $\nu_m f : X_m \cup_{L_m X} L_m Y \longrightarrow Y_m$ is a cofibration since f is a flat cofibration, and a weak equivalence by induction. So its suspension $\nu_m f \wedge S^1 : X_m \wedge S^1 \cup_{L_m X \wedge S^1} L_m Y \wedge S^1 \cong (X_m \cup_{L_m X} L_m Y) \wedge S^1 \longrightarrow Y_m \wedge S^1$ is an acyclic cofibration. By Proposition 2.9 the shifted morphism $\mathrm{sh}\, f : \mathrm{sh}\, X \longrightarrow \mathrm{sh}\, Y$ is again a flat cofibration. Hence the morphism $L_m(\mathrm{sh}\, f) : L_m(\mathrm{sh}\, X) \longrightarrow L_m(\mathrm{sh}\, Y)$ is a cofibration by [...] and a weak equivalence by induction (because $\mathrm{sh}\, f : \mathrm{sh}\, X \longrightarrow \mathrm{sh}\, Y$ is a flat cofibration and a level equivalence). The gluing lemma (1.10) So the morphism $L_{m+1} f : L_{m+1} X \longrightarrow L_{m+1} Y$ of latching objects is a weak equivalence. This morphism is also a cofibration by [...], so its cobase change $X_{m+1} \longrightarrow X_{m+1} \cup_{L_{m+1} X} L_{m+1} Y$ is a cofibration and weak equivalence. The composite of this cobase change with the latching morphism $\nu_{m+1} : X_{m+1} \cup_{L_{m+1} X} L_{m+1} Y \longrightarrow Y_{m+1}$ is the weak equivalence f_{m+1}. So the latching morphism ν_{m+1} is a weak equivalence.

(ii)$\Longrightarrow$(i) We use the relative skeleton filtration (1.13) of the morphism $f : X \longrightarrow Y$ by the intermediate spectra $F^m[f]$. Since the latching morphism $\nu_m f$ is a weak equivalence and cofibration in $\mathcal{C}$, the semifree morphism $G_m(\nu_m f) : G_m(X_m \cup_{L_m X} L_m Y) \longrightarrow G_m Y_m$ is a level cofibration and level equivalence. The morphism $j_m[f] : F^{m-1}[f] \longrightarrow F^m[f]$ is a cobase change of $G_m(\nu_m f)$, compare the pushout square (1.14), so $j_m[f]$ is a level cofibration and level equivalence. Since f is the composite of the countable sequence $j_m[f]$, f itself is a level cofibration and weak equivalence. $\square$

Now we can prove an analogue of Proposition 2.5 for flat cofibrations that are also level equivalences:

Proposition 3.14. *The class of flat cofibrations that are also level equivalences is the smallest class of morphisms in $\mathcal{S}p_{\mathcal{C}}$ that is closed under cobase change and countable composition and contains the semifree* *morphisms $G_n i$ for all $n \geq 0$ and all Σ_n-morphisms j that are cofibrations and weak equivalences after forgetting the Σ_n-action.*

Proof. Let us denote by $[G_n j]$ the smallest class that is closed under cobase change and countable composition and contains the semifree morphisms $G_n j$ for all Σ_n-morphisms $j : L \longrightarrow L'$ that are cofibrations and weak equivalences after forgetting the Σ_n-action. Such semifree morphisms $G_n j$ are flat cofibrations by Example 2.2 and level equivalences. Flat cofibrations are in particular level cofibrations, so the class of flat cofibrations that are level equivalences is also closed under cobase change and countable composition so the class of flat cofibrations contains the class $[G_n j]$.

For the reverse inclusion we consider a flat cofibration $f : X \longrightarrow Y$ that is also a level equivalence. By Proposition 3.13 each latching morphism $\nu_m f$ is a cofibration and weak equivalence after forgetting the group action. The relative skeleton filtration shows that f is a countable composite of cobase changes of the morphisms $\nu_m f$, so f in the class $[G_n j]$. □

Proposition 3.15. *Let f be a level cofibration in $\mathcal{S}p_{\mathcal{C}}$ and let g be a level cofibration in $\mathcal{S}p_{\mathbf{sS}}$. Suppose that f or g is a flat cofibration and suppose that f or g is a level equivalence. Then the pushout product morphism $f\Box g$ is a level equivalence.*

Proof. The first case where f is a flat cofibration and level equivalence and g is a level cofibration is taken care of by Proposition 3.13. **[no longer true]**

In the remaining three cases we apply the same argument as in part (i) of Proposition 2.8. We exploit that the class of morphisms in $\mathcal{S}p_{\mathcal{C}}$ that are level cofibrations and level equivalence is closed under cobase change and countable composition. The justification for this is that colimits in $\mathcal{S}p_{\mathcal{C}}$ are created levelwise after forgetting the symmetric group actions, and because the acyclic cofibrations in the model category $\mathcal{C}$ are closed under cobase change and countable composition.

Case 2: f is a level cofibration and g is a flat cofibration and level equivalence. We consider the class

$$\{\psi \in \mathcal{S}p_{\mathbf{sS}} \mid f\Box\psi \text{ is a level cofibration and level equivalence in } \mathcal{S}p_{\mathcal{C}}\} .$$

This class is closed under cobase change and countable composition by Proposition 2.7. By Proposition 2.6 (ii) this class contains all semifree morphisms $G_n i$, where i is a Σ_n-cofibration and weak Σ_n-equivalence. So by Proposition 3.14 the class contains all flat cofibrations in $\mathcal{S}p_{\mathbf{sS}}$ that are also level equivalences.

Case 3: f is a flat cofibration and g is a level cofibration and level equivalence. We consider the class

$$\{\varphi \in \mathcal{S}p_{\mathcal{C}} \mid \varphi\Box g \text{ is a level cofibration and level equivalence in } \mathcal{S}p_{\mathcal{C}}\} .$$

By Proposition 2.6 this class contains all semifree morphisms $G_n i$, where i is a Σ_n-cofibration. The class is closed under cobase change and countable composition by Proposition 2.7. By Proposition 2.5 every flat cofibration f is in the closure of the class of morphisms $G_n i$ as above under cobase change and countable composition, so $f\Box g$ is a level cofibration and level equivalence.

Case 4: f is a level cofibration and level equivalence g is a flat cofibration. The argument proceeds as in the previous case 3, but with the roles of f and g exchanged. □

4. Stable model structures

In this section we specialize the level model structures of the previous section to symmetric spectra of spaces and simplicial sets. More importantly, we introduce the more important stable model structures on symmetric spectra of spaces and simplicial sets in which the weak equivalences are the stable equivalences as defined in Definition II.4.11. These are the most important model structure on symmetric spectra and the various associated homotopy categories are all equivalent to the stable homotopy category as defined in Section II.1, compare Corollary 4.14.

For easier reference we specialize the G-equivariant models structures (compare Definition 3.1) and the level model structures (compare Definition [...]) of the last section to the categories of based spaces and based simplicial sets.

Definition 4.1. Let G be a group. A morphism $f : X \longrightarrow Y$ of based G-spaces (respectively based G-simplicial sets) is called a

- a *weak G-equivalence* if the underlying map of spaces (respectively simplicial sets) is weak equivalence;
- a *strong G-equivalence* if for every subgroup H of G the map of H-fixed points $f^H : X^H \longrightarrow Y^H$ is a weak equivalence;
- a *weak G-fibration* if the underlying map is a Serre fibration of spaces (respectively a Kan fibration of simplicial sets);
- a *strong G-fibration* if for every subgroup H of G the map of H-fixed points $f^H : X^H \longrightarrow Y^H$ is a Serre fibration of spaces (respectively a Kan fibration of simplicial sets);
- a *strict G-fibration* if it is a strong fibration and for every subgroup H of G the square

$$\begin{tikzcd} X^H \arrow[r] \arrow[d, "f^H"'] & X^{hH} \arrow[d, "f^{hH}"] \\ Y^H \arrow[r] & Y^{hH} \end{tikzcd} \tag{4.2}$$

  is homotopy cartesian;
- a *G-cofibration* if it has the left lifting property for all strong G-acyclic fibrations;
- a *free G-cofibration* if it has the left lifting property for all weak G-acyclic fibrations.

For every group G, the category $G\mathbf{T}$ of based G-spaces and the category $G\mathbf{sS}$ of based G-simplicial sets admit the weak equivariant model structure, i.e., the weak G-equivalences, weak G-fibrations and free G-cofibrations form a model structure. References for this fact include [] in the case of spaces and [**62**,] is the case of simplicial sets.

Similary, $G\mathbf{T}$ and $G\mathbf{sS}$ admit the strong equivariant model structure, i.e., the strong G-equivalences, strong G-fibrations and G-cofibrations form a model structure. References for this fact include [] in the case of spaces and [**62**,] is the case of simplicial sets.

Finally, $G\mathbf{T}$ and $G\mathbf{sS}$ admit the mixed equivariant model structure, i.e., the weak G-equivalences, strict$G$-fibrations and G-cofibrations form a model structure. Proofs can be found in [**77**, Prop. 1.3] in the case of spaces and in [] in the case of simplicial sets.

For every cofibration $i : A \longrightarrow B$ of based spaces (respectively simplicial sets) and every subgroup H of G then the G-morphism $(G/H)^+ \wedge i : (G/H)^+ \wedge A \longrightarrow (G/H)^+ \wedge B$ is a G-cofibration. In the context of spaces, the G-cofibrations can be characterized as the retracts [???] of relative G-CW-complexes, i.e., equivariant cell complexes in which equivariant cells of the form $(G/H \times D^n)^+$, for subgroups H of G and $n \geq 0$, are successively attached along their boundary $(G/H \times S^{n-1})^+$ in the order of increasing dimension. In the simplicial context, the situation is even simpler: a morphism $f : A \longrightarrow B$ of G-simplicial sets is a G-cofibration if and only it it is injective.

For every cofibration $i : A \longrightarrow B$ of based spaces (respectively simplicial sets) free G-morphism $G^+ \wedge i : G^+ \wedge A \longrightarrow G^+ \wedge B$ is a free G-cofibration. Moreover, the G-cofibrations can be characterized as the retracts of free relative G-CW-complexes, i.e., the relative G-CW-complexes built only from free G-cells $(G \times D^n)^+$ (respectively $(G \times \Delta[n])^+$). Moreover, the free G-cofibrations are precisely those G-cofibrations for which G acts freely on the complement of the image. [ref] In the simplicial context, things simplify again: a morphism $f : A \longrightarrow B$ of G-simplicial sets is a free G-cofibration if and only it it is injective and G acts freely on the complement of the image.

Remark 4.3. For any finite group G, the strict G-fibrations can be characterized in at least two other ways as we recall in Proposition A.4.5. For a morphism of G-spaces or G-simplicial sets, the following are equivalent:

(i) f is is a strict G-fibration,

(ii) f is a strong G-fibration and the square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathrm{map}(EG,X) \arrow[d, "\mathrm{map}(EG,f)"] \\ Y \arrow[r] & \mathrm{map}(EG,Y) \end{tikzcd} \tag{4.4}$$

is homotopy cartesian in the strong G-equivariant model structure,

(iii) f has the right lifting property for all G-cofibrations which are weak G-equivalences.

Here $\mathrm{map}(E\Sigma_n, X)$ is the space (resepctively simplicial set) of all maps from the contractible free Σ_n-simplicial set to X, with Σ_n-action by conjugation. [ref] **Does this work in general $\mathcal{C}$?**

[use Cole mixing? weak homotopy equivalences, Hurewicz fibrations and mixed cofibrations; a space is cofibrant if and only if it is homotopy equivalent to a CW-complex; works in G-spaces; Cole mixing [**18**, Thm 2.1] also gives a model structure on $G\mathcal{C}$ with weak G-equivalences, strong G-fibrations and mixed G-cofibrations. X is mixed cofibrant if and only if it is G-cofibrant and G-homotopy equivalent to a free G-cofibrant object]

Now we we specialize the various 'level' notions of cofibrations and fibrations (compare Definition 3.3) to symmetric spectra of spaces and simplicial sets.

Definition 4.5. Let $f : X \longrightarrow Y$ be a morphism of symmetric spectra of spaces (respectively simplicial sets). We call f

- a *level equivalence* if for all $n \geq 0$, the morphism $f_n : X_n \longrightarrow Y_n$ is weak Σ_n-equivalence, i.e., a weak equivalence after forgetting the group action.
- a *level fibration* if for all $n \geq 0$, the morphism $f_n : X_n \longrightarrow Y_n$ is a weak Σ_n-fibration, i.e., a Serre fibration (respectively Kan fibration) after forgetting the group action.
- a *strict fibration* if for all $n \geq 0$, the morphism $f_n : X_n \longrightarrow Y_n$ is a strict Σ_n-fibration.
- a *level cofibration* if the nth level $f_n : X_n \longrightarrow Y_n$ of f is a Σ_n-cofibration for all $n \geq 0$.
- a *flat cofibration* if the latching morphism $\nu_n(f) : X_n \cup_{L_n X} L_n Y \longrightarrow Y_n$ is a Σ_n-cofibration for all $n \geq 0$.
- a *projective cofibration* if the latching morphism $\nu_n(f) : X_n \cup_{L_n X} L_n Y \longrightarrow Y_n$ is a free Σ_n-cofibration for all $n \geq 0$.

By the criterion for flatness given in Proposition I.5.47 a symmetric spectrum A of simplicial sets is flat in the original sense (i.e., $A \wedge -$ preserves level cofibrations [in the weak sense...]) if and only if the unique morphism $* \longrightarrow A$ is a flat cofibration. We call a symmetric spectrum A *projective* if the unique morphism $* \longrightarrow A$ is a projective cofibration or, equivalently, if for every $n \geq 0$ the morphism $\nu_n : L_n A \longrightarrow A_n$ is a free Σ_n-cofibration. For symmetric spectra of simplicial sets a morphism is a level cofibration if and only if it is a categorical monomorphism. In particular, every symmetric spectrum of simplicial sets is level cofibrant.

Clearly every projective cofibration is also a flat cofibration. Flat cofibrations are level cofibrations by the following lemma. Thus we have the following implications for the various kinds of cofibrations:

$$\text{projective cofibration} \implies \text{flat cofibration} \implies \text{level cofibration}$$

All these containments are strict, as the following examples show. A semifree symmetric spectra $G_m L$ is flat whenever L is a base Σ_m-CW-complex (respectively an arbitrary based Σ_m-simplicial set); but for such L $G_m L$ is projective if and only the Σ_m-action is free (away from the base point). The symmetric spectrum $\bar{\mathbb{S}}$ level cofibrant, but it is not flat since its second latching object $L_2\bar{\mathbb{S}}$ is isomorphic to $S^1 \vee S^1$ and the map $L_2\bar{\mathbb{S}} \longrightarrow \bar{\mathbb{S}}_2 = S^2$ is the fold map, which is not injective.

As a special case of Theorem 3.6, the level equivalences, level fibrations and projective cofibrations form the *projective level model structure* on the category of symmetric spectra of spaces (respectively simplicial sets). A morphism f is simultaneously a projective cofibration and a level equivalence if and only if for all

$n \geq 0$ the latching morphism $\nu_n(f)$ is a free Σ_n-cofibration and a weak equivalence of underlying spaces (respectively simplicial sets). Since the Quillen model structure on spaces and simplicial sets are proper, so is the projective level model structure by part (ii) of Theorem 3.6). The projective level model structure is topological (respectively simplicial) and monoidal with respect to the smash product of symmetric spectra [...].

We can also name explicit sets of generating cofibrations and acyclic cofibrations for the projective level model structure. The acyclic fibrations in the Quillen model structure on spaces (respectively simplicial sets) are detected by the cofibrations $\partial D^m \longrightarrow D^m$ for $m \geq 0$. In the context of spaces, D^m is the unit ball in $\mathbb{R}^m$ and ∂D^m is its boundary, and $(m-1)$-dimensional sphere (where ∂D^0 is empty). In the context of simplicial sets, D^m has to be interpreted as the m-simplex $\Delta[m]$ and ∂D^m is its simplicial boundary $\partial\Delta[m]$. By part (iv) of Theorem 3.6, the acyclic level fibrations of symmetric spectra are then detected by the set $I^{\mathrm{lv}}_{\mathrm{proj}}$ of projective cofibrations

$$F_n(\partial D^m)^+ \longrightarrow F_n(D^m)^+$$

for $n, m \geq 0$. Similarly, we obtain a set of acyclic cofibrations which detect level fibrations. The fibrations in the Quillen model structure on spaces are detected by the acyclic cofibrations $D^n \longrightarrow D^n \times [0,1]$ for $n \geq 0$. By part (iii) of Theorem 3.6, the level fibrations of symmetric spectra are then detected by the set $J^{\mathrm{lv}}_{\mathrm{proj}}$ of projective cofibrations

$$F_n(\partial D^m)^+ \longrightarrow F_n(D^m \times [0,1])^+$$

for $n, m \geq 0$. In the context of simplicial sets, the acyclic cofibration $D^m \longrightarrow D^m \times [0,1]$ has to be replaced by the collection of simplicial horn inclusions $\Lambda^k[m] \longrightarrow \Delta[m]$ for $m \geq 1$ and $0 \leq k \leq m$ throughout. Altogether this shows that the projective level model structure is cofibrantly generated with $I^{\mathrm{lv}}_{\mathrm{proj}}$ and $J^{\mathrm{lv}}_{\mathrm{proj}}$ as possible sets of generating cofibrations respectively generating acyclic cofibrations.

It remains to show that the projective level model structure is topological (respectively simplicial) and monoidal. We defer this to Propositions 4.16 and 4.15 below.

As a special case of Theorem 3.8, the level equivalences, strict fibrations and flat cofibrations form the *flat level model structure* on the category of symmetric spectra of spaces (respectively simplicial sets). A morphism f is simultaneously a flat cofibration and a level equivalence if and only if for all $n \geq 0$ the latching morphism $\nu_n(f)$ is a Σ_n-cofibration and a weak equivalence of underlying spaces (respectively simplicial sets).

A morphism $f : X \longrightarrow Y$ is simultaneously a flat fibration and a level equivalence if and only if for all $n \geq 0$ and every subgroup H of Σ_n the map $f^H_n : X^H_n \longrightarrow Y^H_n$ of H-fixed points is a weak equivalence and fibration of spaces (respectively simplicial sets).

Since the Quillen model structure on spaces and simplicial sets are proper, so is the flat level model structure by part (ii) of Theorem 3.8. The flat level model structure is topological (respectively simplicial) and monoidal with respect to the smash product of symmetric spectra [...].

Again we can identify explicit sets of generating cofibrations and generating acyclic cofibrations for the flat level model structure. By part (iv) of Theorem 3.8 the acyclic fibrations in the flat level model structure are then detected by the set $I^{\mathrm{lv}}_{\mathrm{flat}}$ of flat cofibrations

$$G_n(\Sigma_n/H \times \partial D^m)^+ \longrightarrow G_n(\Sigma_n/H \times D^m)^+$$

for $n, m \geq 0$ and all subgroups H of Σ_n. In the context of simplicial sets, D^m again stands for the simplicial m-simplex $\Delta[m]$.

The strict fibrations in the mixed model structure on Σ_n-spaces (respectively simplicial sets) are detected by two different types of Σ_n-maps. First, the maps $\Sigma_n/H \times D^m \longrightarrow \Sigma_n/H \times D^m \times [0,1]$ for all $m \geq 0$ and some subgroup H of Σ_n detect whether the induced map on H-fixed points is a fibration. If this condition is satisfied for all subgroups H of Σ_n, the pushout product

$$C(E\Sigma_n) \times \partial D^m \cup_{E\Sigma_n \times \partial D^m} E\Sigma_n \times D^m \longrightarrow C(E\Sigma_n) \times D^m$$

[reduced?] of the boundary inclusions $\partial D^n \longrightarrow D^n$ with the cone inclusion $E\Sigma_n \longrightarrow C(E\Sigma_n)$ then detects whether all the squares (4.3) are homotopy cartesian [ref]. We let K_n denote the set of mixed Σ_n-acyclic cofibrations consisting of $\Sigma_n/H \times D^m \longrightarrow \Sigma_n/H \times D^m \times [0,1]$ and [...] for all $m \geq 0$ and all subgroups H of Σ_n. By part (iv) of Theorem 3.8, the acyclic level fibrations in the flat model structure of symmetric spectra are then detected by the set

$$J^{\mathrm{lv}}_{\mathrm{flat}} \;=\; \{G_n k \mid n \geq 0,\, k \in K_n\}\ .$$

As before, in the context of simplicial sets, the acyclic cofibration $D^m \longrightarrow D^m \times [0,1]$ has to be replaced by the collection of simplicial horn inclusions $\Lambda^k[m] \longrightarrow \Delta[m]$ for $m \geq 1$ and $0 \leq k \leq m$ throughout. Altogether this shows that the flat level model structure is cofibrantly generated with $I^{\mathrm{lv}}_{\mathrm{flat}}$ and $J^{\mathrm{lv}}_{\mathrm{flat}}$ as possible sets of generating cofibrations respectively generating acyclic cofibrations.

Remark 4.6. For symmetric spectra of simplicial sets, there is yet another level model structure, the *injective level model structure*. Here the cofibrations are the level cofibrations (i.e., monomorphisms) and the injective fibrations are the morphisms which have the right lifting property with respect to all morphisms which are simultaneously level cofibrations and level equivalences. This model structure is also proper, topological (respectively simplicial) and cofibrantly generated; as the classes $I^{\mathrm{lv}}_{\mathrm{inj}}$ (respectively $J^{\mathrm{lv}}_{\mathrm{inj}}$) of generating cofibrations (respectively generating acyclic cofibrations) we can take representatives of the isomorphism classes of monomorphisms f (respectively monomorphisms f which are also level equivalences) with countable target. We do not develop this injective level model structure here, but the details can be found in [**36**, Thm. 5.1.2].

It seems likely that there is also a corresponding injective level model structure for symmetric spectra of spaces, but I do not know a reference and have not tried to prove it.

In the context of simplicial sets we have a few more tools available to characterize the projective and flat cofibrations:

Lemma 4.7. *Let $f : A \longrightarrow B$ be a morphism of symmetric spectra of simplicial sets. Then $f : A \longrightarrow B$ is a projective cofibration if and only if it is a flat cofibration and the cokernel B/A is projective.* [Is a morphism $f : A \longrightarrow B$ is a flat cofibration if and only if it is an injective cofibration (i.e., monomorphism) and the cokernel B/A is flat?]

Proof. This is direct consequence of the definitions since a group acts freely on the complement of the image of an equivariant map $A \longrightarrow B$ if and only if the induced action on the quotient B/A is free away from the basepoint. $\square$

Theorem 4.8 (Positive projective and flat level model structures)**.** (i) *The positive level equivalences, positive level fibrations and positive projective cofibrations form the* positive projective level model structure *on The category of symmetric spectra of spaces (respectively simplicial sets).*

(ii) *The positive level equivalences, positive strict fibrations and positive flat cofibrations form the* positive flat level model structure *on The category of symmetric spectra of spaces (respectively simplicial sets).*

(iii) *The positive projective and the positive flat level model structures are proper, topological (respectively simplicial) cofibrantly generated and monoidal with respect to the smash product of symmetric spectra.*

Now we proceed towards the stable model structures. For every morphism $f : X \longrightarrow Y$ of symmetric spectra the natural morphism $\tilde{\lambda}_X : X \longrightarrow \Omega(\mathrm{sh}\, X)$ adjoint to $\lambda_X : S^1 \wedge X \longrightarrow \mathrm{sh}\, X$ gives rise to a commutative square of symmetric spectra

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\lambda}_X} & \Omega(\mathrm{sh}\, X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Omega(\mathrm{sh}\, f)} \\ Y & \xrightarrow[\tilde{\lambda}_Y]{} & \Omega(\mathrm{sh}\, Y) \end{array} \qquad (4.9)$$

Definition 4.10. A morphism $f : X \longrightarrow Y$ of symmetric spectra is a *stable fibration* if it is a level fibration and the square (4.9) is levelwise homotopy cartesian after forgetting the symmetric group actions. Similarly, f is a *positive stable fibration* if it is a positive level fibration and the square (4.9) is homotopy cartesian after forgetting the symmetric group actions in every positive level.

Theorem 4.11. *The category of symmetric spectra of space (respectively simplicial sets) admits the following stable model structures in which the weak equivalences are the stable equivalences.*

(i) *The stable equivalences, stable fibrations and projective cofibrations form the* absolute projective stable model structure.
(ii) *The stable equivalences, positive stable fibrations and positive projective cofibrations form the* positive projective stable model structure.
(iii) *The stable equivalences, stable and strict fibrations and flat cofibrations form the* absolute flat stable model structure.
(iv) *The stable equivalences, positive stable and positive strict fibrations and positive flat cofibrations form the* positive flat stable model structure.

Moreover, all four stable model structures are proper, topological (respectively simplicial), cofibrantly generated and monoidal with respect to the smash product of symmetric spectra.

Proof. We reduce the proof of the stable model structures to the level model structures by applying a general localization theorem of Bousfield, see Theorem 1.9 of Appendix A. In Proposition I.4.39 we constructed a functor $Q : \mathcal{S}p \longrightarrow \mathcal{S}p$ with values in Ω-spectra and a natural stable equivalence $\alpha_X : X \longrightarrow QX$. We note that a morphism $f : X \longrightarrow Y$ of symmetric spectra is a stable equivalence if and only if $Qf : QX \longrightarrow QY$ is a level equivalence. Indeed, since $\alpha_X : X \longrightarrow QX$ and $\alpha_Y : Y \longrightarrow QY$ are stable equivalences, f is a stable equivalence if and only if Qf is. But Qf is a morphism between Ω-spectra, so it is a stable equivalence if and only if it is a level equivalence.

We now apply Bousfield's Localization Theorem A.1.9 to the flat and projective level model structures, in both the absolute and positive flavors, which are all proper. Axiom (A1) holds because of the commutative square:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[d, "f"'] & QX \arrow[d, "Qf"] \\ Y \arrow[r, "\alpha_Y"'] & QY \end{tikzcd} \tag{4.12}$$

If f is a level equivalences, then Qf is a stable equivalence between Ω-spectra, hence a level equivalence. Axiom (A2) holds: α_{QX} is a stable equivalence between Ω-spectra, hence a level equivalence. Then $Q\alpha_X : QX \longrightarrow QQX$ is a level equivalence since Q takes all stable equivalences, in particular α_X, to level equivalences.

We prove (A3) in the absolute projective level model structure. Since the projective fibrations include the flat fibrations, it then also holds in the absolute flat level model structures. So we are given a pullback square

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "j"'] & Y \end{tikzcd}$$

of symmetric spectra in which X and Y are Ω-spectra (possibly not levelwise Kan), f is levelwise a Kan fibration and j is a stable equivalence. We showed in part (iv) of Proposition I.4.31 that then i is also a stable equivalence. This proves (A3) for the two absolute level model structure. The verification of (A3) in the two positive level model structure is similar [...]

In each of the four cases Bousfield's theorem now provides a model structures with stable equivalences as weak equivalence and with same class of cofibrations as before. Bousfield's theorem also characterizes the fibrations as those morphisms $f : X \longrightarrow Y$ which are fibations in the original model structure and such that the commutative square (4.12) is homotopy cartesian in the original model structure. [fix the rest] So it remains to shows that for a morphism $f : X \longrightarrow Y$ which is levelwise a Kan fibration the square (4.9) is levelwise homotopy cartesian if and only if the square (4.12) is levelwise homotopy cartesian. $\square$

Alternative: we can first establish the absolute stable injective model structure and then use Cole mixing: every level equivalence and every positive level equivalence is a stable equivalence; every flat and projective cofibration, absolute or positive, is a monomorphism. So Cole mixing [**18**, Thm. 2.1] produces model structures with stable equivalences and cofibrations the flat cofibrations, positive flat cofibrations, projective cofibrations and positive projective cofibrations. If the injective stable model structure is not available, we can start from the absolute flat stable model structure instead.

Remark 4.13. For symmetric spectra of simplicial sets, there are at least two more stable model structure, namely the *absolute injective stable model structure* and the *positive injective stable model structure*. The constructions are exactly as in the projective and flat case, but starting from the injective level model structure.

Corollary 4.14. *The following categories are equivalent to each other and to the stable homotopy category $\mathcal{SHC}$ as defined in Section II.1:*

(i) *the homotopy category of projective Ω-spectra of spaces;*
(ii) *the homotopy category of projective positive Ω-spectra of spaces which are trivial in level 0;*
(iii) *the homotopy category of those flat Ω-spectra X of spaces such that for all $n \geq 0$ the map $X_n \longrightarrow \mathrm{map}(E\Sigma_n, X_n)$ is a strong Σ_n-equivalence;*
(iv) *the homotopy category of those flat positive Ω-spectra X of spaces such that X_0 is trivial and for all $n \geq 1$ the map $X_n \longrightarrow \mathrm{map}(E\Sigma_n, X_n)$ is a strong Σ_n-equivalence;*
(v) *the homotopy category of projective Ω-spectra of simplicial sets which are levelwise Kan;*
(vi) *the homotopy category of projective positive Ω-spectra of simplicial sets which are trivial in level 0 and levelwise Kan;*
(vii) *the homotopy category of those flat Ω-spectra X of simplicial sets such that X_n is strictly Σ_n-fibrant for all $n \geq 0$;*
(viii) *the homotopy category of those flat positive Ω-spectra X of simplicial sets such that X_0 is trivial and X_n is strictly Σ_n-fibrant for all $n \geq 1$;*
(ix) *the homotopy category of injective Ω-spectra of simplicial sets;*
(x) *the homotopy category of cofibrant sequential Ω-spectra;*
(xi) *the homotopy category of projective orthogonal Ω-spectra.*

There are many more model for the stable homotopy category that we could add to the list, for example the homotopy categories of of projective unitary Ω-spectra or of cofibrant S-modules.

We still have to show that the level model structures are topological respectively simplicial and that the flat and monoidal with respect to the smash product of symmetric spectra. So we have to verify various forms of the *pushout product property*. We recall that the pushout product of a morphism $i : K \longrightarrow L$ of based spaces (or simplicial sets) or symmetric spectra and a morphism $j : A \longrightarrow B$ of symmetric spectra is the morphism

$$i \wedge j \ : \ L \wedge A \cup_{K \wedge A} K \wedge B \ \longrightarrow \ L \wedge B \ .$$

The first proposition below is about internal smash products of symmetric spectra; the next proposition is about smash products of spaces (simplicial sets) with symmetric spectra.

Proposition 4.15. *Let $i : K \longrightarrow L$ and $j : A \longrightarrow B$ be morphisms of symmetric spectra.*

(i) *If i is a level cofibration and j is a flat cofibration, then $i \wedge j$ is a level cofibration.*
(ii) *If both i and j are flat cofibrations, then so is $i \wedge j$.*
(iii) *If both i and j are projective cofibrations, then so is $i \wedge j$.*
(iv) *If i is a level cofibration, j a flat cofibration and one of i or j a level equivalence, π_*-isomorphism respectively stable equivalence, then $i \wedge j$ is also a level equivalence, π_*-isomorphism respectively stable equivalence.*

Thus the flat and projective level model structures are monoidal *model categories with respect to the smash product of symmetric spectra.*

Proof. Check on generators. □

As a special case [explain] of Proposition 4.15 we obtain:

Proposition 4.16. *Let $i : K \longrightarrow L$ be a morphism of based spaces (respectively simplicial sets) and $j : A \longrightarrow B$ a morphism of symmetric spectra.*

- *Suppose that i is a cofibration of spaces (respectively simplicial sets). If j a level cofibration, flat cofibration respectively projective cofibration, then the pushout product $i \wedge j$ is also a level cofibration, flat cofibration respectively projective cofibration.*
- *If i is a cofibration and weak equivalence of spaces (respectively simplicial sets) and j is a level cofibration, then $i \wedge j$ is a level equivalence of symmetric spectra.*
- *Suppose that i is a cofibration of spaces (respectively simplicial sets) and that j is a level cofibration. If j is a level equivalence, π_*-isomorphism respectively stable equivalence of symmetric spectra, then $i \wedge j$ is also a level equivalence, π_*-isomorphism respectively stable equivalence.*

Thus the flat and projective level model structures are topological (respectively simplicial) model categories.

[State all adjoint forms of the simplicial and monoidal axiom]

5. Operads and their algebras

Proposition 5.1. *Let A be a flat symmetric spectrum and $n \geq 2$. Then the symmetric power spectrum $(A^{\wedge n})/\Sigma_n$ is again flat.*
[is the product of flat spectra flat ? how about A^K and $\mathrm{sh}\, A$?]

Theorem 5.2. *Let $f : X \longrightarrow Y$ be an injective morphism of $\mathbf{\Gamma}$-spaces of simplicial sets. Then the associated morphism $f(\mathbb{S}) : X(\mathbb{S}) \longrightarrow Y(\mathbb{S})$ is a flat cofibration of symmetric spectra. In particular, for every $\mathbf{\Gamma}$-space of simplicial sets X, the associated symmetric spectrum $X(\mathbb{S})$ is flat.*

[how do the BF- and Q-cofibrations of $\mathbf{\Gamma}$-spaces relate to the various cofibrations ?]

Proof. □

Definition 5.3. An *operad* $\mathcal{O}$ of symmetric spectra consists of

- a collection $\{\mathcal{O}(n)\}_{n\geq 0}$ of symmetric spectra,
- an action of the symmetric group Σ_n [on the *right* ?] on the spectrum $\mathcal{O}(n)$ for all $n \geq 0$,
- a *unit morphism* $\iota : \mathbb{S} \longrightarrow \mathcal{O}(1)$ and
- *composition morphisms*

$$\gamma \ : \ \mathcal{O}(n) \wedge \mathcal{O}(i_1) \wedge \ldots \wedge \mathcal{O}(i_n) \ \longrightarrow \ \mathcal{O}(i)$$

for all $n, i_1, \ldots, i_n \geq 0$ where $i = i_1 + \cdots + i_n$.

Moreover, this data has to satisfy the following three (?) conditions:

(Associativity) The square

$$\begin{tikzcd}
\mathcal{O}(n) \wedge \mathcal{O}(i_1) \ldots \mathcal{O}(i_n) \wedge \mathcal{O}(j_1^1) \ldots \mathcal{O}(j_{i_1}^1) \wedge \ldots \wedge \mathcal{O}(j_1^n) \ldots \mathcal{O}(j_{i_n}^n) \arrow[r, "\gamma\wedge\mathrm{Id}"] \arrow[d, "\text{shuffle}"'] & \mathcal{O}(i) \wedge \mathcal{O}(j_1^1) \ldots \mathcal{O}(j_{i_1}^1) \wedge \ldots \wedge \mathcal{O}(j_1^n) \wedge \ldots \wedge \mathcal{O}(\arrow[dd, "\gamma"] \\
\mathcal{O}(n) \wedge \mathcal{O}(i_1) \wedge \mathcal{O}(j_1^1) \ldots \mathcal{O}(j_{i_1}^1) \wedge \ldots \wedge \mathcal{O}(i_n) \wedge \mathcal{O}(j_1^n) \ldots \mathcal{O}(j_{i_n}^n) \arrow[d, "\mathrm{Id}\wedge\gamma\ldots\wedge\gamma"'] & \\
\mathcal{O}(n) \wedge \mathcal{O}(j^1) \wedge \ldots \wedge \mathcal{O}(j^n) \arrow[r, "\gamma"'] & \mathcal{O}(j)
\end{tikzcd}$$

commutes for all $n, i_1, \ldots, i_n, [...] \geq 0$, where the indices run over all natural numbers and $i = i_1 + \cdots + i_n$, $j^k = j_1^k + \cdots j_{i_k}^k$ and $j = j^1 + \cdots j^n$.

(Equivariance)

(Unit) The two composite morphisms

$$\mathcal{O}(n) \cong \mathbb{S} \wedge \mathcal{O}(n) \xrightarrow{\iota \wedge \mathrm{Id}} \mathcal{O}(1) \wedge \mathcal{O}(n) \xrightarrow{\gamma} \mathcal{O}(n)$$

and

$$\mathcal{O}(n) \cong \mathcal{O}(n) \wedge \underbrace{\mathbb{S} \wedge \cdots \wedge \mathbb{S}}_{n} \xrightarrow{\mathrm{Id} \wedge \iota \wedge \ldots \wedge \iota} \mathcal{O}(n) \wedge \underbrace{\mathcal{O}(1) \wedge \cdots \wedge \mathcal{O}(1)}_{n} \xrightarrow{\gamma} \mathcal{O}(n)$$

are the identity for all $n \geq 0$, where the first maps in both composites are unit isomorphisms.

A *morphism* $f : \mathcal{O} \longrightarrow \mathcal{P}$ of operads is a collection of Σ_n-equivariant morphisms of symmetric spectra $f(n) : \mathcal{O}(n) \longrightarrow \mathcal{P}(n)$ for all $n \geq 0$ which preserve the unit morphisms in the sense that $f(1) \circ \iota_{\mathcal{O}} = \iota_{\mathcal{P}}$ and which commute with the structure morphisms in the sense that [...]

Definition 5.4. Given an operad $\mathcal{O}$, an $\mathcal{O}$-*algebra* is a symmetric spectrum A together with morphisms of symmetric spectra

$$\alpha_n \ : \ \mathcal{O}(n) \wedge A^{(n)} \ \longrightarrow \ A$$

for $n \geq 0$ which satisfy the following conditions.

(Associativity) The square

$$\begin{tikzcd}
\mathcal{O}(n) \wedge \mathcal{O}(i_1) \wedge \cdots \wedge \mathcal{O}(i_n) \wedge A^{(i_1)} \wedge \cdots \wedge A^{(i_n)} \arrow[rr, "\gamma \wedge \mathrm{Id}"] \arrow[d, "\text{shuffle}"'] & & \mathcal{O}(i) \wedge A^{(i)} \arrow[dd, "\alpha_i"] \\
\mathcal{O}(n) \wedge \mathcal{O}(i_1) \wedge A^{(i_1)} \wedge \cdots \wedge \mathcal{O}(i_n) \wedge A^{(i_n)} \arrow[d, "\mathrm{Id} \wedge \alpha_{i_1} \wedge \cdots \alpha_{i_n}"] & & \\
\mathcal{O}(n) \wedge A^{(n)} \arrow[rr, "\gamma"'] & & A
\end{tikzcd}$$

commutes for all $n, i_1, \ldots, i_n \geq 0$, where $i = i_1 + \cdots + i_n$.

(Equivariance)

(Unit) The composite

$$A \cong \mathbb{S} \wedge A \xrightarrow{\iota \wedge \mathrm{Id}} \mathcal{O}(1) \wedge A \xrightarrow{\gamma} A$$

is the identity.

A *morphism* $f : A \longrightarrow B$ of $\mathcal{O}$-algebras is a morphism of symmetric spectra which commutes with the action morphisms in the sense that [...]

If we realize geometrically we obtain an operad $|\mathcal{O}|$ in the category of pointed compactly generated spaces, which can similarly act on symmetric spectra of topological spaces.

Remark 5.5. In the special case $n = 1 = i_1$ the associativity condition says that the morphism $\gamma : \mathcal{O}(1) \wedge \mathcal{O}(1) \longrightarrow \mathcal{O}(1)$ is an associative product. Moreover, the unit condition for $n = 1$ says that $\iota : \mathbb{S} \longrightarrow \mathcal{O}(1)$ is unital for $\gamma : \mathcal{O}(1) \wedge \mathcal{O}(1) \longrightarrow \mathcal{O}(1)$. In other words, for any operad $\mathcal{O}$, the object $\mathcal{O}(1)$ is a monoid in the monoidal category $\mathcal{C}$. In the case of operads of symmetric spectra this means that for any operad $\mathcal{O}$, the spectrum $\mathcal{O}(1)$ is a symmetric ring spectrum with unit morphism $\iota : \mathbb{S} \longrightarrow \mathcal{O}(1)$ and multiplication $\gamma : \mathcal{O}(1) \wedge \mathcal{O}(1) \longrightarrow \mathcal{O}(1)$. [$\mathcal{O}(n)$ is an $\mathcal{O}(1)$-module for all $n \geq 0$]

[$\mathcal{O}$-algebras as monoids in $(\mathcal{O}(1)\,\text{mod-}, \Box, I)$? This is only a weak monoidal product...]

The notion of an operad is not restricted to symmetric spectra; indeed, an operad can be defined in any symmetric monoidal category $(\mathcal{C}, \otimes, I)$. In the end we will mainly care for the case of symmetric spectra (with respect to the smash product) but we can use the more general context, among other things, to produce examples of operads of symmetric spectra. [operads in spaces, simplicial sets, categories, sets, and chain complexes]

[Operads can be described as monoids with respect to the circle product of symmetric sequences]

Example 5.6 (Operads of spaces or simplicial sets)**.** ... give rise to operads of symmetric spectra by taking suspension spectra (based and unbased versions)

Important special cases: A_∞-operads (e.g. Stasheff polytopes) and E_∞-operads.

Example 5.7 (Operads of categories)**.**

Example 5.8 (Operads of sets)**.** [gives rise to an operad in any symmetric monoidal $(\mathcal{C}, \otimes, I)$ as long as $\mathcal{C}$ has coproducts]

We can formalize the previous examples as follows. We have a diagram of symmetric monoidal categories and strong monoidal functors

$$\begin{tikzcd}
(\mathbf{set}, \times, *) \arrow[r] & (\text{categories}, \times, *) \arrow[r, "N"] & (\mathbf{sS}, \times, *) \arrow[r, "\Sigma^\infty_+"] \arrow[d, "|\,|"'] & (\mathcal{S}p_{\mathbf{sS}}, \wedge, \mathbb{S}) \arrow[d, "|\,|"] \\
 & & (\mathbf{T}, \times, \{*\}) \arrow[r, "\Sigma^\infty_+"'] & (\mathcal{S}p_{\mathbf{T}}, \wedge, \mathbb{S})
\end{tikzcd}$$

We can (and will) take an operad in any one category and push it forward by applying the respective strong monoidal functor to all objects in the operad.

Example 5.9 (Associative operad)**.** We let $\mathcal{A}ss$ denote the operad of sets with $\mathcal{A}ss(n) = \Sigma_n$ [operad structure]; Because of the equivariance conidition, the action morphism $\alpha_n : \mathcal{A}ss(n) \otimes A^{(n)} = \amalg_{\Sigma_n} A^{(n)} \longrightarrow A$ is completely determined by its restriction to the summand indexed by the identity element of Σ_n.

So the $\mathcal{A}ss$-algebras in the category of sets 'are' then the associative (and unital) monoids. More precisely, the forgetful functor $\mathcal{A}ss$-alg $\longrightarrow$ (monoids) which remembers only the unit morphism and the morphism

$$M \otimes M \xrightarrow{1 \otimes \mathrm{Id} \otimes \mathrm{Id}} \mathcal{A}ss(2) \otimes M \otimes M \longrightarrow M$$

is an isomorphism of categories. [more details for $C = \mathcal{S}p$ in the section on symmetric ring spectra]

In the special case of symmetric spectra under smash product we deduce that the category of $\mathcal{A}ss$-algebras is isomorphic to the category of symmetric ring spectra. [$\mathcal{A}ss$-algebras in the stable homotopy category are homotopy ring spectra]

Example 5.10 (Commutative operad)**.** We let $\mathcal{C}om$ denote the operad of sets with $\mathcal{C}om(n) = *$, the one point set, for all $n \geq 0$, and unique operad structure. This is the terminal operad of sets and the $\mathcal{C}om$-algebras in the category of sets are then the commutative (and associative and unital) monoids. This phenomenon persists to any symmetric monoidal category $(\mathcal{C}, \otimes, I)$ [explain in appendix;reference].

In the special case of symmetric spectra under smash product we deduce that the category of $\mathcal{C}om$-algebras is isomorphic to the category of commutative symmetric ring spectra.

Example 5.11 (A_∞ operads)**.**

Example 5.12 (E_∞ operads)**.** Several different E_∞ operads have been discussed in the literature. Here are some specific examples, where we only recall the spaces $\mathcal{O}(n)$ and refer to the original sources for the remaining structure.

The *Barratt-Eccles operad* is the categorical operad with n-th category given by $\underline{E}\Sigma_n$. The operad is mostly used in its simplicial or topological version, i.e., after taking nerves and possibly also geometric realization.

The *Dold operad* [...] The *surjection operad* [...] The *linear isometries operad* $\mathcal{L}$ [...]

Example 5.13 (Injection operad)**.** The *injection operad* $\underline{\mathcal{M}}$ is the operad of sets defined by letting $\underline{\mathcal{M}}(n)$ be the set of injections from the set $\omega \times \mathbf{n}$ into ω, for $n \geq 0$. Note that for $n = 0$ the source is the empty set, so $\underline{\mathcal{M}}(0)$ has exactly one element, and $\underline{\mathcal{M}}(1)$ is the injection monoid $\mathcal{M}$. The symmetric groups permute the second coordinates in $\omega \times \mathbf{n}$. The operad structure is via disjoint union and composition, i.e., $\mathcal{M}$ is a suboperad of the endomorphism operad of the set ω in the symmetric monoidal category of sets under disjoint union. More precisely, the operad structure morphism

$$\gamma \ : \ \underline{\mathcal{M}}(n) \times \underline{\mathcal{M}}(i_1) \times \cdots \times \underline{\mathcal{M}}(i_n) \ \longrightarrow \ \underline{\mathcal{M}}(i_1 + \cdots + i_n)$$

sends $(\varphi, f_1, \ldots, f_n)$ to $\varphi \circ (f_1 + \cdots + f_n)$.
The injection operad is, in a sense, the discrete analog of the linear isometries operad discussed above. [operad map $\underline{\mathcal{M}} \longrightarrow \mathcal{L}$]

The injection operad came up before because the injection monoid $\mathcal{M} = \underline{\mathcal{M}}(1)$ acts naturally on the naive homotopy groups of a symmetric spectrum. The entire operad injection is relevant because the naive homotopy groups of every symmetric ring spectrum are naturally an algebra over the injection operad, compare Exercise E.I.68.

Example 5.14 (Endomorphism operad)**.** Given any symmetric monoidal category $(\mathcal{C}, \otimes, I)$ and an object X of $\mathcal{C}$, the *endomorphism operad* $\mathcal{E}nd(X)$ is defined as follows.

In the case of symmetric spectra, a monoid with respect to the smash product is a symmetric ring spectrum, and the monoid $\mathcal{E}nd(X)(1)$ coincides with the endomorphism ring spectrum $\mathrm{End}(X)$ as defined in Example 3.41 in Chapter I.

[$\mathcal{O}$-algebra structures on X are the same as operad morphisms $\mathcal{O} \longrightarrow \mathcal{E}nd(X)$]

Example 5.15 (Operads from monoids)**.** As we explained in Remark 5.5, every operad $\mathcal{O}$ gives rise to a monoid $\mathcal{O}(1)$ by neglect of structure. We can also go the other direction: suppse that M is a monoid in the symmetric monoidal category $\mathcal{C}$ with unit morphism $\iota : I \longrightarrow M$ and multiplication morphism $\mu : M \otimes M \longrightarrow M$. We define a operad oM in $\mathcal{C}$ by

$$oM(n) \ = \ \begin{cases} M & \text{for } n = 1, \\ \emptyset & \text{else.} \end{cases}$$

where $\emptyset$ denotes the initial object of $\mathcal{C}$. All symmetric groups act trivially, and the unit morphism $I \longrightarrow uM(1)$ of the operad is the unit morphism $\iota : I \longrightarrow M$ of the monoid. The composition morphism γ is the mutliplication morphism $\mu : M \otimes M \longrightarrow M$ for $n = i_1 = 1$. In all other cases, the source of the unit morphism for oM involves at least one factor which is the initial object $\emptyset$, hence the entire source object is an initial object, which only has one morphism out of it.

The associativity constraint specializes to the associitivity of μ in the case $n = 1$, and it is automatically satisfied in all other cases since then the source is an initial object.

The functor which associates the operad oM to a monoid M is left adjoint to the functor which takes an operad $\mathcal{O}$ to the monoid $\mathcal{O}(1)$.

The terminology is somewhat unfortunate in this particular case: *algebras* over the operad uM are 'the same as' *modules* over the monoid M. More precisely, the forgetful functor oM-alg $\longrightarrow$ M mod- which forgets all action morphisms except $oM(1) \otimes A = M \otimes A \longrightarrow A$ is an isomorphism of categories.

Since simplicial sets act on symmetric spectra in a way compatible with the smash products, we can consider $\mathcal{O}$-algebras in category of symmetric spectra of simplicial sets.

[Operads versus symmetric operads; every operad $\mathcal{O}$ gives rise to a symmetric operad $\Sigma \times \mathcal{O}$ such that the $\mathcal{O}$-algebras coincide with the symmetric algebras over $\Sigma \times \mathcal{O}$]

[Modules over an algebra over an operad; universal envelopping algebra]

6. Model structures for algebras over an operads

In this section we let $\mathcal{O}$ be an operad of symmetric spectra (under smash product), either in the context of spaces or simplicial sets. We will lefi various stable model structure from the category of symmetric spectra to the categorie of $\mathcal{O}$-algebras. This contains the cases of module spectra over a fixed symmetric ring spectrum, symmetric ring spectra, and of commutative symmetric ring spectra in the following sense. These special cases are particularly important, and we devote a seperate chapter (or section?) to each of them [ref...].

Theorem 6.1. *Let $\mathcal{O}$ be an operad of symmetric spectra. The category of $\mathcal{O}$-algebras admits the following* positive stable model structures *in which the weak equivalences are those morphisms of $\mathcal{O}$-algebras which are stable equivalences on underlying symmetric spectra.*

(i) *In the* positive projective stable model structure *the fibrations are those morphisms of $\mathcal{O}$-algebras which are positive projective stable fibrations on underlying symmetric spectra.*
(ii) *In the* positive flat stable model structure *the fibrations are those morphisms of $\mathcal{O}$-algebras which are positive flat stable fibrations on underlying symmetric spectra.*

If the object $\mathcal{O}(n)$ is projective (flat enough?) as a Σ_n-symmetric spectrum for every $n \geq 0$, then the category of $\mathcal{O}$-algebras in R-modules also admits the following absolute stable model structures *in which the*

(i) *In the* absolute flat stable model structure *the fibrations are those morphisms of $\mathcal{O}$-algebras which are absolute flat stable fibrations on underlying symmetric spectra.*
(ii) *In the* absolute projective stable model structure *the fibrations are those morphisms of $\mathcal{O}$-algebras which are absolute projective stable fibrations on underlying symmetric spectra.*

All model structures are cofibrantly generated, simplicial/topological and right proper.

For every (positive resp. absolut, flat) cofibrant $\mathcal{O}$-algebra A the unique morphism $O(0) \longrightarrow A$ from the initial $\mathcal{O}$-algebra is a flat cofibration of underlying symmetric spectra. Thus every cofibrant $\mathcal{O}$-algebra A is flat as a symmetric spectrum. [is this right?]

[free actions on operad of simplicial sets gives projective operad of symmetric spectra]
[explain restriction and extension along an operad morphism]

Theorem 6.2. *Let $f : \mathcal{O} \longrightarrow \mathcal{P}$ be a morphism of operads of symmetric spectra.*

(i) *The functor pair*

$$\mathcal{P}\text{-alg} \quad \underset{f^*}{\overset{f_*}{\rightleftarrows}} \quad \mathcal{O}\text{-alg}$$

is a Quillen adjoint functor pair with respect to the positive projective and the positive flat stable model structures on both sides.
(ii) *If for every $n \geq 0$ the group Σ_n acts freely on $\mathcal{O}(n)$ and $\mathcal{P}(n)$, then (f_*, f^*) is a Quillen adjoint functor pair with respect to the absolut projective and absolute flat stable model structures on both sides.*
(iii) *If for every $n \geq 0$ the map $f(n) : \mathcal{O}(n) \longrightarrow \mathcal{P}(n)$ is a stable equivalence of symmetric spectra after forgetting the Σ_n-action, then the adjoint functor pair (f_*, f^*) is a Quillen equivalences in all the cases whenever it is a Quillen functor pair.*

The Quillen equivalence between commutative and E_∞-ring spectra is a special case of Quillen equivalences associated to weak equivalences of suitable operads[...]

Remark 6.3. Before we prove the theorem, let us explain why the *absolute* stable model structures does not generally lift from symmetric spectra to $\mathcal{O}$-algebras without the freeness assumption on the operad $\mathcal{O}$. We illustrate this for the operad $\mathcal{C}om$ (where the Σ_n-action is certainly not free in general), whose algebras are commutative symmetric ring spectra. So suppose that the forgetful functor from commutative symmetric ring spectra creates a model structure relative to one of the *absolute* (injective, flat or projective) stable model structure on symmetric spectra. Then we could choose a fibrant replacement $\mathbb{S} \longrightarrow \mathbb{S}^f$ of the sphere spectrum in the respective stable model structure. The target is then $\mathbb{S}^f$ a commutative symmetric ring spectrum which is also an Ω-spectrum. Since $\mathbb{S}^f$ is stably equivalent, thus π_*-isomorphic, to the sphere spectrum, its 0-th space $\mathbb{S}^f_0$ has the homotopy type of $QS^0 = \operatorname{colim}_n \Omega^n S^n$. However, the space in level 0 of any commutative symmetric ring spectrum R is a simplicial (or topological) commutative monoid, via $\mu_{0,0} : R_0 \wedge R_0 \longrightarrow R_0$. [in pointed ssets...] If the monoid of components forms a group (which is the case for QS^0), then such a commutative monoid is weakly equivalent to a product of Eilenberg-Mac Lane spaces. So altogether an absolute stable model structure on commutative symmetric ring spectra would imply that space QS^0 is weakly equivalent to a product of Eilenberg-Mac Lane spaces, which is not the case.

From the operad $\mathcal{O}$ we define a functor $\mathcal{O}_n[-] : \mathcal{S}p \longrightarrow \Sigma_n\text{-}\mathcal{S}p$ for $n \geq 0$ to the category of Σ_n-symmetric spectra by

$$\mathcal{O}_n[X] \ = \ \bigvee_{k\geq 0} \mathcal{O}(k+n) \wedge_{\Sigma_k} X^{(k)} \ .$$

In the definition, the symmetric group Σ_k acts on $\mathcal{O}(k+n)$ by restriction along the 'inclusion' $-+1_n : \Sigma_k \longrightarrow \Sigma_{k+n}$ and on $X^{(k)}$ by permuting the smash factors. The symmetric group Σ_n acts on each wedge summand by restriction of the action on $\mathcal{O}(k+n)$ along the monomorphism $1_k + - : \Sigma_n \longrightarrow \Sigma_{k+n}$. As we shall see, the functor $\mathcal{O}[-] = \mathcal{O}_0[-]$ has the structure of a triple on the category of symmetric spectra and it acts from the right on the functors $\mathcal{O}_n[-]$. We define a unit transformation $X \longrightarrow \mathcal{O}[X] = \mathcal{O}_0[X]$ as the composite of

$$X \cong \mathbb{S} \wedge X \ \xrightarrow{\iota\wedge \mathrm{Id}} \ \mathcal{O}(1) \wedge X$$

with the wedge summand inclusion for $k = 1$.

We now define a natural transformation $m : \mathcal{O}_n[\mathcal{O}_0[X]] \longrightarrow \mathcal{O}_n[X]$ of functors to Σ_n-symmetric spectra. First we generalize the operad composition to a morphism of symmetric spectra

$$\mathcal{O}(k+n) \wedge \mathcal{O}(i_1) \wedge \cdots \wedge \mathcal{O}(i_k) \ \xrightarrow{\mathrm{Id}\wedge\iota\wedge\cdots\wedge\iota} \mathcal{O}(k+n) \wedge \mathcal{O}(i_1) \wedge \cdots \wedge \mathcal{O}(i_k) \wedge \underbrace{\mathcal{O}(1) \wedge \cdots \wedge \mathcal{O}(1)}_{n}$$
$$\xrightarrow{\ \gamma\ } \ \mathcal{O}(i_1 + \cdots + i_k + n) \ .$$

For fixed $k \geq 0$ we get a morphism

$$\mathcal{O}(k+n) \wedge \left(\bigvee_{m\geq 0} \mathcal{O}(m) \wedge_{\Sigma_m} X^{(m)} \right)^{(k)} \ \cong \bigvee_{(i_1,\ldots,i_k)\in\mathbb{N}^k} \mathcal{O}(k+n) \wedge \Big(\mathcal{O}(i_1) \wedge \cdots \wedge \mathcal{O}(i_k) \wedge_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_k}} X^{(i_1+\cdots+i_k)} \Big)$$

(6.4)

$$\longrightarrow \bigvee_{(i_1,\ldots,i_k)\in\mathbb{N}^k} \mathcal{O}(i_1+\cdots+i_k+n) \wedge_{\Sigma_{i_1+\cdots+i_k}} X^{(i_1+\cdots+i_k)} \ \longrightarrow \ \bigvee_{m\geq 0} \mathcal{O}(m+n) \wedge_{\Sigma_m} X^{(m)} \ .$$

The morphism (6.4) is Σ_k-equivariant with respect to the diagonal action on the source and the trivial action on the target. The morphism (6.4) is also Σ_n-equivariant for these action. So altogether (6.4) passes to a natural Σ_n-equivariant map

$$\mathcal{O}(k+n) \wedge_{\Sigma_k} \left(\bigvee_{m\geq 0} \mathcal{O}(m) \wedge_{\Sigma_m} X^{(m)} \right)^{(k)} \ \longrightarrow \ \bigvee_{m\geq 0} \mathcal{O}(m+n) \wedge_{\Sigma_m} X^{(m)} \ .$$

As k varies, these morphism add up to the natural transformation $m : \mathcal{O}_n[\mathcal{O}[X]] \longrightarrow \mathcal{O}_n[X]$ of functors with values in Σ_n-symmetric spectra.

These maps m are associative and unital in the sense that the diagrams of functors and natural transformations

$$\begin{tikzcd}
\mathcal{O}_n[\mathcal{O}[\mathcal{O}[X]]] \arrow[r, "\mathcal{O}_n[m_X]"] \arrow[d, "m_{\mathcal{O}[X]}"'] & \mathcal{O}_n[\mathcal{O}[X]] \arrow[d, "m"] \\
\mathcal{O}_n[\mathcal{O}[X]] \arrow[r, "m"'] & \mathcal{O}_n[X]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}_n[X] \arrow[r, "\mathcal{O}_n[\mathrm{unit}]"] \arrow[dr, "\mathrm{Id}"'] & \mathcal{O}_n[\mathcal{O}[X]] \arrow[d, "m"] \\
& \mathcal{O}_n[X]
\end{tikzcd}$$

commute for all $n \geq 0$. These properties ultimately come from associativity and unitality of the operad structure, and we omit the details.

For $n = 0$ we have the additional property that the composite

$$\mathcal{O}[X] \ \xrightarrow{\mathrm{unit}_{\mathcal{O}[X]}} \ \mathcal{O}[\mathcal{O}[X]] \ \xrightarrow{\ m\ } \mathcal{O}[X]$$

is the identity, and so this structure makes the functor $\mathcal{O}[-] = \mathcal{O}_0[-]$ into a triple (also called *monad*) on the category of symmetric spectra. An $\mathcal{O}[-]$-*algebra* is a symmetric spectrum A together with a morphism $\alpha : \mathcal{O}[A] \longrightarrow A$ which is associative and unital in the sense that the following diagrams commute

$$\begin{tikzcd} \mathcal{O}[\mathcal{O}[A]] \arrow[r, "\mathcal{O}[\alpha]"] \arrow[d, "m"'] & A \arrow[d, "\alpha"] \\ \mathcal{O}[A] \arrow[r, "\alpha"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\text{unit}"] \arrow[dr, "\text{Id}"'] & \mathcal{O}[A] \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

Given such an $\mathcal{O}[-]$-algebra (A, α) we can give the symmetric spectrum A the structure of an algebra over the operad $\mathcal{O}$ by declaring the n-th part of the action map to be the composite

$$\mathcal{O}(n) \wedge_{\Sigma_n} A^{(n)} \xrightarrow{\text{incl}} \mathcal{O}[A] \xrightarrow{\alpha} A \ .$$

This assignment is an isomorphism of categories between the algebras over the triple $\mathcal{O}[-]$ and algebras over the operad $\mathcal{O}$. So we will now use these two notions of algebras interchangeably, and just talk about $\mathcal{O}$-algebras even when we think about $\mathcal{O}[-]$-algebras.

One immediate consequence of the comparison between algebras over $\mathcal{O}[-]$ and $\mathcal{O}$ is the following. For every symmetric spectrum X the spectrum $\mathcal{O}[X]$ has the structure of an $\mathcal{O}[-]$-algebra, thus of an $\mathcal{O}$-algebra, such that $X \mapsto \mathcal{O}[X]$ is left adjoint to the forgetful functor from $\mathcal{O}$-algebras to symmetric spectra. By a slight abuse of notation we also denote this free functor by

$$\mathcal{O}[-] \ : \ \mathcal{S}p \ \longrightarrow \ \mathcal{O}\text{-alg} \ .$$

Two examples of this have already come up in Example I.5.27: if $\mathcal{O} = \mathcal{A}ss$ is the associative operad, then $\mathcal{A}ss[X] = TX$ is the tensor algebra generated by the symmetric spectrum X; for the commutative operad $\mathcal{O} = \mathcal{C}om$ we get $\mathcal{C}om[X] = PX$, the symmetric algebra generated by X.

We can also deduce from general principles that the category of $\mathcal{O}$-algebras has limits and colimits.

Proposition 6.5. *For every operad $\mathcal{O}$ of symmetric spectra, the category of $\mathcal{O}$-algebras has [enriched?] limits and colimits. The forgetful functor from $\mathcal{O}$-algebras to symmetric spectra commutes with limits and with filtered colimits.*

Proof. A limit of $\mathcal{O}$-algebras is given by the limit of the underlying symmetric spectra, with a canonical $\mathcal{O}$-algebra structure. Colimits of algebras over a triple are slightly more subtle, but they exist here because the underlying functor of the triple $\mathcal{O}$ preserves filtered colimits. □

The key ingredient in the proof of Theorem 6.1 is a homotopical analysis of certain pushouts of $\mathcal{O}$-algebras. For this we use a certain filtration of such pushout which we now define.

We consider a morphism $f : X \longrightarrow Y$ of symmetric spectra and define a Σ_n-symmetric spectrum $Q^n(f)$ and an equivariant morphism $Q^n(f) \longrightarrow Y^{(n)}$ for $n \geq 0$. To define $Q^n(f)$ we first describe an n-dimensional cube of symmetric spectra; by definition, such a cube is a functor

$$W : \mathcal{P}(\{1, 2, \ldots, n\}) \ \longrightarrow \ \mathcal{S}p$$

from the poset category of subsets of $\{1, 2, \ldots, n\}$ and inclusions. If $S \subseteq \{1, 2, \ldots, n\}$ is a subset, the vertex of the cube at S is defined as

$$W(S) \ = \ C_1 \wedge C_2 \wedge \ldots \wedge C_n$$

with

$$C_i \ = \begin{cases} X & \text{if } i \notin S \\ Y & \text{if } i \in S. \end{cases}$$

For $S \subseteq T$ the morphism $W(S) \longrightarrow W(T)$ is a smash product of copies of the map $f : X \longrightarrow Y$ at all coordinates in $T - S$ with identity maps of X or Y. For example for $n = 2$, the cube is a square and looks like

$$\begin{tikzcd} X \wedge X \arrow[r, "\mathrm{Id} \wedge f"] \arrow[d, "f \wedge \mathrm{Id}"'] & X \wedge Y \arrow[d, "f \wedge \mathrm{Id}"] \\ Y \wedge X \arrow[r, "\mathrm{Id} \wedge f"'] & Y \wedge Y. \end{tikzcd}$$

We denote by $Q^n(f)$ the colimit of the punctured cube, i.e., the cube with the terminal vertex removed. [explain the Σ_n-action] It comes with a natural equivariant morphism $Q^n(f) \longrightarrow W(\{1,\dots,n\}) = Y^{(n)}$ to the terminal vertex of the cube.

Definition 6.6. Let G be a finite group. A morphism $f : A \longrightarrow B$ of G-symmetric spectra is a *flat G-cofibration* (respectively *projective G-cofibration*) if it has the left lifting property with respect to all morphisms of G-symmetric spectra which are flat trivial fibrations (respectively projective trivial fibrations) of underlying spectra.

G-symmetric spectra 'are' modules over the spherical group ring $\mathbb{S}[G]$; under this isomorphism of categories, the flat (projective) G-cofibrations are the same as the cofibrations in the flat (projective) absolute stable model structure on $\mathbb{S}[G]$-modules [see below].

see: Sergey Gorchinskiy, Vladimir Guletskii, Symmetric powers in stable homotopy categories, `arXiv:0907.0730`

Proposition 6.7. *Let $f : X \longrightarrow Y$ be a flat (respectively projective) cofibration of symmetric spectra such that $f_0 : X_0 \longrightarrow Y_0$ is an isomorphism. Then for every $n \geq 0$ the morphism of Σ_n-symmetric spectra*

$$\gamma_n \ : \ Q^n(f) \ \longrightarrow \ Y^{(n)}$$

is a flat (respective projective) Σ_n-cofibration.

We note that hypothesis that f_0 be an isomorphism is essential. Indeed, if K is a non-empty cofibrant space (or simplicial set), then the suspension spectrum $\Sigma^\infty_+ K$ is projective. So the morphism from the trivial spectrum to $\Sigma^\infty_+ K$ is a projective cofibration which violates the conclusion of Proposition 6.7: the smash power $(\Sigma^\infty_+ K)^{(n)}$ is isomorphic to $\Sigma^\infty_+(K^n)$, but for any $n \geq 2$ the cartesian product K^n has Σ_n-fixed points, so it is *not* a free Σ_n-space. Consequently, the spectrum $(\Sigma^\infty_+ K)^{(n)}$ is *not* Σ_n-flat [ref] (although its underlying non-equivariant spectrum is projective, thus flat).

Proof of Proposition 6.7. Let us call a morphism f of symmetric spectra a *power cofibration* if for all $n \geq 0$ the morphism $\gamma_n : Q^n(f) \longrightarrow Y^{(n)}$ is a flat Σ_n-cofibration. We claim that:

(a) The class of power cofibrations is closed under pushouts. Indeed, for every pushout squares of symmetric spectra

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, "g"'] & P \end{tikzcd}$$

the square

$$\begin{tikzcd} Q^n(f) \arrow[r, "\gamma_n(f)"] \arrow[d] & Y^{(n)} \arrow[d] \\ Q^n_{n-1}(g) \arrow[r, "\gamma_n(g)"'] & P^{(n)} \end{tikzcd}$$

is a pushout of Σ_n-spectra. So if f is a power fibration, then so is g, since flat Σ_n-cofibrations are stable under pushouts.

(b) The class of power cofibrations is closed under countable composition. Indeed, if $f_i : X_i \longrightarrow X_{i+1}$ are power cofibrations for all $i \geq 0$ and $f_\infty : X_0 \longrightarrow X_\infty = \operatorname{colim} X_i$ is the canonical morphism, then

(c) For every $m \geq 0$ and every cofibration $\varphi : K \longrightarrow L$ of Σ_m-simplicial sets, the map of semifree symmetric spectra $G_m\varphi : G_mK \longrightarrow G_mL$ is a power cofibrations.

In the general case of a flat cofibration $f : X \longrightarrow Y$ we use the filtration F^mX defined in Section II.**??**. The latching map $\nu_m(f) : X_m \cup_{L_mX} L_mY_m \longrightarrow Y_m$ is a Σ_m-cofibration of based spaces (or simplicial sets), so the morphism $G_m\nu_m(f)$ of semifree spectra is a power cofibration by (c). The pushout squares

$$\begin{tikzcd} G_m(X_m \cup_{L_mX} L_mY_m) \arrow[r, "G_m\nu_m(f)"] \arrow[d] & G_mY_m \arrow[d] \\ X \cup_{F^{m-1}X} F^{m-1}Y \arrow[r, "j_m"'] & X \cup_{F^mX} F^mY \end{tikzcd}$$

[check...] show that the upper horizontal morphism j_m is a power cofibration by (a). The spectrum Y is the colimit of the spectra $X \cup_{F^mX} F^mY$ along the morphisms j_m, so finally the morphism $X = X \cup_{F^{-1}X} F^{-1}Y \longrightarrow Y$ is a power cofibration by (b).

We start with the special case $X = G_mL$ for some $m \geq 0$ and a pointed Σ_m-simplicial set L. We have

$$(G_mL)^{(n)} \ \cong \ G_{mn}(\Sigma^+_{mn} \wedge_{\Sigma_m\times\cdots\times\Sigma_m} L \wedge \ldots \wedge L) \ ;$$

in this description the permutation action of $\gamma \in \Sigma_n$ in level $mn+k$ of $(G_mL)^{(n)}$ is given by

$$\begin{aligned} \gamma_* \ : \ \Sigma^+_{mn+k} \wedge_{(\Sigma_m)^n\times\Sigma_k} L^{(n)} \wedge S^k \ &\longrightarrow \ \Sigma^+_{mn+k} \wedge_{(\Sigma_m)^n\times\Sigma_k} L^{(n)} \wedge S^k \\ \tau \wedge a_1 \wedge \ldots a_n \wedge x \quad &\longmapsto \quad \tau\Delta(\gamma) \wedge a_{\gamma^{-1}(1)} \wedge \ldots a_{\gamma^{-1}(n)} \wedge x \ . \end{aligned}$$

Here $\tau \in \Sigma_{mn+k}$, $a_i \in L$, $x \in S^k$ and $\Delta : \Sigma_n \longrightarrow \Sigma_{mn}$ is the diagonal embedding. The space $\left((G_mL)^{(n)}\right)_{mn+k}$ is a wedge of copies of $L^{(n)} \wedge S^k$ indexed by the cosets of the group $(\Sigma_m)^n \times \Sigma_k$ in Σ_{mn+k}. The diagonal subgroup $\Delta(\Sigma_n)$ normalizes $(\Sigma_m)^n \times \Sigma_k$ inside Σ_{mn+k}, and so Σ_n acts from the right on the set of the cosets $\Sigma_{mn+k}/(\Sigma_m)^n \times \Sigma_k$ by

$$[g\left((\Sigma_m)^n \times \Sigma_k\right)] \cdot \gamma \ = \ g\Delta(\gamma)\left((\Sigma_m)^n \times \Sigma_k\right) \ .$$

By the formula above, this is how the permutation action of Σ_n permutes the wedge summands in level $mn+k$. For $m \geq 1$ the diagonal embedding $\Delta : \Sigma_n \longrightarrow \Sigma_{mn+k}$ is injective and its image intersects the subgroup $\Sigma_{mn+k}/(\Sigma_m)^n \times \Sigma_k$ only in the identity element. Thus the right action of Σ_n on $\Sigma_{mn+k}/(\Sigma_m)^n \times \Sigma_k$ is free, and thus the Σ_n-action on $(G_mL)^{(n)}$ is levelwise free away from the basepoint. □

Construction 6.8. Given an $\mathcal{O}$-algebra $(A, \alpha : \mathcal{O}(A) \longrightarrow A)$ and a number $n \geq 0$ we define a Σ_n-symmetric spectrum $\mathbb{U}_nA$ as the coequalizer, in the category of Σ_n-symmetric spectra, of the two morphisms

$$\mathcal{O}_n(\mathcal{O}(A)) \ \underset{\mathcal{O}_n(\alpha)}{\overset{m_A}{\rightrightarrows}} \ \mathcal{O}_n(A) \ .$$

The role of the spectrum $\mathbb{U}_nA$ is that the underlying symmetric spectrum of an $\mathcal{O}$-algebra coproduct $A \amalg \mathcal{O}(X)$ is isomorphic to

$$\bigvee_{n\geq 0} \mathbb{U}_nA \wedge_{\Sigma_n} X^{\wedge n} \ .$$

For example, we have $\mathbb{U}_0A = A$ since the diagram

$$\mathcal{O}(\mathcal{O}(A)) \ \underset{\mathcal{O}(\alpha)}{\overset{m_A}{\rightrightarrows}} \ \mathcal{O}(A) \ \xrightarrow{\ \alpha\ } \ A$$

is a (split) coequalizer, even as $\mathcal{O}$-algebras. [So we should have $\mathbb{U}_n(\mathcal{O}(*)) = \mathcal{O}(n)$ to recover $\mathcal{O}(X) \cong \mathcal{O}(*) \amalg \mathcal{O}(X)$]

We consider an $\mathcal{O}$-algebra A, a morphism $f : X \longrightarrow Y$ of symmetric spectra and a morphism $g : \mathcal{O}(X) \longrightarrow A$ of $\mathcal{O}$-algebras. We define a filtration of the $\mathcal{O}$-algebra pushout $A \amalg_{\mathcal{O}(X)} \mathcal{O}(Y)$. We set $A_0 = A$

and define a symmetric spectrum A_n inductively as the pushout

$$\begin{tikzcd} \mathbb{U}_n A \wedge_{\Sigma_n} Q^n_{n-1} \arrow[r, "\mathrm{Id} \wedge \gamma_n"] \arrow[d] & \mathbb{U}_n A \wedge_{\Sigma_n} Y^{(n)} \arrow[d] \\ A_{n-1} \arrow[r] & A_n \end{tikzcd} \tag{6.9}$$

The left vertical map is obtained by passage to coequalizers from the composite

$$\mathcal{O}_n(A) \wedge_{\Sigma_n} Q^n_{n-1} \;=\; \bigvee_{k \geq 0} \mathcal{O}(k+n) \wedge_{\Sigma_k \times \Sigma_n} A^{(k)} \wedge Q^n_{n-1} \xrightarrow{\cong}$$

$$\bigvee_{k \geq 0} \mathcal{O}(k+n) \wedge_{\Sigma_k \times \Sigma_n} A^{(k)} \wedge \operatorname{colim}_{S \neq \{1,\dots,n\}} W(S) \xrightarrow{\text{use } f : X \longrightarrow A}$$

Lemma 6.10. *The colimit $A_\infty = \operatorname{colim}_n A_n$ of the sequence of symmetric spectra has the structure of an $\mathcal{O}$-algebra which makes it a pushout in the category of $\mathcal{O}$-algebras of the diagram*

$$A \xleftarrow{g} \mathcal{O}(X) \xrightarrow{\mathcal{O}(f)} \mathcal{O}(Y)$$

in such a way that the canonical morphism of symmetric spectra $A \longrightarrow A_\infty$ becomes the canonical morphism of $\mathcal{O}$-algebras from A to the pushout.

Proof. There are several things to check:

(i) A_∞ is naturally a $\mathcal{O}$-algebra so that
(ii) $A \longrightarrow A_\infty$ is a map of $\mathcal{O}$-algebras and
(iii) A_∞ has the universal property of the pushout in the category of $\mathcal{O}$-algebras.

Define the $\mathcal{O}$-algebra structure $\mathcal{O}(A_\infty) \longrightarrow A_\infty$ as the composite of $A \longrightarrow A_\infty$ with the unit of A. The multiplication of A_∞ is defined from compatible maps $A_n \wedge A_m \longrightarrow A_{n+m}$ by passage to the colimit. These maps are defined by induction on $n + m$ as follows. Note that $A_n \wedge A_m$ is the pushout in mod-R in the following diagram.

$$\begin{tikzcd} Q_n \wedge ((A \wedge L)^m \wedge A) \cup_{(Q_n \wedge Q_m)} ((A \wedge L)^n \wedge A) \wedge Q_m \arrow[r] \arrow[d] & (A \wedge L)^n \wedge A) \wedge ((A \wedge L)^m \wedge A) \arrow[d] \\ (A_{n-1} \wedge A_m) \cup_{(A_{n-1} \wedge A_{m-1})} (A_n \wedge A_{m-1}) \arrow[r] & A_n \wedge A_m \end{tikzcd}$$

The lower left corner already has a map to A_{n+m} by induction, the upper right corner is mapped there by multiplying the two adjacent factors of A followed by the map $(A \wedge L)^{n+m} \wedge A \longrightarrow A_{n+m}$ from the definition of A_{n+m}. We omit the tedious verification that this in fact gives a well defined multiplication map and that the associativity and unital diagrams commute. Hence, A_∞ is a $\mathcal{O}$-algebra. Multiplication in A_∞ was arranged so that $A \longrightarrow A_\infty$ is a $\mathcal{O}$-algebra map.

For (iii), suppose we are given another $\mathcal{O}$-algebra B, a $\mathcal{O}$-algebra morphism $A \longrightarrow B$, and a mod-R-map $L \longrightarrow B$ such that the outer square in

$$\begin{tikzcd} K \arrow[r] \arrow[d] & L \arrow[d] \arrow[ddr] & \\ A \arrow[r] \arrow[drr] & A_\infty \arrow[dr, dotted] & \\ & & B \end{tikzcd}$$

commutes. We have to show that there is a unique $\mathcal{O}$-algebra map $A_\infty \longrightarrow B$ making the entire square commute. These conditions in fact force the behavior of the composite map $W(S) \longrightarrow P_n \longrightarrow A_\infty \longrightarrow B$. Since A_∞ is obtained by various colimit constructions from these basic building blocks, uniqueness follows.

We again omit the tedious verification that the maps $W(S) \longrightarrow B$ are compatible and assemble to a $\mathcal{O}$-algebra map $A_\infty \longrightarrow B$. □

Proposition 6.11. *Let $\mathcal{O}$ be an operad of symmetric spectra, A and $\mathcal{O}$-algebra, $f : X \longrightarrow Y$ be a flat cofibration of symmetric spectra and let $\mathcal{O}(X) \longrightarrow A$ be a morphism of $\mathcal{O}$-algebras. Suppose in addition that*

- *f is an isomorphism in level 0, [or*
- *the symmetric groups Σ_n act freely on $\mathcal{O}(n)$ for all $n \geq 0$].*

Then then the pushout $A \longrightarrow A \cup_{\mathcal{O}(X)} \mathcal{O}(Y)$ is an injective cofibration [flat if source is flat?] of underlying symmetric spectra. If moreover f is a stable equivalence or π_*-isomorphism, then so is the morphism $A \longrightarrow A \cup_{\mathcal{O}(X)} \mathcal{O}(Y)$.

Proof. Since $f : X \longrightarrow Y$ is a flat acyclic cofibration, the morphism $\gamma_n : Q^n \longrightarrow Y^{(n)}$ is also a flat acyclic cofibration by Proposition 6.7. We show by induction on n that the morphisms $A_{n-1} \longrightarrow A_n$ defined in (6.9) are injective stable equivalences.

Since γ_n is a flat acyclic cofibration, the morphism

$$\mathrm{Id} \wedge \gamma_n \ : \ \mathbb{U}_n A \wedge Q^n_{n-1} \ \longrightarrow \ \mathbb{U}_n A \wedge Y^{(n)}$$

is injective and a stable equivalence. In particular, the quotient spectrum $\mathbb{U}_n A \wedge (Y/X)^{(n)}$ is stably contractible. Now we pass to quotients by the Σ_n-action. We deduce that the morphism

$$\mathrm{Id} \wedge_{\Sigma_n} \gamma_n \ : \ \mathbb{U}_n A \wedge_{\Sigma_n} Q^n_{n-1} \ \longrightarrow \ \mathbb{U}_n A \wedge_{\Sigma_n} Y^{(n)}$$

is again injective, and its cokernel is isomorphic to the spectrum

$$\mathbb{U}_n A \wedge_{\Sigma_n} (Y/X)^{(n)} \ .$$

Under the assumption that $f : X \longrightarrow Y$ is an isomorphism in level 0 the symmetric spectrum Y/X is trivial in level 0. Again by Proposition 6.7 the permutation action of Σ_n on the smash power $(Y/X)^{(n)}$ is then free away from the basepoint, hence so is the diagonal action on $\mathbb{U}_n A \wedge (Y/X)^{(n)}$. Since $\mathbb{U}_n A \wedge (Y/X)^{(n)}$ is stably constractible (respectively has trivial homomotopy groups) and has a free action, the quotient spectrum $\mathbb{U}_n A \wedge_{\Sigma_n} (Y/X)^{(n)}$ is again stably contractible (respectively has trivial homomotopy groups) by Proposition 6.12. □

Now we can left the various stable model structures from symmetric spectra to $\mathcal{O}$-algebras.

Proof of Theorem 6.1. We define cofibrations of $\mathcal{O}$-algebras by the lifting property with respect to acylic fibrations of $\mathcal{O}$-algebras.

The category of $\mathcal{O}$-algebras has limits and colimits by Proposition 6.5. The 2-out-of-3 property for stable equivalences of $\mathcal{O}$-algebras and the closure under retracts for stable equivalences and stable fibrations of $\mathcal{O}$-algebras follow from that corresponding properties for symmetric spectra. Cofibrations are defined by a lifting property, so the are closed under retracts.

The factorizations are produced by the small object argument. We define the necessary sets of generating cofibrations and acyclic cofibrations as

$$I_{\mathcal{O}} \ = \ \mathcal{O}(I) \quad \text{and} \quad J_{\mathcal{O}} \ = \ \mathcal{O}(J) \ ,$$

where I and J are the generating cofibrations (respectively acyclic cofibrations) for the [...adjectives...] stable model structure of symmetric spectra, defined in [...].

The key non-trivial step is to verify that every $I_{\mathcal{O}}$-cell complex is also a stable equivalence. This is a special case of Proposition 6.11. □

Is this needed:

Proposition 6.12. *Let G be a finite group and X a G-symmetric spectrum such that the G-action is free away from the basepoint.*

- *If X is stably contractible, then so is the quotient symmetric spectrum.*

• *If X is k-connected for some integer k, then so is the quotient symmetric spectrum.*

PROOF. Since the G-action is free away from the basepoint the morphism $EG^+ \wedge_G X \longrightarrow S^0 \wedge_G X$ induced by the unique map $EG \longrightarrow *$ is a level equivalence. So it suffices to show that the left hand side is stably contractible (respectively k-connected) if X is. We show by induction over n that the symmetric spectrum $E^nG^+ \wedge_G X$ is stably contractible (respectively k-connected), where E^nG is the simplicial n-skeleton of EG. The induction start with $n = -1$ where we interpret $E^{-1}G$ as the empty simplicial set. The quotient $E^nG/E^{n-1}G$ is isomorphic as a G-simplicial set to $G^+ \wedge \Delta[n]/\partial\Delta[n] \wedge G^{(n)}$, where the smash powers are taken with the unit $1 \in G$ as basepoint. So we get a cofibre sequence of symmetric spectra

$$E^{n-1}G^+ \wedge_G X \longrightarrow E^nG^+ \wedge_G X \longrightarrow \Big(G^+ \wedge \Delta[n]/\partial\Delta[n] \wedge G^{(n)}\Big) \wedge_G X \cong \Delta[n]/\partial\Delta[n] \wedge G^{(n)} \wedge X \ .$$

The last term is stably contractible (respectively $(k+n)$-connected) and the first morphism is injective. So the inclusion $E^{n-1}G^+ \wedge_G X \longrightarrow E^nG^+ \wedge_G X$ is a stable equivalence (respectively $(k+n)$-connected) and $E^nG^+ \wedge_G X$ is stably contractible (respectively k-connected) for all $n \geq 0$. Thus the filtered colimit $EG^+ \wedge_G X$ is also stably contractible (respectively k-connected). □

7. Connective covers of structured spectra

Construction 7.1. The category of graded abelian groups forms a symmetric monoidal category under graded tensor product (with sign in the symmetry isomorphism).

Let $\mathcal{O}$ be an operad of symmetric spectra. Then we can define an operad $\pi\mathcal{O}$ in the category of graded abelian groups (under graded tensor product) by

$$(\pi\mathcal{O})(n) \ = \ \pi_*(\mathcal{O}(n)) \ ,$$

the graded abelian group of true homotopy groups of the symmetric spectrum $\mathcal{O}(n)$.

For every operad $\mathcal{O}$ of symmetric spectra and every $\mathcal{O}$-algebra A we now make the true homotopy groups π_*A into a graded $\pi\mathcal{O}$-algebra. For the associative respectively commutative operad this recovers the structure of (commutative) graded ring on the homotopy groups of a semistable (commutative) symmetric ring spectrum.

Example 7.2. Suppose $\mathcal{E}$ is an operad of sets, and let $\mathcal{O} = \Sigma^\infty_+\mathcal{E}$ the operad of symmetric spectra obtained by taking suspensions spectra. Then $\mathcal{O}(n)$ is a wedge of sphere spectrum $\mathbb{S}$, indexed by the elements $\mathcal{E}(n)$, and so by [...] we have $\pi_0\mathcal{O}(n) \cong \mathbb{Z}[\mathcal{E}(n)]$. In other words, the degree part of the operad $\pi\mathcal{O}$ is the free abelian group operad generated by $\mathcal{E}$.

Example 7.3. Algebras over the associative operad 'are' symmetric ring spectra, and the observation above [degree zeo part] reduces to the fact, already observed in Proposition I.6.25, that the homotopy groups of a symmetric ring spectrum naturally form a graded ring.

For the commutative operad, we similar re-obtain that the homotopy groups of a sommutative symmetric ring spectrum naturally form a graded commutative ring.

Example 7.4. We identify the Σ_n-symmetric spectra $\mathbb{U}_nA$ in the case of the initial, the commutative and the associative operad. The key to this identification is the isomorphism

$$A \amalg \mathcal{O}(L) \ \cong \ \bigvee_{n\geq 0} \mathbb{U}_nA \wedge_{\Sigma_n} L^{(n)}$$

which is natural in the $\mathcal{O}$-algebra A and the symmetric spectrum L.

Commutative operad: algebras over the commutative operad are commutative symmetric ring spectra and the coproduct is given by the smash product of the underlying symmetric spectra. So for $\mathcal{O} = \mathcal{C}om$, a symmetric spectrum L and an $\mathcal{O}$-algebra A we have

$$A \amalg \mathcal{O}(L) \ = \ A \wedge PL \ = \ \bigvee_{n\geq 0} A \wedge (L^{(n)}/\Sigma_n) \ .$$

Thus we have $\mathbb{U}_nA = A$ with trivial Σ_n-action. In particular, $\mathbb{U}_nA$ is connective whenver A is.

Associative operad: algebras over the associative operad are symmetric ring spectra and the coproduct of a symmetric ring spectrum A with the tensor algebra TL of a symmetric spectrum L is given

$$A \amalg \mathcal{O}(L) \;=\; \bigvee_{n\geq 0} A \wedge (L\wedge A)^{(n)} \;.$$

Since

$$A \wedge (L\wedge A)^{(n)} \;=\; (\Sigma_n^+ \wedge A^{(n+1)}) \wedge_{\Sigma_n} L^{(n)}$$

we have $\mathbb{U}_n A = \Sigma_n^+ \wedge A^{(n+1)}$. In particular, $\mathbb{U}_n A$ is connective whenver A is and the underlying symmetric spectrum of A is flat, which for example is the case when A is cofibrant as a symmetric ring spectrum.

Initial operad: algebras over the initial operad are symmetric spectra and the coproduct is given by the wedge. So we have

$$A \amalg \mathcal{O}(L) \;=\; A \vee L \;.$$

This means that

$$\mathbb{U}_n(A) \;=\; \begin{cases} A & \text{for } n=0, \\ \mathbb{S} & \text{for } n=1, \text{ and} \\ * & \text{for } n\geq 2. \end{cases}$$

In particular $\mathbb{U}_n A$ is connective whenver A is.

Proposition 7.5. *Let $\mathcal{O}$ be an operad of symmetric spectra such that $\mathcal{O}(n)$ is connective for all $n \geq 0$. Then for every connective cofibrant $\mathcal{O}$-algebra A the Σ_m-symmetric spectrum $\mathbb{U}_m A$ is also connective.*

Recall that a morphism of symmetric spectra is *n-connected*, for some integer n, if it induces isomorphisms of homotopy groups below dimension n and epimorphism on π_n. If the morphism is a level cofibration, then the long exact sequence of homotopy groups shows that the morphism is n-connected if and only if its cokernel is n-connected.

Lemma 7.6. *Let $n \geq 0$. Then for every $(n-1)$-connected flat symmetric spectrum X with $X_0 = *$ and every cofibrant connective $\mathcal{O}$-algebra A the $\mathcal{O}$-algebra morphism $A \longrightarrow A \amalg_{\mathcal{O}(X)} \mathcal{O}(CX)$ is n-connected.*

Proof. In a first step we show that if X is $(n-1)$-connected, then for every $\mathcal{O}$-algebra A the summand inclusion $A \longrightarrow A \amalg \mathcal{O}(X)$ is $(n-1)$-connected. The coproduct is a special case of a pushout along the initial object, so as a special case of Lemma 6.10 the coproduct is isomorphic to

$$\bigvee_{m\geq 0} \mathbb{U}_m A \wedge_{\Sigma_m} X^{(m)} \;.$$

The morphism $A \longrightarrow A \amalg \mathcal{O}(K)$ corresponds to the summand inclusion for $m = 0$, so it suffices to show that the remaining summands are $(n-1)$-connected. Since A is connective and cofibrant the Σ_m-symmetric spectrum $\mathbb{U}_m A$ is also connective. [we only know this part for the associative or the commutative operad...] For $m \geq 1$ the smash power $X^{(m)}$ is again $(n-1)$-connected, flat and has a free Σ_m-action by Proposition 6.7. So the spectrum $\mathbb{U}_m A \wedge X^{(m)}$ is also $(n-1)$-connected and has a free Σ_m-action, and thus by Proposition 6.12 the orbit spectrum $\mathbb{U}_m A \wedge_{\Sigma_m} X^{(m)}$ is $(n-1)$-connected.

The $\mathcal{O}$-algebra $A \amalg_{\mathcal{O}(X)} \mathcal{O}(CX)$ is the realization of the simplicial $\mathcal{O}$-algebra

$$[k] \;\longmapsto\; A \amalg \underbrace{\mathcal{O}(X) \amalg \cdots \amalg \mathcal{O}(X)}_{k} \;=\; A \amalg \mathcal{O}(\underbrace{X \vee \ldots \vee X}_{k}) \;.$$

The object of 0-simplices is exactly A. Since the realization of simplicial $\mathcal{O}$-algebras in performed on underlying symmetric spectra, we argue with the underlying simplicial object of symmetric spectra to deduce that the vertex map $A \longrightarrow A \amalg_{\mathcal{O}(X)} \mathcal{O}(CX)$ is n-connected.

We consider the simplicial spectrum obtained by dimensionwise collapsing the vertex object A is n-connected. By the first part the spectrum

$$\left[A \amalg \mathcal{O}(\underbrace{X \vee \ldots \vee X}_{k}) \right] /A$$

(quotient as symmetric spectra) is $(n-1)$-connected for all $k \geq 1$, and it is trivial for $k = 0$. So the geometric realization is this simplicial quotient spectrum is k-connected, hence so is the vertex morphism $A \longrightarrow A \amalg_{\mathcal{O}(X)} \mathcal{O}(CX)$. $\square$

Proposition 7.7. *Let $\mathcal{O}$ be an operad of symmetric spectra such that $\mathcal{O}(n)$ is connective for all $n \geq 0$ and let $\varphi : A \longrightarrow B$ be a morphism of $\mathcal{O}$-algebras. For every $n \geq 0$ there exists a functorial factorization*

$$A \xrightarrow{g} A^{\langle n \rangle} \xrightarrow{h} B$$

of φ in the category of $\mathcal{O}$-algebras such that:

- *g is a positive projective cofibration of $\mathcal{O}$-algebras and h is a positive projective level fibration;*
- *if A is connective [and cofibrant?], the morphism g induces isomorphisms of homotopy groups below dimension n and an epimorphism on π_n and the morphism h induces a monomorphism on π_n and an isomorphism of homotopy groups above dimension n.*

Proof. The idea is to 'kill homotopy groups' in the world of $\mathcal{O}$-algebras. In other words, we cone off all morphisms from $\mathcal{O}$-algebra spheres $\mathcal{O}(F_m S^{k+m})$ for $k \geq n$ to A and iterate the process. Since we want functoriality, we cannot choose generators of homotopy groups, but we should rather cone off *all* morphism from $\mathcal{O}(F_m S^{k+m})$ to A. The 'small object argument' is exactly the process to achieve this.

We define a set of morphisms of symmetric spectra as $K^{\langle n \rangle} = J^{lv,+}_{\text{proj}} \cup C^{\langle n \rangle}$. Here

$$J^{lv,+}_{\text{proj}} \ = \ \left\{ F_m \Lambda^i[k]^+ \longrightarrow \ F_m \Delta[k]^+ \right\}_{m \geq 1, k \geq 0, 0 \leq i \leq k}$$

of generating acyclic cofibrations for the positive projective level model structure and

$$C^{\langle n \rangle} \ = \ \left\{ F_m S^{k+m} \longrightarrow \ F_m CS^{k+m} \right\}_{m \geq 1, k \geq n}$$

where $S^{k+m} \longrightarrow CS^{k+m}$ is the cone inclusion. A morphism $f : X \longrightarrow Y$ of symmetric spectra has the right lifting property with respect to the set $J^{lv,+}_{\text{proj}}$ if and only if it is a projective level fibration in positive levels, i.e., if and only if the morphisms $f_m : X_m \longrightarrow Y_m$ are Kan fibrations for $m \geq 1$.

If in addition f has the right lifting property for the set $C^{\langle n \rangle}$, then so has its fibre F over the basepoint. This means that for positive m the simplicial set F_m is Kan and has the right lifting property for the cone inclusions $S^{k+m} \longrightarrow CS^{k+m}$ for all $k \geq n$. Thus the homotopy groups of F_m vanish in dimensions $\geq n+m$, and so we have $\pi_k F = 0$ for $k \geq n$. The long exact sequence of homotopy groups shows that the map $\pi_n f : \pi_n X \longrightarrow \pi_n Y$ is injective and f induces isomorphisms of homotopy groups above dimension n.

We now apply the small object argument, in the category of $\mathcal{O}$-algebras, to the morphism $\varphi : A \longrightarrow B$ with respect to the set $\mathcal{O}K^{\langle n \rangle}$. It produces a functorial factorization

$$A \xrightarrow{g} A^{\langle n \rangle} \xrightarrow{h} B$$

of φ in the category of $\mathcal{O}$-algebras such that g is a $\mathcal{O}K^{\langle n \rangle}$-cell complex and h has the right lifting property with respect to the set $\mathcal{O}K^{\langle n \rangle}$. Since $\mathcal{O}$ is left adjoint to the forgetful functor, this means that the underlying morphism of symmetric spectra of h has the right lifting property with respect to the set $K^{\langle n \rangle}$, so by the above, h is a positive projective level fibration, induces a monomorphism on π_n and isomorphisms of homotopy groups above dimensions n.

Since every morphism in the set $K^{\langle n \rangle}$ is a positive projective cofibration of symmetric spectra, every morphism in the set $\mathcal{O}K^{\langle n \rangle}$ is a positive projective cofibration of $\mathcal{O}$-algebras, and so every $\mathcal{O}K^{\langle n \rangle}$-cell complex, in particular the morphsm g, is a positive projective cofibration of $\mathcal{O}$-algebras,

It remains to identify the effect of the morphism $g : A \longrightarrow A^{\langle n \rangle}$ on homotopy groups. Let $f^{\text{lv}} : X \longrightarrow Y$ be a positive projective cofibration which is also a level equivalence. Then $A \longrightarrow A \amalg_{\mathcal{O}(X)} \mathcal{O}(Y)$ is a π_*-isomorphism by Proposition 6.11.

If X is an $(n-1)$-connected projective symmetric spectrum with $X_0 = *$ and $f^c : X \longrightarrow CX$ the cone inclusions, the by Lemma 7.6 the morphism $A \longrightarrow A \amalg_{\mathcal{O}(X)} \mathcal{O}(CX)$ is n-connected.

Every $\mathcal{O}K^{\langle n \rangle}$-cell complex is a countable compositition of pushouts along morphisms $\mathcal{O}(f^{\text{lv}} \vee f^c)$ where f^{lv} and f^c are positive projective cofibrations, f^{lv} is a level equivalence and f^c is a cone inclusion of an $(n-1)$-connected spectrum. Such pushout can be done in two seperate steps, so by the above, the morphism from A to the pushout is n-connected. So every $\mathcal{O}K^{\langle n\rangle}$-cell complex is n-connected. This applies in particular to the morphism $g : A \longrightarrow A^{\langle n\rangle}$ which thus induces a bijection of homotopy groups below dimension n and an epimorphism in dimension n. □

Theorem 7.8. *Let $\mathcal{O}$ be an operad of symmetric spectra such that $\mathcal{O}(n)$ is connective for all $n \geq 0$.*

(i) *There exists a functor*

$$\tau_{\geq 0} \ : \ \mathcal{O}\text{-alg} \ \longrightarrow \ \mathcal{O}\text{-alg}$$

and a natural morphism of $\mathcal{O}$-algebras $\tau_{\geq 0}A \longrightarrow A$ with the following property:

- *the $\mathcal{O}$-algebra $\tau_{\geq 0}A$ is connective*
- *the morphism $\tau_{\geq 0}A \longrightarrow A$ induces an isomorphism on π_k for all $k \geq 0$.*

We refer to $\tau_{\geq 0}A$ as the connective cover *of the $\mathcal{O}$-algebra A.*

(ii) *For every $n \geq 0$ there exists a functor*

$$P_n \ : \ \mathcal{O}\text{-alg} \ \longrightarrow \ \mathcal{O}\text{-alg}$$

and a natural morphism of $\mathcal{O}$-algebras $A \longrightarrow P_nA$ such that for every connective and cofibrant $\mathcal{O}$-algebra A the following properties hold:

- *the homotopy groups $\pi_k(P_nA)$ are trivial for $k > n$*
- *for $k \leq n$ the morphism $A \longrightarrow P_nA$ induces an isomorphism on π_k.*

We refer to P_nA as the n-th Postnikov section *of the connective $\mathcal{O}$-algebra A.*

Proof. For part (i) we first produce a morphism $\varphi : A_+ \longrightarrow A$ whose source is a connective $\mathcal{O}$-algebra and such that φ is surjective on π_0. For example, we can choose a family of maps of pointed spaces $S^{n_j} \longrightarrow A_{n_j}$ with $n_j \geq 1$ whose classes generate π_0A as an abelian group and define A_+ as the free $\mathcal{O}$-algebra

$$A_+ \ = \mathcal{O}\left(\bigvee_j F_{n_j}S^{n_j}\right) \ .$$

The morphism $A_+ \longrightarrow A$ is adjoint to wedge of the adjoints $F_{n_j}S^{n_j} \longrightarrow A$ of the maps $S^{n_j} \longrightarrow A_{n_j}$. We can make A_+ depend functorially on A by using *all* maps $S^j \longrightarrow A_j$ for all $j \geq 1$.

Now we apply Proposition 7.7 with $n = 0$ to the morphism $\varphi : A_+ \longrightarrow A$. Since the source is connective and cofibrant we obtain a functorial $\mathcal{O}$-algebra factorization

$$A_+ \ \xrightarrow{g} \ A^{\langle 0\rangle} \ \xrightarrow{h} \ A$$

of φ such that g induces isomorphisms of negative dimensional homotopy groups and h induces a monomorphism on π_0 and an isomorphism of homotopy groups in positive dimensions. Since A_+ is connective, the first statement says that $A^{\langle 0\rangle}$ is also connective. Since the composite

$$\pi_0(A_+) \xrightarrow{\pi_0 g} \pi_0A^{\langle 0\rangle} \xrightarrow{\pi_0 h} \pi_0A$$

coincides with the surjective map $\pi_0\varphi$, the map $\pi_0h : \pi_0A^{\langle 0\rangle} \longrightarrow \pi_0A$ is not only injective, but in fact bijective. So altogether we conclude that the morphism of $\mathcal{O}$-algebras $h : A^{\langle 0\rangle} \longrightarrow A$ serves as a connective cover.

Part (ii) is the special case of Proposition 7.7 for the unique morphism $A \longrightarrow *$ to the terminal $\mathcal{O}$-algebra. In the factorization

$$A \ \xrightarrow{g} \ A^{\langle n+1\rangle} \ \xrightarrow{h} \ *$$

the morphism g induces an isomorphism of homotopy groups below dimension $n+1$. The morphism h is injective on homotopy group of dimension $n+1$ and above. The terminal $\mathcal{O}$-algebra is a trivial symmetric spectrum, so in this case we deduce that $\pi_kA^{\langle n+1\rangle} = 0$ for $k \geq n+1$. So the morphism of $\mathcal{O}$-algebras $g : A \longrightarrow A^{\langle n+1\rangle}$ serves as the n-th Postnikov section. □

Let $\mathcal{O}$ be an operad of symmetric spectra such that $\mathcal{O}(n)$ is connective for all $n \geq 0$. Then the collection of zeroth homotopy groups $\pi_0\mathcal{O}$ is an operad of abelina groups (under tensor product). For every $\pi_0\mathcal{O}$-algebra A the collection of Eilenberg-Mac Lane spectrum HA (see Example I.1.14) is naturally an algebra over the operad $\mathcal{O}$ [...] [We could also deduce this from an operad morphism $\mathcal{O} \longrightarrow H(\pi_0\mathcal{O})$]

Proposition 7.9 (Uniqueness of Eilenberg-Mac Lane algebras)**.** *Let $\mathcal{O}$ be an operad of symmetric spectra such that $\mathcal{O}(n)$ is connective for all $n \geq 0$. If A is $\mathcal{O}$-algebra with true homotopy groups concentrated in dimension 0, then A is stably equivalent to $H(\pi_0 A)$ as an $\mathcal{O}$-algebra.*

Remark 7.10. It seems worth spelling out the results of this section in the case of the initial operad $o\mathbb{S}$ with objects

$$o\mathbb{S}(n) \;=\; \begin{cases} \mathbb{S} & \text{for } n = 1, \\ * & \text{else.} \end{cases}$$

Then $o\mathbb{S}$-algebra 'are' $\mathbb{S}$-modules, which in turn 'are' symmetric spectra. More precisely, the forgetful functor $o\mathbb{S}\text{-alg} \longrightarrow \mathcal{S}p$ is an isomorphism of categories. Every object in the operad $\mathbb{S}$ is connective, so Theorem 7.8 provides a functor $\tau_{\geq 0} : \mathcal{S}p \longrightarrow \mathcal{S}p$ and a natural morphism of symmetric spectra $\tau_{\geq 0}A \longrightarrow A$ such that $\tau_{\geq 0}A$ is connective and the morphism $\tau_{\geq 0}A \longrightarrow A$ induces an isomorphism on π_k for all $k \geq 0$.

Since the homotopy groups of a $\tau_{\geq 0}A$ depend functorially on the homotopy groups of A, the functor $\tau_{\geq 0}$ and the natural transformation descend to the level of homotopy categories. [ref to universal property] So this section gives a somewhat different way to construct connective covers in the stable homotopy category, which we first discussed in Theorem II.8.1.

In much the same way, the uniqueness result for Eilenberg-Mac Lane algebras is Proposition 7.9 specializes, in the case of the initial operad $o\mathbb{S}$ to the uniqueness result for Eilenberg-Mac Lane spectra in the stable homotopy category as stated in Theorem II.5.25.

Exercises

Exercise 4.1 (Strong level model structure)**.** The purpose of this exercise is to construct, under suitable hypothesis on a model category $\mathcal{C}$ a 'strong' level model structure on the category $\mathcal{S}p_{\mathcal{C}}$ of symmetric spectra in $\mathcal{C}$. The strong model structure has more homotopy types than the flat and projective level model structure.

For every group G the strong G-equivalence, strong G-fibration and G-cofibrations in the category $G\mathcal{C}$ of G-objects in $\mathcal{C}$ were introduced in Definition 3.1. We call a morphism $f : X \longrightarrow Y$ of symmetric spectra in $\mathcal{C}$ a *strong level equivalence* respectively a *strong level fibration* if for every $n \geq 0$ the morphism $f_n : X_n \longrightarrow Y_n$ is a strong Σ_n-equivalence respectively strong Σ_n-fibration.

Assume now that for every symmetric group Σ_n the strong Σ_n-equivalences, strong Σ_n-fibrations and Σ_n-cofibrations form a model structure on the category $\Sigma_n\mathcal{C}$. Show that the flat cofibrations, strong level equivalences Σ_n-model structures form a model structure on the category $\mathcal{S}p_{\mathcal{C}}$. We call this model structure on $\mathcal{S}p_{\mathcal{C}}$ the *strong level model structure*. Define and prove a *positive strong level model structure* on $\mathcal{S}p_{\mathcal{C}}$. [Cole mixing of strong level with projective cofibrations?]

Exercise 4.2. Give instructions on how to prove the projective and flat level and stable model structures for orthogonal spectra. Show that the two adjoint functor pairs

$$\begin{tikzcd} \mathcal{S}p_{\mathbf{T}} \arrow[r, bend left, "\mathbb{P}"] \arrow[r, bend right, "?"'] & \mathcal{S}p^O \arrow[l, "U"'] \end{tikzcd}$$

are Quillen equivalences with respect to the projective stable model structure on $\mathcal{S}p^O$; for the pair $(\mathbb{P}, U)$ we use the projective stable model structure, for $(U, ?)$ the flat stable model structure on symmetric spectra. Need that the forgetful functor takes projective cofibrations in $\mathcal{S}p^O$ to flat cofibrations in $\mathcal{S}p$. Key steps:

(1) A projective symmetric spectrum, X orthogonal Ω-spectrum, then in commutative triangle

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\simeq"'] & & UX \\ & U\mathbb{P}A \arrow[ur, "\hat{\varphi}"'] & \end{tikzcd}$$

the adjunction unit is a stable equivalence [ref], so φ is a stable equivalence of symmetric spectra if and only if its adjoint $\hat{\varphi}$ is a stable equivalence of orthogonal spectra.

(2) B projective orthogonal spectrum, Y flat symmetric Ω-spectrum, then in commutative triangle

$$\begin{tikzcd} UB \arrow[rr, "\hat{\psi}"] \arrow[dr, "U(\psi)"'] & & Y \\ & U?Y \arrow[ur, "\simeq"'] & \end{tikzcd}$$

the adjunction counit is a stable equivalence [ref], so ψ is a stable equivalence of orthogonal spectra if and only if its adjoint $\hat{\psi}$ is a stable equivalence of symmetric spectra.

Same for sequential spectra and unitary spectra.

History and credits

The projective and injective level and stable model structures for symmetric spectra are constructed in the original paper [**36**] of Hovey, Shipley and Smith. The flat model structures show up in the literature under the name of *S-model structure*. (the 'S' refers to the sphere spectrum). The cofibrant objects in this model structure (which we call 'flat' and Hovey, Shipley and Smith call 'S-cofibrant') and parts of the model structures show up in [**36**] and in [**69**], but the first verification of the full model axioms appears in Shipley's paper [**77**]. I prefer the term 'flat' model structure because the cofibrant objects are very analogous to flat modules in algebra and because we can then also use the term 'flat model structure' for algebras over an operad and modules over a symmetric ring spectrum.

The stable model structures for algebras over operads were obtained by different people in successively more general situations. The first special case were the stable model structure for modules over a symmetric ring spectrum and for algebra spectra of a commutative symmetric ring spectrum, which are examples of the general theory of Schwede and Shipley [**71**]. The positive stable model structure for *commutative* symmetric ring spectra was first established by Mandell, May, Schwede and Shipley in [**53**] for the projective cofibration/fibration pair and by Shipley [**77**] for the flat cofibration/fibration pair (called the S-model structure there). [known to Smith, and actually a motivitation for why symmetric spectra are 'right']

CHAPTER IV

Module spectra

1. Model structures for modules

With the symmetric monoidal smash product and a compatible model structure in place, we are ready to explore ring and module spectra. In this section we construct model structures on the category of modules over a symmetric ring spectrum. We restrict our attention to stable model structures and show that the forgetful functor to symmetric spectra 'creates' various such model structure. The forgetful functor also creates various level model structures, but we have no use for that and so will not discuss level model structures for R-modules.

The various stable model structures are also 'stable' in the technical sense that the suspension functor on the homotopy category is an equivalence of categories. As consequence of this is that stable homotopy category of modules over a ring spectrum is a triangulated category. The free module of rank one is a small generator.

We originally defined a symmetric ring spectrum in Definition I.1.3 in the 'explicit' form, i.e., as a family $\{R_n\}_{n\geq 0}$ of pointed simplicial sets with a pointed Σ_n-action on R_n and $\Sigma_p \times \Sigma_q$-equivariant multiplication maps $\mu_{p,q} : R_p \wedge R_q \longrightarrow R_{p+q}$ and two unit maps subject to an associativity, unit and centrality condition. Using the internal smash product of symmetric spectra we saw in Theorem I.5.25 that a symmetric ring spectrum can equivalently be defined as a symmetric spectrum R together with morphisms $\mu : R \wedge R \longrightarrow R$ and $\iota : \mathbb{S} \longrightarrow R$, called the multiplication and unit map, which satisfy certain associativity and unit conditions. In this 'implicit' picture a morphism of symmetric ring spectra is a morphism $f : R \longrightarrow S$ of symmetric spectra commuting with the multiplication and unit maps, i.e., such that $f \circ \mu = \mu \circ (f \wedge f)$ and $f \circ \iota = \iota$.

A *right R-module* was originally defined explicitly, but it can also be given in an implicit form as a symmetric spectrum M together with an action map $M \wedge R \longrightarrow M$ satisfying associativity and unit conditions. A morphism of right R-modules is a morphism of symmetric spectra commuting with the action of R. We denote the category of right R-modules by mod-R.

The unit $\mathbb{S}$ of the smash product is a ring spectrum in a unique way, and $\mathbb{S}$-modules are the same as symmetric spectra. The smash product of two ring spectra is naturally a ring spectrum. For a ring spectrum R the opposite ring spectrum R^{op} is defined by composing the multiplication with the twist map $R \wedge R \longrightarrow R \wedge R$ (so in terms of the bilinear maps $\mu_{p,q} : R_p \wedge R_q \longrightarrow R_{p+q}$, a block permutation appears). The definitions of left modules and bimodules is hopefully clear; left R-modules and R-T-bimodule can also be defined as right modules over the opposite ring spectrum R^{op}, respectively right modules over the ring spectrum $R^{\mathrm{op}} \wedge T$.

A formal consequence of having a closed symmetric monoidal smash product of symmetric spectra is that the category of R-modules inherits a smash product and function objects. The smash product $M \wedge_R N$ of a right R-module M and a left R-module N can be defined as the coequalizer, in the category of symmetric spectra, of the two maps

$$M \wedge R \wedge N \rightrightarrows M \wedge N$$

given by the action of R on M and N respectively. Alternatively, one can characterize $M \wedge_R N$ as the universal example of a symmetric spectrum which receives a bilinear map from M and N which is R-*balanced*, i.e., all the diagrams

$$\begin{tikzcd} M_p \wedge R_q \wedge N_r \arrow[r, "\mathrm{Id} \wedge \alpha_{q,r}"] \arrow[d, "\alpha_{p,q} \wedge \mathrm{Id}"'] & M_p \wedge N_{q+r} \arrow[d, "\iota_{p,q+r}"] \\ M_{p+q} \wedge N_r \arrow[r, "\iota_{p+q,r}"'] & (M \wedge N)_{p+q+r} \end{tikzcd} \tag{1.1}$$

commute. If M happens to be a T-R-bimodule and N an R-S-bimodule, then $M \wedge_R N$ is naturally a T-S-bimodule. If R is a commutative ring spectrum, the notions of left and right module coincide and agree with the notion of a symmetric bimodule. In this case $\wedge_R$ is an internal symmetric monoidal smash product for R-modules. There are also symmetric function spectra $\mathrm{Hom}_R(M, N)$ defined as the equalizer of two maps

$$\mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(R \wedge M, N) .$$

The first map is induced by the action of R on M, the second map is the composition of $R \wedge - : \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(R \wedge M, R \wedge N)$ followed by the map induced by the action of R on N. The internal function spectra and function modules enjoy the 'usual' adjointness properties with respect to the various smash products. [spell out]

Proposition 1.2. *Given a morphism $f : R \longrightarrow S$ of symmetric ring spectra, the functor $f^* : \text{mod-}S \longrightarrow \text{mod-}R$ of restriction of scalar has a left and a right adjoint, and hence commutes with limits and colimits.*

In particular, for every symmetric ring spectrum R the forgetful functor to symmetric spectra has a left and a right adjoint, and hence commutes with limits and colimits.

Theorem 1.3. *Let R be a symmetric ring spectrum of topological spaces or simplicial sets. The category of right R-modules admits the following four* stable model structures *in which the weak equivalences are those morphisms of R-modules which are stable equivalences on underlying symmetric spectra.*

(i) *In the* absolute projective stable model structure *the fibrations are those morphisms of R-modules which are absolute projective stable fibrations on underlying symmetric spectra.*
(ii) *In the* positive projective stable model structure *the fibrations are those morphisms of R-modules which are positive projective stable fibrations on underlying symmetric spectra.*
(iii) *In the* absolute flat stable model structure *the fibrations are those morphisms of R-modules which are absolute flat stable fibrations on underlying symmetric spectra.*
(iv) *In the* positive flat stable model structure *the fibrations are those morphisms of R-modules which are positive flat stable fibrations on underlying symmetric spectra.*

Moreover we have:

- *All four stable model structures are proper, simplicial and cofibrantly generated.*
- *If R is commutative then all four stable model structures are monoidal with respect to the smash product over R.*

If underlying symmetric spectrum of R is flat, then the category of right R-modules admits the following two injective stable model structures *in which the weak equivalences are those morphisms of R-modules which are stable equivalences on underlying symmetric spectra.*

(i) *In the* absolute injective stable model structure *the fibrations are those morphisms of R-modules which are absolute injective stable fibrations on underlying symmetric spectra.*
(ii) *In the* positive injective stable model structure *the fibrations are those morphisms of R-modules which are positive injective stable fibrations on underlying symmetric spectra.*

Moreover, both injective stable model structures are proper, simplicial and cofibrantly generated.

In all six model structures, a cofibration of R-modules is a monomorphism of underlying symmetric spectra.

PROOF. In the language of Definition 1.4 of Appendix A we claim that in all of the six cases the forgetful functor from R-modules to symmetric spectra creates a model structure on R-modules. In Theorem A.1.5 we can find sufficient conditions for this, which we will now verify.

The category of R-modules is complete, cocomplete and simplicial; in fact all limits, colimits, tensors and cotensors with simplicial sets are created on underlying symmetric spectra. In particular the forgetful functor preserves filtered colimits. The forgetful functor has a left adjoint free functor, given by smashing with R. [Smallness]

It remains to check the condition which in practice is often the most difficult one, namely that every $(J \wedge R)$-cell complex is a weak equivalence. We claim that in all six cases the free functor $X \mapsto X \wedge R$ takes stable acyclic cofibrations of symmetric spectra of the respective kind to stable equivalences of R-modules which are monomorphisms. In the first four cases (where we have no assumption on R) this uses that every generating acyclic cofibration $i : A \longrightarrow B$ is in particular a flat cofibration, so $i \wedge \mathrm{Id} : A \wedge R \longrightarrow B \wedge R$ is injective and a stable equivalence by parts (i) and (iv) of Proposition 4.15. In the 'injective' cases (v) and (vi) the argument is slightly different; then the assumption that R is flat assures that for every injective stable equivalence $i : A \longrightarrow B$ the morphism $i \wedge \mathrm{Id} : A \wedge R \longrightarrow B \wedge R$ is again injective (by the definition of flatness) and a stable equivalence (by Proposition II.5.50).

So in all the six cases, the free functor $- \wedge R$ takes the generating stable acyclic cofibrations to injective stable equivalences of R-modules. Since colimits of R-modules are created on underlying symmetric spectra, the class of injective stable equivalences is closed under wedges, cobase change and transfinite composition. So every $(J \wedge R)$-cell complex is a stable equivalence. So we have verified the hypothesis of Theorem 1.5, which thus shows that the forgetful functor creates the six model structure. It also shows that the model structures are simplicial and right proper.

[left proper] [monoidal if R commutative] [preservation of cofibrations] □

[Is there an 'strongly injective' stable model structure in which cofibrations are the monomorphisms of R-modules ? make exercise?]

Proposition 1.4. *A morphism $f : M \longrightarrow N$ of right R-modules is a flat cofibration if and only if for every morphism $g : V \longrightarrow W$ of left R-modules whose underlying morphism of symmetric spectra is a level cofibration the pushout product map*

$$f \wedge_R g \ : \ M \wedge_R W \cup_{M \wedge_R V} N \wedge_R W \ \longrightarrow \ N \wedge_R W$$

is an level cofibration of symmetric spectra.

There are also characterizations of flat and projective cofibrations in terms of 'R-module latching objects', see Exercise E.IV.2.

As we just proved, cofibrations of R-modules are always monomorphisms of underlying symmetric spectra, but sometimes more is true. As the special case $S = \mathbb{S}$ of Theorem 1.5 (iii) below we will see that if R is flat as a symmetric spectrum, then every flat cofibration of R-modules is also a flat cofibration on underlying symmetric spectra. Similarly, if R is projective as a symmetric spectrum, then every projective cofibration of R-modules is also a projective cofibration on underlying symmetric spectra.

For a morphism $f : S \longrightarrow R$ of symmetric ring spectra, there is are two adjoint functor pairs relating the modules over S and R. The functors are analogous to restriction and extension respectively coextension of scalars. Every R-module becomes an S-module if we let S act through the homomorphism f; more precisely, given an R-module M we define an S-module f^*M as the same underlying symmetric spectrum as M and with S-action given by the composite

$$(f^*M) \wedge S \ = \ M \wedge S \xrightarrow{\mathrm{Id} \wedge f} M \wedge R \xrightarrow{\alpha} M \ .$$

We call the resulting functor $f^* : \text{mod-}R \longrightarrow \text{mod-}S$ *restriction of scalars* along f and note that it has both a left and right adjoint. We call the left adjoint *extension of scalars* and denote it by $f_* : \text{mod-}S \longrightarrow \text{mod-}R$. The left adjoint takes an S-module N to the R-module $f_*N = N \wedge_S R$, where S is a left R-module via f, and with right R-action through the right multiplication action of R on itself. We call the right adjoint of f^* the *coextension of scalars* and denote it by $f_! : \text{mod-}S \longrightarrow \text{mod-}R$. The right adjoint takes an S-module

N to the R-module $f_! N = \mathrm{Hom}_{\text{mod-}S}(R, N)$, where S is a right R-module via f, and with right R-action through the left multiplication action of R on itself.

Theorem 1.5. *Let $f : S \longrightarrow R$ be a homomorphism of symmetric ring spectra.*

(i) *The functor pair*

$$\text{mod-}S \;\; \underset{f^*}{\overset{f_*}{\rightleftarrows}} \;\; \text{mod-}R$$

is a Quillen adjoint functor pair with respect to the absolute projective, the positive projective, the absolute flat and the positive flat stable model structures on both sides.

(ii) *If S and R are flat as symmetric spectra then (f_*, f^*) is a Quillen adjoint functor pair with respect to the absolute injective and the positive injective stable model structures on both sides.*

(iii) *Suppose that the morphism $f : S \longrightarrow R$ makes R into a flat (respectively projective) right S-module. Then the functor pair*

$$\text{mod-}R \;\; \underset{f_!}{\overset{f^*}{\rightleftarrows}} \;\; \text{mod-}S$$

is a Quillen adjoint functor pair with respect to the absolute and positive flat stable (respectively absolute and positive projective stable) model structures on both sides. In particular, the restriction of scalars f^* then takes flat (respectively projective) cofibrations of R-modules to flat (respectively projective) cofibrations of S-modules.

(iv) *If the homomorphism $f : S \longrightarrow R$ is a stable equivalence, then the adjoint functor pairs (f_*, f^*) and $(f^*, f_!)$ are a Quillen equivalences in all the cases when they are Quillen adjoint functors.*

Proof. (i) In each case, the weak (i.e., stable) equivalences and the various kinds of fibrations are defined on underlying symmetric spectra, hence the restriction functor preserves fibrations and acyclic fibrations. By adjointness, the extension functor preserves cofibrations and trivial cofibrations.

(iv) If $f : S \longrightarrow R$ is a stable equivalence, then for every flat right S-module N the morphism

$$N \;\cong\; N \wedge_S S \;\longrightarrow\; N \wedge_S R \;=\; f_* N$$

is a stable equivalence. Thus if Y is a fibrant left R-module, an S-module map $N \longrightarrow Y$ is a weak equivalence if and only if the adjoint R-module map $f_* N \longrightarrow Y$ is a weak equivalence. □

1.1. The derived category of a ring spectrum. For every model category $\mathcal{C}$ the full subcategory of cofibrant objects form a cofibration category. Hence Theorem 1.3 immediately implies [except for 'stable']

The arguments involved in the verification of the axioms (T1)–(T4) are quite general and we find it convenient to produce triangulated categories more generally. We do this in the axiomatic framework of *cofibration categories* and show that the homotopy category of any stable cofibration category is triangulated in a natural way. Besides symmetric spectra, we wil later also apply this to modules spectra over a symmetric ring spectrum. We also give various exercises that show how more triangulations can be constructed using the cofibration category framework.

Definition 1.6. A *cofibration category* is a category $\mathcal{C}$ equipped with two classes of morphisms, called *cofibrations* respectively *weak equivalences*, that satisfy the following axioms (C1)–(C5).

In the statements, an *acyclic cofibration* is a morphism that is both a cofibration and a weak equivalence. An object is *fibrant* if it has the extension property with respect to all acyclic cofibrations.

(C1) All isomorphisms are cofibrations and weak equivalences. Cofibrations are stable under composition. $\mathcal{C}$ has an initial object $\emptyset$ and for every object A the unique morphism $\emptyset \longrightarrow A$ is a cofibration.

(C2) Given two composable morphisms f, g, in $\mathcal{C}$, then if two of the three morphisms f, g and gf are weak equivalences, so is the third.

(C3) Given a cofibration $i : A \longrightarrow B$ and any morphism $f : A \longrightarrow C$, there exists a pushout square

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \\
B \arrow[r] & P
\end{tikzcd}
\tag{1.7}
$$

in $\mathcal{C}$ and the morphism j is a cofibration. If additionally i is a weak equivalence, then so is j.

(C4) Every morphism in $\mathcal{C}$ can be factored as the composite of a cofibration followed by a weak equivalence.

(C5) For every object A there exists an acyclic cofibration $p : A \longrightarrow \omega A$ such that ωA is fibrant.

We note that in a cofibration category the coproduct $B \amalg C$ of any two objects in $\mathcal{C}$ exists by (C3) with $A = \emptyset$ an initial object, and the canonical morphisms from B and C to $B \amalg C$ are cofibrations.

Example 1.8. The category of symmetric spectra of simplicial sets has the structure of a cofibration category with respect to the stable equivalences as weak equivalences and the monomorphisms as cofibrations [justify axioms]

Moreover, the 'concrete' homotopy relation using homotopies defined on $\Delta[1]^+ \wedge A$ coincides with the model category theoretic homotopy relation using abstract cylinder objects. Thus the stable homotopy category as introduced above turns out to be the homotopy category, in the sense of model category theory, with respect to the injective stable model structure.

Remark 1.9. The axioms of a cofibration category are strictly weaker than those of a model category. In the case of symmetric spectra, the cofibration structure is underlying a model structure, the *injective stable model structure*; every symmetric spectrum is cofibrant and the fibrant objects are precisely the injective Ω-spectra.

A *cylinder object* for an object A in a cofibration category is a quadrupel (I, i_0, i_1, p) consisting of an object I and morphisms $i_0, i_1 : A \longrightarrow I$ and a weak equivalence $p : I \longrightarrow A$ satisfying $pi_0 = pi_1 = \mathrm{Id}_A$ and such that $i_0 + i_1 : A \amalg A \longrightarrow I$ is a cofibration. Every object has a cylinder object by axiom (C4). A *left homotopy* between two morphisms $f, g : A \longrightarrow B$ is a choice of cylinder object for A and a morphism $H : I \longrightarrow B$ such that $f = Hi_0$ and $g = Hi_1$. We say that f is left homotopic to g if such a left homotopy exists.

Lemma 1.10. *Let $\mathcal{C}$ be a cofibration category.*

(1) Every morphism $f : A \longrightarrow B$ can be factored as $f = qj$ where $j : A \longrightarrow B'$ is a cofibration and $r : B' \longrightarrow B$ is a weak equivalence that is left inverse to an acyclic cofibration.

(2) Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor that sends acyclic cofibrations to isomorphisms. Then F sends all weak equivalences to isomorphisms.

Proof. (i) We choose a cylinder object (I, i_0, i_1, p) for A as in (C4). We define B' as the pushout:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_1"'] & B \arrow[d, "s"] \arrow[dr, bend left, "\mathrm{Id}"] & \\
I \arrow[r, "F"] \arrow[rr, bend right, "fp"'] & B' \arrow[r, dashed, "r"] & B
\end{tikzcd}
$$

Then the morphism s is an acyclic cofibration since i_1 is. The morphism $fp : I \longrightarrow B$ and the identity of B glue to a morphism $r : B' \longrightarrow B$ with $rs = \mathrm{Id}_B$, and r is a weak equivalence since s is. We define $j = Fi_0$, and then have $rj = rFi_0 = fpi_0 = f$. It remains to show that j is a cofibration. For this we observe that the square

$$\begin{tikzcd} A \amalg A \arrow[r, "\mathrm{Id} \amalg f"] \arrow[d, "i_0+i_1"'] & A \amalg B \arrow[d, "j+s"] \\ I \arrow[r, "F"'] & B' \end{tikzcd}$$

is also a pushout. Since $i_0 + i_1$ is a cofibration, so is $j + s$, and hence also j since the canonical map $A \longrightarrow A \amalg B$ is a cofibration.

(ii) Let $\varphi : A \longrightarrow B$ be any weak equivalence. Part (i) lets us choose a factorization $\varphi = rj$ where $j : A \longrightarrow \bar{A}$ is a cofibration and $r : \bar{A} \longrightarrow A'$ is a weak equivalence that is left inverse to an acyclic cofibration $s : A' \longrightarrow \bar{A}$. The morphisms Fs is an isomorphism in $\mathcal{D}$ by assumption and the relation $(Fr)(Fs) = F(rs) = \mathrm{Id}$ implies that Fr is the inverse of Fs, and hence itself an isomorphism. Since φ and r are weak equivalences, so is j. Since j is an acyclic cofibration, Fj is an isomorphisms in $\mathcal{D}$ by assumption. Hence $Ff = (Fr)(Fj)$ is an isomorphism. □

As we shall see in the next proposition, 'left homotopy' is an equivalence relation on the set of morphisms between any pair of objects. We denote by $[A, B]$ the set of left homotopy classes of morphisms from A to B.

Proposition 1.11. *Let A and B be objects of a cofibration category.*

(i) *'Left homotopy' is an equivalence relation on the set of morphisms from A to B.*
(ii) *If $f, g : A \longrightarrow B$ are left homotopic morphism and $\varphi : A' \longrightarrow A$ and $\psi : B \longrightarrow \bar{B}$ any morphisms, then $f\varphi$ is left homotopic to $g\varphi$ and ψf is left homotopic to ψg.*
(iii) *If Z is fibrant and IA is a cylinder object for an object A and $f, g : A \longrightarrow Z$ are left homotopic morphisms, then there is a homotopy from f to g defined on IA.*
(iv) *Let $i : A \longrightarrow B$ be a cofibration and Z a fibrant object. Let $f : A \longrightarrow Z$ and $g : B \longrightarrow Z$ be morphisms such that f is left homotopic to gi. Then there is a morphism $g' : B \longrightarrow Z$ which is left homotopic to g and such that $g'i = f$.*
(v) *If Z is fibrant and $\varphi : A \longrightarrow B$ is a weak equivalence, then the induced map $[\varphi, Z] : [A', Z] \longrightarrow [A, Z]$ on homotopy classes of morphisms into Z is bijective.*
(vi) *Every weak equivalence between fibrant objects is a homotopy equivalence.*

Proof. (i) For every morphism $f : A \longrightarrow B$ and every cylinder object (IA, i_0, i_1, p) for A the morphism $fp : IA \longrightarrow B$ is a homotopy from f to f, so 'left homotopy' is reflexive. If f is left homotopic to $g : A \longrightarrow B$ via a homotopy $H : IA \longrightarrow B$ with respect to a cylinder object (IA, i_0, i_1, p), then the same morphisms $H : IA \longrightarrow B$ is a homotopy from g to f with respect to a different cylinder object, namely (IA, i_1, i_0, p) (i.e., the two 'end inclusions' $i_0, i_1 : A \longrightarrow IA$ have changed their roles). So the relation 'left homotopy' is symmetric. Given three morphisms $f, g, h : A \longrightarrow B$, a homotopy $H : IA \longrightarrow B$ from f to g with respect to a cylinder object (IA, i_0, i_1, p) and a homotopy $H' : I'A \longrightarrow B$ from g to h with respect to a cylinder object $(I'A, i'_0, i'_1, p')$, we construct a homotopy from f to h as follows. We define $I''A$ as the pushout:

$$\begin{tikzcd} A \arrow[r, "i'_0"] \arrow[d, "i_1"'] & I'A \arrow[d, "j_1"] \\ IA \arrow[r, "j_0"'] & I''A \end{tikzcd}$$

Since the morphisms $i'_0 : A \longrightarrow I'A$ and $i_1 : A \longrightarrow IA$ are acyclic cofibrations, so are the morphisms $j_0 : IA \longrightarrow I''A$ and $j_1 : I'A \longrightarrow I''A$. The morphisms $p : IA \longrightarrow A$ and $p' : I'A \longrightarrow A$ glue to a morphism $q : I''A \longrightarrow A$ which is a weak equivalence since $qj_0 = p$ and j_0 and p are weak equivalences. The two morphisms $j_0 i_0, j_1 i'_1 : A \longrightarrow I''A$ are composites of cofibrations, hence cofibrations. [need that $j_0 i_0 + j_1 i'_1 : A \amalg A \longrightarrow I''A$ is a cofibration]

(ii) If $H : IA \longrightarrow B$ is a homotopy from f to g, then $\psi H : IA \longrightarrow \bar{B}$ is a homotopy from ψf to ψg. [...]

(iii) Let $H : I'A \longrightarrow Z$ be a homotopy from f to g with respect to a cylinder object $(I'A, i'_0, i'_1, p')$. We define SA as the pushout:

$$\begin{tikzcd} A \amalg A \arrow[r, "i'_0+i'_1"] \arrow[d, "i_0+i_1"'] & I'A \arrow[d, "j_1"] \\ IA \arrow[r, "j_0"'] & SA \end{tikzcd}$$

The morphisms $p : IA \longrightarrow A$ and $p' : I'A \longrightarrow A$ glue to a morphism $q : SA \longrightarrow A$ which we can factor as $q = sk$ where $k : SA \longrightarrow DA$ is a cofibration and $s : DA \longrightarrow A$ is a weak equivalence. The composite $kj_1 : I'A \longrightarrow DA$ is a cofibration, and a weak equivalence since $s(kj_1) = qj_1 = p'$ and s and p' are weak equivalences. Since Z is fibrant the homotopy $H : I'A \longrightarrow Z$ has an extension $K : DA \longrightarrow Z$ such that $Kkj_1 = H$. The restriction $Kkj_0 : IA \longrightarrow Z$ is then the required homotopy from f to g through the cylinder object IA.

(iv) Let (IA, i_0, i_1, p) be a cylinder object for A, $\bar{f} : B \longrightarrow Z$ an extension-up-to-homotopy and $H : IA \longrightarrow Z$ a left homotopy from f to gi. We choose a pushout

$$\begin{tikzcd} A \amalg A \arrow[r, "i \amalg i"] \arrow[d, "i_0+i_1"'] & B \amalg B \arrow[d] \\ IA \arrow[r] & B \cup_A IA \cup_A B \end{tikzcd}$$

and a factorization $\mathrm{Id} \cup ip \cup \mathrm{Id} = qj$ as a cofibration $j : B \cup_A IA \cup_A B \longrightarrow IB$ followed by a weak equivalence $q : IB \longrightarrow B$. The quadrupel (IB, i'_0, i'_1, q) is then a cylinder object for B.

Moreover [...] the map $IA \cup_A B \longrightarrow B \cup_A IA \cup_A B$ is a cofibration, hence so is the composite $j(-) : IA \cup_A B \longrightarrow IB$. This map is also a weak equivalence since the composite with the weak equivalence $i_1 : B \longrightarrow IA \cup_A B$ is right inverse to the weak equivalence q, and hence a weak equivalence. So $j(-) : IA \cup_A B \longrightarrow IB$ is an acyclic cofibration, and thus the morphism $H \cup g : IA \cup_A B \longrightarrow Z$ admits an extension $\bar{H} : IB \longrightarrow Z$. The morphism $\bar{g} = \bar{H} i_0 : B \longrightarrow Z$ is then homotopic to g and an extension of f.

(v) We show that for every acyclic cofibration $j : A \longrightarrow B$ the induced map $[j, Z] : [B, Z] \longrightarrow [A, Z]$ is bijective. Lemma 1.10 (ii), applied to the set valued functor $[-, Z]$ then shows the claim. Since Z is fibrant, every morphism $f : A \longrightarrow Z$ has an extension $\bar{f} : B \longrightarrow Z$ with $\bar{f} j = f$; so $[j, Z]$ is surjective. Now suppose that $f, g : B \longrightarrow Z$ are two morphisms such that $fj, gj : A \longrightarrow Z$ are left homotopic via a homotopy $H : IA \longrightarrow Z$. We consider the pushout:

$$\begin{tikzcd} A \amalg A \arrow[r, "i_0+i_1"] \arrow[d, "j \amalg j"'] & IA \arrow[d, "J"] \arrow[ddr, bend left, "jp"] & \\ A' \amalg A' \arrow[r, "i'_0+i'_1"'] \arrow[drr, bend right, "\mathrm{Id}+\mathrm{Id}"'] & IA' \arrow[dr, dotted, "p'"] & \\ & & A' \end{tikzcd}$$

Since $i_0 + i_1$ is a cofibration, so is $i'_0 + i'_1$. The morphism $jp : IA \longrightarrow B$ and the fold map glue to a morphism $p' : IB \longrightarrow B$. Since $j \amalg j$ is an acyclic cofibration, so is the morphism $J : IA \longrightarrow IB$. Since $p'J = jp$ and J, j and p are weak equivalence, so is p'. Hence (IB, i'_0, i'_1, p') is a cylinder object for B. The homotopy $H : IA \longrightarrow Z$ and the morphism $f + g : B \amalg B \longrightarrow Z$ glue to a homotopy $H' : IB \longrightarrow Z$ from f to g. Altogether this shows that the the map $[j, Z]$ is injective.

(vi) This part is a formal consequence of (v). □

Now we introduce the homotopy category of a cofibration category. This construction is a direct generalization of the construction of the stable homotopy category $\mathcal{SHC}$ in Section 1. For each object Y of the cofibration category $\mathcal{C}$ we choose a 'fibrant replacement', i.e., an acyclic cofibration $p_Y : Y \longrightarrow \omega Y$ to a fibrant object. We insist that if Y is already fibrant, then $\omega Y = Y$ and p_Y is the identity.

Definition 1.12. The *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of a cofibration category $\mathcal{C}$ has the same objects as $\mathcal{C}$. For two objects, the morphisms from X to Y in $\mathrm{Ho}(\mathcal{C})$ are given by $[X, \omega Y]$, the set of left homotopy classes of morphisms from X to the chosen fibrant replacement ωY. If $f : X \longrightarrow \omega Y$ is a morphism in $\mathcal{C}$ we denote by $[f] : X \longrightarrow Y$ its homotopy class, considered as a morphism in $\mathrm{Ho}(\mathcal{C})$.

Composition in the homotopy category is defined as follows. Let $f : X \longrightarrow \omega Y$ and $g : Y \longrightarrow \omega Z$ be $\mathcal{C}$-morphism which represent morphism from X to Y respectively from Y to Z in $\mathrm{Ho}(\mathcal{C})$. Since $p_Y : Y \longrightarrow \omega Y$ is an acyclic cofibration and ωZ is fibrant Proposition 1.26 (v). provides a morphism $\bar{g} : \omega Y \longrightarrow \omega Z$, unique up to left homotopy, such that $\bar{g} \circ p_Y = g$. Moreover, the homotopy class of the extension $\bar{g}$ depends only on the homotopy class of g. So we get a well-defined composite of $[f] \in \mathrm{Ho}(\mathcal{C})(X,Y)$ and $[g] \in \mathrm{Ho}(\mathcal{C})(Y,Z)$ by

$$[g] \circ [f] \;=\; [\bar{g} \circ f] \;\in\; \mathrm{Ho}(\mathcal{C})(X,Z) \;.$$

To see that composition in $\mathrm{Ho}(\mathcal{C})$ is associative we consider four objects X, Y, Z and W and three morphisms $f : X \longrightarrow \omega Y$, $g : Y \longrightarrow \omega Z$ and $h : Z \longrightarrow \omega W$ in $\mathcal{C}$. We choose extensions $\bar{g} : \omega Y \longrightarrow \omega Z$ and $\bar{h} : \omega Z \longrightarrow \omega W$ such that $\bar{g} \circ p_Y = g$ and $\bar{h} \circ p_Z = h$. Then $\bar{h}\bar{g} : \omega Y \longrightarrow \omega W$ is an extension of $\bar{h}g$, so we have

$$([h] \circ [g]) \circ [f] \;=\; [\bar{h}g] \circ [f] \;=\; [(\bar{h}\bar{g})f] \;=\; [\bar{h}(\bar{g}f)] \;=\; [h] \circ [\bar{g} \circ f] \;=\; [h] \circ ([g] \circ [f]) \;.$$

It is straightforward to check that $[p_X]$, the homotopy class of the chosen fibrant replacement $p_X : X \longrightarrow \omega X$, is a two-sided unit for composition, so p_X represents the identity of X in $\mathrm{Ho}(\mathcal{C})$.

The construction of the homotopy category comes with a functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ from the cofibration category which is the identity on objects. For a morphism $\varphi : X \longrightarrow Y$ in $\mathcal{C}$ we set

$$\gamma(\varphi) \;=\; [p_Y \circ \varphi] \quad \text{in} \quad \mathrm{Ho}(\mathcal{C})(X,Y) \;,$$

where $p_Y : Y \longrightarrow \omega Y$ is the fibrant replacement. Note that we have $\gamma(p_Y) = [p_Y]$ since $p_{\omega Y} = \mathrm{Id}$ by convention. Thus for every $\mathcal{C}$-morphism $f : X \longrightarrow \omega Y$ we have the relation $\gamma(p_Y) \circ [f] = \gamma(f)$ as morphisms from X to ωY in the $\mathrm{Ho}(\mathcal{C})$. Since $\gamma(p_Y) = [p_Y]$ is an isomorphism with inverse $[\mathrm{Id}_{\omega Y}]$, this can also be rewritten as

$$[f] \;=\; \gamma(p_Y)^{-1} \circ \gamma(f) \;\in \mathrm{Ho}(\mathcal{C})(X,Y) \;. \tag{1.13}$$

In other words, every morphism in the homotopy category can be written as a 'fraction', i.e., the composite of a $\mathcal{C}$-morphism with the inverse of a weak equivalence.

We also note that for morphisms $\alpha : W \longrightarrow X$ and $f : X \longrightarrow \omega Y$ we have the relation

$$[f] \circ \gamma(\alpha) \;=\; [f\alpha] \;\in \mathrm{Ho}(\mathcal{C})(W, \omega Y) \;. \tag{1.14}$$

Indeed, if $\bar{f} : \omega X \longrightarrow \omega Y$ is such that $\bar{f} p_X$ is left homotopic to f, then $\bar{f} p_X \alpha$ is left homotopic to $f\alpha$ and so

$$[f] \circ \gamma(\alpha) \;=\; [f] \circ [p_X \circ \alpha] \;=\; [\bar{f} \circ p_X \circ \alpha] \;=\; [f\alpha] \;.$$

Since the homotopy class of p_Y is the identity of Y in $\mathrm{Ho}(\mathcal{C})$, γ preserves identities. For composable $\mathcal{C}$-morphism $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ we have

$$\gamma(\psi)\gamma(\varphi) \;=\; [p_Z \circ \psi]\gamma(\varphi) \;=\; [p_Z \circ (\psi\varphi)] \;=\; \gamma(\psi\varphi)$$

by (1.14). So γ is indeed a functor.

We now show that the functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ is a localization of the cofibration category at the class of weak equivalences, i.e., a universal example of a functor which takes weak equivalences to isomorphisms.

Theorem 1.15. *Let $\mathcal{C}$ be a cofibration category.*

(i) *The functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ takes weak equivalences to isomorphisms.*
(ii) *For every functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ which takes weak equivalences to isomorphisms, there exists a unique functor $\bar{F} : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathcal{D}$ such that $\bar{F}\gamma = \varepsilon$.*
(iii) *The homotopy category $\mathrm{Ho}(\mathcal{C})$ has coproducts and the functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ preserves coproducts.*

(iv) *Every morphism* $\alpha : X \longrightarrow Y$ *in* $\mathrm{Ho}(\mathcal{C})$ *can be written as* $\alpha = \gamma(s)^{-1}\gamma(i)$ *with* $i : X \longrightarrow Z$ *a cofibration and* $s : Y \longrightarrow Z$ *an acyclic cofibration.*

Proof. (i) By Lemma 1.10 (ii) it suffices to show that γ takes every acyclic cofibration $i : X \longrightarrow Y$ to an isomorphism. Since ωX is fibrant we can choose an extension $r : Y \longrightarrow \omega X$ satisfying $ri = p_X$. We claim that the class of r in $[Y, \omega X] = \mathrm{Ho}(\mathcal{C})(Y, X)$ is inverse to $\gamma(i)$. Indeed, we have

$$[r] \circ \gamma(i) \;=\; [r] \circ [p_Y i] \;=\; [ri] \;=\; [p_X] \;=\; \mathrm{Id}_X$$

in $\mathrm{Ho}(\mathcal{C})$. To evaluate the other composite we choose a morphism $I : \omega X \longrightarrow \omega Y$ such that $p_Y i$ is left homotopic to $Ip_X = Iri$. Since i is a weak equivalence, the morphisms p_Y and $Ir : Y \longrightarrow \omega X$ are left homotopic by Proposition 1.26 (v). So we have

$$\gamma(i) \circ [r] \;=\; [p_Y i] \circ [r] \;=\; [Ir] \;=\; [p_Y] \;=\; \mathrm{Id}_Y$$

in $\mathrm{Ho}(\mathcal{C})$.

(ii) We consider a functor $G : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathcal{D}$ and prove the uniqueness property by showing that G is completely determined by the composite functor $G \circ \gamma : \mathcal{C} \longrightarrow \mathcal{D}$. This is clear on objects since γ is the identity on objects. If $f : X \longrightarrow \omega Y$ is a $\mathcal{C}$-morphism which represents a morphism $[f] : X \longrightarrow Y$ in $\mathrm{Ho}(\mathcal{C})$, then we can apply G to the equation (1.13) and obtain

$$G([f]) \;=\; (G \circ \gamma)(p_Y)^{-1} \circ (G \circ \gamma)(f) \;.$$

Thus also the behavior of G on morphisms is determined by the composite $G \circ \gamma$.

Now we tackle the existence property. Given a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ which takes weak equivalences to isomorphisms we set $\bar{F}(X) = F(X)$ on objects. Given a morphism $f : X \longrightarrow \omega Y$, the uniqueness argument tells us that we have to define the value of $\bar{F}$ on $[f]$ by

$$\bar{F}([f]) \;=\; F(p_Y)^{-1} \circ F(f) \;.$$

We have to check that this is well-defined and functorial.

To see that the assignment is well-defined we have to show that the $\mathcal{D}$-morphism $F(f)$ only depends on the left homotopy class of $f : X \longrightarrow \omega Y$. Indeed, if (IX, i_0, i_1, p) is a cylinder object for X, then p is a weak equivalence and so $F(p)$ is an isomorphism in $\mathcal{D}$. The two morphisms $i_0, i_1 : X \longrightarrow IX$ satisfy $p \circ i_0 = \mathrm{Id}_X = p \circ i_1$, so we have

$$F(p) \circ F(i_0) \;=\; \mathrm{Id}_{FX} \;=\; F(p) \circ F(i_1) \;.$$

Since $F(p)$ is an isomorphism, we deduce $F(i_0) = F(i_1)$. Now suppose that $f, g : X \longrightarrow \omega Y$ are left homotopic morphisms via some homotopy $H : IX \longrightarrow \omega Y$. Then we have

$$F(f) \;=\; F(H) \circ F(i_0) \;=\; F(H) \circ F(i_1) \;=\; F(g) \;,$$

which proves that $\bar{F}([f])$ is well-defined.

By the various definitions we have

$$\bar{F}(\mathrm{Id}_X) \;=\; \bar{F}([p_X]) \;=\; F(p_X)^{-1} \circ F(p_X) \;=\; \mathrm{Id}_{F(X)}$$

so $\bar{F}$ is unital. For associativity we consider two morphisms $f : X \longrightarrow \omega Y$ and $g : Y \longrightarrow \omega Z$ as well as a morphism $\bar{g} : \omega Y \longrightarrow \omega Z$ such that $\bar{g} \circ p_Y$ is left homotopic to g. Then we have

$$\begin{aligned}
\bar{F}([g] \circ [f]) \;&=\; \bar{F}([\bar{g} \circ f]) \;=\; F(p_Z)^{-1} \circ F(\bar{g} \circ f) \\
&=\; F(p_Z)^{-1} \circ F(\bar{g} \circ p_Y) \circ F(p_Y)^{-1} \circ F(f) \\
&=\; (F(p_Z)^{-1} \circ F(g)) \circ (F(p_Y)^{-1} \circ F(f)) \;=\; \bar{F}(g) \circ \bar{F}(f)
\end{aligned}$$

where we used functoriality of F and homotopy invariance of F. Thus $\bar{F}$ is a functor.

Finally, we have to check the relation $\bar{F} \circ \gamma = F$. On objects this holds by definition. For a morphism $\varphi : X \longrightarrow Y$ in $\mathcal{C}$ we have

$$\bar{F}(\gamma(\varphi)) \;=\; \bar{F}([p_Y \circ \varphi]) \;=\; F(p_Y)^{-1} \circ F(p_Y \circ \varphi) \;=\; F(\varphi) \;,$$

which finishes the proof.

(iii) Since $\mathcal{C}$ has coproducts and every object of $\mathrm{Ho}(\mathcal{C})$ is in the image of γ, it suffices to show that γ preserves coproducts. After unraveling the definitions, this comes down to the following: Let $i_A : A \longrightarrow A \amalg B$ and $i_B : B \longrightarrow A \amalg B$ be two $\mathcal{C}$-morphisms which make $A \amalg B$ a coproduct of A and B. Then for every object Y the map

$$[A \amalg B, \omega Y] \longrightarrow [A, \omega Y] \times [B, \omega Y] , \qquad [f] \quad \longmapsto \quad \big([fi_A], [fi_B]\big)$$

between sets of left homotopy classes of morphisms is bijective. The map is surjective by the universal property of the coproduct. For injectivity we consider two morphisms $f, g : A \amalg B \longrightarrow \omega Y$ such that fi_A is left homotopic to gi_A and fi_B is left homotopic to gi_B. We choose cylinder objects $(IA, i_0.i_1, p)$ for A and $(IB, j_0.j_1, q)$ for B and homotopies $H : IA \longrightarrow \omega Y$ from fi_A to gi_A and $K : IB \longrightarrow \omega Y$ from fi_B to gi_B. Then $IA \amalg IB$ is a cylinder object for $A \amalg B$, and $H + K : IA \amalg IB \longrightarrow \omega Y$ is a homotopy from f to g.

(iv) Let $f : X \longrightarrow \omega Y$ represent φ. We choose a factorization $(f + p_Y) = vu$ of the morphism $f + p_Y : X \amalg Y \longrightarrow \omega Y$ as a cofibration $u : X \amalg Y \longrightarrow Z$ followed by a weak equivalence $v : Z \longrightarrow \omega Y$. Then $u = i + s$ for uniquely defined morphisms $i : X \longrightarrow Z$ and $s : Y \longrightarrow Z$; the morphisms i and s are cofibrations since u is. Since $vs = p_Y$ and v and p_Y are weak equivalences, s is also a weak equivalence. Finally, we have

$$\alpha \;=\; \gamma(p_Y)^{-1}\gamma(f) \;=\; \big(\gamma(p_Y)^{-1}\gamma(v)\big)\big(\gamma(v)^{-1}\gamma(f)\big) \;=\; \gamma(s)^{-1}\gamma(i) \;. \qquad \square$$

A cofibration category is *pointed* if it has a zero object (i.e., if every initial object is also terminal). We denote any zero object by $*$ and write also write $* : A \longrightarrow B$ for the zero morphism, i.e., unique morphism that factors through a zero object. In a pointed setting we write $A \vee B$ for the coproduct of two objects.

A *cone functor* for a pointed cofibration category is a functor $C : \mathcal{C} \longrightarrow \mathcal{C}$ together with a natural transformation $i : \mathrm{Id} \longrightarrow C$ such that for every object A of $\mathcal{C}$ the morphism $i_A : A \longrightarrow CA$ is a cofibration and the object CA is weakly contractible.

Given a cone functor on a pointed cofibration category $\mathcal{C}$, we define an associated *suspension functor* $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$ by setting $\Sigma A = CA/A$, the cokernel of the 'cone inclusion' $i_A : A \longrightarrow CA$,, i.e., a pushout:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & CA \arrow[d, "p"] \\ * \arrow[r] & \Sigma A \end{tikzcd}$$

By the gluing lemma [ref] the suspension functor takes weak equivalences to weak equivalences, so the universal property of the localization functor $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ provides a unique functor $\Sigma : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$ such that $\Sigma \circ \gamma = \gamma \circ \Sigma$.

We observe that the suspension functor preserves coproducts in $\mathrm{Ho}(\mathcal{C})$. Indeed, if $i_A : A \longrightarrow CA$, $i_B : B \longrightarrow CB$ and $i_{A \vee B} : A \vee B \longrightarrow C(A \vee B)$ are the chosen cones for two objects A, B and a coproduct $A \vee B$, then $i_A \vee i_B : A \vee B \longrightarrow CA \vee CB$ is another cone for $A \vee B$. So the is a unique isomorphism $\psi : (i_A \vee i_B) \cong i_{A \vee B}$ in $\mathrm{Ho}(\mathrm{Cone}\,\mathcal{C})$ between the two cones such that $\mathrm{Ho}(S)(\psi)$ is the identity of $A \vee B$ in $\mathrm{Ho}(\mathcal{C})$. Applying the derived quotient functor $\mathrm{Ho}(-/-)$ to ψ produces an isomorphism from $\mathrm{Ho}(i_A \vee i_B) = \Sigma A \vee \Sigma B$ to $\mathrm{Ho}(i_{A \vee B}) = \Sigma(A \vee B)$.

Definition 1.16. A pointed cofibration category $\mathcal{C}$ [with cone functor...] is *stable* if the suspension functor $\Sigma : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$ is an equivalence of categories.

Remark 1.17. It is not strictly necessary to assume the extra structure a cone functor on the pointed cofibration category if we want to establish the homotopy category $\mathrm{Ho}(\mathcal{C})$ as a triangulated category, since the suspension functor and the distinguished triangles on $\mathrm{Ho}(\mathcal{C})$ can be defined without functorial cones on $\mathcal{C}$. However, even though functorial cones are not logically necessary, they simplify the arguments a lot; since the examples we care about have functorial cones, we only develop the theory in this context. The necessary changes in the absence of a cone functor are explain in Exercises [...]

We will now show that the homotopy category of a stable cofibration category is additive. For this purpose we use extra structure on a suspension, namely a certain *collapse morphism* $\kappa_A : \Sigma A \longrightarrow \Sigma A \vee \Sigma A$

in $\mathrm{Ho}(\mathcal{C})$. To define it, we consider a pushout $CA \cup_A CA$ of two copies of the cone CA along i_A. The gluing lemma [!] guarantees that the map $0 \cup \mathrm{Id} : CA \cup_A CA \longrightarrow CA/A = \Sigma A$ induced on horizontal pushouts of the left commutative diagram

$$\begin{tikzcd} CA \arrow[d, "\sim"'] & A \arrow[l, "i_A"'] \arrow[r, "i_A"] \arrow[d, equal] & CA \arrow[d, equal] \\ * & A \arrow[l] \arrow[r, "i_A"'] & CA \end{tikzcd} \qquad \begin{tikzcd} CA \arrow[d, "p"'] & A \arrow[l, "i_A"'] \arrow[r, "i_A"] \arrow[d] & CA \arrow[d, "p"] \\ \Sigma A & * \arrow[l] \arrow[r] & \Sigma A \end{tikzcd}$$

is a weak equivalence. We define the κ_A as the composite

$$\Sigma A \xrightarrow{\gamma(0\cup p)^{-1}} CA \cup_A CA \xrightarrow{\gamma(p\cup p)} \Sigma A \vee \Sigma A$$

where the second morphism is the image of the $\mathcal{C}$-morphism induced on horizontal pushouts of the right commutative diagram above. Since the cone is functorial and the cofibration $i_A : A \longrightarrow CA$ is a natural transformation, the collapse morphism is natural, i.e., for every morphism $a : A \longrightarrow B$ we have $(\Sigma a \vee \Sigma a) \circ \kappa_A = \kappa_B \circ (\Sigma a)$.

Proposition 1.18. *The morphism $\kappa_A : \Sigma A \longrightarrow \Sigma A \vee \Sigma A$ satisfies the relations*

$$(0 + \mathrm{Id})\kappa_A \;=\; \mathrm{Id} \quad \textit{and} \quad (\mathrm{Id} + \mathrm{Id})\kappa_A \;=\; 0$$

as endomorphisms of ΣA in $\mathrm{Ho}(\mathcal{C})$. The endomorphism

$$m_A \;=\; (\mathrm{Id} + 0) \circ \kappa_A$$

is an involution, i.e., $m_A^2 = \mathrm{Id}$.

Proof. We observe that $(0 + \mathrm{Id}) \circ (p \cup p) = 0 \cup \mathrm{Id}$ as $\mathcal{C}$-morphisms $CA \cup_A CA \longrightarrow \Sigma A$, so

$$(0 + \mathrm{Id})\kappa_A \;=\; (0 + \mathrm{Id}) \circ \gamma(p \cup p) \circ \gamma(0 \cup \mathrm{Id})^{-1} \;=\; \gamma(0 \cup \mathrm{Id}) \circ \gamma(0 \cup \mathrm{Id})^{-1} \;=\; \mathrm{Id}\ .$$

The square

$$\begin{tikzcd} CA \cup_A CA \arrow[r, "p\cup p"] \arrow[d, "\mathrm{Id}\cup\mathrm{Id}"'] & \Sigma A \vee \Sigma A \arrow[d, "\mathrm{Id}+\mathrm{Id}"] \\ CA \arrow[r, "p"'] & \Sigma A \end{tikzcd}$$

commutes in $\mathcal{C}$, so the morphism $(\mathrm{Id} + \mathrm{Id})\kappa_A = (\mathrm{Id} + \mathrm{Id}) \circ \gamma(p \cup p) \circ \gamma(0 \cup \mathrm{Id})^{-1}$ factors through the cone CA, which is a zero object in $\mathrm{Ho}(\mathcal{C})$. Thus $(\mathrm{Id} + \mathrm{Id})\kappa_A = 0$.

For the next relation we denote by τ the involution of $CA \cup_A CA$ that interchanges the two cones. Then we have

$$m_A \;=\; (\mathrm{Id} + 0) \circ \gamma(p \cup p) \circ \gamma(0 \cup p)^{-1} \;=\; \gamma(p \cup 0) \circ \gamma(0 \cup p)^{-1} \;=\; \gamma(0 \cup p) \circ \gamma(\tau) \circ \gamma(0 \cup p)^{-1}\ .$$

Since $\tau^2 = \mathrm{Id}$ this leads to $m_A^2 = \mathrm{Id}$. □

By Theorem 1.15 (iii) the coproduct in any cofibration category $\mathcal{C}$ descends to a coproduct in the homotopy category $\mathcal{C}$. We will now show that for stable $\mathcal{C}$ the coproduct $X \vee Y$ is also a product of X and Y in $\mathrm{Ho}(\mathcal{C})$ with respect to the morphisms $p_X = \mathrm{Id} + 0 : X \vee Y \longrightarrow X$ and $p_Y = 0 + \mathrm{Id} : X \vee Y \longrightarrow Y$. So we have to show that for every object B of $\mathcal{C}$ the map

$$[B, X \vee Y] \;\longrightarrow\; [B, X] \times [B, Y]\ , \quad \varphi \;\longmapsto\; (p_X\varphi,\, p_Y\varphi) \tag{1.19}$$

is bijective.

Proposition 1.20. *Let $\mathcal{C}$ be a pointed cofibration category.*

(i) *If the object B is a suspension, then the map* (1.19) *is surjective.*
(ii) *Let $\varphi, \psi : B \longrightarrow X \vee Y$ be morphisms in $\mathrm{Ho}(\mathcal{C})$ such that $p_X\varphi = p_X\psi$ and $p_Y\varphi = p_Y\psi$. Then $\Sigma\varphi = \Sigma\psi$.*

(iii) *If $\mathcal{C}$ is stable, then the homotopy category* $\mathrm{Ho}(\mathcal{C})$ *is additive and for every object A of $\mathcal{C}$ the morphism $m_A : \Sigma A \longrightarrow \Sigma A$ is the negative of the identity of ΣA.*

PROOF. (i) Given two morphisms $\alpha : \Sigma A \longrightarrow X$ and $\beta : \Sigma A \longrightarrow Y$ in $\mathrm{Ho}(C)$ we consider the morphism $((\alpha \circ m_A) \vee \beta) \circ \kappa_A : \Sigma A \longrightarrow X \vee Y$. This morphism then satisfies

$$p_X \circ ((\alpha \circ m_A) \vee \beta) \circ \kappa_A \ = \ \alpha \circ m_A \circ (\mathrm{Id} + 0) \circ \kappa_A \ = \ \alpha \circ m_A^2 \ = \ \alpha$$

and similarly $p_Y \circ ((\alpha \circ m_A) \vee \beta) \circ \kappa_A = \beta$. So the map (1.19) is surjective.

(ii) We first show that the composite

$$\Sigma(X \vee Y) \xrightarrow{\kappa_{X\vee Y}} \Sigma(X \vee Y) \vee \Sigma(X \vee Y) \xrightarrow{(\Sigma i_X) m_X (\Sigma p_X) + (\Sigma i_Y)(\Sigma p_Y)} \Sigma(X \vee Y) \tag{1.21}$$

is the identity of $\Sigma(X \vee Y)$. Indeed, after precomposition with $\Sigma i_X : \Sigma X \longrightarrow \Sigma(X \vee Y)$ we have

$$\begin{aligned} &((\Sigma i_X) m_X (\Sigma p_X) + (\Sigma i_Y)(\Sigma p_Y)) \circ \kappa_{X \vee Y} \circ (\Sigma i_X) \\ &\quad = \ ((\Sigma i_X) m_X (\Sigma p_X) + (\Sigma i_Y)(\Sigma p_Y)) \circ (\Sigma i_X \vee \Sigma i_X) \circ \kappa_X \\ &\quad = \ ((\Sigma i_X) m_X + 0) \circ \kappa_X \ = \ ((\Sigma i_X) m_X) \circ (\mathrm{Id}_{\Sigma X} + 0) \circ \kappa_X \\ &\quad = \ (\Sigma i_X) \circ m_X^2 \ = \ \Sigma i_X \end{aligned}$$

Similarly, we have $((\Sigma i_X) m_X (\Sigma p_X) + (\Sigma i_Y)(\Sigma p_Y)) \circ \kappa_{X \vee Y} \circ (\Sigma i_Y) = \Sigma i_Y$. Since the suspension functor preserves coproducts, a morphism out of $\Sigma(X \vee Y)$ is determined by precomposition with Σi_X and Σi_Y. This proves that the composite (1.21) is the identity. For $\varphi : B \longrightarrow X \vee Y$ we then have

$$\begin{aligned} \Sigma\varphi \ &= \ ((\Sigma i_X) m_X (\Sigma p_X) + (\Sigma i_Y)(\Sigma p_Y)) \circ \kappa_{X \vee Y} \circ (\Sigma\varphi) \\ &= \ ((\Sigma i_X) m_X (\Sigma p_X) + (\Sigma i_Y)(\Sigma p_Y)) \circ (\Sigma\varphi \vee \Sigma\varphi) \circ \kappa_B \\ &= \ ((\Sigma i_X) m_X \Sigma(p_X \varphi) + (\Sigma i_Y)\Sigma(p_Y \varphi)) \circ \kappa_B \ . \end{aligned}$$

So $\Sigma\varphi$ is determined by the composites $p_X \varphi$ and $p_Y \varphi$, and this proves the claim.

(iii) Since $\mathcal{C}$ is stable, every object is isomorphic to a suspension, so the map (1.19) is always surjective by part (i). Moreover, suspension is faithful, so the map (1.19) is always injective by part (ii). Thus the map (1.19) is bijective for all objects B, X and Y, and so coproducts in $\mathrm{Ho}(\mathcal{C})$ are also products.

As we explained in Proposition 1.12 above, the fact that $\mathrm{Ho}(\mathcal{C})$ has isomorphic coproducts and products (in the sense that the map (1.19) is always bijective) implies that the morphisms sets have a natural structure of abelian monoid: given $f, g : X \longrightarrow Z$, let $f \perp g : X \longrightarrow Z \vee Z$ be the unique morphism such that $(\mathrm{Id} + 0)(f \perp g) = f$ and $(0 + \mathrm{Id})(f \perp g) = g$. Then the assignment $f + g = (\mathrm{Id} + \mathrm{Id})(f \perp g)$ is an associative, commutative and binatural operation on the set $\mathrm{Ho}(\mathcal{C})(X, Z)$ with neutral element given by the zero morphism.

The collapse map $\kappa_A : \Sigma A \longrightarrow \Sigma A \vee \Sigma A$ satisfies $(\mathrm{Id} + 0)\kappa = m_A$ and $(0 + \mathrm{Id})\kappa_A = \mathrm{Id}$, and so $\kappa_A = m_A \perp \mathrm{Id}$. So we have $m_A + \mathrm{Id} = (\mathrm{Id} + \mathrm{Id})\kappa_A = 0$. This shows that the morphism m_A is the additive inverse of the identity of ΣA. In particular, the abelian monoid $\mathrm{Ho}(\mathcal{C})(\Sigma A, Z)$ has inverses, and is thus an abelian group. Since every object is isomorphic to a suspension, the abelian monoid $\mathrm{Ho}(\mathcal{C})(B, Z)$ is a group for all objects B and Z, and so $\mathrm{Ho}(\mathcal{C})$ is an additive category. □

Now we introduce and discuss the class of distinguished triangles. Given a cofibration $i : A \longrightarrow B$ in a pointed cofibration category $\mathcal{C}$, we define the *connecting morphism* $\delta(i) : B/A \longrightarrow \Sigma A$ in $\mathrm{Ho}(\mathcal{C})$ as

$$\delta(i) \ = \ \gamma(p \cup 0) \circ \gamma(0 \cup q)^{-1} \ : B/A \ \longrightarrow \ \Sigma A \ . \tag{1.22}$$

Here $p \cup 0 : C(i) = CA \cup_i B \longrightarrow \Sigma A$ is the morphism that collapses B and $0 \cup q : C(i) \longrightarrow B/A$ is the weak equivalence that collapses CA. Since the cone is functorial and the cofibration $i_A : A \longrightarrow CA$ is a natural transformation, the connecting morphism (1.22) is natural, i.e., for every commutative square in $\mathcal{C}$ on the left

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A' \arrow[r, "i'"'] & B' \end{tikzcd} \qquad \begin{tikzcd} B/A \arrow[r, "\delta(i)"] \arrow[d, "\gamma(\beta/\alpha)"'] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\ B'/A' \arrow[r, "\delta(i')"'] & \Sigma A' \end{tikzcd}$$

such that i and i' are cofibrations, the square on the right commutes in $\mathrm{Ho}(\mathcal{C})$.

The *elementary distinguished triangle* associated to a cofibration $i : A \longrightarrow B$ is the sequence

$$A \xrightarrow{\gamma(i)} B \xrightarrow{\gamma(q)} B/A \xrightarrow{\delta(i)} \Sigma A\ .$$

A *distinguished triangle* is any triangle in the homotopy category which is isomorphic to the elementary distinguished triangle associated to a cofibration in $\mathcal{C}$.

Theorem 1.23. *The suspension functor and the class of distinguished triangles make the homotopy category* $\mathrm{Ho}(\mathcal{C})$ *of a stable cofibration category into a triangulated category.*

The category of symmetric spectra of simplicial sets if a pointed cofibration category with respect to monomorphisms as cofibrations and stable equivalences as weak equivalences. As a cone functor we can take the functor $A \mapsto \Delta[1] \wedge A$, and then [...] shows that this cofibration category structure is stable.

Proof of Theorem 1.23. We have see in Proposition 1.20 (iii) that the homotopy category of a stable cofibration category is additive. So it remains to prove the axioms (T1) – (T4).

(T1) The unique morphism for any zero object to X is a cofibration with quotient morphism the identity of X. The triangle $0 \longrightarrow X \xrightarrow{\mathrm{Id}} X \longrightarrow 0$ is the associated elementary distinguished triangle.

(T2 – Rotation) We start with a distinguished triangle (f, g, h) and want to show that the triangle $(g, h, -\Sigma f)$ is also distinguished. It suffices to consider the elementary distinguished triangle $(\gamma(i), \gamma(q), \delta(i))$ associated to a cofibration $i : A \longrightarrow B$. In the diagram on the left

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "i_A"'] & B \arrow[r] \arrow[d, "j"'] & * \arrow[d] \\ CA \arrow[r] & C(i) \arrow[r, "p\cup 0"'] & \Sigma A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\gamma(j)"] \arrow[d, equal] & Ci \arrow[r, "\gamma(p\cup 0)"] \arrow[d, "\gamma(0\cup q)"', "\cong"] & \Sigma A \arrow[r, "\delta(j)"] \arrow[d, equal] & \Sigma B \arrow[d, equal] \\ B \arrow[r, "\gamma(q)"'] & B/A \arrow[r, "\delta(i)"'] & \Sigma A \arrow[r, "\Sigma\gamma(i)\circ\delta(i_A)"'] & \Sigma B \end{tikzcd}$$

the left square and the composite outer square are pushouts; so the right square is also a pushout and the morphism $p \cup 0 : Ci \longrightarrow \Sigma A$ is the quotient projection associated to the cofibration $j : B \longrightarrow C(i)$. Moreover, both i_A and j are cofibrations, so by naturality of the connecting morphisms we get $\delta(j) \circ \mathrm{Id}_{\Sigma A} = \Sigma\gamma(i) \circ \delta(i_A)$. Hence the diagram on the right commutes. The upper row is the elementary distinguished triangle of the cofibration $j : B \longrightarrow C(i)$, and all vertical maps are isomorphisms, so the lower triangle is distinguished, as claimed. By definition the connecting morphism $\delta(i_A)$ conincides with the involution m_A of ΣA. In the stable context, m_A is the negative of the identity, so $(\Sigma f) \circ \delta(i_A) = -\Sigma f$.

(T3 – Completion of triangles) We are given two distinguished triangles (f, g, h) and (f', g', h') and two morphisms a and b in $\mathrm{Ho}(\mathcal{C})$ satisfying $bf = f'a$ as in the diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, dashed, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A' \end{tikzcd}$$

We have to extend this data to a morphism of triangles, i.e., find a morphism c making the entire diagram commute. If we can solve the problem for isomorphic triangles, then we can also solve it for the original triangles. We can thus assume that the triangle (f,g,h) and (f',g',h') are the elementary distinguished triangle arising from two cofibrations $i : A \longrightarrow B$ and $i' : A' \longrightarrow B'$.

We start with the special case where $a = \gamma(\alpha)$ and $b = \gamma(\beta)$ for $\mathcal{C}$-morphisms $\alpha : A \longrightarrow A'$ and $\beta : B \longrightarrow B'$. Then $\gamma(i'\alpha) = \gamma(\beta i)$, so the composite maps

$$p_{B'} i'\alpha \ , \ p_{B'}\beta i \ : \ A \ \longrightarrow \ \omega(B')$$

are homotopic. Proposition 1.26 (iv), applied to $f = p_{B'} i'\alpha$, provides a morphism $\beta' : B \longrightarrow \omega(B')$, homotopic to $p_{B'}\beta$, such that $\beta' i = p_{B'} i'\alpha$.

The following diagram on the left commutes in $\mathcal{C}$, so the diagram of elementary distinguished triangles on the right commutes in $\mathrm{Ho}(\mathcal{C})$ by the naturality of the connecting morphisms:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta'"] \\
A' \arrow[r, "p_{B'} i'"] \arrow[d, equal] & \omega(B') \\
A' \arrow[r, "i'"'] & B' \arrow[u, "p_{B'}"', "\sim"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\gamma(i)"] \arrow[d, "\gamma(\alpha)"'] & B \arrow[r, "\gamma(q)"] \arrow[d, "\gamma(\beta')"'] & B/A \arrow[r, "\delta(i)"] \arrow[d, "\gamma(\beta'/\alpha)"] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\
A' \arrow[r, "\gamma(p_{B'} i')"] \arrow[d, equal] & \omega(B') \arrow[r, "\gamma(\bar{q})"] & \omega(B')/A' \arrow[r, "\delta(p_{B'} i')"] & \Sigma A' \arrow[d, equal] \\
A' \arrow[r, "\gamma(i')"'] & B' \arrow[r, "\gamma(q')"'] \arrow[u, "\gamma(p_{B'})", "\cong"'] & B'/A' \arrow[r, "\delta(i')"'] \arrow[u, "\cong", "\gamma(t/A')"'] & \Sigma A'
\end{tikzcd}$$

Since $\gamma(p_{B'})^{-1} \circ \gamma(\beta') = b$, the morphism $c = \gamma(p_{B'}/A')^{-1} \circ \gamma(\beta'/\alpha) : B/A \longrightarrow B'/A'$ is the desired filler.

In the general case we write $a = \gamma(s)^{-1}\gamma(\alpha)$ where $\alpha : A \longrightarrow \bar{A}$ is a $\mathcal{C}$-morphism and $s : A' \longrightarrow \bar{A}$ is an acyclic cofibration. We choose a pushout

$$\begin{tikzcd}
\bar{A} \arrow[r, "j"] & \bar{A} \cup_{A'} B' \\
A' \arrow[r, "i'"'] \arrow[u, "s", "\simeq"'] & B' \arrow[u, "\simeq", "s'"']
\end{tikzcd}$$

We write $\gamma(s')b = \gamma(t)^{-1}\gamma(\beta) : A \longrightarrow \bar{A} \cup_{A'} B'$ where $\beta : B \longrightarrow \bar{B}$ is a $\mathcal{C}$-morphism and $t : \bar{A} \cup_{A'} B' \longrightarrow \bar{B}$ is an acyclic cofibration. We then have

$$\gamma(tj)\gamma(\alpha) \ = \ \gamma(tj)\gamma(s)a \ = \ \gamma(ts')\gamma(i')a \ = \ \gamma(ts')b\gamma(i) \ = \ \gamma(\beta)\gamma(i) \ ,$$

so by the special case, applied to the cofibrations $i : A \longrightarrow B$ and $tj : \bar{A} \longrightarrow \bar{B}$ and the morphisms $\alpha : A \longrightarrow \bar{A}$ and $\beta : B \longrightarrow \bar{B}$, there exists a morphism $c : B/A \longrightarrow \bar{B}/\bar{A}$ in $\mathrm{Ho}(\mathcal{C})$ making the diagram

$$\begin{tikzcd}
A \arrow[r, "\gamma(i)"] \arrow[d, "\gamma(\alpha)"'] & B \arrow[r, "\gamma(q)"] \arrow[d, "\gamma(\beta)"'] & B/A \arrow[r, "\delta(i)"] \arrow[d, dotted, "c"'] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\
\bar{A} \arrow[r, "\gamma(tj)"] & \bar{B} \arrow[r, "\gamma(\bar{q})"] & \bar{B}/\bar{A} \arrow[r, "\delta(tj)"] & \Sigma\bar{A} \\
A' \arrow[r, "\gamma(i')"'] \arrow[u, "\gamma(s)"] & B' \arrow[r, "\gamma(q')"'] \arrow[u, "\gamma(ts')"] & B'/A' \arrow[r, "\delta(i')"'] \arrow[u, "\gamma(ts'/s)"'] & \Sigma A' \arrow[u, "\Sigma\gamma(s)"']
\end{tikzcd}$$

commute (the lower part commutes by naturality of connecting morphisms). Since s is an acyclic cofibration so is its cobase change s'. By the gluing lemma the weak equivalences $s : A' \longrightarrow \bar{A}$ and $ts' : B' \longrightarrow \bar{B}$ induce a weak equivalence $ts'/s : B'/A' \longrightarrow \bar{B}/\bar{A}$ on quotients and the composite

$$B/A \ \xrightarrow{\ c\ } \ \bar{B}/\bar{A} \ \xrightarrow{\gamma(ts'/s)^{-1}} \ B'/A'$$

in $\mathrm{Ho}(\mathcal{C})$ thus solves the original problem.

(T4 - Octahedral axiom) We start with the special case where $f = \gamma(i)$ and $f' = \gamma(j)$ for cofibrations $i : A \longrightarrow B$ and $j : B \longrightarrow D$. Then the composite $ji : A \longrightarrow D$ is a cofibration with $\gamma(ji) = f'f$. The diagram

$$\begin{tikzcd}
A \arrow[r, "\gamma(i)"] \arrow[d, equal] & B \arrow[r, "\gamma(q_i)"] \arrow[d, "\gamma(j)"'] & B/A \arrow[r, "\delta(i)"] \arrow[d, "\gamma(j/A)"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(ji)"'] & D \arrow[r, "\gamma(q_{ji})"'] \arrow[d, "\gamma(q_j)"'] & D/A \arrow[r, "\delta(ji)"'] \arrow[d, "\gamma(D/i)"] & \Sigma A \arrow[d, "\Sigma\gamma(i)"] \\
& D/B \arrow[r, equal] \arrow[d, "\delta(j)"'] & D/B \arrow[r, "\delta(j)"'] \arrow[d, "\delta(j/A)=(\Sigma\gamma(c_i))\delta(j)"] & \Sigma B \\
& \Sigma B \arrow[r, "\Sigma\gamma(c_i)"'] & \Sigma B/A &
\end{tikzcd}$$

then commutes by naturality of connecting morphisms. Moreover, the four triangles in question are the elementary distinguished triangles of the cofibrations i, j, ji and $j/A : B/A \longrightarrow D/A$.

In the general case we write $f = \gamma(s)^{-1}\gamma(a)$ for a $\mathcal{C}$-morphism $a : A \longrightarrow B'$ and a weak equivalence $s : B \longrightarrow B'$. Then a can be factored as $a = pi$ for a cofibration $i : A \longrightarrow \bar{B}$ and a weak equivalence $p : \bar{B} \longrightarrow B'$. Altogether we then have $f = \varphi \circ \gamma(i)$ where $\varphi = \gamma(s)^{-1} \circ \gamma(p) : \bar{B} \longrightarrow B$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. We can apply the same reasoning to the morphism $f'\varphi : \bar{B} \longrightarrow D$ and write it as $f' \circ \varphi = \psi \circ \gamma(j)$ for a cofibration $j : \bar{B} \longrightarrow \bar{D}$ in $\mathcal{C}$ and an isomorphism $\psi : \bar{D} \longrightarrow D$ in $\mathrm{Ho}(\mathcal{C})$. The special case can then be applied to the cofibrations $i : A \longrightarrow \bar{B}$ and $j : \bar{B} \longrightarrow \bar{D}$. The resulting commutative diagram that solves (T4) for $(\gamma(i), \gamma(j))$ can then be translated back into a commutative diagram that solves (T4) for (f, f') by conjugating with the isomorphisms $\varphi : \bar{B} \longrightarrow B$ and $\psi : \bar{D} \longrightarrow D$. This completes the proof of the octahedral axiom (T4), and hence the proof of Theorem 1.23. □

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Now we discuss how exact functors between stable cofibration cateories give rise to exact functors between the triangulated homotopy categories. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between cofibration categories is *exact* if it preserves cofibrations, weak equivalences and the particular pushouts (1.7) along cofibrations that are guaranteed by axiom (C3). Since F preserves weak equivalences, the composite functor $\gamma^{\mathcal{D}} \circ F : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{D})$ takes weak equivalences to isomorphisms and the universal property of the homotopy category provides a unique *derived functor* $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ such that $\mathrm{Ho}(F) \circ \gamma^{\mathcal{C}} = \gamma^{\mathcal{D}} \circ F$.

We will now explain that for pointed cofibration categories $\mathcal{C}$ and $\mathcal{D}$ the derived functor $\mathrm{Ho}(F)$ commutes with suspension, in the sense that there is a preferred natural isomorphism

$$\tau^F \ : \ (LF) \circ \Sigma \ \xrightarrow{\cong} \ \Sigma \circ (LF) \tag{1.24}$$

of functors from $\mathrm{Ho}(\mathcal{C})$ to $\mathrm{Ho}(\mathcal{D})$. Our construction is somewhat indirect, using universal properties. This abstract approach to the construction of the isomorphism removed the necessity to deal with the choices of cones and suspensions, and avoids the construction of homotopies.

The exact functor F prolongs to a functor $\mathrm{Cone}\, F : \mathrm{Cone}\, \mathcal{C} \longrightarrow \mathrm{Cone}\, \mathcal{D}$ between the cone categories: for example, on objects we have $(\mathrm{Cone}\, F)(i : A \longrightarrow C) = (Fi : FA \longrightarrow FC)$. The prolonged functor $\mathrm{Cone}\, F$ is exact since F is, so it has a derived functor $\mathrm{Ho}(\mathrm{Cone}\, F) : \mathrm{Ho}(\mathrm{Cone}\, \mathcal{C}) \longrightarrow \mathrm{Ho}(\mathrm{Cone}\, \mathcal{D})$. We have $S \circ (\mathrm{Cone}\, F) = F \circ S : \mathrm{Cone}\, \mathcal{C} \longrightarrow \mathcal{D}$ (equality, not just isomorphism), so equality $\mathrm{Ho}(S) \circ \mathrm{Ho}(\mathrm{Cone}\, F) = \mathrm{Ho}(F) \circ \mathrm{Ho}(S)$ also holds as functors $\mathrm{Ho}(\mathrm{Cone}\, \mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$.

As we explained in [...] the individual choices of cones for the objects of $\mathcal{C}$ and $\mathcal{D}$ assemble uniquely into cone functors

$$C^{\mathcal{C}} \ : \ \mathcal{C} \ \longrightarrow \ \mathrm{Ho}(\mathrm{Cone}\, \mathcal{C}) \quad \text{and} \quad C^{\mathcal{D}} \ : \ \mathcal{D} \ \longrightarrow \ \mathrm{Ho}(\mathrm{Cone}\, \mathcal{D})$$

characterized by $\mathrm{Ho}(S) \circ C^{\mathcal{C}} = \gamma^{\mathcal{C}}$ respectively $\mathrm{Ho}(S) \circ C^{\mathcal{D}} = \gamma^{\mathcal{D}}$. The two functors

$$\mathrm{Ho}(\mathrm{Cone}\, F) \circ C^{\mathcal{C}} \ , \ C^{\mathcal{D}} \circ F \ : \ \mathcal{C} \ \longrightarrow \ \mathrm{Ho}(\mathrm{Cone}\, \mathcal{D})$$

become equal to $\mathrm{Ho}(F) \circ \gamma^{\mathcal{C}} = \gamma^{\mathcal{D}} \circ F$ after composition with the equivalence $\mathrm{Ho}(S) : \mathrm{Ho}(\mathrm{Cone}\, \mathcal{D}) \longrightarrow \mathrm{Ho}(\mathcal{D})$. So there is a unique natural isomorphism

$$\Theta \ : \ \mathrm{Ho}(\mathrm{Cone}\, F) \circ C^{\mathcal{C}} \ \longrightarrow \ C^{\mathcal{D}} \circ F$$

of functors $\mathcal{C} \longrightarrow \mathrm{Ho}(\mathrm{Cone}\, \mathcal{D})$ such that $\mathrm{Ho}(S) \circ \Theta$ is the identity natural transformation of the functor $\mathrm{Ho}(F) \circ \gamma^{\mathcal{C}} = \gamma^{\mathcal{D}} \circ F : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{D})$. We compose the natural isomorphism Θ with the derived quotient functor $\mathrm{Ho}(-/-) : \mathrm{Ho}(\mathrm{Cone}\, \mathcal{D}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ and arrive at a natural isomorphism $\mathrm{Ho}(-/-) \circ \Theta$. The source functor of $\mathrm{Ho}(-/-) \circ \Theta$ is equal to

$$\begin{aligned} \mathrm{Ho}(-/-) \circ \mathrm{Ho}(\mathrm{Cone}\, F) \circ C^{\mathcal{C}} \ &= \ \mathrm{Ho}((-/-) \circ \mathrm{Cone}\, F) \circ C^{\mathcal{C}} \ \cong \ \mathrm{Ho}(F \circ (-/-)) \circ C^{\mathcal{C}} \\ &= \ \mathrm{Ho}(F) \circ \mathrm{Ho}(-/-) \circ C^{\mathcal{C}} \ = \ \mathrm{Ho}(F) \circ \Sigma^{\mathcal{C}} \circ \gamma^{\mathcal{C}}; \end{aligned}$$

the target functor of $\mathrm{Ho}(-/-) \circ \Theta$ is equal to

$$\mathrm{Ho}(-/-) \circ C^{\mathcal{D}} \circ F \ = \ \Sigma^{\mathcal{D}} \circ \gamma^{\mathcal{D}} \circ F \ = \ \Sigma^{\mathcal{D}} \circ \mathrm{Ho}(F) \circ \gamma^{\mathcal{C}} \ .$$

So $\mathrm{Ho}(-/-) \circ \Theta$ descends to a unique natural isomorphism $\tau_F : \mathrm{Ho}(F) \circ \Sigma^{\mathcal{C}} \longrightarrow \Sigma^{\mathcal{D}} \circ \mathrm{Ho}(F)$ of functors $\mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ such that $\tau_F \circ \gamma^{\mathcal{D}} = \mathrm{Ho}(-/-) \circ \Theta$.

Proposition 1.25. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an exact functor between stable cofibration categories. Then the derived functor $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ is an exact functor of triangulated categories with respect to the natural isomorphism $\tau_F : \mathrm{Ho}(F) \circ \Sigma^{\mathcal{C}} \longrightarrow \Sigma^{\mathcal{D}} \circ \mathrm{Ho}(F)$.*

Proof. Any exact functor preserves finite coproducts; since coproducts descend to the homotopy category, the derived functor of an exact functor again preserves coproducts. So in the stable context, $\mathrm{Ho}(F)$ is an additive functor.

We have to show the for every distinguished triangle (f, g, h) in $\mathrm{Ho}(\mathcal{C})$ the triangle $(\mathrm{Ho}(F)(f), \mathrm{Ho}(F)(g), \tau_F \circ \mathrm{Ho}(F)(h))$ is distinguished in $\mathrm{Ho}(\mathcal{D})$. If suffices to consider the case when the triangle is the elementary distinguished triangle associated to a cofibration $i : A \longrightarrow Y$ in $\mathcal{C}$. Since F is exact, $F(i)$ is a cofibration in $\mathcal{D}$, $F(*)$ is a zero object and the square

$$\begin{tikzcd} F(A) \arrow[r, "f(i)"] \arrow[d] & F(B) \arrow[d, "F(q)"] \\ F(*) \arrow[r] & F(B/A) \end{tikzcd}$$

is a pushout. We claim that the connecting morphism of the cofibration $F(i)$ equals $\tau_A^F \circ \mathrm{Ho}(F)(\delta(i))$ [...] The cofibration $F(i_A) : F(A) \longrightarrow F(CA)$ and $i_{F(A)} : F(A) \longrightarrow C(F(A))$ are two cones of $F(A)$. Since the source functor $S : \mathrm{Ho}(\mathrm{Cone}\, \mathcal{D}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ is an equivalence of categories, there is a unique isomorphism $\varphi : F(i_A) \longrightarrow i_{F(A)}$ in the homotopy category of cones that is the identity on the source A. [...]

This established the claim and shows that the triangle

$$F(A) \xrightarrow{\gamma(F(i))} F(B) \xrightarrow{\gamma(F(q))} F(B/A) \xrightarrow{\tau_F(A) \circ \mathrm{Ho}(F)(\delta(i))} \Sigma^{\mathcal{D}} F(A)$$

is the elementary distinguished triangle of the cofibration $F(i)$. Because $\mathrm{Ho}(F) \circ \gamma = \gamma \circ F$, this triangle is also the image of the triangle $(\gamma(i), \gamma(q), \delta(i))$, and this concludes the proof. □

Proposition 1.26. *Let $\mathcal{C}$ be a cofibration category with functorial cylinders.*

(i) *If $f, g : A \longrightarrow B$ are homotopic morphism and $\varphi : A' \longrightarrow A$ and $\psi : B \longrightarrow \bar{B}$ any morphisms, then $f\varphi$ is homotopic to $g\varphi$ and ψf is homotopic to ψg.*

(ii) *If Z is fibrant and A an arbitrary object of $\mathcal{C}$, then the homotopy relation on the set of morphisms from A to Z is an equivalence relation.*

(iii) *Let $i : A \longrightarrow B$ be a cofibration and Z a fibrant object. Let $f : A \longrightarrow Z$ and $g : B \longrightarrow Z$ be morphisms such that f is homotopic to gi. Then there is a morphism $g' : B \longrightarrow Z$ which is homotopic to g and such that $g'i = f$.*

(iv) *If Z is fibrant and $\varphi : A \longrightarrow B$ is a weak equivalence, then the induced map $[\varphi, Z] : [A', Z] \longrightarrow [A, Z]$ on homotopy classes of morphisms into Z is bijective.*

(v) *Every weak equivalence between fibrant objects is a homotopy equivalence.*

Proof. (i) If $H : IA \longrightarrow B$ is a homotopy from f to g, then $\psi H : IA \longrightarrow \bar{B}$ is a homotopy from ψf to ψg and $H(I\varphi) : IA' \longrightarrow B$ is a homotopy from $f\varphi$ to $g\varphi$.

(ii) The homotopy relation is reflexive because $fp : IA \longrightarrow Z$ is a homotopy from a morphism $f : A \longrightarrow Z$ to itself.

For the symmetry relation we define SA as the pushout in the left square

$$\begin{tikzcd}
A \amalg A \arrow[r, "i_1+i_0"] \arrow[d, "i_0+i_1"'] & IA \arrow[d, "j_1"] \arrow[dr, "p"] & \\
IA \arrow[r, "j_0"] \arrow[rr, bend right, "p"'] & SA \arrow[r, dotted, "q"] & A
\end{tikzcd}$$

(observe that i_0 and i_1 occur in different orders in the two maps). The morphisms $p : IA \longrightarrow A$ on both copies of IA glue to a morphism $q : SA \longrightarrow A$ which we can factor as $q = sk$ where $k : SA \longrightarrow DA$ is a cofibration and $s : DA \longrightarrow A$ is a weak equivalence. The composite $kj_1 : IA \longrightarrow DA$ is a cofibration, and a weak equivalence since $s(kj_1) = qj_1 = p$ and s and p are weak equivalences. Since Z is fibrant the homotopy $H : IA \longrightarrow Z$ has an extension $K : DA \longrightarrow Z$ such that $Kkj_1 = H$. The restriction $Kkj_0 : IA \longrightarrow Z$ is then a homotopy from g to f, so the homotopy relation is symmetric.

For transitivity we consider three morphisms $f, g, h : A \longrightarrow Z$, a homotopy $H : IA \longrightarrow Z$ from f to g and a homotopy $\bar{H} : IA \longrightarrow Z$ from g to h. We construct a homotopy from f to h as follows. We define $IA \cup_A IA$ as the pushout:

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i_1"'] & IA \arrow[d, "j_1"] \\
IA \arrow[r, "j_0"'] & IA \cup_A IA
\end{tikzcd}$$

Since the morphisms $i_0 : A \longrightarrow IA$ and $i_1 : A \longrightarrow IA$ are acyclic cofibrations, so are the morphisms $j_0, j_1 : IA \longrightarrow IA \cup_A IA$. The morphisms $p : IA \longrightarrow A$ on both copies of IA glue to a morphism $p \cup p : IA \cup_A IA \longrightarrow A$ which is a weak equivalence since $(p \cup p)j_0 = p$ and j_0 and p are weak equivalences. The two morphisms $j_0 i_0, j_1 i_1' : A \longrightarrow IA \cup_A IA$ are composites of cofibrations, hence cofibrations. [need that $j_0 i_0 + j_1 i_1' : A \amalg A \longrightarrow I''A$ is a cofibration]

(iii) Let $H : IA \longrightarrow Z$ be a homotopy from f to gi. We choose a pushout

$$\begin{tikzcd}
A \amalg A \arrow[r, "i \amalg i"] \arrow[d, "i_0+i_1"'] & B \amalg B \arrow[d] \\
IA \arrow[r] & B \cup_A IA \cup_A B
\end{tikzcd}$$

and a factorization $\mathrm{Id} \cup ip \cup \mathrm{Id} = qj$ as a cofibration $j : B \cup_A IA \cup_A B \longrightarrow IB$ followed by a weak equivalence $q : IB \longrightarrow B$. The quadrupel (IB, i_0', i_1', q) is then a cylinder object for B.

Moreover [...] the map $IA \cup_A B \longrightarrow B \cup_A IA \cup_A B$ is a cofibration, hence so is the composite $j(-) : IA \cup_A B \longrightarrow IB$. This map is also a weak equivalence since the composite with the weak equivalence $i_1 : B \longrightarrow IA \cup_A B$ is right inverse to the weak equivalence q, and hence a weak equivalence. So $j(-) : IA \cup_A B \longrightarrow IB$ is an acyclic cofibration, and thus the morphism $H \cup g : IA \cup_A B \longrightarrow Z$ admits an extension $\bar{H} : IB \longrightarrow Z$. The morphism $\bar{g} = \bar{H} i_0 : B \longrightarrow Z$ is then homotopic to g and an extension of f.

(iv) We show that for every acyclic cofibration $j : A \longrightarrow B$ the induced map $[j, Z] : [B, Z] \longrightarrow [A, Z]$ is bijective. Lemma 1.10 (ii), applied to the set valued functor $[-, Z]$ then shows the claim. Since Z is fibrant, every morphism $f : A \longrightarrow Z$ has an extension $\bar{f} : B \longrightarrow Z$ with $\bar{f}j = f$; so $[j, Z]$ is surjective. Now suppose that $f, g : B \longrightarrow Z$ are two morphisms such that $fj, gj : A \longrightarrow Z$ are left homotopic via a homotopy $H : IA \longrightarrow Z$. We consider the pushout:

$$\begin{tikzcd}
A \amalg A \arrow[r, "i_0+i_1"] \arrow[d, "j \amalg j"'] & IA \arrow[d, "J"] \arrow[ddr, bend left, "jp"] & \\
A' \amalg A' \arrow[r, "i'_0+i'_1"'] \arrow[drr, bend right, "\mathrm{Id}+\mathrm{Id}"'] & IA' \arrow[dr, dotted, "p'"'] & \\
 & & A'
\end{tikzcd}$$

Since i_0+i_1 is a cofibration, so is $i'_0+i'_1$. The morphism $jp : IA \longrightarrow B$ and the fold map glue to a morphism $p' : IB \longrightarrow B$. Since $j \amalg j$ is an acyclic cofibration, so is the morphism $J : IA \longrightarrow IB$. Since $p'J = jp$ and J, j and p are weak equivalence, so is p'. Hence (IB, i'_0, i'_1, p') is a cylinder object for B. The homotopy $H : IA \longrightarrow Z$ and the morphism $f + g : B \amalg B \longrightarrow Z$ glue to a homotopy $H' : IB \longrightarrow Z$ from f to g. Altogether this shows that the the map $[j, Z]$ is injective.

(v) This part is a formal consequence of (iv). $\square$

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Proposition 1.27. *For every symmetric ring spectrum R the category of flat R-modules is a stable cofibration category with respect to the stable equivalences and the flat cofibrations.*

Define $\mathcal{D}(R)$ as the homotopy category of flat R-module spectra. [recall] In the special case $R = \mathbb{S}$ of the sphere spectrum the R-modules are simply symmetric spectra (up to isomorphism of categories). So $\mathcal{D}(\mathbb{S})$ is the homotopy category of flat symmetric spectra. By [...] the inclusion $\mathcal{S}p^\flat \longrightarrow \mathcal{S}p$ of flat symmetric spectra into all symmetic spectra induces an equivalence of homotopy categories $\mathcal{D}(\mathbb{S}) = \mathrm{Ho}(\mathcal{S}p^\flat) \longrightarrow \mathcal{SHC}$ between the derived category of the sphere spectrum and the stable homotopy category. In this sense the derived category $\mathcal{D}(\mathbb{S})$ 'is' the stable homotopy category.

Theorem 1.23 the speciallizes to

Theorem 1.28. *Let R be a symmetric ring spectrum. Then the suspension functor and the distinguished triangles make the category $\mathcal{D}(R)$ into a triangulated category. The image in $\mathcal{D}(R)$ of the free R-module of rank one is a compact weak generator of $\mathcal{D}(R)$. If R is commutative, then the derived smash product over R makes $\mathcal{D}(R)$ into a tensor triangulated category. For every morphism $f : R \longrightarrow S$ of symmetric ring spectra, the functor $- \wedge_R S : (\text{mod-}R)^\flat \longrightarrow (\text{mod-}S)^\flat$ of extension of scalars is exact; its derived functor*

$$- \wedge_R^L S \ : \ \mathcal{D}(R) \ \longrightarrow \mathcal{D}(S)$$

is exact and has a left adjoint.

Remark 1.29. Theorem 1.28 says that modules over a symmetric ring spectrum form a stable model category with single compact generator. The converse is also true, at least up to Quillen equivalence and under some technical hypothesis. More precisely, let $\mathcal{C}$ be a stable model category which is proper, cofibrantly generated and simplicial. Theorem 3.1.1 of [**72**] says that if the triangulated homotopy category has a single compact generator, then $\mathcal{C}$ is Quillen equivalent to the stable model category of modules over a symmetric ring spectrum. A ring spectrum which does the job can be obtained as a suitable endomorphism ring spectrum (in a similar sense as in Example I.3.41) of a cofibrant-fibrant weak generator.

Corollary 1.30 (of the appendix)**.** *Let R be a connective symmetric ring spectrum. Then R-modules have Postnikov section, and this gives a t-structure on $\mathcal{D}(R)$ whose heart is equivalent, via the functor π_0 to the abelian category of modules over the ring $\pi_0 R$.*

2. Toda brackets

Construction 2.1. Let $\mathcal{T}$ be a triangulated category and $\alpha : Y \longrightarrow Z$, $\beta : X \longrightarrow Y$ and $\gamma : W \longrightarrow X$ three composable morphisms which satisfy $\alpha\beta = 0 = \beta\gamma$. We define the *Toda bracket* $\langle\alpha, \beta, \gamma\rangle$, a subset of the morphism group $\mathcal{T}(\Sigma W, Z)$, as follows.

We start by choosing a distinguished triangle

$$X \xrightarrow{\beta} Y \xrightarrow{i} C(\beta) \xrightarrow{p} \Sigma X \ . \tag{2.2}$$

Since $\alpha\beta = 0$ there exists a morphism $\bar{\alpha} : C(\beta) \longrightarrow Z$ such that $\bar{\alpha}i = \alpha$; since $\beta\gamma = 0$ there exists a morphism $\underline{\gamma} : \Sigma W \longrightarrow C(\beta)$ such that $p\underline{\gamma} = \Sigma\gamma$, compare the commutative diagram.

$$\begin{tikzcd}
 & \Sigma W \arrow[d, "\underline{\gamma}"'] \arrow[dr, "\Sigma\gamma"] & \\
Y \arrow[r, "i"] \arrow[dr, "\alpha"'] & C(\beta) \arrow[r, "p"] \arrow[d, "\bar{\alpha}"] & \Sigma X \\
 & Z &
\end{tikzcd}$$

The bracket $\langle\alpha, \beta, \gamma\rangle$ then consists of all morphisms of the form $\bar{\alpha}\underline{\gamma} : \Sigma W \longrightarrow Z$ for varying $\bar{\alpha}$ and $\underline{\gamma}$. We recall that any two choices of triangles (2.2) are isomorphic, and so the bracket $\langle\alpha, \beta, \gamma\rangle$ does not depend on this choice.

Proposition 2.3. *Let $\alpha : Y \longrightarrow Z$, $\beta : X \longrightarrow Y$ and $\gamma : W \longrightarrow X$ be morphisms in a triangulated category $\mathcal{T}$ satisfying $\alpha\beta = 0 = \beta\gamma$.*

(i) *The Toda bracket $\langle\alpha, \beta, \gamma\rangle \subseteq \mathcal{T}(\Sigma W, X)$ is a coset of the subgroup $(\alpha \circ \mathcal{T}(\Sigma W, Y)) + (\mathcal{T}(\Sigma X, Z) \circ \Sigma\gamma)$.*

(ii) *If $\delta : V \longrightarrow W$ is another morphism such that $\gamma\delta = 0$, then the relation*

$$\alpha \circ \langle\beta, \gamma, \delta\rangle \ = \ -\langle\alpha, \beta, \gamma\rangle \circ (\Sigma\delta)$$

holds as subsets of $\mathcal{T}(\Sigma V, Z)$.

(iii) *For every exact functor $(F, \tau) : \mathcal{T} \longrightarrow \mathcal{T}'$ of triangulated categories the relation*

$$F(\langle\alpha, \beta, \gamma\rangle) \ \subseteq \ \langle F\alpha, F\beta, F\gamma\rangle \circ \tau_W$$

holds as subsets of $\mathcal{T}'(F(\Sigma W), F(Z))$.

(iv) *Let*

$$W \xrightarrow{\gamma} X \xrightarrow{i'} C(\gamma) \xrightarrow{p'} \Sigma W$$

be a distinguished triangle and $\bar{\beta} : C(\gamma) \longrightarrow Y$ a morphism such that $\bar{\beta}i' = \beta$. Then the Toda bracket $\langle\alpha, \beta, \gamma\rangle$ contains all morphisms $t : \Sigma W \longrightarrow Z$ that satisfy $tp' = \alpha\bar{\beta}$.

Proof. (i) Let $\underline{\gamma'} : \Sigma W \longrightarrow C(\beta)$ be another morphism satisfying $p\underline{\gamma'} = \Sigma\gamma$. Then $p(\underline{\gamma'} - \underline{\gamma}) = 0$, so there is a morphism $u : \Sigma W \longrightarrow Y$ such that $iu = (\underline{\gamma'} - \underline{\gamma})$. We have

$$\bar{\alpha}\underline{\gamma} - \bar{\alpha}\underline{\gamma'} \ = \ \bar{\alpha}(\underline{\gamma'} - \underline{\gamma}) \ = \ \bar{\alpha}iu \ = \ \alpha u \ .$$

So different lifts for γ change the bracket representative by an element in $\alpha \circ \mathcal{T}(\Sigma W, Y)$. The analogous argument shows that extensions of α change the bracket representative by an element in $\mathcal{T}(\Sigma X, Z) \circ (\Sigma\delta)$. So the bracket $\langle\alpha, \beta, \gamma\rangle$ lies in a single coset of the subgroup $(\alpha \circ \mathcal{T}(\Sigma W, Y)) + (\mathcal{T}(\Sigma X, Z) \circ \Sigma\gamma)$.

Conversely, let $\bar{\alpha} : C(\beta) \longrightarrow Z$ and $\underline{\gamma} : \Sigma W \longrightarrow C(\beta)$ satisfy $\bar{\alpha}i = \alpha$ and $p\underline{\gamma} = \gamma$, so that $\bar{\alpha}\underline{\gamma} \in \langle\alpha, \beta, \gamma\rangle$. Given arbitrary morphisms $v : \Sigma X \longrightarrow Z$ and $u : \Sigma W \longrightarrow Y$, then $\bar{\alpha} + vp$ is another extension of α and $\underline{\gamma} + iu$ is another lift of γ. Hence

$$(\bar{\alpha} + vp)(\underline{\gamma} + iu) \ = \ \bar{\alpha}\underline{\gamma} + \bar{\alpha}iu + vp\underline{\gamma} \ = \ \bar{\alpha}\underline{\gamma} + \alpha u + v(\Sigma\gamma)$$

is also an element of the bracket $\langle\alpha, \beta, \gamma\rangle$. So $\langle\alpha, \beta, \gamma\rangle$ indeed consists of the entire coset.

(ii) We choose distinguished triangles

$$X \xrightarrow{\beta} Y \xrightarrow{i} C(\beta) \xrightarrow{p} \Sigma X \qquad \text{and} \qquad W \xrightarrow{\gamma} X \xrightarrow{i'} C(\gamma) \xrightarrow{p'} \Sigma W \ .$$

The triangle

$$W \xrightarrow{0} Y \xrightarrow{\binom{1}{0}} Y \oplus \Sigma W \xrightarrow{(0,1)} \Sigma W$$

is also distinguished, so the 'strong form' of the octahedral axiom (compare Proposition 2.10 (iv)) provides morphisms $x : C(\gamma) \longrightarrow Y \oplus \Sigma W$ and $y : Y \oplus \Sigma W \longrightarrow C(\beta)$ such that the triangle $(x, y, (\Sigma i') \circ p)$ is distinguished and the diagram

$$\begin{tikzcd}
W \arrow[r, "\gamma"] \arrow[d, equal] & X \arrow[r, "i'"] \arrow[d, "\beta"'] & C(\gamma) \arrow[r, "p'"] \arrow[d, dashed, "x"] & \Sigma W \arrow[d, equal] \\
W \arrow[r, "0"'] & Y \arrow[r, "\binom{1}{0}"'] \arrow[d, "i"'] & Y \oplus \Sigma W \arrow[r, "(0,1)"'] \arrow[d, dashed, "y"] & \Sigma W \arrow[d, "\Sigma\gamma"] \\
 & C(\beta) \arrow[r, equal] \arrow[d, "p"'] & C(\beta) \arrow[r, "p"'] \arrow[d, "(\Sigma i')\circ p"] & \Sigma X \\
 & \Sigma X \arrow[r, "\Sigma i'"'] & \Sigma C(\gamma) &
\end{tikzcd}$$

commutes. Then $x = \binom{\bar\beta}{p'}$ for a unique morphism $\bar\beta : C(\gamma) \longrightarrow Y$ such that $\bar\beta i' = \beta$; similarly, $y = (i, \underline{\gamma})$ for a unique morphism $\underline{\gamma} : \Sigma W \longrightarrow C(\beta)$ such that $p\underline{\gamma} = \Sigma\gamma$. We have $yx = 0$ as consecutive morphisms in a distinguished triangle, and this implies that

$$\begin{aligned}
\alpha \circ \bar\beta \circ \underline{\delta} \ + \ \bar\alpha \circ \underline{\gamma} \circ (\Sigma\delta) \ &= \ (\alpha, \bar\alpha\underline{\gamma}) \circ \begin{pmatrix} \bar\beta\underline{\delta} \\ \Sigma\delta \end{pmatrix} \ = \ \bar\alpha \circ (i, \underline{\gamma}) \circ \begin{pmatrix} \bar\beta \\ p' \end{pmatrix} \circ \underline{\delta} \\
&= \ \bar\alpha \circ y \circ x \circ \underline{\delta} \ = \ 0 \ .
\end{aligned}$$

Hence the morphism

$$\alpha \circ \bar\beta \circ \underline{\delta} \ = \ -\bar\alpha \circ \underline{\gamma} \circ (\Sigma\delta)$$

is in the intersection of the sets $\alpha \circ \langle \beta, \gamma, \delta \rangle$ and $-\langle \alpha, \beta, \gamma \rangle \circ (\Sigma\delta)$. Since both sets are cosets for the same subgroup

$$\alpha \circ \mathcal{T}(\Sigma W, Y) \circ (\Sigma\delta) \quad \text{of} \quad \mathcal{T}(\Sigma V, Z) \ ,$$

they must coincide.

(iii) For any choice of distinguished triangle (2.2) the image triangle

$$FX \xrightarrow{F\beta} FY \xrightarrow{Fi} F(C(\beta)) \xrightarrow{\tau_X \circ Fp} \Sigma(FX)$$

is distinguished in $\mathcal{T}'$, so it can be used to determine the bracket $\langle F(\alpha), F(\beta), F(\gamma) \rangle$. If moreover $\bar\alpha : C(\beta) \longrightarrow Z$ and $\underline{\gamma} : \Sigma W \longrightarrow C(\beta)$ satisfy $\bar\alpha i = \alpha$ respectively $p\underline{\gamma} = \Sigma\gamma$, then $F\bar\alpha : F(C(\beta)) \longrightarrow FZ$ and $F\underline{\gamma} \circ \tau_W^{-1} : \Sigma(FW) \longrightarrow F(C(\beta))$ satisfy $F\bar\alpha \circ Fi = F\alpha$ respectively

$$(\tau_X \circ Fp) \circ (F\underline{\gamma} \circ \tau_W^{-1}) \ = \ \tau_X \circ F(p\underline{\gamma}) \circ \tau_W^{-1} \ = \ \tau_X \circ F(\Sigma\gamma) \circ \tau_W^{-1} \ = \ \Sigma(F\gamma) \circ \tau_W \circ \tau_W^{-1} \ = \ \Sigma(F\gamma)$$

by naturality of τ. So the morphism

$$F(\bar\alpha \circ \underline{\gamma}) \ = \ F\bar\alpha \circ (F\underline{\gamma} \circ \tau_W^{-1}) \circ \tau_W$$

belongs to the set $\langle F\alpha, F\beta, F\gamma \rangle \circ \tau_W$. This proves the relation $F\langle \alpha, \beta, \gamma \rangle \subseteq \langle F\alpha, F\beta, F\gamma \rangle \circ \tau_W$. $\square$

Remark 2.4. The subgroup $(\alpha \circ \mathcal{T}(\Sigma W, Y)) + (\mathcal{T}(\Sigma X, Z) \circ \Sigma\gamma)$ of the group $\mathcal{T}(\Sigma W, X)$ that comes up in part (i) of the previous proposition is called the *indeterminacy* of the Toda bracket $\langle \alpha, \beta, \gamma \rangle$. The relation in part (ii) of the previous proposition is sometimes referred to as a *juggling formula*.

In part (iii) above, the indeterminacy of the right hand side may be larger than the image of the indeterminacy of the bracket $\langle \alpha, \beta, \gamma \rangle$, which is why in general we only have containment, not necessarily equality, as subsets of $\mathcal{T}'(F(\Sigma W), F(Z))$.

Construction 2.5 (Toda brackets for ring spectra)**.** The homotopy groups of a symmetric ring spectrum have more structure than that of a graded ring, namely 'secondary' (and higher. . .) forms of multiplications, also called *Toda brackets*. Toda brackets are the homotopical analogues of Massey products in differential graded algebras, and they satisfy similar relations.

We will restrict ourselves to the simplest kind of such brackets, namely triple brackets (as opposed to four-fold, five-fold,. . .) with single entries (as opposed to 'matric' Toda brackets). We construct these Toda brackets as a special case of the Toda brackets for triangulated categories.

So we let R be a symmetric ring spectrum and M a right R-module. Evaluation at the suspended unit is an isomorphism of abelian groups

$$\mathrm{ev} \ : \ \mathcal{D}(R)(R, M)_k \ \longrightarrow \ \pi_k M \ , \quad \alpha \longmapsto (\pi_k \alpha)(\iota_k \cdot 1)$$

and this map is multiplicative in the sense that it takes composition in the derived category of R-modules to the action of $\pi_* R$ on $\pi_* M$; more precisely, the square

$$\begin{tikzcd}
\mathcal{D}(R)(R,M)_k \otimes \mathcal{D}(R)(R,R)_l \arrow[r, "\circ"] \arrow[d, "\mathrm{ev}\otimes\mathrm{ev}"'] & \mathcal{D}(R)(R,M)_{k+l} \arrow[d, "\mathrm{ev}"] \\
\pi_k M \otimes \pi_l R \arrow[r, "\cdot"'] & \pi_{k+l} M
\end{tikzcd}$$

commutes for all integers k, l [ref...]. We can use this multiplicative isomorphism to translate the Toda brackets in $\mathcal{D}(R)$ into Toda brackets in $\pi_* R$ as follows. We consider homogeneous elements $x \in \pi_k M$, $y \in \pi_l R$ and $z \in \pi_j R$ which satisfies the relations $xy = 0 = yz$. We let $\bar{x} \in \mathcal{D}(R)(\mathbb{S}^k \wedge R, M)$, $\bar{y} \in \mathcal{D}(R)(\mathbb{S}^l \wedge R, R)$ and $\bar{z} \in \mathcal{D}(R)(\mathbb{S}^j \wedge R, R)$ be the morphisms that satisfy $\mathrm{ev}(\bar{x}) = x$, $\mathrm{ev}(\bar{y}) = y$ respectively $\mathrm{ev}(\bar{z}) = z$. By the multiplicativity of the evaluation the two consecutive composites in the sequence

$$\mathbb{S}^{k+l+j} \wedge R \ \xrightarrow{\mathbb{S}^{k+l} \wedge \bar{z}} \ \mathbb{S}^{k+l} \wedge R \ \xrightarrow{\mathbb{S}^k \wedge \bar{y}} \ \mathbb{S}^k \wedge R \ \xrightarrow{\bar{x}} \ M$$

are zero. So the Toda bracket

$$\langle \bar{x}, \ \mathbb{S}^k \wedge \bar{y}, \ \mathbb{S}^{k+l} \wedge \bar{z} \rangle \ \subseteq \ \mathcal{D}(R)(\Sigma(\mathbb{S}^{k+l+j} \wedge R), \ M)$$

is defined. If we evaluate all elements in this bracket at the homotopy class $S^1 \wedge (\iota_{k+l+j} \cdot 1)$ of $\pi_{1+k+l+j}(\Sigma(\mathbb{S}^{k+l+j} \wedge R))$ we end up with a subset

$$\langle x, y, z \rangle \ \subseteq \ \pi_{1+k+l+j} M$$

which defines the Toda bracket $\langle x, y, z \rangle$.

The three parts of Proposition 2.3 then specialize to corresponding properties for the Toda brackets of symmetric ring spectra.

Proposition 2.6. *Let R be a symmetric ring spectrum and M a right R-module. Let $x \in \pi_k M$, $z \in \pi_l R$ and $z \in \pi_j R$ be homotopy classes that satisfies $xy = 0 = yz$.*

(i) *The Toda bracket $\langle x, y, z \rangle \subseteq \pi_{1+k+l+j} M$ is a coset of the subgroup $x \cdot \pi_{1+l+j} R \ + \ \pi_{1+k+l} M \cdot z$.*

(ii) *If $u \in \pi_i R$ is another homotopy class such that $z \cdot u = 0$, then the relation*

$$x \cdot \langle y, z, u \rangle \ = \ (-1)^{k+1} \cdot \langle x, y, z \rangle \cdot u \tag{2.7}$$

holds as subsets of $\pi_{1+k+l+j+i} M$.[check the sign]

(iii) *For every morphism of symmetric ring spectra* $f : R \longrightarrow S$*, every* S*-module* N *and every morphism of* R*-modules* $\varphi : M \longrightarrow f^*N$ *the relation*

$$\varphi_*(\langle x,y,z\rangle) \ \subseteq\ \langle \varphi_*(x),\, f_*(y),\, f_*(z)\rangle$$

holds as subsets of $\pi_{1+k+l+j}N$*.*

The subgroup $x\cdot\pi_{1+l+j}R \,+\, \pi_{1+k+l}M\cdot z$ is called the *indeterminacy* of the Toda bracket $\langle x,y,z\rangle$. The relation in part (ii) of the previous proposition is sometimes referred to as a *juggling formula*; one can think of the juggling formula as a kind of 'higher form of associativity'.

The indeterminacy of the right hand side may be larger than the image of the indeterminacy of the bracket $\langle x,y,z\rangle$, which is why in general we only have containment, not necessarily equality, as subsets of $\pi_{k+l+j+1}N$.

As we explain in Exercise E.IV.16, the Toda brackets of the form $\langle x,y,z\rangle$ for varying y and z contain significant information about the structure of $\pi_*(M/xR)$ as a graded π_*R-module, where M/xR is the mapping cone of a morphism of R-modules which realizes left multiplication by x in homotopy.

We have given a rather abstract and indirect definition of Toda brackets in the homotopy ring of a symmetric spectrum, via the triangulated derived category. However, in many cases, Toda brackets can be calculated more explicitly and directly from geometric representative of homotopy classes. We explain this in detail now.

Construction 2.8 (Toda brackets from geometric representatives)**.** We let R be a symmetric ring spectrum, M a right R-module and we consider based maps

$$f \ :\ S^{k+n} \ \longrightarrow\ M_n\ ,\quad g\ :\ S^{l+m}\ \longrightarrow\ R_m \qquad\text{and}\qquad h\ :\ S^{j+p}\ \longrightarrow\ R_p\ .$$

These elements represent naive homotopy classes $[f]\in\hat\pi_kM$, $[g]\in\hat\pi_lR$ respectively $[h]\in\hat\pi_jR$, and we denote by

$$\langle f\rangle\ \in\ \pi_kM\ ,\quad \langle g\rangle\ \in\ \pi_lR \qquad\text{respectively}\qquad \langle h\rangle\ \in\ \pi_jR$$

the corresponding true homotopy classes, i.e., the images under the tautological maps $c:\hat\pi_*M\longrightarrow\pi_*M$ and $c:\hat\pi_*R\longrightarrow\pi_*R$. We recall that the 'geometric product' $f\cdot g$ is the composite

$$S^{k+n+l+m}\ \xrightarrow{\ f\wedge g\ }\ M_n\wedge R_m\ \xrightarrow{\ a_{n+m}\ }\ M_{n+m}$$

where $a_{n,m}$ is the action of R on M. We assume now that the 'geometric products' $f\cdot g: S^{k+n+l+m}\longrightarrow M_{n+m}$ and $g\cdot h: S^{l+m+j+p}\longrightarrow R_{m+j}$ are nullhomotopic in the sense that there exist extensions $H: C(S^{k+n+l+m})\longrightarrow M_{n+m}$ and $\bar H: C(S^{l+m+j+p})\longrightarrow R_{m+j}$ to the respective cones. Proposition I.6.25 (iii) tells us that then

$$\langle f\rangle\cdot\langle g\rangle\ =\ (-1)^{nl}\cdot\langle f\cdot g\rangle\ =\ 0 \qquad\text{and}\qquad \langle g\rangle\cdot\langle h\rangle\ =\ (-1)^{mj}\cdot\langle g\cdot h\rangle\ =\ 0\ .$$

So the Toda bracket $\langle\langle f\rangle,\langle g\rangle,\langle h\rangle\rangle$ is defined in $\pi_{1+k+l+j}M$, and we'll give an explicit constructruction of a geometric representative.

To construct this representative we multiply the nullhomotopies H and $\bar H$ with the appropriate spheres on the left respecively right to arrive a two based maps

$$H\cdot S^{j+p}\colon C(S^{k+n+l+m})\wedge S^{j+n}\ \longrightarrow\ M_{n+m+j}\quad\text{respectively}\quad S^{k+n}\cdot\bar H\colon S^{k+n}\wedge C(S^{l+m+j+p})\ \longrightarrow\ M_{n+m+j}\ .$$

If we restrict $H\cdot S^{j+p}$ or $S^{k+n}\cdot\bar H$ to $S^{k+n+l+m+j+p}$, the associativity of the action of R on M guarantees that we get the same map in both cases, namely

$$f\cdot g\cdot h\ :\ S^{k+n+l+m+j+p}\ \longrightarrow\ M_{n+m+j}\ .$$

So the maps glue to a map

$$H\cdot S^{j+p}\cup S^{k+n}\cdot\bar H\ :\ C(S^{k+n+l+m})\wedge S^{j+n}\cup_{S^{k+n+l+m+j+p}} S^{k+n}\wedge C(S^{l+m+j+p})\ \longrightarrow\ M_{n+m+j}\ .$$

The source of this resulting map is a sphere of dimension $1+k+n+m+l+j+p$, and after making an identification with the standard sphere $S^{1+k+n+m+l+j+p}$, the combined map represents a homotopy class

$$\langle (H\cdot S^{j+p})\cup(S^{k+n}\cdot\bar H)\rangle\ \in\ \pi_{1+k+l+j}M\ .$$

Proposition 2.9. *Let H be a nullhomotopy of $f \cdot g$ and $\bar{H}$ a nullhomotopy of $g \cdot h$. Then the homotopy class*

$$\pm \langle (H \cdot S^{j+p}) \cup (S^{k+n} \cdot \bar{H}) \rangle$$

is in the Toda bracket $\langle \langle f \rangle, \langle g \rangle, \langle h \rangle \rangle$.

Let us now turn to some calculations with Toda brackets. We start with a general relation, due to Toda.

Proposition 2.10. *Let R be a symmetric ring spectrum and x a homotopy class of R such that $2 \cdot x = 0$. Then the relation*

$$\eta \cdot x \ \in \ \langle 2, x, 2 \rangle \tag{2.11}$$

holds, where $\eta \in \pi_1 R$ is the Hurewicz image of the Hopf class $\eta \in \pi_1 \mathbb{S}$.

Proof. We choose a distinguished triangle in the stable homotopy category

$$\mathbb{S} \xrightarrow{\cdot 2} \mathbb{S} \xrightarrow{j} \mathbb{S}/2 \xrightarrow{\delta} \Sigma\mathbb{S}$$

that defines the mod-2 Moore spectrum $\mathbb{S}/2$. The derived extension of scalars functor $R \wedge^L - : \mathcal{SHC} \longrightarrow \mathcal{D}(R)$ takes this to a distinguished triangle

$$\mathbb{S}^k \wedge R \xrightarrow{\cdot 2} \mathbb{S}^k \wedge R \xrightarrow{j \wedge R} \mathbb{S}^k \wedge \mathbb{S}/2 \wedge R \xrightarrow{\delta \wedge R} \Sigma(\mathbb{S}^k \wedge R) \ .$$

in the derived category of R-modules. We let $\bar{x} : \mathbb{S}^k \wedge R \longrightarrow R$ be the morphism in $\mathcal{D}(R)(R, R)_k$ that satisfies $\mathrm{ev}(\bar{x}) = x$ in $\pi_k R$. Since $2 \cdot x = 0$ the morphism $\bar{x}$ admits an extension $y : \mathbb{S}^k \wedge \mathbb{S}/2 \wedge R \longrightarrow R$ such that $y(j \wedge R) = \bar{x}$. Using the factorization (6.50) of $2 \cdot \mathrm{Id}_{\mathbb{S}/2}$ as $j\eta\delta$ we obtain

$$2 \cdot y \ = \ y \circ ((j\eta\delta) \wedge R) \ = \ y \circ (j \wedge R)) \circ (\eta \wedge R) \circ (\delta \wedge R) \ = \ \bar{x} \circ (\eta \wedge R) \circ (\delta \wedge R) \ .$$

By Proposition 2.3 (iv) the morphism $\bar{x} \circ (\eta \wedge R) : \Sigma(\mathbb{S}^k \wedge R) \longrightarrow R$ is thus contained in the Toda bracket

$$\langle 2, \bar{x}, 2 \rangle \ \subseteq \ \mathcal{D}(R)(\Sigma(\mathbb{S}^k \wedge R), R) \ .$$

If we evaluate on the homotopy class $\Sigma(\iota_k \cdot 1) \in \pi_{1+k} R$ this yields

$$\eta \cdot x \ = \ (\bar{x} \circ (\eta \wedge R))_*(\Sigma(\iota_k \cdot 1)) \in \ \langle 2, x, 2 \rangle \ . \qquad \square$$

The previous proposition is in fact a special case of a relation between Toda brackets and power operations in the homotopy ring of a commutative symmetric ring spectrum R. Indeed, for a homotopy class $x \in \pi_k R$ of even dimension there is a certain class $\mathrm{Sq}_1(x) \in \pi_{2k+1} R$ called the '$\cup_1$-oconstruction' such that for every homotopy class $y \in \pi_l R$ that satisfies $x \cdot y = 0$ the relation

$$\mathrm{Sq}_1(x) \cdot y \ \in \ \langle x, y, x \rangle$$

holds. Moreover, the $\cup_1$-operation is natural for homomorphism between commutative symmetric ring spectra and we have $\mathrm{Sq}_1(2) = \eta$ in $\pi_1 \mathbb{S}$, which gives back Proposition 2.10.

Example 2.12. Here are some examples of non-trivial Toda brackets. In the stable stems, i.e., the homotopy groups of the sphere spectrum (compare the table in Example 1.11) we have

$$\begin{array}{rll rll}
\eta^2 & \in \langle 2, \eta, 2 \rangle & \mod (0) & 6\nu & \in \langle \eta, 2, \eta \rangle & \mod (12\nu) \\
\nu^2 & \in \langle \eta, \nu, \eta \rangle & \mod (0) & 40\sigma & \in \langle \nu, 24, \nu \rangle & \mod (0) \\
\eta\sigma + \epsilon & \in \langle \nu, \eta, \nu \rangle & \mod (0) & \epsilon & \in \langle \eta, \nu, 2\nu \rangle & \mod (\eta\sigma)
\end{array}$$

The first bracket is an instance of Toda's relation (2.11). The next two brackets are special instances of the relation

$$3\nu \cdot x \ \in \ \langle \eta, x, \eta \rangle$$

that holds for every homotopy classes x in a symmetric ring spectrum R that satisfies $\eta \cdot x = 0$. One proof of this relation proceeds along the same lines as the proof of Proposition 2.10, but instead of the relation $2 \cdot \mathrm{Id}_{\mathbb{S}/2} = j\eta\delta$ one uses the factorization

$$\eta \cdot \mathrm{Id}_{C(\eta)} \ = \ 3 \cdot j\nu\delta$$

where

$$\mathbb{S}^1 \xrightarrow{\eta} \mathbb{S} \xrightarrow{j} C(\eta) \xrightarrow{\delta} \Sigma\mathbb{S}^1$$

is a distinguished triangle defining a cone of η. One should beware though that this patter does *not* continue: multiplication by ν on the mapping cone of ν does *not* factor through the Hopf map σ.

With the help of juggling formula (2.7), we can use Toda brackets to deduce relations which themselves don't refer to brackets. For example,

$$\eta \cdot \langle 2, \eta, 2\rangle \;=\; \langle \eta, 2, \eta\rangle \cdot 2$$

holds in π_3^{s} with zero indeterminacy (because $2\eta = 0$). By Toda's relation (2.11), the left hand side contains η^3 while by the table above, the right hand side contains 12ν. So we get the multiplicative relation

$$\eta^3 \;=\; 12\nu$$

as a consequence of the Toda brackets involving 2 and η. Since $2\eta = 0$, multiplying this relation by 2 respectively η yields the relations

$$24 \cdot \nu \;=\; 0 \qquad \text{and} \qquad \eta^4 \;=\; 0 \; .$$

Another example of a non-trivial Toda bracket is $u \in \langle 2, \eta, 1\rangle$ (modulo $2u$) in $\pi_2(ku)$. Here the complex topological K-theory spectrum ku (compare Example I.**??**) is viewed just as a symmetric spectrum (and not as a ring spectrum). Here η and 2 have to be viewed as elements in the stable stems, while the unit 1 and the Bott class u are homotopy classes of ku.

Since the relation $\eta^2 \in \langle 2, \eta, 2\rangle$ holds in π_2^{s}, it also holds in the second homotopy group of every ring spectrum (with possibly bigger indeterminacy, and possibly with $\eta^2 = 0$); In the homotopy of real topological K-theory ko the class η^2 is non-zero (compare the table in Example I.1.20), so we get a non-trivial bracket $\eta^2 \in \langle 2, \eta, 2\rangle$ (modulo 0) in $\pi_2(ko)$. We also have $\xi \in \langle 2, \eta, \eta^2\rangle$ (modulo 2ξ) in $\pi_4(ko)$ and $2\beta \in \langle \xi, \eta, \eta^2\rangle$ (modulo 4β) in $\pi_8(ko)$. [prove. Also $\eta^2\beta = \mathrm{Sq}_1(\xi) \in \langle \xi, \eta, \xi\rangle$ (modulo 0) in $\pi_{10}(ko)$]

In Example 6.39 and Remark 6.40 of Chapter I we explained how we can 'kill' the action of a homotopy class $x \in \pi_l R$ on an R-module M as long as x acts injectively on $\pi_* M$. With the help of Toda brackets one can show that in general not all graded modules over $\pi_* R$ are realized by R-module spectra.

Proposition 2.13. *Let R be a symmetric ring spectrum and $x \in \pi_k R$ a homotopy class. If there is a right R-module whose homotopy groups are isomorphic to $\pi_* R/(x \cdot \pi_{*-k} R)$ as a graded right $\pi_* R$-module, then for all homogeneous homotopy classes y, z with $xy = 0 = yz$ the Toda bracket $\langle x, y, z\rangle$ contains 0.*

Proof. We choose a right R-module N and a π_*-linear epimorphism $p : \pi_* R \longrightarrow \pi_* N$ with kernel $x \cdot (\pi_{*-k} R)$. The bracket $\langle p(1), x, y\rangle$ is then defined with indeterminacy $p(1) \cdot \pi_{k+l+1} R + \pi_{k+1} N \cdot y$. Since $p(1)$ generates $\pi_* N$ as a graded $\pi_* R$-module, the indeterminacy of the bracket $\langle p(1), x, y\rangle$ is the entire group $\pi_{k+l+1} N$, and so we have $\langle p(1), x, y\rangle = \pi_{k+l+1} N$.

We choose an element $a \in \pi_{k+l+j+1} R$ of the bracket $\langle x, y, z\rangle$. Using the juggling formula (2.7) we get

$$p(a) \;=\; p(1) \cdot a \;\in\; p(1) \cdot \langle x, y, z\rangle \;=\; -\langle p(1), x, y\rangle \cdot z \;=\; \pi_{l+1} N \cdot z \;=\; p(\pi_{l+1} R \cdot z) \; .$$

Since the kernel of p equals $x \cdot \pi_{*-k} R$ we deduce that a lies in $x \cdot \pi_{l+j+1} R + \pi_{k+l+1} R \cdot z$. This is precisely the indetermincy of the bracket $\langle x, y, z\rangle$, so the bracket also contains 0. □

Example 2.14. Here are some non-realizability results which can be obtained with the help of Proposition 2.13. We have the relation $\eta^2 \in \langle 2, \eta, 2\rangle$ in π_2^{s} with zero indeterminacy. So the bracket $\langle 2, \eta, 2\rangle$ does not contain zero and hence there is no symmetric spectrum whose homotopy groups realize $\mathbb{Z}/2 \otimes \pi_*^{\mathrm{s}}$ as a graded π_*^{s}-module. For an odd prime p, the bracket $\langle p, \alpha_1, \alpha_1\rangle$ in π_{4p-5}^{s} does not contain zero, hence $\mathbb{Z}/p \otimes \pi_*^{\mathrm{s}}$ is not realizable as the homotopy of a spectrum.

In Example 2.12 we also exhibited various nonzero triple Toda brackets in the homotopy of the real topological K-theory spectrum ko, such as

$$\eta^2 \in \langle 2, \eta, 2\rangle \; , \quad \xi \in \langle 2, \eta, \eta^2\rangle \quad \text{and} \quad 2\beta \in \langle \xi, \eta, \eta^2\rangle \; .$$

Moreover, in all three cases, the class on the left is not in the indeterminacy group, so the three brackets do not contain zero. Proposition 2.13 lets us conclude that the $\pi_*(ko)$-modules

$$\mathbb{Z}/2 \otimes \pi_*(ko) \ , \quad \pi_*(ko)/(\eta^2 \cdot \pi_*(ko)) \quad \text{and} \quad \pi_*(ko)/(\xi \cdot \pi_*(ko))$$

are not realizable as the homotopy of any ko-module spectrum. In Exercise E.I.42 we show more generally that the only cyclic π_*KO-modules which are realizable as the homotopy of a KO-module spectrum are the free module, the trivial module and $\mathbb{Z}/n \otimes \pi_*KO$ for n an odd integer. [work for connective also?]

[show that any ko-morphism $ko/\eta \longrightarrow ku$ which extends the complexification map $ko \longrightarrow ku$ is a stable equivalence]

Exercises

Exercise E.IV.1 (Modules as continuous functors)**.** There is way to interpret modules over a symmetric ring spectrum R as continuous functors on a based topological (or simplicial) category $\mathbf{\Sigma}_R$ which generalizes the isomorphism between the categories of symmetric spectra and continuous functors $\mathbf{\Sigma} \longrightarrow \mathbf{T}$.

Given a symmetric ring spectrum of topological spaces R we define a based topological category $\mathbf{\Sigma}_R$ as follows. The objects of $\mathbf{\Sigma}_R$ are the natural numbers $0, 1, 2, \dots$ and the based space of morphisms from n to m is given by $\mathbf{\Sigma}_R(n,m) = \Sigma_m^+ \wedge_{1\times\Sigma_{m-n}} R_{m-n}$, which is to be interpreted as a one-point space if $m < n$. Composition is defined by $\circ : \mathbf{\Sigma}_R(m,k) \wedge \mathbf{\Sigma}_R(n,m) \longrightarrow \mathbf{\Sigma}_R(n,k)$ is defined by

$$[\tau \wedge z] \circ [\gamma \wedge y] \ = \ [\tau(\gamma \times 1) \wedge \mu_{n,m}(y \wedge z)]$$

where $\tau \in \Sigma_k$, $\gamma \in \Sigma_m$, $z \in R_{k-m}$ and $y \in R_{m-n}$. The identity in $\mathbf{\Sigma}(n,n) = \Sigma_n^+ \wedge R_0$ is the identity of Σ_n smashed with the identity element of R_0.

Show that $\mathbf{\Sigma}_R$ is a category and construct an isomorphism between the category of R-modules and the category of based continuous functors from $\mathbf{\Sigma}_R$ to the category $\mathbf{T}$ of based compactly generated spaces. [Analog for simplicial sets]

Exercise E.IV.2. Let R by a symmetric ring spectrum. We define an R-bimodule $\bar{R}$ by

$$\bar{R}_n \ = \ \begin{cases} * & \text{for } n = 0 \\ R_n & \text{for } n \geq 1. \end{cases}$$

We define the *n-latching object* $L_n^R M$ of a right R-module M by $L_n^R M = (M \wedge_R \bar{R})_n$. [use previous exercise for latching objects] The latching object has a left action of the symmetric group Σ_n and a right action of the pointed monoid R_0. The inclusion $\bar{R} \longrightarrow R$ is a morphism of R-bimodules and thus induces a morphism of Σ_n-R_0 simplicial bisets

$$\nu_n \ : \ L_n^R M = (M \wedge_R \bar{R})_n \ \longrightarrow \ (M \wedge_R R)_n \cong M_n \ .$$

Show:

(i) A morphism $f : M \longrightarrow N$ is a flat cofibration of R-modules if and only if the maps $\nu_n(f) : L_n^R N \cup_{L_n^R M} M_n \longrightarrow N_n$ are cofibrations of right R_0-simplicial sets.

(ii) A morphism $f : M \longrightarrow N$ is a projective cofibration of R-modules if and only if the maps $\nu_n(f) : L_n^R N \cup_{L_n^R M} M_n \longrightarrow N_n$ are cofibrations of Σ_n-R_0-simplicial bisets.

(Hint: define a suitable R-module analog of the filtration $F^m A$ of a symmetric spectrum A so that the proof of Proposition II.5.47 can be adapted.)

Exercise E.IV.3. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between cofibration categories is *exact* if it preserves weak equivalences, cofibrations, initial objects and pushouts along cofibrations.

Let $\mathcal{C}$ and $\mathcal{D}$ be pointed cofibration categories and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an exact functor. Construct a natural isomorphism $\tau : \operatorname{Ho} F \circ \Sigma_{\operatorname{Ho}(\mathcal{C})} \longrightarrow \Sigma_{\operatorname{Ho}(\mathcal{D})} \circ \operatorname{Ho} F$ of functors $\operatorname{Ho}(\mathcal{C}) \longrightarrow \operatorname{Ho}(\mathcal{D})$ such that for every distinguished triangle (f, g, h) in $\operatorname{Ho}(\mathcal{C})$ the triangle

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \xrightarrow{\tau_C \circ Fh} \Sigma FA$$

is distinguished. In particular, if $\mathcal{C}$ and $\mathcal{D}$ are stable cofibration categories, then the induced functor $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ is an exact functor of triangulated categories.

Exercise E.IV.4 (Approximation theorem)**.** An exact functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between cofibration categories, has the *approximation property* if it satisfies the following two conditions:

(App1) A morphism α in $\mathcal{C}$ is a weak equivalence if and only if its image $F\alpha$ is a weak equivalence in $\mathcal{D}$.
(App2) For every object A of $\mathcal{C}$, every fibrant object Z of $\mathcal{D}$ and every morphism $\varphi : FA \longrightarrow Z$ in $\mathcal{D}$ there exists a morphism $\alpha : A \longrightarrow A'$ in $\mathcal{C}$ and a weak equivalence $\psi : FA' \longrightarrow Z$ in $\mathcal{D}$ such that $\varphi' \circ F\alpha = \varphi$, i.e., the following triangle commutes:

$$\begin{tikzcd} FA \arrow[d, "F\alpha"'] \arrow[dr, "\varphi"] & \\ FA' \arrow[r, "\psi"'] & Z \end{tikzcd}$$

This approximation property was introduced by Waldhausen in his foundational work on algebraic K-theory [**88**, Sec. 1.6] (with the minor difference that Waldhausen does not consider fibrant objects); in that context the approximation property serves as a sufficient condition for an exact functor to induce an equivalence of K-theory spaces.

(i) Show that for every exact functor F with the approximation property the induced functor $\mathrm{Ho}\, F : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ is an equivalence of homotopy categories.
(ii) Let $L : \mathcal{C} \longrightarrow \mathcal{D}$ be a left Quillen functor between model categories. Then the restriction $L^c : \mathcal{C}^c \longrightarrow \mathcal{D}^c$ of L to the full subcategories of cofibrant objects is an exact functor of cofibration categories. Show that if L is a left Quillen equivalence, then L^c satisfies the approximation property.

Exercise E.IV.5. For a cofibration category $\mathcal{C}$ we let $\mathrm{cof}\,\mathcal{C}$ be the category of cofibrations in $\mathcal{C}$: the objects of $\mathrm{cof}\,\mathcal{C}$ are the cofibrations in $\mathcal{C}$ and a morphism from a cofibration $i : A \longrightarrow B$ to a cofibration $i' : A' \longrightarrow B'$ is a pair $(\alpha : A \longrightarrow A', \beta : B \longrightarrow B')$ of morphisms such that $\beta i = i'\alpha$. A morphism (α, β) is a weak equivalence in $\mathrm{cof}\,\mathcal{C}$ if α and β are weak equivalences in $\mathcal{C}$, and $(\alpha.\beta)$ is a cofibration in $\mathrm{cof}\,\mathcal{C}$ if α and $i' \cup \beta : A' \cup_A B \longrightarrow B'$ are cofibrations in $\mathcal{C}$. Since $\beta = (i' \cup \beta) \circ (i_*\alpha)$ the morphism β is then also a cofibration in $\mathcal{C}$. Show that these definitions make $\mathrm{cof}\,\mathcal{C}$ into a cofibration category.

Exercise E.IV.6. Let $\mathcal{C}$ be a pointed cofibration category. We denote by $\mathrm{Cone}(\mathcal{C})$, the *category of cones* in $\mathcal{C}$, the full subcategory of $\mathrm{cof}\,\mathcal{C}$ spanned by those cofibrations $i : A \longrightarrow C$ whose target C is weakly contractible.

(i) Show that $\mathrm{Cone}(\mathcal{C})$ is a cofibration category by restriction of the cofibrations structure of Exercise E.IV.5 on the category $\mathrm{cof}\,\mathcal{C}$.
(ii) Show that the forgetful functor
$$U \ : \ \mathrm{Cone}(\mathcal{C}) \ \longrightarrow \ \mathcal{C} \ , \quad U(i : A \longrightarrow C) = A$$
is exact and has the approximation property. Conclude that this functor induces an equivalence of homotopy categories $\mathrm{Ho}(\mathrm{Cone}(\mathcal{C})) \cong \mathrm{Ho}(\mathcal{C})$.
(iii) Let $F : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathrm{Cone}(\mathcal{C}))$ be a quasi-inverse to the equivalence categories from (ii). Show that the composite
$$\mathrm{Ho}(\mathcal{C}) \ \xrightarrow[\cong]{F} \ \mathrm{Ho}(\mathrm{Cone}(\mathcal{C})) \ \xrightarrow{(i:A \longrightarrow C)\mapsto C/A} \ \mathrm{Ho}(\mathcal{C})$$
is naturally isomorphic to the suspension functor arsing from any cone functor. Conclude that $\mathcal{C}$ is stable if and only if the functor $\mathrm{Cone}(\mathcal{C}) \longrightarrow \mathcal{C}$ that sends $(i : A \longrightarrow C)$ to the quotient C/A induces an equivalence of homotopy categories from $\mathrm{Ho}(\mathrm{Cone}(\mathcal{C}))$ to $\mathrm{Ho}(\mathcal{C})$.

Exercise E.IV.7. Let $\mathcal{C}$ be a stable cofibration category and let $\mathcal{T}$ be a full triangulated subcategory of the homotopy category $\mathrm{Ho}(\mathcal{C})$, i.e., $\mathcal{T}$ is non-empty, closed under isomorphisms, two-out-of-three and distinguished triangles. Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ spanned by those objects that belong to $\mathcal{T}$. Show that the restricted notions of cofibrations and weak equivalence make $\mathcal{C}'$ into a stable cofibration category and that the inclusion $\mathcal{C}' \longrightarrow \mathcal{C}$ induces a fully faithful functor $\mathrm{Ho}(\mathcal{C}') \longrightarrow \mathrm{Ho}(\mathcal{C})$ whose image equals $\mathcal{T}$.

Exercise E.IV.8. Let $\mathcal{C}$ be a cofibration category and $\tau : F \longrightarrow G$ a natural transformation between functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ which takes weak equivalences to isomorphisms. Show that there exists a unique natural transformation $\bar{\tau} : \bar{F} \longrightarrow \bar{G}$ between the induced functors $\bar{F}, \bar{G} : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathcal{D}$ such that $\bar{\tau}\gamma = \tau$. Show that $\bar{\tau}$ is an isomorphism whenever τ is.

Exercise E.IV.9. Let $\mathcal{C}$ be a cofibration category and let $\varphi : A \longrightarrow B$ be any $\mathcal{C}$-morphism. Show that $\gamma(\varphi)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$ if and only if there are cofibrations $f : B \longrightarrow B'$ and $f' : B' \longrightarrow B''$ such that $f'\varphi$ and $f''f'$ are weak equivalences.

Exercise E.IV.10 (Additive categories with translation)**.** Let $\mathcal{A}$ be additive category equipped with an additive endofunctor $T : \mathcal{A} \longrightarrow \mathcal{A}$. A *differential object* in $(\mathcal{A}, T)$ is a pair (X, d) consisting of an object X of $\mathcal{A}$ and a morphism $d : X \longrightarrow TX$, the *differential*. (Beware that there is no condition on the composite $(Td)d : X \longrightarrow TTX$, i.e., we are *not* asking for a 'complex'). A morphism $f : (X, d) \longrightarrow (X', d')$ of differential objects is an $\mathcal{A}$-morphism $f : X \longrightarrow X'$ satisfying $d'f = (Tf)d : X \longrightarrow TX'$. Two morphisms $f, g : (X, d) \longrightarrow (X', d')$ are *homotopic* if there exists an $\mathcal{A}$-morphism $s : TX \longrightarrow X'$ (the *homotopy*) such that $d's + (Ts)(Td) = Tf - Tg$ as morphisms $TX \longrightarrow TX'$. A morphism $f : (X, d) \longrightarrow (X', d')$ of differential objects is a *homotopy equivalence* if there is a morphism $g : (X', d') \longrightarrow (X, d)$ of differential objects such that fg and gf are homotopic to the respective identity maps. A morphism f of differential objects is a *cofibration* if the underlying map in $\mathcal{A}$ is a split monomorphism, i.e., if there is an $\mathcal{A}$-morphism $g : C \longrightarrow X'$ such that $f + g : X \oplus C \longrightarrow X'$ is an isomorphism.

(1) Show that the cofibrations and homotopy equivalences make the category of differential objects in $(\mathcal{A}, T)$ into a cofibration category in which every object is fibrant.
(2) Show that the notion of 'homotopy' is an additive equivalence relation compatible with composition. Show that the homotopy category of the cofibration structure in (i) is the category $\mathbf{K}(\mathcal{A}, T)$ whose objects are differential objects in $(\mathcal{A}, T)$ and whose morphisms are homotopy classes of morphisms.
(3) The *shift* of a differential object is given by $(X, d)[1] = (TX, -Td)$ on objects and by $f[1] = Tf$ on morphisms. Show that the shift functor on the category of differential object passes to a shift functor on the homotopy category $\mathbf{K}(\mathcal{A}, T)$. Show that for a suitably chosen cone functor, this induced shift functor on $\mathbf{K}(\mathcal{A}, T)$ is the suspension functor of the cofibration structure from (i).
(4) The *mapping cone* of a morphism $f : (X, d) \longrightarrow (X', d')$ of differential objects is the object
$$Cf \;=\; \left(X \oplus TX',\; \begin{pmatrix} d & f \\ 0 & -Td' \end{pmatrix}\right) .$$
Show that the image of the sequence
$$(X, d) \xrightarrow{\;f\;} (X', d') \xrightarrow{(1,0)} Cf \xrightarrow{\binom{0}{1}} (X, d)[1] \tag{E.IV.11}$$
is a distinguished triangle in the homotopy category $\mathbf{K}(\mathcal{A}, T)$.
(5) Suppose now that the translation functor T is an auto-equivalence. Show that the cofibration structure in (i) is then stable. Conclude that the homotopy category $\mathbf{K}(\mathcal{A}, T)$ is a triangulated category. Show that a triangle in $\mathbf{K}(\mathcal{A}, T)$ is distinguished if and only if it is isomorphic to the image of a diagram of the form (E.IV.11) for some morphism $f : (X, d) \longrightarrow (X', d')$ of differential objects.

Exercise E.IV.12 (Cochain complexes in additive categories)**.** For any additive category $\mathcal{A}$ we denote by $C(\mathcal{A})$ the category of chain complex in $\mathcal{A}$; the homotopy category $\mathbf{K}(C(\mathcal{A}))$ of chain complexes in $\mathcal{A}$ has as objects the chain complexes and as morphisms the chain homotopy classes of chain morphisms. Define a structure of stable cofibration category on $C(\mathcal{A})$ in which the weak equivalences are the chain homotopy equivalences. Show that the homotopy category $\mathbf{K}(C(\mathcal{A}))$ has a triangulated structure with suspension functor giving by the shift of a complex. (Hint: reduce to Exercise E.IV.10)

Exercise E.IV.13 (Modules over Frobenius rings)**.** A *(right) Frobenius ring* is a ring R such that the class of projective right R-modules coincides with the class of injective right R-modules. In this exercise, all R-modules are right R-modules.

Two morphisms of R-modules $f, g : M \longrightarrow N$ are *homotopic* if the difference $f - g$ factors through a projective R-module. A morphism $f : M \longrightarrow N$ of R-modules is a *stable equivalence* if there is a morphism $g : N \longrightarrow M$ such that fg and gf are homotopic to the respective identity maps.

(i) Show that the monomorphisms and stable equivalences make the category of right R-modules into a stable cofibration category in which every object is fibrant.

(ii) Show that the notion of 'homotopy' is an additive equivalence relation compatible with composition. Let $\mathbf{S}(\text{Mod-}R)$ denote the *stable category* whose objects are the right R-modules and whose morphisms are homotopy classes of R-linear maps. Show that $\mathbf{S}(\text{Mod-}R)$ is a homotopy category of the cofibration structure in (i). Conclude that the stable category $\mathbf{S}(\text{Mod-}R)$ is a triangulated category.

(iii) Let M be an R-module and $i : M \longrightarrow I$ an injective hull, i.e., a monomorphism with injective target. Show that the quotient I/M can be taken as the suspension of M in $\mathbf{S}(\text{Mod-}R)$.

(iv) Let
$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} Q \longrightarrow 0$$
be a short exact sequence of R-modules and let $i : M \longrightarrow I$ be an injective hull. We choose an extension $j : N \longrightarrow I$, i.e., a homomorphism such that $j \circ f = i$. Then we define a map $\delta : Q \longrightarrow I/M$ by
$$\delta(q) \;=\; j(\bar{q}) \;+\; i(M) \;,$$
where $\bar{q} \in N$ satisfies $g(\bar{q}) = q$. Show that δ is well-defined and R-linear. Show that the image of the morphism δ in $\mathbf{S}(\text{Mod-}R)$ is independent of the extension j. Show that the triangle
$$M \xrightarrow{f} N \xrightarrow{g} Q \xrightarrow{\delta} I/M = \Sigma M$$
in the stable category $\mathbf{S}(\text{Mod-}R)$ is distinguished. Show that a triangle in $\mathbf{S}(\text{Mod-}R)$ is distinguished if and only if is isomorphic to a triangle arising in this way from a short exact sequence of R-modules.

Exercise E.IV.14. [adapt to new defn of homotopy category; check if works] Let R be a symmetric ring spectrum of simplicial sets. An R-module M is *strongly injective* is it has the extension property for all homomorphisms of R-modules which are levelwise injective and a weak equivalence of underlying simplicial sets. We define the *derived category* $\mathcal{D}(R)$ of the ring spectrum R as the homotopy category of those strongly injective R-modules whose underlying symmetric spectra are Ω-spectra.

(i) Suppose that R is flat as a symmetric spectrum. Show that then the underlying symmetric spectrum of a strongly injective R-modules is injective. Give an example showing that the converse is not true.

(ii) Show that the derived category $\mathcal{D}(R)$ has the structure of a triangulated category with shift and distinguished triangles defined after forgetting the R-action.

(iii) Show that $\mathcal{D}(R)$ is the target of a universal functor from R-modules which takes stable equivalences to isomorphisms.

(iv) Let $f : R \longrightarrow S$ be a homomorphism of symmetric ring spectra which makes R a flat right S-module. Show that restriction of scalars from S-modules to R-modules passes to an exact functor of triangulated categories $f^* : \mathcal{D}(S) \longrightarrow \mathcal{D}(R)$.

(v) Suppose that the underlying symmetric spectrum of R is semistable. Show that then R, considered as a module over itself, has a strongly injective Ω-spectrum replacement γR as an R-module. Show that the map
$$[\gamma R, \gamma R]_k^{\mathcal{D}(R)} \;\cong\; \pi_k(\gamma R) \cong \pi_k R$$
is an isomorphism of graded rings, where the first map is evaluation at the unit $1 \in \pi_0(\gamma R) \cong \pi_k(\gamma R)[k]$. Show that the map
$$[\gamma R, M]_k^{\mathcal{D}(R)} \;\cong\; \pi_k M$$
is an isomorphism of graded modules over $\pi_* R$ for every strongly injective Ω-R-module M. Show that γR is a compact weak generator of the triangulated category $\mathcal{D}(R)$.

We shall see later that for $R = HA$ the Eilenberg-Mac Lane ring spectrum associated to a ring A (compare Example I.1.14) the derived category $\mathcal{D}(HA)$ is triangle equivalent to the unbounded derived category of the ring A. In fact, the equivalence of triangulated categories will come out as a corollary of a Quillen equivalence of model categories.

Exercise E.IV.15. For commutative symmetric ring spectrum, get a cocycle on $\mathrm{Pic}(R)$, the group of isomorphism classes of invertible R-modules. Should be related to the first multiplicative k-invariant.

Exercise E.IV.16. Let R be a symmetric ring spectrum, M a left R-module and $x \in \pi_k M$ a homogeneous homotopy class in the image of $c : \hat{\pi}_* M \longrightarrow \pi_* M$. Denote by M/xR the mapping cone of a R-homomorphism $\lambda_x : F_n S^{k+n} \wedge R \longrightarrow M$ which takes the unit class to x. The morphism λ_x should be thought of as 'left multiplication by x' (and indeed, this is its effect in homotopy). Then the Toda brackets of the form $\langle x, y, z\rangle$ for varying y and z contain significant information about the structure of $\pi_*(M/xR)$ as a graded $\pi_* R$-module.

(i) Show that there is a short exact sequence of graded right $\pi_* R$-modules

$$0 \longrightarrow \pi_* M / x \cdot \pi_{*-k} R \xrightarrow{j} \pi_*(M/xR) \xrightarrow{\delta} \mathrm{ann}_{*-k-1}(x) \longrightarrow 0$$

where $\mathrm{ann}_*(x) = \{y \in \pi_* R \mid x \cdot y = 0\}$ is the annihilator (right) ideal of x.

(ii) Let $y \in \pi_l R$ and $z \in \pi_j R$ be homogeneous classes which satisfy $xy = 0 = yz$. Show that the Toda bracket $\langle x, y, z\rangle$ equals the set of all elements α of $\pi_{k+l+j+1} M$ for which there exists $\tilde{y} \in \pi_{k+l+1}$ with

$$\delta(\tilde{y}) = y \quad \text{and} \quad j(\alpha + x \cdot \pi_{*-k} R) = \tilde{y} \cdot z \ .$$

History and credits

Shipley [**77**] calls the flat model structure for modules over a symmetric ring spectrum R the 'R-model structure'.

APPENDIX A

Miscellaneous tools

In more detail, a *triple* is a pointed functor $T : \mathrm{set}_* \longrightarrow \mathrm{set}_*$ together with natural transformations $m : T \circ T \longrightarrow T$ and $\eta : \mathrm{Id} \longrightarrow T$ such that the following diagrams commute

$$\begin{tikzcd} T \circ T \circ T \arrow[r, "\mathrm{Id} \circ m"] \arrow[d, "m \circ \mathrm{Id}"'] & T \circ T \arrow[d, "m"] \\ T \circ T \arrow[r, "m"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\mathrm{Id} \circ \eta"] \arrow[dr, "="'] & T \circ T \arrow[d, "m"] & T \arrow[l, "\eta \circ \mathrm{Id}"'] \arrow[dl, "="] \\ & T & \end{tikzcd}$$

An *algebra* over a triple T is a pointed set A together with a pointed map $\alpha : TA \longrightarrow A$ such that $\alpha \circ \eta_A$ is the identity and the following associativity diagram commutes:

$$\begin{tikzcd} T(TA) \arrow[r, "T(\alpha)"] \arrow[d, "m_A"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd}$$

For every pointed set K, the pointed set TK is a T-algebra with structure map $m_K : T(TK) \longrightarrow TK$; the T-algebra (TK, m_K) will be denoted $f^T K$ and it is the *free T-algebra* generated by K in the sense that the free functor $f^T : \mathrm{set}_* \longrightarrow T\text{-alg}$ is left adjoint to the forgetful functor f_T from T-algebras to pointed sets. Note that we have $T = f_T \circ f^T$ and that the structure maps of T-algebras provides a natural transformation $\alpha : f^T \circ f_T \longrightarrow \mathrm{Id}$.

1. Model category theory

The main references for model categories are Quillen's original book [**62**], the modern introduction by Dwyer and Spalinski [**23**] and Hovey's monograph [**35**].

Definition 1.1. Let $i : A \longrightarrow B$ and $g : X \longrightarrow Y$ be morphisms in some category $\mathcal{C}$. We say that i has the *left lifting property* for g (or g has the *right lifting property* for i, or the pair (i, g) has the *lifting property*) if for every commutative square (solid arrows only)

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists a *lifting*, i.e., a morphism $B \longrightarrow X$ (dotted arrow) which makes both resulting triangles commute.

1.1. Cofibrantly generated model categories and a lifting theorem.

In this section we review cofibrantly generated model categories and a general method for creating model category structures. If a model category is cofibrantly generated, its model category structure is completely determined by a set of cofibrations and a set of acyclic cofibrations. The transfinite version of Quillen's small object argument allows functorial factorization of maps as cofibrations followed by acyclic fibrations and as acyclic cofibrations followed by fibrations. Most of the model categories in the literature are cofibrantly generated, e.g. topological spaces and simplicial sets, as are all model structures involving symmetric spectra which we discuss in this book.

The only complicated part of the definition of a cofibrantly generated model category is formulating the definition of relative smallness. For this we need to consider the following set theoretic concepts. The reader might keep in mind the example of a compact topological space which is $\aleph_0$-small relative to closed embeddings.

Ordinals and cardinals. An *ordinal* γ is an ordered isomorphism class of well ordered sets; it can be identified with the well ordered set of all preceding ordinals. For an ordinal γ, the same symbol will denote the associated poset category. The latter has an initial object $\emptyset$, the empty ordinal. An ordinal κ is a *cardinal* if its cardinality is larger than that of any preceding ordinal. A cardinal κ is called *regular* if for every set of sets $\{X_j\}_{j\in J}$ indexed by a set J of cardinality less than κ such that the cardinality of each X_j is less than that of κ, then the cardinality of the union $\bigcup_J X_j$ is also less than that of κ. The successor cardinal (the smallest cardinal of larger cardinality) of every cardinal is regular.

Transfinite composition. Let $\mathcal{C}$ be a cocomplete category and γ a well ordered set which we identify with its poset category. A functor $V:\gamma \longrightarrow \mathcal{C}$ is called a γ-sequence if for every limit ordinal $\beta < \gamma$ the natural map $\mathrm{colim} V|_\beta \longrightarrow V(\beta)$ is an isomorphism. The map $V(\emptyset) \longrightarrow \mathrm{colim}_\gamma V$ is called the transfinite composition of the maps of V. A subcategory $\mathcal{C}_1 \subset \mathcal{C}$ is said to be closed under transfinite composition if for every ordinal γ and every γ-sequence $V:\gamma \longrightarrow \mathcal{C}$ with the map $V(\alpha) \longrightarrow V(\alpha+1)$ in $\mathcal{C}_1$ for every ordinal $\alpha < \gamma$, the induced map $V(\emptyset) \longrightarrow \mathrm{colim}_\gamma V$ is also in $\mathcal{C}_1$. Examples of such subcategories are the cofibrations or the acyclic cofibrations in a closed model category.

Relatively small objects. Consider a cocomplete category $\mathcal{C}$ and a subcategory $\mathcal{C}_1 \subset \mathcal{C}$ closed under transfinite composition. If κ is a regular cardinal, an object $C \in \mathcal{C}$ is called *κ-small relative to* $\mathcal{C}_1$ if for every regular cardinal $\lambda \geq \kappa$ and every functor $V:\lambda \longrightarrow \mathcal{C}_1$ which is a λ-sequence in $\mathcal{C}$, the map

$$\mathrm{colim}_\lambda \mathrm{Hom}_{\mathcal{C}}(C, V) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, \mathrm{colim}_\lambda V)$$

is an isomorphism. An object $C \in \mathcal{C}$ is called *small relative to* $\mathcal{C}_1$ if there exists a regular cardinal κ such that C is κ-small relative to $\mathcal{C}_1$.

I-injectives, I-cofibrations and I-cell complexes. Given a cocomplete category $\mathcal{C}$ and a class I of maps, we denote

- by I-inj the class of maps which have the right lifting property with respect to the maps in I. Maps in I-inj are referred to as *I-injectives*.
- by I-cof the class of maps which have the left lifting property with respect to the maps in I-inj. Maps in I-cof are referred to as *I-cofibrations*.
- by I-cell $\subset I$-cof the class of the (possibly transfinite) compositions of pushouts (cobase changes) of maps in I. Maps in I-cell are referred to as *I-cell complexes*.

In [**62**, p. II 3.4] Quillen formulates his *small object argument*, which immediately became a standard tool in model category theory. In our context we will need a transfinite version of the small object argument, so we work with the 'cofibrantly generated model category', which we now recall. Note that here I has to be a *set*, not just a class of maps. The obvious analogue of Quillen's small object argument would seem to require that coproducts are included in the I-cell complexes. In fact, any coproduct of an I-cell complex is already an I-cell complex, see [**35**, 2.1.6].

Lemma 1.2. *Let $\mathcal{C}$ be a cocomplete category and I a set of maps in $\mathcal{C}$ whose domains are small relative to* I-cell. *Then*

- *there is a functorial factorization of any map f in $\mathcal{C}$ as $f = qi$ with $q \in I$-*inj *and $i \in I$-*cell *and thus*
- *every I-cofibration is a retract of an I-cell complex.*

Definition 1.3. A model category $\mathcal{C}$ is called *cofibrantly generated* if it is complete and cocomplete and there exists a set of cofibrations I and a set of acyclic cofibrations J such that

- the fibrations are precisely the J-injectives;
- the acyclic fibrations are precisely the I-injectives;
- the domain of each map in I (resp. in J) is small relative to I-cell (resp. J-cell).

Moreover, here the (acyclic) cofibrations are the I (J)-cofibrations.

For a specific choice of I and J as in the definition of a cofibrantly generated model category, the maps in I (resp. J) will be referred to as generating cofibrations (resp. generating acyclic cofibrations). In cofibrantly generated model categories, a map may be functorially factored as an acyclic cofibration followed by a fibration and as a cofibration followed by an acyclic fibration.

Definition 1.4. Let $\mathcal{C}$ be a model category

$$R : \mathcal{D} \longrightarrow \mathcal{C}$$

a functor. We say that R *creates a model structure* on the category $\mathcal{D}$ if the following definitions make $\mathcal{D}$ into a model category: a morphism f in $\mathcal{D}$ is a

- weak equivalence if the morphism $R(f)$ is a weak equivalence in $\mathcal{C}$,
- fibration if the morphism $R(f)$ is a fibration in $\mathcal{C}$,
- cofibration if it has the left lifting property with respect to all morphisms in $\mathcal{D}$ which are both fibrations and weak equivalences.

Theorem 1.5. *Let $\mathcal{C}$ be a model category, $\mathcal{D}$ a category which is complete and cocomplete and let*

$$R : \mathcal{D} \longrightarrow \mathcal{C} : L$$

*be a pair of adjoint functors such that R commutes with filtered colimits. Let I (J) be a set of generating cofibrations (resp. acyclic cofibrations) for the cofibrantly generated model category $\mathcal{C}$. Let LI (resp. LJ) be the image of these sets under the left adjoint L. Assume that the domains of LI (LJ) are small relative to LI-*cell *(LJ-*cell*). Finally, suppose every LJ-cell complex is a weak equivalence. Then $R : \mathcal{D} \longrightarrow \mathcal{C}$ creates a model structure on $\mathcal{D}$ which is cofibrantly generated with LI (LJ) a generating set of (acyclic) cofibrations.*

If the model category $\mathcal{C}$ is right proper, then so is the model structure on $\mathcal{D}$.

If $\mathcal{C}$ and $\mathcal{D}$ are simplicially enriched, the adjunction (L, R) is simplicial, and the model structure of $\mathcal{C}$ is simplicial, then the model structure on $\mathcal{D}$ is again simplicial.

If $\mathcal{C}$ and $\mathcal{D}$ are topologically enriched, the adjunction (L, R) is continuous, and the model structure of $\mathcal{C}$ is topological, then the model structure on $\mathcal{D}$ is again topological.

Proof. Model category axiom MC1 (limits and colimits) holds by hypothesis. Model category axioms MC2 (saturation) and MC3 (closure properties under retracts) are clear. One half of MC4 (lifting properties) holds by the definition of cofibrations in $\mathcal{D}$.

The proof of the remaining axioms uses the transfinite small object argument (Lemma 1.2), which applies because of the hypothesis about the smallness of the domains. We begin with the factorization axiom, MC5. Every map in LI and LJ is a cofibration in $\mathcal{D}$ by adjointness. Hence every LI-cofibration or LJ-cofibration is a cofibration in $\mathcal{D}$. By adjointness and the fact that I is a generating set of cofibrations for $\mathcal{C}$, a map is LI-injective precisely when the map becomes an acyclic fibration in $\mathcal{C}$ after application of R, i.e., an acyclic fibration in $\mathcal{D}$. Hence the small object argument applied to the set LI gives a (functorial) factorization of any map in $\mathcal{D}$ as a cofibration followed by an acyclic fibration.

The other half of the factorization axiom, MC5, needs the hypothesis. Applying the small object argument to the set of maps LJ gives a functorial factorization of a map in $\mathcal{D}$ as an LJ-cell complex followed by a LJ-injective. Since J is a generating set for the acyclic cofibrations in $\mathcal{C}$, the LJ-injectives are precisely the fibrations among the $\mathcal{D}$-morphisms, once more by adjointness. We assume that every LJ-cell complex is a weak equivalence. We noted above that every LJ-cofibration is a cofibration in $\mathcal{D}$. So we see that the factorization above is an acyclic cofibration followed by a fibration.

It remains to prove the other half of MC4, i.e., that any acyclic cofibration $A \longrightarrow B$ in $\mathcal{D}$ has the left lifting property with respect to fibrations. In other words, we need to show that the acyclic cofibrations are contained in the LJ-cofibrations. The small object argument provides a factorization

$$A \longrightarrow W \longrightarrow B$$

with $A \longrightarrow W$ a LJ-cofibration and $W \longrightarrow B$ a fibration. In addition, $W \longrightarrow B$ is a weak equivalence since $A \longrightarrow B$ is. Since $A \longrightarrow B$ is a cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r] \arrow[d] & W \arrow[d, "\sim"] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus $A \longrightarrow B$ is a retract of a LJ-cofibration, hence it is a LJ-cofibration. □

In cofibrantly generated model categories fibrations can be detected by checking the right lifting property against a *set* of maps, the generating acyclic cofibrations, and similarly for acyclic fibrations. This is in contrast to general model categories where the lifting property has to be checked against the whole class of acyclic cofibrations. Similarly, in cofibrantly generated model categories, the pushout product axiom and the monoid axiom only have to be checked for a set of generating (acyclic) cofibrations:

Lemma 1.6. *Let $\mathcal{C}$ be a cofibrantly generated model category endowed with a closed symmetric monoidal structure. If the pushout product axiom holds for a set of generating cofibrations and a set of generating acyclic cofibrations, then it holds in general.*

Proof. For the first statement consider a map $i : A \longrightarrow B$ in $\mathcal{C}$. Denote by $G(i)$ the class of maps $j : K \longrightarrow L$ such that the pushout product

$$A \wedge L \cup_{A \wedge K} B \wedge K \ \longrightarrow \ B \wedge L$$

is a cofibration. This pushout product has the left lifting property with respect to a map $f : X \longrightarrow Y$ if and only if j has the left lifting property with respect to the map

$$p : [B, X] \ \longrightarrow \ [B, Y] \times_{[A,Y]} [A, X].$$

Hence, a map is in $G(i)$ if and only if it has the left lifting property with respect to the map p for all $f : X \longrightarrow Y$ which are acyclic fibrations in $\mathcal{C}$.

$G(i)$ is thus closed under cobase change, transfinite composition and retracts. If $i : A \longrightarrow B$ is a generating cofibration, $G(i)$ contains all generating cofibrations by assumption; because of the closure properties it thus contains all cofibrations, see Lemma 1.2. Reversing the roles of i and an arbitrary cofibration $j : K \longrightarrow L$ we thus know that $G(j)$ contains all generating cofibrations. Again by the closure properties, $G(j)$ contains all cofibrations, which proves the pushout product axiom for two cofibrations. The proof of the pushout product being an acyclic cofibration when one of the constituents is, follows in the same manner. □

We now spell out the small object argument for symmetric spectra.

Theorem 1.7 (Small object argument)**.** *Let I be a set of morphisms of symmetric spectra based on simplicial sets. Then there exists a functorial factorization of morphisms as I-cell complexes followed by I-injective morphisms.*

Proof. In the first step we construct a functor F from the category of morphisms of symmetric spectra to symmetric spectra as follows. Given a morphism $f : X \longrightarrow Y$ and a morphism $i : S_i \longrightarrow T_i$ in the set I we let D_i denote the set of all pairs $(a : S_i \longrightarrow X, b : T_i \longrightarrow Y)$ of morphisms satisfying $fa = bi$, i.e., which make the square

$$\begin{tikzcd} S_i \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T_i \arrow[r, "b"'] & Y \end{tikzcd}$$

commute. We define $F(f)$ as the pushout in the diagram

$$\begin{tikzcd} \bigvee_{i\in I}\bigvee_{D_i} S_i \arrow[r, "\vee a"] \arrow[d, "\vee i"'] & X \arrow[d, "j"] \\ \bigvee_{i\in I}\bigvee_{D_i} T_i \arrow[r] & F(f) \end{tikzcd}$$

The morphisms $b : T_i \longrightarrow Y$ and $f : X \longrightarrow Y$ glue to a morphism $p : F(f) \longrightarrow Y$ such that $pj = f$. The factorization we are looking for is now obtained by iterating this construction infinitely often, possibly transfinitely many times.

We define functors $F^n : Ar(\mathcal{S}p) \longrightarrow$ Spec and natural transformations $X \xrightarrow{j_n} F^n(f) \xrightarrow{p_n} Y$ for every ordinal n by transfinite induction. We start with $F^0(f) = X$, $j_0 = \mathrm{Id}$ and $p_0 = f$. For successor ordinal we set $F^{n+1}(f) = F(p_n : F^n(f) \longrightarrow Y)$ with the morphisms $j_{n+1} = j \circ j_n$ respectively $p_{n+1} = p(p_n)$. For limit ordinals λ we set $F^\lambda(f) = \mathrm{colim}_{\mu<\lambda} F^\mu(f)$ with morphisms induced by the j_μ and p_μ. By construction, all morphisms $j_n : X \longrightarrow F^n(f)$ are I-cell complexes.

We claim that there exists a limit ordinal κ, depending on the set I, such that for every morphism f the map $p_\kappa : F^\kappa(f) \longrightarrow Y$ is I-injective. Then $f = p_\kappa j_\kappa$ is the required factorization.

We prove the claim under the simplifying hypothesis that for each morphism $i \in I$ the source S_i is *finitely presented* as a symmetric spectrum, i.e., for every sequence $Z_0 \longrightarrow Z_1 \longrightarrow Z_2 \longrightarrow \dots$ the natural map

$$\mathrm{colim}_n\ \mathcal{S}p(S_i, Z_n) \ \longrightarrow\ \mathcal{S}p(S_i, \mathrm{colim}_n Z_n)$$

is bijective. In that case, the first infinite ordinal ω will do the job. Indeed, $F^\omega(f)$ is the colimit over the sequence

$$X = F^0(f) \xrightarrow{j_1} F^1(f) \xrightarrow{j_2} F^2(f) \cdots .$$

Given a morphism $i \in I$ and a lifting problem

(1.8)
$$\begin{tikzcd} S_i \arrow[r, "a"] \arrow[d, "i"'] & F^\omega(f) \arrow[d, "p_\omega"] \\ T_i \arrow[r, "b"'] & Y \end{tikzcd}$$

there exists a factorization $a = ca_n$ for some $n \geq 0$ and some morphism $a_n : S_i \longrightarrow F^n(f)$ since S_i is finitely presented (where $c : F^n(f) \longrightarrow F^\omega(f)$ is the canonical morphism to the colimit). The commutative square

$$\begin{tikzcd} S_i \arrow[r, "a_n"] \arrow[d, "i"'] & F^n(f) \arrow[d, "p_\omega c = p_n"] \\ T_i \arrow[r, "b"'] & Y \end{tikzcd}$$

is an element in the set D_i which is used to define $F^{n+1}(f) = F(p_n)$. Thus the canonical morphism $C : T_i \longrightarrow F^{n+1}(f)$ makes the diagram

$$\begin{tikzcd} S_i \arrow[r, "a_n"] \arrow[d, "i"'] & F^{n+1}(f) \arrow[d, "p_{n+1}"] \\ T_i \arrow[r, "b"'] \arrow[ur, "C"] & Y \end{tikzcd}$$

commute. Then the composite of C with the canonical morphism $F^{n+1}(f) \longrightarrow F^\omega(f)$ solves the lifting problem (1.8). $\square$

1.2. Bousfield's localization theorem.

Theorem 1.9 (Bousfield). *Let $\mathcal{C}$ be a proper model category with a functor $Q : \mathcal{C} \longrightarrow \mathcal{C}$ and a natural transformation $\alpha : 1 \longrightarrow Q$ such that the following three axioms hold:*

(A1) *if $f : X \longrightarrow Y$ is a weak equivalence, then so is $Qf : QX \longrightarrow QY$;*

(A2) *for each object X of $\mathcal{C}$, the maps $\alpha_{QX}, Q\alpha_X : QX \longrightarrow QQX$ are weak equivalences;*

(A3) *for a pullback square*

$$\begin{tikzcd} V \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "h"'] & Y \end{tikzcd}$$

in $\mathcal{C}$, if f is a fibration between fibrant objects such that $\alpha : X \longrightarrow QX$, $\alpha : Y \longrightarrow QY$ and $Qh : QW \longrightarrow QY$ are weak equivalences, then $Qk : QV \longrightarrow QX$ is a weak equivalence.

Then the following notions define a proper model structure on $\mathcal{C}$: a morphisms $f : X \longrightarrow Y$ is a Q-cofibration if and only if it is a cofibration, a Q-equivalence if and only if $Qf : QX \longrightarrow QY$ is a weak equivalence, and Q-fibration if and only if f is a fibration and the commutative square

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & QX \arrow[d, "Qf"] \\ Y \arrow[r, "\alpha"'] & QY \end{tikzcd}$$

is homotopy cartesian.

The reference is [**11**, Thm. 9.3]. [note: if Q preserves pullbacks and fibrations, then (A3) is automatic]

1.3. Some useful lemmas.

Lemma 1.10 (Gluing lemma). *We consider a commutative diagram*

$$\begin{tikzcd} A \arrow[d, "\alpha"'] & B \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\ A' & B' \arrow[l, "f'"] \arrow[r, "g'"'] & C' \end{tikzcd}$$

in a model category $\mathcal{C}$ in which the morphisms $\alpha : A \longrightarrow A'$ and $g' \cup \gamma : B' \cup_B C \longrightarrow C'$ are cofibrations respectively acyclic cofibrations. then the induced morphism on pushouts $\alpha \cup \gamma : A \cup_B C \longrightarrow A' \cup_{B'} C'$ is a cofibration respectively acyclic cofibration.

Proof. The map $\alpha \cup \gamma$ factors as the composite

$$A \cup_B C \xrightarrow{\alpha \cup C} A' \cup_B C \xrightarrow{A' \cup \gamma} A' \cup_{B'} C' .$$

The first map $\alpha \cup C$ is a cofibration (resp. acyclic cofibration) since α is. The second map $A' \cup \gamma$ is a cofibration (resp. acyclic cofibration) by the assumption of $g' \cup \gamma$ and since

$$\begin{tikzcd} B' \cup_B C \arrow[r, "g' \cup \gamma"] \arrow[d, "f \cup C"'] & C' \arrow[d] \\ A' \cup_B C \arrow[r, "A' \cup \gamma"'] & A' \cup_{B'} C' \end{tikzcd}$$

is a pushout. $\square$

2. Compactly generated spaces

An continuous map $f : A \longrightarrow B$ of spaces is an *h-cofibration* if it has the homotopy extension property, i.e., given a continuous map $\varphi : B \longrightarrow X$ and a homotopy $H : [0,1] \times A \longrightarrow X$ such that $H(0,-) = \varphi f$, there is a homotopy $\bar{H} : [0,1] \times B \longrightarrow X$ such that $\bar{H} \circ ([0,1] \times f) = H$ and $\bar{H}(0,-) = \varphi$. An equivalent condition is that the map $[0,1] \times A \cup_{0 \times f} B \longrightarrow [0,1] \times B$ has a retraction.

Lemma 2.1. *Let $f : A \longrightarrow B$ be an h-cofibration and suppose that a is contractible to a point $a \in A$, relative to a. Then the quotient map $q : B \longrightarrow B/A$ that collapses the image of A to a point is a based homotopy equivalence, where B is based at $f(a)$.*

Proof. We let $H : [0,1] \times A \longrightarrow A$ be a homotopy from the identity to the constant map at a that satisfies $H(t,a) = a$ for all $t \in [0,1]$. The homotopy extension property provides a continuous map $\bar{H} : [0,1] \times B \longrightarrow B$ extending $f \circ H$ and the identity of B. The map $\bar{H}(1,-) : B \longrightarrow B$ then sends A to the basepoint $f(a)$, so it factors over a continuous map $s : B/A \longrightarrow B$, i.e., which satisfies $sq = \bar{H}(1,-)$. We claim that s is homotopy inverse, in the based sense, to the quotient map q. The map $\bar{H}$ is a homotopy from the identity to qs, and it is based since $\bar{H}(t, f(a)) = f(H(t,a)) = f(a)$. So it remains to find a homotopy for the other composite.

Since the interval is compact, the map $[0,1] \times q : [0,1] \times B \longrightarrow [0,1] \times B/A$ is a quotient map and so $q\bar{H} : [0,1] \times B \longrightarrow B/A$ factors over a continuous map $K : [0,1] \times B/A \longrightarrow B/A$ such that $K([0,1] \times q) = q\bar{H}$. Then we have $K(0,-)q = q\bar{H}(0,-) = q$; since q is surjective, this shows that the homotopy K starts from the identity of B/A. Similarly we have $K(1,-)q = q\bar{H}(1,-) = qsq$; since q is surjective, this shows that the homotopy K ends in the map qs. Finally the homotopy is based because $K(t, q(f(a))) = q(\bar{H}(t, f(a))) = q(H(t,a)) = q(a)$ for all $t \in [0,1]$. □

We let $f : A \longrightarrow B$ be a continuous maps of based spaces. The *reduced mapping cone* $C(f)$ of f is defined as the space

$$C(f) \ = \ ([0,1] \wedge A) \cup_f B \ .$$

The unit interval $[0,1]$ is pointed by $0 \in [0,1]$, so that $[0,1] \wedge A$ is the reduced cone of A. An equivalent definition of $C(f)$ is as the pushout:

$$\begin{tikzcd} A \arrow[r, "1\wedge -"] \arrow[d, "f"'] & {[0,1] \wedge A} \arrow[d, "j"] \\ B \arrow[r, "i"'] & C(f) \end{tikzcd}$$

The cone $[0,1] \wedge A$ of every based space A is contractible to its base point, relative to the basepoint. For example, the homotopy

$$[0,1] \times ([0,1] \wedge A) \ \longrightarrow \ [0,1] \wedge A \ , \quad (t, s \wedge a) \ \longmapsto \ ((1-t)s) \wedge a$$

does the job. If the map f is an h-cofibration, then so is its cobase change $j : [0,1] \wedge A \longrightarrow C(f)$. So Lemma 2.1, applied to the map j, yields:

Corollary 2.2. *For every h-cofibration $f : A \longrightarrow B$ the map $C(f) \longrightarrow B/A$ that collapses the image of the cone of A to a point is a based homotopy equivalence.*

In this section we review some properties of compactly generated spaces. The most comprehensive reference for this category is Appendix A of Lewis' thesis [**44**] which, although widely circulated, is unpublished. So during this section we depart from our standing assumption that every space of compactly generated.

Let us fix some terminology. A topological space is *compact* if it is quasi-compact (i.e., every open cover has a finite subcover) and satisfies the Hausdorff separation property (i.e., every pair of distinct points can be separated by disjoint open subsets).

Definition 2.3. Let X be a topological space.

- X is *weak Hausdorff* if for every compact space K and every continuous map $f : K \longrightarrow X$ the image $f(K)$ is closed in X.
- A subset U of X is *compactly closed* if for every compact space K and every continuous map $f : K \longrightarrow X$, the inverse image $f^{-1}(U)$ is closed in K.
- X is a *Kelley space* if every compactly open subset is open.
- X is a *compactly generated* space if it is a Kelley space and weak Hausdorff.

We denote by **Spc** the category of topological spaces and continuous maps, by **K** its full subcategory of Kelley spaces and by **T** the full subcategory of **Spc** and **K** of compactly generated spaces. We collect some immediate observations. If K is compact and $f : K \longrightarrow X$ continuous, then the image $f(K)$ is always quasi-compact; if X is Hausdorff, then any quasi-compact subset such as $f(K)$ is closed. In other words, Hausdorff spaces are also weak Hausdorff spaces. Any point of any space is the continuous image of a compact space. So in weak Hausdorff spaces, all points and thus all finite subsets are closed. If X is weak Hausdorff, K compact and $f : K \longrightarrow X$ continuous, then the image $f(K)$ is compact [**44**, Lemma 1.1].

Every closed subset is also compactly closed. One can similarly define *compactly open* subsets of X by demanding that for every compact space K and every continuous map $f : K \longrightarrow X$, the inverse image is open in K. A subset is then compactly open if and only if its complement is compactly closed. Thus Kelley spaces can equivalently be defined by the property that all compactly open subsets are open.

All compact spaces are compactly generated. [also locally compact ? see Lewis] If X is weak Hausdorff (respectively a Kelley space, respectively compactly generated) and K is compact, then the $X \times K$ with the product topology is also weak Hausdorff (respectively a Kelley space, respectively compactly generated). [check; ref or proof]

[ref. to Lewis' thesis]

If X is any topological space we let kX be the space which has the same underlying set as X, but such that the open subsets of kX are the compactly open subsets of X. This indeed defines a topology which makes kX into a Kelley space and such that the identity $\mathrm{Id} : kX \longrightarrow X$ is continuous. Part (i) of the next proposition is a fancy way of saying that any continuous map $Y \longrightarrow X$ whose source Y is a Kelley space is also continuous when viewed as a map to kX. If X is weak Hausdorff, then so is kX.

If X is any space we let wX denote the maximal weak Hausdorff quotient of X [defined; does this preserve Kelley spaces ?].

Proposition 2.4.
- *The functor $k : \mathbf{Spc} \longrightarrow \mathbf{K}$ is left inverse and right adjoint to the inclusion. Unit and counit of the adjunction are the identity maps.*
- *The functor $w : \mathbf{K} \longrightarrow \mathbf{T}$ is right inverse and left adjoint to the inclusion.*

The construction of the left adjoint $w : \mathbf{K} \longrightarrow \mathbf{T}$ to the inclusion is not particularly instructive, as it is obtained by Freyd's adjoint functor theorem. [give the proof]

There is a useful criterion, due to Lewis [**44**, App. A, Prop. 3.1], for when a Kelley space X is weak Hausdorff: if and only if the diagonal subset in $X \times X$ is closed in the **K**-topology (i.e., compactly closed in the usual product topology).

It follows formally from part (i) of this proposition that the category **K** of Kelley spaces has small limits and colimits. Colimits can be calculated in the ambient category of all topological spaces; equivalently, any colimits of Kelley spaces is again a Kelley space. To construct limits, we can first take a limit in the ambient category of all topological spaces; this 'ambient limit' need not be a Kelley space, but applying the functor $k : \mathbf{Spc} \longrightarrow \mathbf{K}$ yields a limit in **K**. Since k does not change the underlying set, the categories **K** and **Spc** share the property that the forgetful functor to sets preserves all limits and colimits. More loosely speaking, the underlying set of a limit or colimit in **K** is what one first thinks of.

An important example where a limit in **K** resp. **T** can differ from the limit in **Spc** is the product of two CW-complexes X and Y. All CW-complexes are compactly generated [ref?], and the product $X \times Y$ with the usual product topology is a Hausdorff space which comes with a natural filtration $(X \times Y)_{(n)} = \cup_{p+q=n} X_{(p)} \times Y_{(q)}$, where $X_{(p)}$ is the p-skeleton of the CW structure on X. If X or Y is locally finite, then the product topology is compactly generated, and then the above filtration makes $X \times Y$ into a CW-complex. In general, however, $X \times Y$ may not be a Kelley space, and hence cannot have a CW structure. But the product in the category **K**, i.e., the space $k(X \times Y)$, is always compactly generated and a CW-complex via the above filtration.

It follows formally from the above and part (ii) of the proposition that the category **T** of compactly generated spaces has small limits and colimits. Limits can be calculated in the category **K** of Kelley spaces as explained in the previous paragraph. To construct colimits, we can first take a colimit in the category **K** of Kelley spaces (or equivalently in **Spc**); while a Kelley space, this colimit need not be weak Hausdorff, but applying the functor $w : \mathbf{K} \longrightarrow \mathbf{T}$ yields a colimit in **T**. The 'maximal weak Hausdorff quotient' functor $w : \mathbf{K} \longrightarrow \mathbf{T}$ is not particularly explicit and may change the underlying set; so one has to be especially careful with general colimits in **T**: unlike for **Spc** of **K**, the forgetful functor from **T** to sets need not preserve colimits. More loosely speaking, the underlying set of colimit in **T** may be smaller than one first thinks.

It will be convenient to know some particular instances of diagrams in **T** where it makes no difference if we calculate the colimit in the category **T** or in **K** respectively **Spc**.

We call a continuous map $f : X \longrightarrow Y$ between topological spaces a *closed embedding* if f is injective, the image $f(X)$ is closed in Y and f is a homeomorphism onto its image. The basechange, in **Spc** or **K**, of a closed embedding is again a closed embedding. (Note that there is an ambiguity with the meaning of 'embedding' in general, due to the fact that a general subset of a Kelley space, endowed with the subspace topology, need not be a Kelley space, and so one may or may not want to apply 'Kelleyfication' $k : \mathbf{Spc} \longrightarrow \mathbf{K}$ to the subspace topology. However, closed subsets of Kelley spaces are again Kelley spaces with the usual subspace topology [this is OK for subsets which are the intersection of an open and a closed subset], so there is not such ambiguity with the notion of 'closed embedding'.)

A *partially ordered set* is a set P equipped with a binary relation '$\leq$' which is reflexive (i.e., $x \leq x$ for all $x \in P$), antisymmetric (i.e., $x \geq y$ and $y \leq x$ imply $x = y$) and transitive (i.e., $x \geq y$ and $y \leq z$ imply $x \leq z$). The partially ordered set P is *filtered* if for every pair of elements $x, y \in P$ there exists an element $z \in P$ such that $x \leq z$ and $y \leq z$. [non-empty?] We will routinely interpret a partially ordered set P as a category without change in notation. In the associated category, the objects are the elements of P and there is a unique morphism from x to y if $x \leq y$, and no morphism from x to y otherwise. Via this interpretation we can consider functors defined on partially ordered sets.

Proposition 2.5. (i) *Given a pushout in the category* **K** *of Kelley spaces*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \arrow[d] \\ Y \arrow[r, "g"'] & Y \cup_f Z \end{tikzcd}$$

such that f is a closed embedding, then g is also a closed embedding. If moreover, X, Y and Z are compactly generated, the so is $Y \cup_f Z$, and hence the diagram is a pushout in **T**.

(ii) *Let P be a filtered partially ordered set and $F : P \longrightarrow \mathbf{T}$ a functor from the associated poset category. Let F_∞ be the colimit of F in the category* **K** *of Kelley spaces (or, equivalently, in* **Spc***) and $\kappa_i : F(i) \longrightarrow F_\infty$ the canonical map. If for every $i \leq j$ in P the map $F(i) \longrightarrow F(j)$ is injective, then the maps κ_i are also injective and the colimit F_∞ is weak Hausdorff, thus a colimit of F in the category* **T***. If moreover, all maps $F(i) \longrightarrow F(j)$ are closed embedding, then so are the maps $\kappa_i : F(i) \longrightarrow F_\infty$.*

(iii) *Let λ be a regular (?) cardinal and $X : \lambda \longrightarrow \mathbf{T}$ a λ-sequence of injective maps. Then the colimit* $\mathrm{colim}_\lambda X$ *in the category* **Spc** *is again compactly generated and thus a colimit of X in* **T**.

PROOF. (i) is [**44**, Prop. 7.5], (ii) is [**44**, Prop. 9.3] and (iii) is in Hovey [**35**,]. □

An example of Lewis [**44**,] shows that a pushout in **T** along a non-closed embedding need not even be injective.

The following proposition says that compact spaces are 'small with respect to closed embeddings'. Note that if all spaces in the λ-sequence are compactly generated, then by Proposition 2.5 (ii) it makes no difference whether the colimit is calculated in the category **Spc**, **K** or **T**.

Proposition 2.6. [**35**, Prop. 2.4.2] *Let λ be an ordinal and $X : \lambda \longrightarrow \mathbf{T}$ a λ-sequence of closed embeddings [def? what happens at limit cardinals ?]. Then for every compact space K the natural map*

$$\operatorname{colim}_{i>\lambda} C(K, X_i) \longrightarrow C(K, \operatorname{colim}_\lambda X))$$

is bijective.

It is not in general true that $C(K, -)$ commutes with filtered colimits over closed embeddings. An example from Lewis' theses (which he credits to Myles Tierney) is the unit interval $[0, 1]$. A subset of $[0, 1]$ is closed if and only if its intersection with every countable closed subset J of $[0, 1]$ is closed in J, which implies that $[0, 1]$ is the filtered colimit of its countable closed subsets, ordered by inclusion. Since $[0, 1]$ is uncountable, the identity map of $[0, 1]$ does not factor through any of the spaces in the filtered system.

[is there a criterion in terms of the poset P when for every compact space K the natural map

$$\operatorname{colim}_{i\in P} C(K, F(i)) \longrightarrow C(K, \operatorname{colim}_P F))$$

is bijective. ? [or even a homeomorphism ?]]

Proposition 2.7. *Let $\{X_i\}_{i\in I}$ be a family of based compactly generated spaces. Then the wedge (one-point union) $\bigvee_{i\in I} X_i$ is compactly generated, thus the coproduct of the family in $\mathbf{T}$. Moreover, for every compact space K and every continuous map $f : K \longrightarrow \bigvee_{i\in I} X_i$ there is a finite subset J of I such that f factors through the sub-wedge $\bigvee_{i\in J} X_i$.*

PROOF. □

Corollary 2.8. *Let λ be an ordinal and $X : \lambda \longrightarrow \mathbf{T}$ a λ-sequence of compactly generated spaces. If all maps in the sequence are closed embeddings [def? what happens at limit cardinals ?], then for every point $x \in X_\emptyset$ and every $n \geq 0$ the natural map*

$$\operatorname{colim}_{i>\lambda} \pi_n(X_i, x) \longrightarrow \pi_n(\operatorname{colim}_\lambda X, x)$$

is bijective. If in addition all maps in the sequence are weak equivalences, then so is the transfinite composite $X_\emptyset \longrightarrow \operatorname{colim}_\lambda X$.

[check out also Lemma 9.3 of [**80**]]

There is a suitable version of the compact open topology which gives mapping spaces in the category $\mathbf{T}$. For spaces X and Y, we let $C(X, Y)$ denote the set of continuous maps from X to Y. A subbasis for a topology is given by all sets $S(U, f : K \longrightarrow X)$ where U is an open subset of Y, K a compact space and f a continuous map; the set $S(U, f)$ consists of all those continuous $\varphi : X \longrightarrow Y$ such that $\varphi(f(k)) \subset U$. If X and Y are compactly generated, then the space $C(X, Y)$ is weak Hausdorff, but not necessarily a Kelley space. So the mapping space $\operatorname{map}(X, Y)$ is defined as $kC(X, Y)$, which is then a compactly generated space. [ref to Lewis]

Theorem 2.9. *The category of compactly generated spaces is cartesian closed, i.e., the natural map*

$$\operatorname{map}(X \times Y, Z) \cong \operatorname{map}(X, \operatorname{map}(Y, Z))$$

is a homeomorphism for all compactly generated spaces X, Y and Z, where the product on the left hand side is taken in the category $\mathbf{T}$.

[ref to Lewis ?]

A continuous map $f : X \longrightarrow Y$ of topological spaces is a *weak equivalence* if f induces a bijection on path components and for every point $x \in X$ and $n \geq 1$ the map $\pi_n(f) : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism. An equivalent condition is that for every CW-complex A (possibly empty) the induced map $[A, f] : [A, X] \longrightarrow [A, Y]$ on homotopy classes of continuous maps is bijective.

The map f is a *cofibration* if and only if it is a retract of a relative cell complex [spell out]. The map f is a *fibration* if it is a Serre fibration, i.e., has the right lifting property with respect to the inclusions $D^n \longrightarrow D^n \times [0, 1], x \mapsto (x, 0)$ for all $n \geq 0$.

Theorem 2.10. [**35**, Thm. 2.4.25] *The cofibrations, weak equivalences and (Serre) fibrations make the category* $\mathbf{T}$ *of compactly generated spaces into a proper model category. The model structure is monoidal with respect to the cartesian product. [generators]*

Although this will not be relevant for us we feel obliged to mention that the weak equivalences, cofibrations and (Serre) fibrations also define model structures on the categories **Spc** of all topological spaces (this is originally due to Quillen [**62**, II.3, Thm. 1], proofs can also be found in [**23**, Prop. 8.3] or [**35**, Thm. 2.4.19]) and $\mathbf{K}$ of Kelley spaces [**35**, Thm. 2.4.23]. Since weak equivalences are defined by taking homotopy classes of continuous maps out of compact spaces the identity $kX \longrightarrow X$ is a weak equivalence for every topological space X. So weak equivalences don't see any difference between the categories **Spc** and $\mathbf{K}$. A more structured way to say this is that the left of the two adjoint functor pairs

$$\mathbf{Spc} \;\underset{k}{\overset{\text{incl.}}{\leftrightarrows}}\; \mathbf{K} \;\underset{\text{incl.}}{\overset{w}{\rightleftarrows}}\; \mathbf{T}$$

is a Quillen equivalence. The second pair is also a Quillen equivalence because the right adjoint preserves and detects weak equivalences and fibrations and because for every cofibrant object A of $\mathbf{K}$ the adjunction unit $A \longrightarrow wA$ is an isomorphism, thus a weak equivalence. However, the category $\mathbf{T}$ is the only one among these three model categories which is monoidal, and that is the reason why we work in $\mathbf{T}$. [no: $\mathbf{K}$ is also closed symmetric monoidal...] [Strom model structures a la Cole]

[based versions]

We consider a sequence

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

of based continuous maps between based topological spaces. We want to define the *reduced mapping telescope* $\mathrm{tel}_i\, X_i$ of the sequence. We first define the 'partial telescopes' F_j and based homotopy equivalences $\alpha_j : X_j \longrightarrow \mathrm{tel}_{[0,j]}\, X_i$ by induction on $j \geq 0$. We start with $\mathrm{tel}_{[0,0]}\, X_i = X_0$ and $\alpha_0 = \mathrm{Id}$. The next partial telescope $\mathrm{tel}_{[0,j+1]}\, X_i$ is defined as the pushout of the diagram

$$F_j \xleftarrow{\alpha_j} X_j \xrightarrow{x \mapsto x \wedge 0} X_j \wedge [0,1]^+ \cup_{f_j} X_{j+1} \ .$$

The map α_{j+1} is the composite of the homotopy equivalence $X_{j+1} \longrightarrow X_j \wedge [0,1]^+ \cup_{f_j} X_{j+1}$ and the canonical map from the mapping cylinder $X_j \wedge [0,1]^+ \cup_{f_j} X_{j+1}$ to the pushout F_{j+1}. The composite $\alpha_{j+1} \circ f_j : X_j \longrightarrow F_{j+1}$ is homotopic, in a basepoint preserving fashion, to the composite of α_j and the canonical map $F_j \longrightarrow F_{j+1}$. We can now define the reduced mapping telescope as the colimit of the sequence

$$F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots \ .$$

The composite maps $\alpha'_j : X_j \longrightarrow F_j \longrightarrow \mathrm{tel}_i\, X_i$ then have the property that $\alpha'_{j+1} \circ f_j : X_j \longrightarrow F_{j+1}$ is based homotopic to α'_j. Thus for every $n \geq 0$ the induced maps on homotopy groups (or sets, for $n = 0$) satisfy

$$\pi_n(\alpha'_{j+1}) \circ \pi_n(f_j) \ = \ \pi_n(\alpha'_j)$$

and so they assemble into a map

$$\mathrm{colim}_i\, \pi_n(X_i, x_i) \ \longrightarrow \ \pi_n(\mathrm{tel}_i\, X_i, x_\infty) \ .$$

Proposition 2.11. *For every sequence of based compactly generated spaces* X_i *and based continuous maps* $f_i : X_i \longrightarrow X_{i+1}$ *the reduced mapping telescope* $\mathrm{tel}_i\, X_i$ *is again compactly generated and the natural map*

$$\mathrm{colim}_i\, \pi_n(X_i, x_i) \ \longrightarrow \ \pi_n(\mathrm{tel}_i\, X_i, x_\infty)$$

is bijective for every n.

Proof. We claim that if all the spaces X_i are compactly generated, then the maps $F_j \longrightarrow F_{j+1}$ are all closed embeddings and all the partial telescopes F_j are compactly generated. Proposition 2.5 (ii) then shows that the telescope is compactly generated and Corollary 2.8 shows that the canonical map

$$\mathrm{colim}_j\, \pi_n(F_j, x_\infty) \ \longrightarrow \ \pi_n(\mathrm{tel}_i\, X_i, x_\infty)$$

is bijective for every n. Since $\alpha_j : X_j \longrightarrow F_j$ is a homotopy equivalence we can replace the group (or set) $\pi_n(\mathrm{tel}_{[0,j]} X^i, x_\infty)$ by the isomorphic group $\pi_n(X_j, x_j)$ to obtain the isomorphism we want.

It remains to prove the claim. The map from X_j to the reduced mapping cone $X_j \wedge [0,1]^+ \cup_{f_j} X_{j+1}$ which sends x to the class of $x \wedge 0$ is a closed embedding [...] So its basechange $F_j \longrightarrow F_{j+1}$ is a closed embedding and we can use Proposition 2.5 (i) to argue inductively that F_{j+1} is compactly generated. □

3. Simplicial sets

The books by May [**55**], Lamotke [**43**] and Goerss-Jardine [**30**] or Chapter 3 of [**35**] can serve as general references for simplicial sets and their homotopy theory. [search all these for references...]

We denote by $\mathbf{\Delta}$ the *simplex category* whose objects are the partially ordered sets $[n] = \{0, \dots, n\}$ (with the usual order) and whose morphisms are weakly monotone maps. A *simplicial set* is a contravariant functor X from $\mathbf{\Delta}$ to the category of sets. We often denote the value $X([n])$ of X at the object $[n]$ by X_n, and refer to its elements as the *n-simplices* of X. [introduce d_i and s_i]

[morphisms] We denote the category of based simplicial sets by **sS**

Some important examples of simplicial sets are the representable simplicial sets $\Delta[n] = \mathbf{\Delta}(-, [n])$ called the *standard n-simplex* and its *boundary* $\partial\Delta[n]$ defined by

$$(\partial\Delta[n])_m \;=\; \{\alpha : [m] \longrightarrow [n] \mid \alpha \text{ is not surjective}\} \;.$$

Moreover, for every $0 \leq k \leq n$ there is a simplicial subset $\Lambda^k[n]$ of $\partial\Delta[n]$ called its *k-th horn* and given by

$$(\Lambda^k[n])_m \;=\; \{\alpha : [m] \longrightarrow [n] \mid k \notin \alpha([m])\} \;.$$

Every small category $\mathcal{C}$ give rise to a simplicial set $N\mathcal{C}$, called the *nerve* of $\mathcal{C}$. To define the nerve we introduce the category $[[n]]$ associated to the ordered set $[n]$; so the object set of $[[n]]$ is $\{0, \dots, n\}$ and there is a unique morphism from i to j if and only if $i \leq j$. Every weakly monotone map $\alpha : [k] \longrightarrow [n]$ is the object function of a unique functor $[\alpha] : [[k]] \longrightarrow [[n]]$. In total this gives a fully faithful functor $[-] : \mathbf{\Delta} \longrightarrow \mathbf{Cat}$ from the simplex category $\mathbf{\Delta}$ to the category $\mathbf{Cat}$ of small categories. We can then define the nerve of $\mathcal{C}$ by $N\mathcal{C} = \mathbf{Cat}([-], \mathcal{C})$. In more detail, the k-simplices of the nerve $N\mathcal{C}$ are given by

$$(N\mathcal{C})_k \;=\; \mathbf{Cat}([[k]], \mathcal{C}) \;,$$

the set of functors from $[[k]]$ to $\mathcal{C}$. The structure map $\alpha^* : (N\mathcal{C})_n \longrightarrow (N\mathcal{C})_k$ associated to a weakly monotone map $\alpha : [k] \longrightarrow [n]$ is given by precomposition with the functor $[\alpha]$. We note that a functor $\varphi : [[k]] \longrightarrow \mathcal{C}$ is determined by a string of k composable morphisms

$$i_0 \xrightarrow{a_1} i_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} i_k$$

in $\mathcal{C}$. In particular, $(N\mathcal{C})_0$ is the set of objects of $\mathcal{C}$ and $(N\mathcal{C})_1$ is the set of morphisms of $\mathcal{C}$.

[BM for a monoid M]

A simplicial set can be thought of as a combinatorially defined CW-complex. This is made precise by the functor of *geometric realization*. For a simplicial set X the geometric realization $|X|$ is the topological space

$$|X| \;=\; \int_{[n]\in\Delta} X_n \times \underline{\Delta}[n] \;.$$

This need some explanation. We denote by

$$\underline{\Delta}[n] \;=\; \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum_{i=0}^{n} x_i = 1\}$$

the topological n-simplex. As n varies, we get a covariant functor $\underline{\Delta}[-] : \mathbf{\Delta} \longrightarrow \mathbf{Spc}$ by sending a weakly monotone map $\alpha : [k] \longrightarrow [n]$ to the affine linear map $\alpha_* : \underline{\Delta}[k] \longrightarrow \underline{\Delta}[n]$ given by

$$\alpha_*(x_0, \dots, x_k) \;=\; (y_0, \dots, y_n)$$

with $y_j = \sum_{\alpha(i)=j} x_i$. [write as coequalizer, quotient space]

For every simplicial set X, the geometric realization $|X|$ comes with a natural filtration whose p-th term is the image of the space $\amalg_{n=0}^{p} X_n \times \underline{\Delta}[n]$ under the quotient map. This filtration gives $|X|$ the structure of a CW-complex; in particular, the geometric realization of a simplicial set is a compactly generated space. Thus we can, and will, view geometric realization as a functor $|-| : \mathbf{sS} \longrightarrow \mathbf{T}$ to the category of compactly generated spaces.

A homeomorphism

$$\underline{\Delta}[n] \xrightarrow{\cong} |\Delta[n]| \tag{3.1}$$

from the topological n-simplex to the geometric realization of the standard n-simplex is given by the composite of the summand inclusion $\underline{\Delta}[n] \longrightarrow \amalg_m \Delta[n]_m \times \underline{\Delta}[m]$ indexed by $\mathrm{Id}_{[n]} \in \Delta[n]_n$ and the quotient map to $|\Delta[n]|$. Under this homeomorphism, the realization $|\partial\Delta[n]|$ of the boundary of the simplex corresponds to the geometric boundary of the topological n-simplex, i.e., the subspace of tuples $(x_0, \ldots, x_n) \in \underline{\Delta}[n]$ such that $x_i = 0$ for at least one coordinate X_i. Moreover, the realization $|\Lambda^k[n]|$ of the k-th horn corresponds to the union of all except the k-th faces of the topological n-simplex, i.e., the subspace of tuples $(x_0, \ldots, x_n) \in \underline{\Delta}[n]$ such that there exists an $i \neq k$ with $x_i = 0$.

The geometric realization $|N\mathcal{C}|$ of the nerve is often called the *classifying space* of the small category $\mathcal{C}$.

Theorem 3.2. *The geometric realization functor $|-| : \mathbf{sS} \longrightarrow \mathbf{T}$ has the following properties.*

(i) *For all simplicial sets K and L the natural map*

$$|K \times L| \longrightarrow |K| \times |L|$$

is a homeomorphism where the target is the product in the category $\mathbf{T}$ of compactly generated spaces.

(ii) *For all based simplicial sets K and L the natural map*

$$|K \wedge L| \longrightarrow |K| \wedge |L|$$

is a homeomorphism where the target is the smash product in the category $\mathbf{T}$ of compactly generated based spaces.

(iii) *If A and X are simplicial sets and X is fibrant, then the natural map*

$$|\operatorname{map}(A, X)| \longrightarrow \operatorname{map}(|A|, |X|)$$

is a weak equivalence.

(iv) *If A and X are based simplicial sets and X is fibrant, then the natural map*

$$|\operatorname{map}(A, X)| \longrightarrow \operatorname{map}(|A|, |X|)$$

[this time based; sort out the notation...] is a weak equivalence.

(v) *The geometric realization of a Kan fibration between simplicial sets is a Serre fibration of spaces.*

References for the proofs: (i): [**35**, Lemma 3.2.4], plus the fact that limits in the category $\mathbf{T}$ of compactly generated spaces can be formed in the category $\mathbf{K}$ of Kelley spaces. [also: Ch. 3 of Gabriel-Zisman [**29**] ?] (iii): the original reference is [**63**], but see also [**35**, Cor. 3.6.2] [is this in Gabriel-Zisman [**29**] ?]. □

As a special case of part (ii') above we observe that for a fibrant based simplicial set X the natural map

$$|\Omega X| \longrightarrow \Omega|X|$$

is a weak equivalence.

The geometric realization functor has an adjoint, the *singular complex* functor $\mathcal{S} : \mathbf{Spc} \longrightarrow \mathbf{sS}$. For a topological space X the n-simplices of the simplicial set $\mathcal{S}(X)$ are given by

$$\mathcal{S}(X)_n \ : \ \mathbf{Spc}(\underline{\Delta}[n], X) ,$$

the set of continuous maps from the topological n-simplex to X. For weakly monotone map $\alpha : [k] \longrightarrow [n]$ the induced map $\alpha^* : \mathcal{S}(X)_n \longrightarrow \mathcal{S}(X)_n$ is precomposition with the affine linear map $\alpha_* : \underline{\Delta}[k] \longrightarrow \underline{\Delta}[n]$. The adjunction bijection

$$\mathbf{sS}(A, \mathcal{S}(X)) \ \cong \ \mathbf{T}(|A|, X)$$

is quite tautological. A morphism of simplicial sets $f : A \longrightarrow \mathcal{S}(X)$ gives, for every n-simplex $a \in A_n$, a continuous map $f(a) : \underline{\Delta}[n] \longrightarrow X$. As the n-simplices vary, these maps give a continuous map

$$A_n \times \underline{\Delta}[n] \longrightarrow X \ , \ (a, y) \mapsto f(a)(y) \ .$$

These maps are compatible as n varies, so they assemble into a continuous map $\hat{f} : |A| \longrightarrow X$.

[Since the simplex $\underline{\Delta}[n]$ is a compact space, the identity $\mathrm{Id} : kX \longrightarrow X$ induces an isomorphism after taking singular complexes. Same for $X \longrightarrow wX$?]

[The singular complex functor commuted with filtered colimits over closed embeddings.]

[compare combinatorial homotopy groups with homotopy groups of realization whenever X is Kan] [realization of simplicial object is diagonal]

A morphism $f : X \longrightarrow Y$ of simplicial sets is a *weak equivalence* if the geometric realization $|f|$ is a weak equivalence of topological spaces. [equivalent characterizations: $|f|$ is homotopy equivalence; Ex^∞ is homotopy equivalence of ssets; $[f, X]$ is bijection for all Kan simplicial sets X]

The map f is a *cofibration* if it is dimensionwise injective, or, equivalently, a categorical monomorphism. The map f is a *fibration* if it is a Kan fibration, i.e., has the right lifting property with respect to the inclusions $\Lambda^k[n] \longrightarrow \Delta[n]$ of all horns into simplices. [equivalently: right lifting property for all injective weak equivalences.]

For every topological space X (in **Spc**, **K** or **T**?) the adjunction counit $|\mathcal{S}(X)| \longrightarrow X$ is a weak equivalence and for every simplicial set Y, the adjunction unit $Y \longrightarrow \mathcal{S}(|Y|)$ is a weak equivalence. [refs... the second is a formal consequence of the first by the def'n of weak equivalences in **sS**]

The following model structure on the category of simplicial sets is due to Quillen [**62**, II.3, Thm. 3]; proofs can also be found in [**35**, Thm. 3.6.5] and [**30**, Thm I.11.3]. The Quillen equivalence can be obtained by combining Theorems 2.4.25 and 3.6.7 of [**35**]; however the equivalence of the homotopy categories of CW-complexes and simplicial sets has been know since the 1950's [ref's: Gabriel-Zisman [**29**] ? May [**55**]? according to Goerss-Jardine also Kan's [**40**]]

Theorem 3.3. *The cofibrations, weak equivalences and Kan fibrations make the category* **sS** *of simplicial sets into a proper model category. The model structure is monoidal with respect to the cartesian product. The adjoint functors of geometric realization and singular complex are a Quillen equivalence*

$$\mathbf{sS} \ \underset{\mathcal{S}}{\overset{|-|}{\rightleftarrows}} \ \mathbf{T} \ .$$

[generators]

Proposition 3.4. *Let B be a countable simplicial set and A a simplicial subset.*

Proposition 3.5. *Let B be a simplicial set and A a simplicial subset of B such that the inclusion $A \longrightarrow B$ is a weak equivalence. Let C be a countable simplicial subset of B. Then there is a countable simplicial subset E of B that contains C and such that the inclusion $E \cap A \longrightarrow E$ is a weak equivalence.*

Proof. For a pair of simplicial sets $K \subseteq L$ and a basepoint $x \in K_0$ we consider the relative homotopy set $\pi_k(L, K, x)$, which for $k = 0$ is defined as the quotient set $\pi_0 L/\pi_k K$. The inclusion $K \longrightarrow L$ is then a weak equivalence if and only the set $\pi_k(L, K, x)$ consists of a single element for all $k \geq 0$.

The simplicial set B is the filtered union of the simplicial subsets $C \cup K$ as K runs through all finite simplicial subsets of B containing a given basepoint $a \in A_0$. The relative homotopy set $\pi_k(B, A, a)$ is then the filtered colimit of the sets $\pi_k(C \cup K, (C \cup K) \cap A, a)$; but $\pi_k(B, A, a)$ is trivial since $A \longrightarrow B$ is a weak equivalence. So for every $n \geq 1$ and every relative homotopy class $\alpha \in \pi_k(C, C \cap A)$ there is a finite simplicial subsets K_α of B, containing a, such that the image of α under the map

$$\pi_k(C, C \cap A) \longrightarrow \pi_k(C \cup K_\alpha, (C \cup K_\alpha) \cap A)$$

is trivial. We let F be the union of C and the K_α for all $k \geq 0$ and all $\alpha \in \pi_k(C, C \cap A)$. Since C is countable the set $\pi_k(C, C \cap A)$ is also countable [ref]. So F is a countable simplicial subset of B with $C \subseteq F$ and such that the map

$$\pi_k(C, C \cap A) \longrightarrow \pi_k(F, F \cap A)$$

is constant. We iterate this construction with F instead of C, etc, and end up with an ascending sequence

$$C \subseteq F \subseteq F^2 \subseteq F^3 \subseteq \cdots$$

of countable simplicial subsets of B. The union $E = \cup_{n\geq 0} F^n$ is then again countable and the relative homotopy group

$$\pi_k(E, E\cap A) \;\cong\; \mathrm{colim}_n\, \pi_k(F^n, F^n \cap A)$$

is trivial since all maps the colimit is taken over are constant. Hence the inclusion $E \cap A \longrightarrow E$ is a weak equivalence. □

Proposition 3.6. *Let G be a finite group, B a G-simplicial set and A a G-invariant simplicial subset of B such that the inclusion $A \longrightarrow B$ is a weak equivalence. Let C be a countable simplicial subset of B (not necessarily G-invariant). Then there is a countable G-invariant simplicial subset E of B that contains C and such that the inclusion $E \cap A \longrightarrow E$ is a weak equivalence.*

PROOF. We construct a sequence of countable simplicial subsets E_n of B starting with $E_0 = C$. For $n > 0$ we observe that $G \cdot E_{n-1}$, the union of all G-translates of E_{n-1}, is again countable because E_{n-1} is. Proposition 3.5 lets us choose a countable simplicial subset E_n of B that contains $G \cdot E_{n-1}$ and such that the inclusion $E_n \cap A \longrightarrow E_n$ is a weak equivalence. We let E be the union of the ascending sequence of simplicial subsets E_n. Since all the E_n are countable, so is E. Since $G \cdot E_n \subseteq E_{n+1}$, the union E is G-invariant. Since the inclusions $E_n \cap A \longrightarrow E_n$ are weak equivalences for all $n \geq 1$, so is their union $E \cap A \longrightarrow E$. □

[based version; realization of simplicial spaces and bisimplicial sets]

The *augmented simplicial category* $\mathbf{\Delta}_+$ is the category with objects the finite ordered sets $[n] = \{0, \dots, n\}$ for $n \geq -1$, where $[-1] = \emptyset$ is the empty set, and all weakly monotone maps as morphisms. Thus $\mathbf{\Delta}_+$ contains the simplicial category $\mathbf{\Delta}$ as a full subcategory and has one additional object $[-1]$ which is initial and receives no morphisms from $[n]$ for $n \geq 0$. An *augmented simplicial object* is a contravariant functor from the augmented simplicial category $\mathbf{\Delta}_+$ to $\mathcal{C}$. An augmented simplicial object X determines, and is determined by

- the simplicial object obtained by restriction of X to $\mathbf{\Delta}$,
- the object $X_{-1} = X([-1])$ and
- the morphism $d_0 : X_0 \longrightarrow X_{-1}$, called the *augmentation*, induced by the unique morphism $[-1] \longrightarrow [0]$ in $\mathbf{\Delta}_+$ which has to coequalize the two morphism $d_0, d_1 : X_1 \longrightarrow X_0$.

The augmentation $d_0 : X_0 \longrightarrow X_{-1}$ of an augmented simplicial space X gives rise to a continuous map

$$|uX| \;\longrightarrow\; X_{-1} \; .$$

Proposition 3.7. *Let X be an augmented simplicial space which admits extra degeneracies. Then the map $|res(X)| \longrightarrow X_{-1}$ induced by the augmentation is a based homotopy equivalence.*

PROOF. We let cX_{-1} denote the constant simplicial space with values X_{-1}. We can define a morphism of simplicial spaces $s_{-1} : cX_{-1} \longrightarrow uX$ in simplicial dimension k by the composite of $s_{-1} : X_{-1} \longrightarrow X_0$ with the degeneracy morphism $s : X_0 \longrightarrow X_k$. We can define a morphism of simplicial spaces $d : uX \longrightarrow cX_{-1}$ in simplicial dimension k by the unique face morphism $X_n \longrightarrow X_{-1}$. These morphisms induce based continuous maps on geometric realizations

$$|s_{-1}| : |cX_{-1}| \longrightarrow |uX| \quad \text{respectively} \quad |d| : |uX| \longrightarrow |cX_{-1}| \; .$$

Since the composite $ds_{-1} : cX_{-1} \longrightarrow cX_{-1}$ is the identity, so is the composite of the realizations of d and s_{-1}. A homotopy of the other composite $|s_{-1}| \circ |d| : |uX| \longrightarrow |uX|$ is given by the realization of the morphism of simplicial spaces

$$H \;:\; uX \times \Delta[1] \;\longrightarrow\; uX$$

defined in simplicial dimension k by [...] □

Proposition 3.8. *Let X be an augmented simplicial abelian group which admits extra degeneracies. Then the chain complex*

$$\cdots \longrightarrow X_{n+1} \xrightarrow{\partial} X_n \xrightarrow{\partial} \cdots X_1 \xrightarrow{\partial} X_0 \xrightarrow{\partial} X_{-1} \longrightarrow 0$$

is exact.

Proof. □

4. Equivariant homotopy theory

In this section we review some facts about equivariant homotopy theory over finite groups, with emphasis on model structures. We let G be a finite group, the most important case for us will be when $G = \Sigma_n$ is a symmetric group.

We denote by $\mathbf{T}^G$ respectively $\mathbf{sS}^G$ the categories of G-objects in $\mathbf{T}$ or $\mathbf{sS}$. So in the first case an object is a based compactly generated spaces equipped with a continuous, based left G-action. In the second case an object is a based simplicial X set equipped with an associative and unital action morphism $G \times X \longrightarrow X$ which fixes the basepoint; the category $\mathbf{sS}^G$ can equivalently be described as the category of simplicial objects of left G-sets equipped with G-invariant basepoints. Morphisms in $\mathbf{T}^G$ respectively $\mathbf{sS}^G$ are those morphisms in $\mathbf{T}$ or $\mathbf{sS}$ which commute with the G-action. [limits and colimits in underlying categories $\mathbf{T}$ respectively $\mathbf{sS}$; symmetric monoidal closed structure...]

The following terminology is useful for naming the various model structures. A morphism $f : X \longrightarrow Y$ of based G-spaces is a

- *G-cell complex* [...]
- *G-cofibration* if it is a retract of a G-cell complex;
- *free G-cofibration* if it is a retract of a free G-cell complex;
- *weak G-equivalence* (respectively *weak G-fibration*) if the underlying map of spaces is a weak equivalence (respectively fibration) after forgetting the group action;
- *strong G-equivalence* (respectively *strong G-fibration*) if for every subgroup K of G the map of K-fixed points $f^K : X^K \longrightarrow Y^K$ is a weak equivalence (respectively fibration) of spaces.

We note that relative G-CW complexes are G-cell complexes, and thus G-cofibrations, but in a G-cell complex the cells need not be attached in the order of dimension.

A morphism $f : X \longrightarrow Y$ of based G-simplicial sets is a

- *G-cofibration* if it is a monomorphism, i.e., injective in every simplicial dimension;
- *free G-cofibration* if it is a monomorphism a nd in every simplicial dimension n the action of G on Y_n is free away from the image $f(X_n)$;
- *weak G-equivalence* (respectively *weak G-fibration*) if the underlying map of simplicial sets is a weak equivalence (respectively fibration) after forgetting the group action;
- *strong G-equivalence* (respectively *strong G-fibration*) if for every subgroup K of G the map of K-fixed points $f^K : X^K \longrightarrow Y^K$ is a weak equivalence (respectively fibration) of simplicial sets.

Theorem 4.1 (Weak equivariant model structure)**.** *For every finite group G the weak G-equivalences, weak G-fibrations and free G-cofibrations make the categories $\mathbf{T}^G$ of based G-spaces and $\mathbf{sS}^G$ of based G-simplicial sets into proper model categories. [generators, monoidal]*

The weak equivariant model structure is a special case of Quillen's model structure [**62**, Ch. II.4, Thm. 4] on the category of simplicial objects in a category with sufficiently many projectives. [other refs ? case of spaces ?]

Theorem 4.2 (Strong equivariant model structure)**.** *For every finite group G the strong G-equivalences, strong G-fibrations and G-cofibrations the G-cofibrations, strong G-fibrations and strong G-equivalences make the categories $\mathbf{T}^G$ of based G-spaces and $\mathbf{sS}^G$ of based G-simplicial sets into proper model categories. [generators, monoidal]*

[can do this more generally for a family of subgroups...]

[for $\mathbf{T}^G$: Strom model structures a la Cole]

There is another pair of equivariant model structure which is relevant for the study of symmetric spectra, namely the *mixed model structure*. [better name ?] The word 'mixed' refers to the fact that this model structure has the same weak equivalences as the weak equivariant model structure, but the same cofibrations as the strong equivalent model structure. There has to be a new class of fibrations then, which we now define. The *homotopy fixed points* of a G-space (or G-simplicial set) X is the space (simplicial set) $X^{hG} = \mathrm{map}(EG, X)^G$ of G-equivariant maps from the free contractible G-space EG to X. The unique map $EG \longrightarrow *$ is equivariant and induces a natural map $X^G = \mathrm{map}(*, X)^G \longrightarrow \mathrm{map}(EG, X)^G = X^{hG}$ from the fixed points to the homotopy fixed points. A morphism $f : X \longrightarrow Y$ of bases G-spaces or based G-simplicial sets is a *strict G-fibration* if it is a strong G-fibration and for every subgroup H of G the square of spaces (simplicial sets)

$$\begin{tikzcd} X^H \arrow[r] \arrow[d, "f^H"'] & X^{hH} \arrow[d, "f^{hH}"] \\ Y^H \arrow[r] & Y^{hH} \end{tikzcd} \tag{4.3}$$

is homotopy cartesian.

Theorem 4.4 (Mixed equivariant model structure)**.** *For every finite group G the weak G-equivalences, strict G-fibrations and G-cofibrations make the categories $\mathbf{T}^G$ of based G-spaces and $\mathbf{sS}^G$ of based G-simplicial sets into proper model categories. [generators, monoidal]*

In the case of G-simplicial sets, the mixed model structure was established by Shipley in [**77**, Prop. 1.3]. However, Shipley defines the mixed fibrations as the morphism with the right lifting property for G-cofibration which are also weak G-equivalences, so we have to prove that this class coincides with our definition of mixed G-fibration.

Proposition 4.5. *Let G be a finite group and $f : X \longrightarrow Y$ a morphism of based G-spaces or based G-simplicial sets. Then the following are equivalent:*

(i) *f is is a strict G-fibration;*

(ii) *f is a strong G-fibration and the square*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathrm{map}(EG, X) \arrow[d, "\mathrm{map}(EG,f)"] \\ Y \arrow[r] & \mathrm{map}(EG, Y) \end{tikzcd} \tag{4.6}$$

is homotopy cartesian in the strong G-equivariant model structure;

(iii) *f has the right lifting property for all G-cofibrations which are weak equivalences after forgetting the group action.*

Proof. [fix the proof... and generalize to general ground model category] (i)$\Longrightarrow$(ii) Strict G-fibrations are strong G-fibrations by definition, so it remains to show that the square 4.6 is strongly G-homotopy cartesian. Since f is a strong G-fibration and EG is G-cofibrant, the map $\mathrm{map}(EG, f)$ is a strong G-fibration by the [adjoint of] the pushout product property. So we may show that the map $X \longrightarrow Y \times_{\mathrm{map}(EG,Y)} \mathrm{map}(EG, X)$ is a strong G-equivalence.

We fix a G-equivariant based map $f : X \longrightarrow Y$. Let us first assume that f has the right lifting property for all G-cofibrations which are weak G-equivalences. Then in particular f has the right lifting property for all G-cofibrations which are strong G-equivalences, so f is a strong G-fibrations, and it remains to show that the square (4.3) is homotopy cartesian for every subgroup H of G.

We claim that for every G-cofibration $K \longrightarrow L$ which is also a weak G-equivalence the induced map

$$\mathrm{map}(L, X) \longrightarrow \mathrm{map}(K, X) \times_{\mathrm{map}(K,Y)} \mathrm{map}(L, Y) \tag{4.7}$$

is a strong G-fibration and strong G-equivalence. To prove this, we note that for all subgroups H of G and all boundary inclusions, the G-equivariant pushout product morphism

$$G/H \times (L \times \partial\Delta[n] \cup_{K \times \partial\Delta^i[n]} K \times \Delta[n]) \longrightarrow G/H \times L \times \Delta[n]$$

is a G-cofibration and weak G-equivalence. Since f has the right lifting property for such maps, by adjointness the H-fixed points of the map (4.7) have the right lifting property for all boundary inclusions, so they are acyclic fibrations of spaces (simplicial sets). Since this holds for all subgroups H of G, the map (4.7) is a strong G-acyclic fibration.

[...fix this...] Now we show that the square of property (ii) is G-homotopy cartesian. Since f is G-fibration, so is the morphism $\mathrm{map}(EG, f)$, and hence it suffices to show that the morphism

$$X \longrightarrow Y \times_{\mathrm{map}(EG,Y)} \mathrm{map}(EG, X)$$

is a G-weak equivalence. The inclusion $EG \longrightarrow C(EG)$ of EG into its cone is G-equivariant and an injective weak equivalence of underlying simplicial sets (but not a G-weak equivalence !). So by the previous paragraph the induced morphism

$$\mathrm{map}(C(EG), X) \longrightarrow \mathrm{map}(EG, X) \times_{\mathrm{map}(EG,Y)} \mathrm{map}(C(EG), Y)$$

is a G-acyclic fibration. In the commutative square

$$\begin{tikzcd}
X = \mathrm{map}(*, X) \arrow[r] \arrow[d] & \mathrm{map}(EG, X) \times_{\mathrm{map}(EG,Y)} \mathrm{map}(*, Y) \arrow[d] \\
\mathrm{map}(C(EG), X) \arrow[r] & \mathrm{map}(EG, X) \times_{\mathrm{map}(EG,Y)} \mathrm{map}(C(EG), Y)
\end{tikzcd}$$

the vertical maps are induced by the unique morphism $C(EG) \longrightarrow *$ which is a G-equivariant homotopy equivalence, so induces a homotopy equivalence on mapping spaces. So the top horizontal map is a G-weak equivalence since the other three maps are.

For the other direction we assume that f is a strict G-fibration. Let $i : K \longrightarrow L$ be a G-cofibration which is also a weak G-equivalence; we have to show that the pair (i, f) has the lifting property. Then i is a cofibration in the strong equivariant model structure. Since that model structure is monoidal, the induced map

$$\mathrm{map}(L, X) \longrightarrow \mathrm{map}(K, X) \times_{\mathrm{map}(K,Y)} \mathrm{map}(L, Y)$$

is a strong G-fibration. We show that it is also a strong G-equivalence, [thus a G-acyclic fibration]. Since the square is G-homotopy cartesian, we can replace the G-fibration $f : X \longrightarrow Y$ be the G-fibration $\mathrm{map}(EG, f)$ and show that the G-fibration.

$$\mathrm{map}(L, \mathrm{map}(EG, X)) \longrightarrow \mathrm{map}(K, \mathrm{map}(EG, X)) \times_{\mathrm{map}(K,\mathrm{map}(EG,Y))} \mathrm{map}(L, \mathrm{map}(EG, Y))$$

is a G-weak equivalence. This map is isomorphic to

$$\mathrm{map}(L \times EG, X) \longrightarrow \mathrm{map}(K \times EG, X) \times_{\mathrm{map}(K \times EG,Y)} \mathrm{map}(L \times EG, Y) .$$

What we have gained now is that the morphism $i \times \mathrm{Id} : K \times EG \longrightarrow L \times EG$ is a G-equivariant weak equivalence between *free* G-simplicial sets, thus a G-weak equivalence. So the latter morphism is a G-acyclic fibration by the adjoint of the pushout product property.

By taking G-fixed points we then get an acyclic fibration of simplicial sets

$$\mathrm{map}_G(L, X) \longrightarrow \mathrm{map}_G(K, X) \times_{\mathrm{map}_G(K,Y)} \mathrm{map}_G(L, Y)$$

which is in particular surjective on vertices. This exactly means that $f : X \longrightarrow Y$ has the right lifting property with respect to $i : K \longrightarrow L$. □

The fact that the strong and mixed equivariant model structures share the same class of cofibrations implies that they also share the same acyclic fibrations. In other words, for a morphism $f : X \longrightarrow Y$ in $\mathbf{T}^G$ or $\mathbf{sS}^G$ the following are equivalence:

- f is a strong G-equivalence and strong G-fibration;

- f is a weak G-equivalence and strong G-fibration and for every subgroup K of G the square (4.3) of spaces (simplicial sets) is homotopy cartesian;
- f is a weak G-equivalence and mixed G-fibration.

For every finite group G the category of based G-spaces (compactly generated and weak Hausdorff) and the category of based G-simplicial sets admit three model structures in which equivalence are the *weak equivariant equivalences*, i.e., those equivariant morphisms which are weak equivalences of underlying spaces (respectively simplicial sets). We call these three model structure the *weak*, *tight* and *mixed* model structures. Propositions **??** can be applied to each of these and yields the projective, tight respective flat level model structure of Theorem **??** respectively of Theorem **??**. We discuss each of the three cases in slightly more detail.

Example 4.8 (Projective level model structure)**.** For every finite group G the category of based G-spaces (compactly generated and weak Hausdorff) and the category of based G-simplicial sets admit the following *weak equivariant model structures*. An equivariant morphism $f : A \longrightarrow B$ of G-spaces (G-simplicial sets) is a weak equivalence (resp. fibration) if and only if it is a weak equivalence (resp. Serre/Kan fibrations) of spaces (simplicial sets) after forgetting the G-action. In topological context the morphism f is a cofibration if it is a retract of a free G-cell complex. In the simplicial context, f is a cofibration if and only if it is injective and the action of G on B is dimensionwise free away from the image of f.

In both contexts, the cofibrations can equivalently be described as those retracts G-cell complexes G-CW-complex such that for every non-trivial subgroup K of G the induced map on K-fixed points $f^K : A^K \longrightarrow B^K$ is an isomorphism. [simplicial...] [More details in the appendix...] [First reference: Quillen [**62**]] Since fibrations and weak equivalences are defined on underlying spaces (simplicial sets), restriction along every group homomorphism $f : H \longrightarrow G$ preserves fibrations and weak equivalences, so f^* is a right Quillen functor. So as G-varies, the weak equivariant model structures define a Σ-model structure on the category of based spaces respectively simplicial sets.

Proposition **??** provides an associated level model structure on symmetric spectra of spaces (resp. simplicial sets), which is exact the *projective level model structure* of Theorem **??**. The weak equivariant model structures are proper and have functorial factorizations, so the projective level model structure are also proper and have functorial factorizations. This proves part (i) of Theorem **??** respectively of Theorem **??**.

Example 4.9 (Tight level model structure)**.** For every finite group G the category of based G-spaces (compactly generated and weak Hausdorff) and the category of based G-simplicial sets admit the following *tight equivariant model structures*. The weak equivalences are again the equivariant weak equivalences (equivariant morphisms which are weak equivalences on underlying spaces or simplicial sets). A morphism $f : A \longrightarrow B$ of G-spaces (G-simplicial sets) is a fibration if and only if it is a strong equivariant fibration, i.e., for every subgroup K of G the induced map on K-fixed points $f^K : A^K \longrightarrow B^K$ is a Serre fibration (respectively Kan fibration). The cofibrations are those retracts G-cell complexes G-CW-complex such that for every non-trivial subgroup K of G the induced map on K-fixed points $f^K : A^K \longrightarrow B^K$ is a weak equivalence of spaces (resp. simplicial sets). [simplicial...] [More details in the appendix...] [First reference: Cole [**18**]] Since the weak equivalences are defined on underlying spaces (simplicial sets), restriction along every group homomorphism $f : H \longrightarrow G$ preserves weak equivalences. [Fibrations...], so f^* is a right Quillen functor. So as G-varies, the tight equivariant model structures define a Σ-model structure on the category of based spaces respectively simplicial sets.

Proposition **??** provides an associated level model structure on symmetric spectra of spaces (resp. simplicial sets), which is exact the *tight level model structure* of Theorem **??**. The tight equivariant model structures are proper and have functorial factorizations, so the tight level model structure are also proper and have functorial factorizations. This proves part (ii) of Theorem **??** respectively of Theorem **??**.

Example 4.10 (Flat level model structure)**.** [change projective to flat...] For every finite group G the category of based G-spaces (compactly generated and weak Hausdorff) and the category of based G-simplicial sets admit the following *weak equivariant model structures*. An equivariant morphism $f : A \longrightarrow B$ of

G-spaces (G-simplicial sets) is a weak equivalence (resp. fibration) if and only if it is a weak equivalence (resp. Serre/Kan fibrations) of spaces (simplicial sets) after forgetting the G-action. In topological context the morphism f is a cofibration if it is a retract of a free G-cell complex. In the simplicial context, f is a cofibration if and only if it is injective and the action of G on B is dimensionwise free away from the image of f.

In both contexts, the cofibrations can equivalently be described as those retracts G-cell complexes G-CW-complex such that for every non-trivial subgroup K of G the induced map on K-fixed points $f^K : A^K \longrightarrow B^K$ is an isomorphism. [simplicial...] [More details in the appendix...] [First reference: Quillen [**62**]] Since fibrations and weak equivalences are defined on underlying spaces (simplicial sets), restriction along every group homomorphism $f : H \longrightarrow G$ preserves fibrations and weak equivalences, so f^* is a right Quillen functor. So as G-varies, the weak equivariant model structures define a Σ-model structure on the category of based spaces respectively simplicial sets.

Proposition **??** provides an associated level model structure on symmetric spectra of spaces (resp. simplicial sets), which is exact the *projective level model structure* of Theorem **??**. The weak equivariant model structures are proper and have functorial factorizations, so the projective level model structure are also proper and have functorial factorizations. This proves part (i) of Theorem **??** respectively of Theorem **??**.

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Now for some homotopy theory. Given enriched functors $\alpha : \mathbf{J} \longrightarrow \mathbf{J}'$ and $X : \mathbf{J} \longrightarrow \mathbf{T}$ where $\mathbf{J}$ and $\mathbf{J}'$ are small, we construct left and right *homotopy Kan extensions* of X along α. For this purpose we first define a simplicial object $B_\bullet(\alpha, \mathbf{J}, X)$ in the category of enriched functors from $\mathbf{J}'$ to $\mathbf{T}$, called the *two-sided bar construction.* The functor of k-simplices is given by

$$B_k(\alpha, \mathbf{J}, X) \ = \bigvee_{(i_0,\dots,i_k)\in(\mathrm{ob}\mathbf{J})^{k+1}} F^{\mathbf{J}'}_{i_k} \wedge \mathbf{J}(i_{k-1}, i_k) \wedge \cdots \wedge \mathbf{J}(i_0, i_1) \wedge X_{i_0} \ .$$

The face maps are given by the coaction of $\mathbf{J}$ on $F^{\mathbf{J}'}_{i_k}$ through α, composition in $\mathbf{J}$ respectively the action on X [spell out]. The degeneracy maps are given by the identity morphisms in $\mathbf{J}$. We note that the coequalizer of the two face map $d_0.d_1 : B_1(\alpha, \mathbf{J}, X) \longrightarrow B_0(\alpha, \mathbf{J}, X)$ is precisely the coend $F^{\mathbf{J}'}_\bullet \wedge_{\mathbf{J}} X_\bullet$, which is the (ordinary) left Kan extension of X along α.

We define the *homotopy Kan extension* $\alpha^h_*(X)$ as the realization of this simplicial functor, i.e.,

$$\alpha^h_*(X) \ = \ |B_\bullet(\alpha, \mathbf{J}, X)| \ .$$

[define augmented simplicial object with $B_{-1}(\alpha, \mathbf{J}, X) = X$] We thus obtain a natural morphism $\alpha^h_*(X) \longrightarrow \alpha_*(X)$.

Now consider the special case where $\mathbf{J} = J^+$ and $\mathbf{J}' = (J')^+$ arise from an ordinary categories J by giving then the discrete topology and adding disjoint basepoints to the morphism sets, and where $\alpha : \mathbf{J} \longrightarrow \mathbf{J}'$ arises similarly from an ordinary functor $a : J \longrightarrow J'$. Then the homotopy Kan extensions $\alpha^h_*(X)$ can be presented in a slightly different form. In fact the $\mathbf{J}'$-functor $B_k(\alpha, \mathbf{J}, X)$ is then isomorphic to

$$\bigvee_{(a_1,\dots,a_k)\in(NJ)_k} F^{\mathbf{J}'}_{i_k} \wedge X_{i_0} \ ;$$

this wedge is indexed over all k-simplices of the nerve of the category J, i.e., strings of k composable morphisms

$$i_0 \xrightarrow{a_1} i_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} i_k$$

in J. The face maps are given by the coaction of $\mathbf{J}$ on $F^{\mathbf{J}'}_{i_k}$ through α, composition in $\mathbf{J}$ respectively the action on X [spell out]. The degeneracy maps are given by the identity morphisms in $\mathbf{J}$.

We discuss two important special cases of homotopy Kan extensions.

Homotopy colimits. We can take the target category $\mathbf{J}'$ as the trivial category $*$, i.e., with only one object and one identity morphism. If $\alpha : \mathbf{J} \longrightarrow *$ [should the target have morphisms S^0?] is the unique functor (which is automatically enriched), then the (ordinary) Kan extension $\alpha_* X$ of an enriched functor

$X : \mathbf{J} \longrightarrow \mathbf{T}$ is the enriched colimit $\mathrm{colim}_{\mathbf{J}} X$ [explain]. If we now use the 'homotopy' version of the Kan extension we obtain the *homotopy colimit* $\alpha_*^h(X)$. Since the functor α is uniquely determined it is customary to use a different notation and write $\mathrm{hocolim}_{\mathbf{J}} X$ for $\alpha_*^h(X)$.

In the case where $\mathbf{J} = J^+$ arises from an ordinary category J by giving it the discrete topology and adding disjoint basepoints to the morphism sets, the enriched colimit $\mathrm{colim}_{\mathbf{J}} X$ is just the ordinary colimit $\mathrm{colim}_J X$ of X over J. The homotopy colimit $\mathrm{hocolim}_{\mathbf{J}} X$ is then also written $\mathrm{hocolim}_J X$. [nerve and classifying space as hocolims]

Homotopy coends. Let $G : \mathbf{J}^{op} \longrightarrow \mathbf{T}$ and $X : \mathbf{J} \longrightarrow \mathbf{T}$ be continuous functors. We can define a simplicial object $B_\bullet(X, \mathbf{J}, \alpha)$ in the category of enriched functors from $\mathbf{J}'$ to $\mathbf{T}$, called the *two-sided bar construction.* The functor of k-simplices is given by

$$B_k(G, \mathbf{J}, X) \ = \bigvee_{(i_0,\dots,i_k)\in(\mathrm{ob}\mathbf{J})^{k+1}} G^{i_k} \wedge \mathbf{J}(i_{k-1}, i_k) \wedge \cdots \wedge \mathbf{J}(i_0, i_1) \wedge X_{i_0} \ .$$

The face maps are given by the coaction of $\mathbf{J}$ on G^{i_k} through α, composition in $\mathbf{J}$ respectively the action on X [spell out]. The degeneracy maps are given by the identity morphisms in $\mathbf{J}$. We note that the coequalizer of the two face map $d_0.d_1 : B_1(G, \mathbf{J}, X) \longrightarrow B_0(G, \mathbf{J}, X)$ is precisely the coend $G \wedge_{\mathbf{J}} X$.

We define the *homotopy coend* $G \wedge^h_{\mathbf{J}} X$ as the realization of this simplicial functor, i.e.,

$$G \wedge^h_{\mathbf{J}} X \ = \ |B_\bullet(G, \mathbf{J}, X)| \ .$$

[define augmented simplicial object with $B_{-1}(G, \mathbf{J}, X) = G \wedge_{\mathbf{J}} X$] We thus obtain a natural morphism $G \wedge^h_{\mathbf{J}} X \longrightarrow G \wedge^h_{\mathbf{J}} X$ from the homotopy coend to the coend.

[under cofibrancy conditions, weak equivalences in G or X induce weak equivalences of homotopy coends]

Proposition 4.11. (i) *Homotopy coend commutes with colimits and smash product with a based space in both variables.*

(ii) *Natural isomorphism $G \wedge^h_{\mathbf{J}} X \cong X^{op} \wedge^h_{\mathbf{J}^{op}} G^{op}$*

(iii) *Let $g : G \longrightarrow G'$ be a natural weak equivalence of enriched $\mathbf{J}^{op}$-functors and let $\varphi : X \longrightarrow X'$ be a natural weak equivalence of enriched $\mathbf{J}$-functors. Suppose in addition that G, G', X and X' are objectwise cofibrant as based spaces [+cofibrancy in $\mathbf{J}$]. Then the induced map $g \wedge^h_{\mathbf{J}} \varphi : G \wedge^h_{\mathbf{J}} X \longrightarrow G' \wedge^h_{\mathbf{J}} X'$ of homotopy coends is a weak equivalence.*

(iv) *Let X be an enriched $\mathbf{J}$-functor and j an object of $\mathbf{J}$. Then the augmentation*

$$\mathbf{J}(-, j) \wedge^h_{\mathbf{J}} X \ \longrightarrow \ X_j$$

is a homotopy equivalence.

(v) *Let G be an enriched $\mathbf{J}^{op}$-functor and j an object of $\mathbf{J}$. Then the augmentation*

$$G \wedge^h_{\mathbf{J}} \mathbf{J}(j, -) \ \longrightarrow \ G^j$$

is a homotopy equivalence.

Proof. (?) Use the 'orientation reversal' automorphism of the simplicial category $\mathbf{\Delta}$.

(ii) The space of k-simplices of the simplicial space $B_\bullet(\mathbf{J}(-, j), \mathbf{J}, X)$ is given by

$$B_k(\mathbf{J}(-, j), \mathbf{J}, X) \ = \bigvee_{(i_0,\dots,i_k)\in(\mathrm{ob}\mathbf{J})^{k+1}} \mathbf{J}(i_k, j) \wedge \mathbf{J}(i_{k-1}, i_k) \wedge \cdots \wedge \mathbf{J}(i_0, i_1) \wedge X_{i_0} \ .$$

Since j is fixed we can define 'extra degeneracy maps'

$$s_{-1} \ : \ B_k(\mathbf{J}(-, j), \mathbf{J}, X) \ \longrightarrow \ B_{k+1}(\mathbf{J}(-, j), \mathbf{J}, X)$$

by including into the summand where $i_k = j$ via the identity morphism of the object j. For $k \geq -1$, where $B_{-1}(\mathbf{J}(-, j), \mathbf{J}, X) = X_j$. The extra degeneracies satisfy the relations [...] which means that the realize to a homotopy equivalence between $\mathbf{J}(-, j) \wedge^h_{\mathbf{J}} X = |B_\bullet(\mathbf{J}(-, j), \mathbf{J}, X)|$ and X_j.

(iii) If we replace $\mathbf{J}$ by the opposite category $\mathbf{J}^{op}$, this becomes an instance of (ii). □

Bar resolutions. We have a bifunctor $\mathbf{J}(-,-)^{op}_{\mathbf{J}} \wedge \mathbf{J} \longrightarrow \mathbf{T}$; if we fix a particular value in one if the two variables, we obtain co- respectively contravariant representable functors. For fixed j, the homotopy coend $\mathbf{J}(-,j) \wedge^h_{\mathbf{J}} X$ yields a based space, and as j varies we obtain another covariant functor $X^\natural : \mathbf{J} \longrightarrow \mathbf{T}$ given by $(X^\natural)_j = \mathbf{J}(-,j) \wedge^h_{\mathbf{J}} X$. The augmentations provide a natural transformation of $\mathbf{J}$-functors $X^\natural \longrightarrow X$ which is objectwise a homotopy equivalence by Proposition 4.11.

[Can also view this as a special case of the homotopy Kan extension, $X^\natural = \mathrm{Id}^h_*(X)$, by taking α as the identity functor the category $\mathbf{J}$. Then restriction along α does not do anything, and so the ordinary left and right Kan extension functors along $\alpha = \mathrm{Id}_{\mathbf{J}}$ do not do anything either.] Homotopy Kan extension, however, do have an effect. We refer to $X^\natural$ as the *bar resolution* of X. The advantage of the resolution $X^\natural$ over the original functor is that $X^\natural$ tends to be 'free' (or 'projective', or 'cofibrant'), see for example [...]

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