GEOMETRY OF DIFFERENTIAL EQUATIONS

BORIS KRUGLIKOV, VALENTIN LYCHAGIN

Abstract. We review geometric and algebraic methods of investigations of systems of partial differential equations. Classical and modern approaches are reported.

Contents

Key words and phrases. System of PDEs, characteristics, symbols, Spencer cohomology, compatibility, integrability.

Introduction

In this paper we present an overview of geometric and algebraic methods in the study of differential equations. The latter are considered as co-filtered submanifolds in the spaces of jets, possibly with singularities. Investigation of singularities is a very subtle question, so that we will be mainly assuming regularity.

Jets are formal substitutions to actual derivatives and certain geometric structure retains this meaning, namely the Cartan distribution. Thus geometry enters differential equations. Differential-geometric methods, in particular connections and curvatures, are our basic tools.

Since differential operators form a module, (differential) algebra is also an essential component in the study of differential equations. This algebra is non-commutative, but the associated graded object is commutative, and so commutative algebra plays a central role in the investigation.

Thus we get the main ingredients and the theory is based on the interplay between them. Our exposition will center around compatibility theory, followed by formal/local (and only eventually global) integrability. So we are mainly interested in the cases, when the number of independent variables is at least two. Therefore we consider systems of partial differential equations (PDEs) and discuss methods of investigation of their compatibility, solvability or integrability.

Part of the theory is trivial for ODEs, but some methods are useful for establishing exact solutions, discovering general solutions and analysis of their singularities also for this case.

The exposition is brief and we don't prove or try to explain the results in details. The reader is referred to the cited papers/books. We have not covered some important topics, like transformation theory, equivalence problem, complete integrability, integro-differential equations etc. However our short panorama of the general theory of differential equations should help in understanding the modern progress and a possible future development.

1. Geometry of jets-spaces

1.1. **Jet spaces.** Let us fix a smooth manifold E of dimension $m+n$ and consider submanifolds in E of the fixed codimension m. We say that two such submanifolds N_1 and N_2 are k-equivalent at a point $a \in N_1 \cap N_2$ if they are tangent (classically "have contact") of order $k \geq 0$ at this point.

Denote by $[N]_a^k$ the k-equivalence class of a submanifold $N \subset E$ at the point $a \in N$. This class is called *k-jet*[1] of N at a. Let $J_a^k(E,m)$ be the space of all k-jets of all submanifolds of codimension m at the point a and let $J^k(E,m) = \cup_{a\in E} J_a^k(E,m)$ be the space of all k-jets.

The reductions k-jets to l-jets $[N]_a^k \mapsto [N]_a^l$ gives rise the natural projections $\pi_{k,l} : J^k(E,m) \to J^l(E,m)$ for all $k > l \geq 0$. The jet spaces carry a structure of smooth manifolds and the projections $\pi_{k,l}$ are smooth bundles.

For small values of k these bundles have a simple description. Thus $J^0(E,m) = E$ and $J^1(E,m) = \mathrm{Gr_n}(TE)$ is the Grassman bundle over E.

For each submanifold $N \subset E$ of $\operatorname{codim} N = m$ there is the natural embedding $j_k : N \to J^k(E,m)$, $N \ni a \mapsto [N]_a^k \in J^k(E,m)$ and $\pi_{k,l} \circ j_k = j_l$. The submanifolds $j_k(N) \subset J^k(E,m)$ are called the k-jet extensions of N.

Let $\pi : E_\pi \to M$ be a rank m bundle over an n-dimensional manifold. Local sections $s \in C^\infty_{\mathrm{loc}}(\pi)$ are submanifolds of the total space E_π of codimension m that are transversal to the fibres of the projection π. Let $[s]_x^k$ denote k-jet of the submanifold $s(M)$ at the point $a = s(x)$, which is also called k-jet of the section s at x.

Denote by $J_x^k(\pi) \subset \cup_{a\in\pi^{-1}(x)} J_a^k(E_\pi, m)$ the space of all k-jets of the local sections at the point $x \in M$ and by $J^k(\pi) \subset J^k(E_\pi, m)$ the space of all k-jets. $J^k(\pi)$ is an open dense subset of the latter space and thus the projections $\pi_{k,l} : J^k\pi \to J^l\pi$ form smooth fiber bundles for all $k > l$.

Projection to the base will be denoted by $\pi_k : J^k\pi \to M$. Then local sections $s \in C^\infty_{\mathrm{loc}}(M)$ have k-jet extensions $j_k(s) \in C^\infty_{\mathrm{loc}}(\pi_k)$ defined as $j_k(s)(x) = [s]_x^k$. Points of $J^k\pi$ will be also denoted by x_k and then their projections are: $\pi_{k,l}(x_k) = x_l$, $\pi_k(x_k) = x$.

If we assume π is a smooth vector bundle (without loss of generality for our purposes we'll be doing it in the sequel), then $\pi_{k,l}$ are vector bundles and the following sequences are exact:

$$0 \to S^k T^* \otimes \pi \to J^k(\pi) \xrightarrow{\pi_{k,k-1}} J^{k-1}(\pi) \to 0,$$

where $T^* = T^*M$ is the cotangent bundle of M.

Smooth maps $f : M \to N$ can be identified with sections s_f of the trivial bundle $\pi : E = M \times N \to M$ and k-jets of maps $[f]_x^k$ are k-jets of these sections. We denote the space of all k-jets of maps f by $J^k(M,N)$.

For small values of k we have: $J^0(M,N) = M \times N$ and $J^1_{(x,y)}(M,N) = \mathrm{Hom}(T_xM, T_yN)$.

For $N = \mathbb{R}$ we denote $J^k(M,\mathbb{R}) = J^k(M)$. In this case $J^1(M) = T^*M \times \mathbb{R}$. In the dual case $J^1(\mathbb{R}, M) = TM \times \mathbb{R}$. The spaces $J^k(\mathbb{R}, M)$ are manifolds of "higher velocities".

[1] The notion of jet was introduced by Ehresmann [Eh], though was essentially in use already in S.Lie's time [Lie_1].

1.2. **Differential groups and affine structures.** Let us denote by $\mathbf{G}^k_{x,y}$ the subset of k-jets of local diffeomorphisms in $J^k_{(x,y)}(M,M)$. Then composition of diffeomorphisms defines the group structure on $\mathbf{G}^k_{x,x}$. This Lie group is called a complete *differential group of order* k (this construction is a basic example of groupoid and is fundamental for the notion of pseudogroups, see §4.5). The group $\mathbf{G}^1_{x,x}$ is the linear group $\mathrm{GL}(T_xM)$.

From §1.1 we deduce the group epimorphisms $\pi_{k,l}:\mathbf{G}^k_{x,x}\to\mathbf{G}^l_{x,x}$ for $l<k$ and exact sequences of groups for $k\geq 2$:

$$0\to S^kT^*_x\otimes T_x\to\mathbf{G}^k_{x,x}\stackrel{\pi_{k,k-1}}{\longrightarrow}\mathbf{G}^{k-1}_{x,x}\to 1.$$

In other words, groups $\mathbf{G}^k_{x,x}$ are extensions of the general linear group $\mathrm{GL}(T_xM)$ by means of abelian groups $S^kT^*_x\otimes T_x$, $k>1$.

The differential groups $\mathbf{G}^k_{x,x}(M)$ act naturally on the jet spaces:

$$\mathbf{G}^k_{x,x}(M)\times J^k_x(M,m)\to J^k_x(M,m),\quad [F]^k_x\times[N]^k_x\mapsto[F(N)]^k_x.$$

The kernel $S^kT^*_x\otimes T_x$ of the projection $\mathbf{G}^k_{x,x}\to\mathbf{G}^{k-1}_{x,x}$ is the abelian group, which acts transitively on the fibre $F(x_{k-1})=\pi^{-1}_{k,k-1}(x_{k-1})$, $x_{k-1}=[N]^{k-1}_x$, of the projection $\pi_{k,k-1}:J^k_x(M,m)\to J^{k-1}_x(M,m)$. Therefore, the fibre $F(x_{k-1})$ is an affine space. The associated vector space for the fibre is

$$S^kT^*_xN\otimes\nu_x(N),$$

where $\nu_x(N)=T_xM/T_xN$ is the *normal space* to N at the point $x\in N$.

Thus, the bundle $\pi_{k,k-1}:J^k(M,m)\to J^{k-1}(M,m)$ has a canonical *affine structure* for $k\geq 2$. Moreover, each local diffeomorphism $F:M\to M$ has the natural lifts to local diffeomorphisms $F^{(k)}:J^k(M,m)\to J^k(M,m)$, $[N]^k_x\mapsto[F(N)]^k_{F(x)}$ preserving the affine structures.

For a vector bundle $\pi:E_\pi\to M$ the affine structure in the fibers of $J^k\pi\to J^{k-1}\pi$ coincides with the structure induced by the vector bundle structure. If π is a fiber bundle, the preceding construction provides the affine structure. This gives rise the following construction.

Let $M\subset E$ be a submanifold of codimension m and $U\supset M$ be its neighborhood, which is transversally foliated, so that the projection along the fibers $\pi:U\to M$ can be identified with the normal bundle. We can denote $U=E_\pi$. Then the embedding $\varkappa:E_\pi\subset E$ induces the embedding $\varkappa^{(k)}:J^k(\pi)\hookrightarrow J^k(E,m)$ with $M^{(k)}$ being the zero section. The image is an open neighborhood and the affine structure on $J^k(\pi)\to J^{k-1}(\pi)$ induces the affine structure on $J^k(E,m)\to J^{k-1}(E,m)$, so that both projections agree and are denoted by the same symbol $\pi_{k,k-1}$. These neighborhoods $J^k(\pi)\hookrightarrow J^k(E,m)$ together with the maps $\varkappa^{(k)}$ are called *affine charts*.

Remark 1. *Usage of affine charts in general jet-spaces is the exact analog of independence condition in exterior differential systems. Most of the theory works for spaces $J^k(E,m)$, though for simplicity we will often restrict to the case of jets of sections $J^k\pi$.*

1.3. **Cartan distribution.** In addition to the affine structure on the co-filtration $\pi_{k,k-1} : J^k(E,m) \to J^{k-1}(E,m)$, the space $J^k(E,m)$ bears an additional structure, which allows to distinguish submanifolds $j_k(N) \subset J^k(E,m)$, $N \subset E$, among all submanifolds in E of dimension $n = \dim N$. To describe it denote

$$L(x_{k+1}) = T_{x_k}[j_k(N)] \subset T_{x_k}J^k(\pi), \qquad x_{k+1} = [N]_x^{k+1}$$

(this subspace does not depend on a particular choice of N, but only on x_{k+1}). Define the *Cartan distribution* on the space $J^k(E,m)$ by the formula:

$$\mathcal{C}_k(x_k) = \operatorname{span}\{L(x_{k+1}) : x_{k+1} \in \pi_{k+1,k}^{-1}(x_k)\} = (d\pi_{k,k-1})^{-1}L(x_k).$$

Submanifolds of the form $N^{(k)}$ are clearly integral manifolds of the Cartan distributions such that $\pi_{k,0} : N^{(k)} \to E$ are embeddings. The inverse is also true: if $W \subset J^k(E,m)$ is an integral submanifold of dimension n of the Cartan distribution such that $\pi_{k,0} : W \to E$ is an embedding, then $W = N^{(k)}$ for the submanifold $N = \pi_{k,0}(W) \subset E$. In other words, the Cartan distribution gives a geometrical description for the jet-extensions.

In a similar way one can construct the Cartan distributions for the jet spaces $J^k(\pi)$. Moreover, any affine chart $\varkappa^{(k)} : J^k(\pi) \to J^k(E,m)$ sends the Cartan distribution on $J^k(\pi)$ to the Cartan distribution on $J^k(E,m)$. By using this observation we can restrict ourselves to Cartan distributions on the jet-spaces of sections.

For the case $J^k\pi$, there is a description of the Cartan distribution in terms of differential forms. Namely, let us denote by $\Omega_0^r(J^k\pi)$ the module of π_k-horizontal forms, that is, such differential r-forms ω that $i_X\omega = 0$ for any π_k-vertical vector field X: $d\pi_k(X) = 0$.

These forms can be clearly identified with non-linear differential operators[2] $\operatorname{diff}_k(\pi, \Lambda^r T^*M)$ acting from sections of π to differential r-forms on the manifold M. Indeed the space of such non-linear operators is nothing else than the space of smooth maps $C^\infty(J^k\pi, \Lambda^r T^*M)$.

The composition with the exterior differential $d : \Omega^r(M) \to \Omega^{r+1}(M)$ generates the *total differential* $\widehat{d} : \Omega_0^r(J^k\pi) \to \Omega_0^{r+1}(J^{k+1}\pi)$. The total differential is a differentiation of degree 1 and it satisfies the property $\widehat{d}^2 = 0$.

Hence any function $f \in C^\infty(J^{k-1}\pi)$ defines two differential forms on the jet-space $J^k(\pi)$: $\widehat{d}f \in \Omega_0^1(J^k\pi)$ and $d(\pi_{k,k-1}^* f) = \pi_{k,k-1}^*(df) \in \Omega^1(J^k\pi)$. Both of them coincide on k-jet prolongations $j_k(s)$. Their difference:

$$U(f) = d(\pi_{k,k-1}^* f) - \widehat{d}f \in \Omega^1(J^k\pi)$$

is called the *Cartan form* associated with function $f \in C^\infty(J^{k-1}\pi)$.

The annihilator of the Cartan distribution on $J^k\pi$ is generated by the Cartan forms: $\operatorname{Ann}\mathcal{C}_k(x_k) = \operatorname{span}\{U(f)_{x_k} : f \in C^\infty(J^{k-1}\pi)\}$.

As an example consider the case $m = 1$, $k = 1$. Then the Cartan distribution on $J^1(E,1)$ is the classical contact structure on the space of contact

[2]These will be treated in §2.4. We introduce here only a minor part of the theory.

elements. It is known that it cannot be defined by one differential 1-form. On the other hand, for the affine chart $J^1(M) = T^*M \times \mathbb{R}$ the Cartan distribution (=the standard contact structure) can be defined by one Cartan form $U(u) = du - p\,dq$, where $u : J^1(M) \to \mathbb{R}$ is the natural projection and $p\,dq$ is the Liouville form on T^*M.

1.4. **Lie transformations.** Any local diffeomorphism $F : E \to E$ has prolongations $F^{(k)} : J^k(E, m) \to J^k(E, m)$, $[N]^k_x \mapsto [F(N)]^k_{F(x)}$, and they satisfy: $(F \circ G)^{(k)} = F^{(k)} \circ G^{(k)}$, $\pi_{k,k-1} \circ F^{(k)} = F^{(k-1)} \circ \pi_{k,k-1}$. Moreover, by the construction, the diffeomorphisms $F^{(k)}$ are symmetries of the Cartan distribution, i.e. they preserve $\mathcal{C}_k$.

For $m = 1$ the Cartan distribution on the 1-jet space $J^1(E, 1)$ defines the contact structure, and not all contact diffeomorphisms have the form $F^{(1)}$, where $F : E \to E$. Let $\phi : J^1(E, 1) \to J^1(E, 1)$ be a contact local diffeomorphism and let $x_k = [N]^k_x$. We can consider this point as $(k-1)$-jet of an integral manifold $N^{(1)}$ at the point $x_1 = [N]^1_x$. Then $\phi(N^{(1)})$ is an integral manifold of the contact structure, and it has the form $N^{(1)}_\phi$ for some submanifold $N_\phi \subset E$ if $\pi_{1,0} : \phi(N^{(1)}) \to E$ is an embedding.

Denote by $\Sigma_\phi \subset J^1(E, 1)$ the set of points x_1, where the last condition is not satisfied. Then, for the points $x_k \in J^k(E, 1)$, such that projections $x_1 = \pi_{k,1}(x_k)$ belong to the compliment Σ^c_ϕ, we can define the lift $\phi^{(k-1)}$: $J^k(E, 1) \to J^k(E, 1)$, $[N]^k_x \mapsto [\phi(N^{(1)})]^{(k-1)}_{\phi(x_1)}$. As before we get:

$$(\phi \circ \psi)^{(k-1)} = \phi^{(k-1)} \circ \psi^{(k-1)}, \quad \pi_{k,k-1} \circ \phi^{(k-1)} = \phi^{(k-2)} \circ \pi_{k,k-1}.$$

Diffeomorphisms $F : E \to E$ are also called *point transformations*. So the local diffeomorphisms $F^{(k)}$ and $\phi^{(k-1)}$ are called prolongations of the point transformation F or the contact transformation ϕ respectively.

A local diffeomorphism of $J^k(E, m)$ preserving the Cartan distribution is called a *Lie transformation*. The following theorem is known as Lie-Backlund theorem on prolongations, see [KLV].

Theorem 1. *Any Lie transformation of $J^k(E, m)$ is the prolongation of*

$m \ge 2$: *Local point transformation $F : E \to E$,*
$m = 1$: *Local contact diffeomorphism $\phi : J^1(E, 1) \to J^1(E, 1)$.*

In the same way one can construct prolongations of vector fields on E and contact vector fields on $J^1(E, 1)$ to $J^k(E, m)$ or $J^k(E, 1)$ respectively and the prolongations preserve the Cartan distribution. A vector field on $J^k(E, m)$, which preserves the Cartan distribution, is called a *Lie vector field*. The Lie-Backlund theorem claims that Lie vector fields are prolongations of vector fields on E if $m \ge 2$ or contact vector fields on $J^1(E, 1)$ if $m = 1$.

The same statements hold for $E = E_\pi$, when the jet-space is $J^k\pi$.

Remark 2. *Prolongations $F^{(k)}$ of the point transformations preserve the affine structure for any $k \geq 2$, i.e. starting from the 2^{nd} jets. The prolongations $\phi^{(k)}$ of the contact transformations also preserve the affine structure for $k \geq 2$, but this means starting from the 3^{rd} jets.*

Let us briefly introduce here systems of PDEs[3]. Such a system of pure order k is represented as a smooth subbundle $\mathcal{E} \subset J^k(\pi)$. It is possible to use more general jet-spaces $J^k(E, m)$; exteriour differential systems concern the case $k = 1$. Scalar PDEs correspond to the trivial bundle $E_\pi = M \times \mathbb{R}$ and $\mathcal{E} \subset J^k(M)$.

Solutions of $\mathcal{E}$ on an open set $U_M \subset M$ are sections $s \in C^\infty_{(\text{loc})}(\pi)$ such that $j_k(s)(U_M) \subset \mathcal{E}$. Generalized solutions are n-dimensional integral manifolds W^n of the Cartan distribution such that $W \subset \mathcal{E}$ (in this form there's no difference with equations in the general jet-space $J^k(E, m) \supset \mathcal{E}$). If $\pi_{k,0} : W \to M$ is not an embedding, we call such solution multi-valued or singular.

Another description of generalized solutions are n-dimensional integral manifolds of the induced Cartan distribution $\mathcal{C}_\mathcal{E} = \mathcal{C}_k \cap T\mathcal{E}$. Then internal Lie transformations (finite or infinitesimal) are (local) diffeomorphisms of $\mathcal{E}$ that are symmetries of $\mathcal{C}_\mathcal{E}$ (they transform generalized solutions to generalized solutions [Lie$_2$, LE]).

In general there exist higher internal Lie transformations, which are not prolongations from lower-order jets. But for certain type of systems $\mathcal{E}$ we have the exact analog of Lie-Backlund theorem, see [KLV].

1.5. **Calculations.** A coordinate system (x^i, u^j) on E_π, subordinated to the bundle structure, induces coordinates (x^i, p^j_σ) on $J^k\pi$, where multiindex $\sigma = (i_1, \dots, i_n)$ has length $|\sigma| = i_1 + \dots + i_n \leq k$ and $p^j_\sigma\Bigl([s]^k_x\Bigr) = \dfrac{\partial^{|\sigma|} s^j}{\partial x^\sigma}(x)$.

For a vector field $X \in \mathcal{D}(M)$ the operator of total derivative along X is $\mathcal{D}_X = i_X \circ \widehat{d} : C^\infty(J^k\pi) \to C^\infty(J^{k+1}\pi)$ (this is just a post-composition of a differential operator with Lie derivative along X) and it has the following expression. Let $X = \sum \xi^i \partial_{x^i}$. Then $\mathcal{D}_X = \sum \xi^i \mathcal{D}_i$, where the basis total derivation operator $\mathcal{D}_i = \mathcal{D}_{\partial_{x^i}}$ is given by infinite series

$$\mathcal{D}_i = \partial_{x^i} + \sum p_{\sigma + 1_i} \partial_{p_\sigma}.$$

If in the above sum we restrict $|\sigma| < k$ we get vector fields $\mathcal{D}^{(k)}_i$ on $J^k\pi$. In terms of them the Cartan distribution on J^kM is given by

$$\mathcal{C}_k = \langle \mathcal{D}^{(k)}_i, \partial_{p_\sigma} \rangle_{1 \leq i \leq n, |\sigma| = k} \ .$$

To write it via differential forms note that the operator of total derivative equals $\widehat{d} = \sum \mathcal{D}_i \otimes dx^i$. Thus for $f \in C^\infty(J^{k-1}\pi)$ we get expression for the Cartan forms $U(f) = \sum_{i,j;|\sigma|<k} \bigl((\partial_{p^j_\sigma} f)\, dp^j_\sigma + (\partial_{x^i} f - \mathcal{D}_i f)\, dx^i\bigr)$.

[3] Main definitions come only in §2.3,§3.4 after development of algebraic machinery.

In particular, the differential forms $\omega^j_\sigma = U(p^j_\sigma) = dp^j_\sigma - \sum p^j_{\sigma+1_i} dx^i$ span the annulator of the Cartan distribution, i.e. $\mathcal{C}_k = \operatorname{Ker}\{\omega_\sigma\}_{0\le|\sigma|<k}$.

Finally let us express in coordinates Lie infinitesimal transformations.

Vector field $X = \sum_i a^i(x,u)\,\partial_{x^i} + \sum_j b^j(x,u)\,\partial_{u^j}$ on $E = J^0\pi$ (point transformation) prolongs to

$$X^{(k)} = \sum_i a^i(x,u)\,\mathcal{D}_i^{(k+1)} + \sum_{j;|\sigma|\le k} \mathcal{D}_\sigma(\varphi^j)\,\partial_{p^j_\sigma}, \tag{1}$$

where $\varphi^j = b^j - \sum_{i=1}^n a^i p^j_i$ are components of the so-called *generating function* $\varphi = (\varphi^1,\dots,\varphi^r)$. Though the coefficients of (1) depend seemingly on the $(k+1)$-jets, the Lie field belongs in fact to $\mathcal{D}(J^k\pi)$.

A contact vector field $X^{(1)} = X_\varphi$ on $J^1\pi$ is determined by generating scalar-valued function $\varphi = \varphi(x^i,u,p_i)$ via the formula

$$X^{(1)} = \sum_i \Bigl[\mathcal{D}_i^{(1)}(\varphi)\,\partial_{p_i} - \partial_{p_i}(\varphi)\,\mathcal{D}_i^{(1)}\Bigr] + \varphi\,\partial_u.$$

The prolongation of this field to $J^k\pi$ is given by the formula similar to (1):

$$X^{(k)} = -\sum_i \partial_{p_i}(\varphi)\,\mathcal{D}_i^{(k+1)} + \sum_{|\sigma|\le k} \mathcal{D}_\sigma(\varphi)\,\partial_{p_\sigma}. \tag{2}$$

Again this is a vector field on $J^k\pi$, coinciding with X_φ for $k=1$.

1.6. **Integral Grassmanians.** Denote

$$I_0(x_k) = \{L(x_{k+1}) : x_{k+1} \in F(x_k)\} \subset \mathrm{Gr}_n(T_{x_k}J^k)$$

the Grassmanian of all tangent planes to jet-sections through x_k. The letter I indicates that this can represented as the space of integral elements. Consider for simplicity the space of jets of sections of a vector bundle π.

The map $C^\infty(J^{k-1}\pi) \ni f \mapsto dU(f)|_{\mathcal{C}(x_k)} \in \Lambda^2(\mathcal{C}^*(x_k))$ is a derivation and therefore defines a linear map $\Omega_{x_k} : T^*_{x_{k-1}}(J^{k-1}\pi) \to \Lambda^2(\mathcal{C}^*(x_k))$. Since the latter vanishes on $\mathrm{Im}(d\pi^*_{k-1,k-2})$ it descends to the linear map

$$\Omega_{x_k} : S^{k-1}T_x \otimes \pi^*_x \to \Lambda^2\mathcal{C}(x_k)^*,$$

which is called the *metasymplectic structure* on the Cartan distribution. We treat Ω_{x_k} as a 2-form on $\mathcal{C}(x_k)$ with values in $F(x_{k-2}) \simeq S^{k-1}T^*_x \otimes \pi_x$.

Remark that for the trivial rank 1 bundle $\pi = \mathbf{1}$ and $k=1$ the metasymplectic structure Ω_{x_1} on $J^1(M)$ coincides with the symplectic structure on the Cartan distribution induced by the contact structure.

Call a subspace $L \subset \mathcal{C}(x_k)$ *integral* if $\Omega_{x_k}|_L = 0$. Then $I(x_k)$ consists of all integral n-dimensional spaces for Ω_{x_k} and $I(x_k) \supset I_0(x_k)$. Denote

$$I_l(x_k) = \{L \in I(x_k) : \dim(\pi_k)_*(L) = n - l\} = \{L : \dim \operatorname{Ker}((\pi_k)_*|_L) = l\}.$$

Elements $I_0(x_k)$ are called *regular*; they correspond to tangent spaces of the usual smooth solutions (jet-extensions of sections). The others are tangent spaces of singular (multi-valued) solutions.

The difference $\dim I_0(x_k) - \dim I_l(x_k)$ depends on m, k, l only. We denote this number by $c_{m,k,l}$ and call *formal codimension* of $I_l(x_k)$. Usually this number is negative. The only cases when $m \leq 4$ and $c > 0$ are listed in the following tables:

m = 1

$k \diagdown l$	1	2	3	4
1	1	3	6	10
2	1	2	1	**-4**
3	1	1	**-6**	**-29**
4	1	0	**-15**	**-48**
5	1	**-1**	**-26**	**-124**

m = 2

$k \diagdown l$	1	2	3	4
1	1	2	3	4
2	1	0	**-7**	**-24**
3	1	**-2**	**-21**	**-74**

m = 3

$k \diagdown l$	1	2	3	4
1	1	1	0	**-2**
2	1	**-2**	**-15**	**-44**

m = 4

$k \diagdown l$	1	2	3
1	1	0	**-3**
2	1	**-4**	**-23**

These tables show that the regular cell $I_0(x_k)$, as a rule, has smaller dimension than $I(x_k)$. This means that most elements of $I(x_k)$ are not tangent planes to multi-valued jet-extensions $j_k s$ with singularities of projection on the set of measure zero.

To avoid paradoxical integral planes we introduce the notion of R-Grassmanian and R-spaces. By R-Grassmanian $RI(x_k)$ we mean the closure of the regular cell $I_0(x_k)$ in $I(x_k)$. Its elements are called R-spaces.

When $RI(x_k) \neq I(x_k)$ (which is often the case by the above mentioned dimensional reasons), then there are integral manifolds, which represent singular solutions such that "all small deformations in the class of integral manifolds" have singularities too!

This leads to the notion of an R-manifold, which is an n-dimensional integral manifold N of the Cartan distribution with all tangent spaces $T_{x_k}N$, $x_k \in N$, being R-spaces. For a more detailed description of R-spaces and R-manifolds based on the associated Jordan algebra structures consult [L_2].

Remark 3. *This discussion makes important distinction between exterior differential systems and systems of PDEs embedded in jets. While with the first approach a system is given just as a subbundle in a Grassmanian, the second case keeps algebraic structures visible, in particular structure of integral manifolds is graspable and stratification of singularities is prescribed.*

Notice that $c_{m,k,l} = l^2 - mk\binom{l+k-1}{k+1}$ ([L_2]). Since $\binom{l+k-1}{k+1} \sim \frac{1}{(k+1)!}\, l^{k+1}$, we observe that only for $k = 1$, $m \leq 2$ the formal codimensions of I_l are non-negative for all l. These numbers are $c_{1,1,l} = \frac{l(l-1)}{2}$ and $c_{2,1,l} = l$.

For $k = m = 1$ we have Legendrian Grassmanian $I(x_1) \subset \mathrm{Gr}_n(T_{x_1} J^1(E, 1))$. Its restriction to the affine chart $\tilde{I}(x_1) \subset \mathrm{Gr}_n(T_{x_1}(T^*M \times \mathbb{R}))$ induces via projection the Lagrangian Grassmanian $\mathrm{LG}(a) \subset \mathrm{Gr}_n(T_a(T^*M))$, $a = (x, p)$.

The case $k = 1$, $m = 2$ includes complex Grassmanians. All other integral Grassmanians are singular (with regard to the stratum $I_0(x_k)$).

The topological structure of integral Grassmanians is important in investigation of singularities of solutions.

Theorem 2. [L$_2$]. *The cohomology ring $H^*(I(x_k), \mathbb{Z}_2)$ is isomorphic to the polynomial ring $\mathbb{Z}_2[w_1^k, \dots, w_n^k]$ up to dimension n, where $w_1^k, \dots, w_n^k$ are Stiefel-Whitney classes of the tautological bundle over $I(x_k)$.*

For a system of differential equations $\mathcal{E}$ of order k its integral Grassmanian is $I\mathcal{E}(x_k) = I(x_k) \cap \mathrm{Gr}_n(T_{x_k}\mathcal{E})$. In other words, if $\mathcal{C}_{\mathcal{E}}$ is the Cartan structure on $\mathcal{E}$ and $\Omega_{\mathcal{E}} = \Omega_{x_k}|_{T_{x_k}\mathcal{E}} \in \Lambda^2 \mathcal{C}_{\mathcal{E}}^* \otimes F(x_{k-2})$ is the restriction of the metasymplectic structure, then $I\mathcal{E}(x_k)$ coincides with the space of all integral n-dimensional planes for $\Omega_{\mathcal{E}}$.

Note that the tangent spaces to solutions of the system are integral spaces. Thus description of integral Grassmanians of systems of PDEs is important for investigation of solutions (remark that fixation of the subbundle $I\mathcal{E}(x_k)$ is essentially the starting point in EDS approach). We shall return to this problem in §3.7.

2. Algebra of differential operators

2.1. **Linear differential operators.** Denote by $\mathbf{1}$ the trivial one-dimensional bundle over M . Let $\mathcal{A}_k = \mathrm{Diff}_k(\mathbf{1}, \mathbf{1})$ be the $C^\infty(M)$-module of scalar linear differential operators of order $\le k$ and $\mathcal{A} = \cup_k \mathcal{A}_k$ be the corresponding filtered algebra, $\mathcal{A}_k \circ \mathcal{A}_l \subset \mathcal{A}_{k+l}$.

Notice that the associated graded algebra $\mathrm{gr}(\mathcal{A}) = \oplus \mathcal{A}_{k+1}/\mathcal{A}_k$ is the symmetric power of the tangent bundle:

$$\mathrm{gr}(\mathcal{A}) = ST = \oplus_i S^i T, \quad \text{where } T = T_x M.$$

Consider two linear vector bundles π and ν. Denote by $\mathrm{Diff}(\pi, \nu) = \cup_k \mathrm{Diff}_k(\pi, \nu)$ the filtered module of all linear differential operators from $C^\infty(\pi)$ to $C^\infty(\nu)$. We have the natural pairing

$$\mathrm{Diff}_k(\rho, \nu) \times \mathrm{Diff}_l(\pi, \rho) \to \mathrm{Diff}_{k+l}(\pi, \nu)$$

given by the composition of differential operators.

In particular, $\mathrm{Diff}(\pi, \mathbf{1})$ is a filtered left $\mathcal{A}$-module, $\mathrm{Diff}(\mathbf{1}, \pi)$ is a filtered right $\mathcal{A}$-module and they have an $\mathcal{A}$-valued $\mathcal{A}$-linear pairing

$$\Delta \in \mathrm{Diff}_l(\pi, \mathbf{1}), \ \nabla \in \mathrm{Diff}_k(\mathbf{1}, \pi) \mapsto \langle \Delta, \nabla \rangle = \Delta \circ \nabla \in \mathcal{A}_{k+l},$$

with $\langle \theta \circ \Delta, \nabla \rangle = \theta \circ \langle \Delta, \nabla \rangle$, $\langle \Delta, \nabla \circ \theta \rangle = \langle \Delta, \nabla \rangle \circ \theta$ for $\theta \in \mathcal{A}$.

Each linear differential operator $\Delta : C^\infty(\pi) \to C^\infty(\nu)$ of order l induces a right $\mathcal{A}$-homomorphism $\phi_\Delta : \mathrm{Diff}(\mathbf{1}, \pi) \to \mathrm{Diff}(\mathbf{1}, \nu)$ by the formula:

$$\mathrm{Diff}_k(\mathbf{1}, \pi) \ni \nabla \mapsto \Delta \circ \nabla \in \mathrm{Diff}_{k+l}(\mathbf{1}, \nu).$$

Its $\langle\,,\rangle$-dual is a left $\mathcal{A}$-homomorphism $\phi^\Delta : \mathrm{Diff}(\nu,\mathbf{1}) \to \mathrm{Diff}(\pi,\mathbf{1})$ given by

$$\mathrm{Diff}_k(\nu,\mathbf{1}) \ni \square \mapsto \square\circ\Delta \in \mathrm{Diff}_{k+l}(\pi,\mathbf{1}).$$

Denoting by $\mathcal{J}^k(\pi) = C^\infty(\pi_k)$ the space of (non-holonomic) sections of the jet-bundle we have:

$$\mathrm{Diff}_k(\pi,\nu) = \mathrm{Hom}_{C^\infty(M)}(\mathcal{J}^k(\pi), C^\infty(\nu)), \tag{3}$$

and differential operators Δ of order k are in bijective correspondence with morphisms $\psi^\Delta : J^k(\pi) \to \nu$ via the formula $\Delta = \psi^\Delta\circ j_k$, where $j_k : C^\infty(\pi) \to \mathcal{J}^k(\pi)$ is the jet-section operator.

The prolongation $\psi^\Delta_l : J^{k+l}(\pi) \to J^l(\nu)$ of $\psi^\Delta = \psi^\Delta_0$ is conjugated to the $\mathcal{A}$-homomorphism $\phi^\Delta : \mathrm{Diff}_l(\nu,\mathbf{1}) \to \mathrm{Diff}_{k+l}(\pi,\mathbf{1})$ via isomorphism (3). This makes a geometric interpretation of the differential operator Δ as the bundle morphism.

Similarly one can interpret the $\mathcal{A}$-homomorphism $\phi_\Delta : \mathrm{Diff}_l(\mathbf{1},\pi) \to \mathrm{Diff}_{k+l}(\mathbf{1},\nu)$, see [KLV]. More generally lift of the operator Δ, obtained via post-composition, is $\hat\Delta : \mathrm{Diff}_l(\xi,\pi) \to \mathrm{Diff}_{k+l}(\xi,\nu)$.

2.2. **Prolongations, linear PDEs and formal integrability.** A system $\mathcal{E}$ of PDEs of order k associated to an operator $\Delta \in \mathrm{Diff}_k(\pi,\nu)$ is, by definition, the subbundle $\mathcal{E}_k = \mathrm{Ker}(\psi^\Delta) \subset J^k(\pi)$. Its prolongation is $\mathcal{E}_{k+l} = \mathcal{E}_k^{(l)} = \mathrm{Ker}(\psi^\Delta_l) \subset J^{k+l}(\pi)$.

If $\nu = r\cdot\mathbf{1}$ is the trivial bundle of $\mathrm{rank}\,\nu = r$, we can identify $\Delta = (\Delta_1,\dots,\Delta_r)$ to be a collection of scalar operators. Then the system $\mathcal{E}_{k+l}$ is given by the equations $\mathcal{D}_\sigma\circ\Delta_j\,[u(x)] = 0$, where $1\le j\le r$, $\sigma = (i_1,\dots,i_n)$ is a multi-index of length $|\sigma| = \sum i_s \le l$ and $\mathcal{D}_\sigma = \mathcal{D}_1^{i_1}\cdots\mathcal{D}_n^{i_n}$.

Points of $\mathcal{E}_k$ can be identified as k-jet solutions (not k-jets of solutions!) of the system $\Delta = 0$ and the points of $\mathcal{E}_{k+l}$ are $(k+l)$-jet solutions of the l-prolonged system. Formal solutions are points of $\mathcal{E}_\infty = \varprojlim \mathcal{E}_i$.

Not all the points from $\mathcal{E}_k$ can be prolonged to $(k+l)$-jet solutions, but only those from $\pi_{k+l,k}(\mathcal{E}_{k+l}) \subset \mathcal{E}_k$. Investigation of these as well as formal solutions can be carried successively in l and we arrive to

Definition 1. *System $\mathcal{E}$ is formally integrable if $\mathcal{E}_i$ are smooth manifolds and the maps $\pi_{i+1,i} : \mathcal{E}_{i+1} \to \mathcal{E}_i$ are vector bundles.*

Define the dual $\mathcal{E}^* = \mathrm{Coker}\,\phi^\Delta$ as the collection of spaces $\mathcal{E}^*_i$ given by the exact sequence:

$$\mathrm{Diff}_k(\nu,\mathbf{1}) \xrightarrow{\phi^\Delta_k} \mathrm{Diff}_{k+l}(\pi,\mathbf{1}) \to \mathcal{E}^*_{k+l} \to 0.$$

We endow the dual $\mathcal{E}^*$ with natural maps $\pi^*_{i+1,i} : \mathcal{E}^*_i \to \mathcal{E}^*_{i+1}$. But it becomes an $\mathcal{A}$-module only when these maps are injective.

However in general we can define the inductive limit $\mathcal{E}^\Delta = \varinjlim \mathcal{E}^*_i$. It is a filtered left $\mathcal{A}$-module. Thus we can consider the system as a module over differential operators ($\mathcal{D}$-module).

The dual $\mathcal{E}_\Delta = \mathrm{Ker}(\phi_\Delta) \subset \mathrm{Diff}(\mathbf{1},\pi)$ is a right $\mathcal{A}$-module and we have the pairing $\mathcal{E}^\Delta \times \mathcal{E}_\Delta \to \mathcal{A}$. For formally-integrable systems this pairing is non-degenerate, as follows from the following statement:

Proposition 3. *A system $\mathcal{E} = \mathrm{Ker}(\psi^\Delta)$ is formally integrable iff $\mathcal{E}_i^*$ are projective $C^\infty(M)$-modules and the maps $\pi_{i+1,i}^* : \mathcal{E}_i^* \to \mathcal{E}_{i+1}^*$ are injective.*

Proof. The projectivity condition is equivalent to regularity (constancy of rank of the projection $\pi_{i+1,i}$), while invjectivity of $\pi_{i+1,i}^*$ is equivalent to surjectivity of $\pi_{i+1,i}$. □

If we have several differential operators $\Delta_i \in \mathrm{Diff}(\pi,\nu_i)$ of different orders k_i, $1 \le i \le t$, then their sum is no longer a differential operator of pure order $\Delta = (\Delta_1,\dots,\Delta_t) : C^\infty(\pi) \to C^\infty(\nu)$, $\nu = \oplus\nu_i$. Thus ϕ^Δ is not an $\mathcal{A}$-morphism, unless we put certain weights to the graded components ν_i.

Namely if we introduce weight k_i^{-1} for the operator Δ_i (equivalently to the bundle ν_i), then the operator Δ becomes an $\mathcal{A}$-homomorphism of degree 1. This allows to treat formally the systems of different orders via the same algebraic machinery as for the systems of pure order k. Geometric approach will be explained in the next section.

The prolongation theory wholly transforms for systems of PDEs $\mathcal{E}$ of different orders. In particular for formally integrable systems we have left $\mathcal{A}$-module $\mathcal{E}^*$. It is not a bi-module, but one can investigate sub-algebras $\mathcal{S} \subset \mathcal{A}$, which act on $\mathcal{E}^*$ from the right. It will be clear from §4.4 that they correspond to symmetries of the system $\mathcal{E}$.

2.3. **Symbols, characteristics and non-linear PDEs.** Consider symbolic analogs of the above modules (we will write sometimes $T = T_xM$ for brevity). Since $ST\otimes\pi^* = \oplus S^iT\otimes\pi^*$ is the graded module associated to the filtrated $C^\infty(M)$-module $\mathrm{Diff}(\pi,\mathbf{1}) = \cup\,\mathrm{Diff}_i(\pi,\mathbf{1})$, the bundle morphism ϕ^Δ produces the graded homomorphisms, called symbols of our differential operator Δ:

$$\sigma_\Delta : ST\otimes\nu^* \to ST\otimes\pi^*.$$

The value $\sigma_{\Delta,x}$ of σ_Δ at $x\in M$ is a homomorphism of ST-modules.

The ST-module $\mathcal{M}_\Delta = \mathrm{Coker}(\sigma_{\Delta,x})$ is called the *symbolic module* at the point $x \in M$ ([GS]). Its annihilator is called the *characteristic ideal* $I(\Delta) = \oplus I_q$, where I_q are homogeneous components. The set of covectors $p \in T^*\setminus\{0\}$ annihilated by $I(\Delta)$ is the *characteristic variety* $\mathrm{Char}_{\mathrm{aff}}(\Delta)$. We will consider projectivization of this conical affine variety $\mathrm{Char}_x(\Delta) \subset \mathbb{P}T^*$.

It is often convenient to work over complex numbers. If we complexify the symbolic module, we get the complex characteristic variety

$$\mathrm{Char}_x^{\mathbb{C}}(\Delta) = \{p \in \mathbb{P}^{\mathbb{C}}T^* \mid f(p^q) = 0\,\forall f \in I_q, \forall q\}.$$

Proposition 4. [Go, S$_2$]. *For $p \in T_x^*M \setminus \{0\}$ let $\mathfrak{m}(p) = \oplus_{i>0}S^iT \subset ST$ be the maximal ideal of homogeneous polynomials vanishing at p. Then covector p is characteristic iff the localization $(\mathcal{M}_\Delta)_{\mathfrak{m}(p)} \neq 0$.*

The set of localizations $(\mathcal{M}_\Delta)_{\mathfrak{m}(p)} \neq 0$ for characteristic covectors p form the *characteristic sheaf* $\mathcal{K}$ over the characteristic variety $\mathrm{Char}^{\mathbb{C}}_x(\Delta)$.

The above definitions work for systems $\mathcal{E}$ of different order PDEs if we impose the weight-convention of the previous section. However it will be convenient to interpret this case geometrically and such approach works well even in non-linear situation.

A system of PDEs of pure order k is represented as a smooth subbundle $\mathcal{E}_k \subset J^k(\pi)$, non-linear case corresponds to fiber-bundles (in regular situation; in general the fibers $\pi^{-1}_{k,k-1}(*)\cap\mathcal{E}_k$ are not diffeomorphic and $\mathcal{E}_k$ is just a submanifold in $J^k\pi$). Prolongations are defined by the formula

$$\mathcal{E}_k^{(1)} = \{x_{k+1} = [s]_x^{k+1} \in J^{k+1}\pi : T_{x_k}[j_k s(M)] \subset T_{x_k}\mathcal{E}\}.$$

Higher prolongations can be defined successively in regular case, but in general $\mathcal{E}_k^{(l)}$ equals the set of all points $x_{k+l} = [s]_x^{k+l} \in J^{k+l}\pi$ with the property that $j_k s(M)$ is tangent to $\mathcal{E}$ at x_k with order $\geq l$.

To cover the case of several equations of different orders we modify the usual definition. By a differential equation/system of maximal order k we mean a sequence $\mathcal{E} = \{\mathcal{E}_i\}_{-1\leq i\leq k}$ of submanifolds $\mathcal{E}_i \subset J^i(\pi)$ with $\mathcal{E}_{-1} = M$, $\mathcal{E}_0 = J^0M = E_\pi$ such that for all $0 < i \leq k$ the following conditions hold:

(1) $\pi^{\mathcal{E}}_{i,i-1} : \mathcal{E}_i \to \mathcal{E}_{i-1}$ are smooth fiber bundles.
(2) The first prolongations $\mathcal{E}^{(1)}_{i-1}$ are smooth subbundles of π_i and $\mathcal{E}_i \subseteq \mathcal{E}^{(1)}_{i-1}$.

Consider a point $x_k \in \mathcal{E}_k$ with $x_i = \pi_{k,i}(x_k)$ for $i < k$ and $x = x_{-1}$. It determines the collection of symbols $g_i(x_k) \subset S^iT^*_xM \otimes N_{x_0}$, where $N_{x_0} = T_{x_0}\left[\pi^{-1}(x)\right]$, by the formula

$$g_i(x_k) = T_{x_i}\left[\pi^{-1}_{i,i-1}(x_{i-1})\right] \cap T_{x_i}\mathcal{E}_i \subset S^iT^*_xM \otimes N_{x_0} \quad \text{for } i \leq k.$$

For $i > k$ the symbolic spaces g_i are defined as symbols of the prolongations $\mathcal{E}_i = \mathcal{E}_k^{(i-k)}$, and they still depend on the point $x_k \in \mathcal{E}_k$.

In this situation $g^*(x_k) = \oplus g_i^*$ is a graded module over the algebra $\mathcal{R} = ST_xM$ of homogeneous polynomials on the cotangent space T^*_xM. It is called the *symbolic module* of $\mathcal{E}$ at the point x_k and for systems of linear PDEs $\mathcal{E} = \mathrm{Ker}(\Delta)$ this coincides with the previously defined module $\mathcal{M}_\Delta$.

The *characteristic ideal* is defined by $I_{x_k}(\mathcal{E}) = \mathrm{ann}(g^*) \subset \mathcal{R}$ (in the symbolic context denoted by $I(g)$). The *characteristic variety* is the (projectivized/complexified) set of non-zero covectors $v \in T^*$ such that for every i there exists a vector $w \in N\setminus\{0\}$ with $v^i \otimes w \in g_i$. If the system is of maximal order k, it is sufficient for this definition to take $i = k$ only. We denote it by $\mathrm{Char}^{\mathbb{C}}_{x_k}(\mathcal{E}) \subset \mathbb{P}^{\mathbb{C}}T^*_xM$ (variants: $\mathrm{Char} \subset \mathbb{P}T^*_xM$, $\mathrm{Char}^{\mathbb{C}}_{\mathrm{aff}} \subset {}^{\mathbb{C}}T^*_xM$ etc).

Denote by $\mathrm{diff}(\pi,\nu)$ the space of all non-linear differential operators (linear included) between sections of bundles π and ν. Let $F \in \mathrm{diff}(\pi,\nu)$ determine the system $\mathcal{E}$. Then its symbol at $x_k \in \mathcal{E}_k$ resolves the symbolic module $g^*(x_k)$:

$$\cdots \to ST_xM \otimes \nu_x^* \stackrel{\sigma_F(x_k)}{\longrightarrow} ST_xM \otimes \pi_x^* \to g^*(x_k) \to 0.$$

Here we use the weight convention in order to make the symbol map $\sigma_F(x_k)$ into $\mathcal{R}$-homomorphism. Its value at covector $p \in \operatorname{Char}^{\mathbb{C}}_{x_k}(\mathcal{E})$ is the fiber of the characteristic sheaf: $\mathcal{K}_p = \operatorname{Coker}\bigl[\sigma_F(x_k)(p)\bigr]$.

Precise form of the above free resolution in various cases allows to investigate the system $\mathcal{E}$ in details. In particular, results of §3.6 are based on the Buchsbaum-Rim resolution [BR].

Working with symbolic modules we inherit various concepts from commutative algebra (consult e.g. [E]). Some of them are of primary importance for PDEs. For instance

$$\dim_{\mathcal{R}} g^* = \dim_{\mathbb{C}} \operatorname{Char}^{\mathbb{C}}_{\text{aff}}(\mathcal{E}) = \dim_{\mathbb{C}} \operatorname{Char}^{\mathbb{C}}(\mathcal{E}) + 1$$

is the Chevalley dimension of $g^* = g^*(x_k)$.

System $\mathcal{E}$ is called a *Cohen-Macaulay system* if the corresponding symbolic module g^* is Cohen-Macaulay, i.e. $\operatorname{depth} g^* = \dim g^*$ (see [BH]). Other notions like grade and height turns to be important in applications to differential equations as well ([KL$_4$]).

Castlnuovo-Mumford regularity of g^* is closely related to the notion of involutivity (we'll give definition via Spencer δ-cohomology in §3.2). It is instructive to notice that, even though quasi-regular sequences are basic for both classes, involutive systems exhibit quite unlike properties compared to Cohen-Macaulay systems (some apparent duality is shown in [KL$_9$]).

2.4. **Non-linear differential operators.** Let $\mathfrak{F} = C^\infty(J^\infty\pi)$ be the filtered algebra of smooth functions depending on finite jets of π, i.e. $\mathfrak{F} = \cup_i \mathfrak{F}_i$ with $\mathfrak{F}_i = C^\infty(J^i\pi)$.

Denote $\mathfrak{F}_i^{\mathcal{E}} = C^\infty(\mathcal{E}_i)$. The projections $\pi_{i+1,i} : \mathcal{E}_{i+1} \to \mathcal{E}_i$ induce the maps $\pi_{i+1,i}^* : \mathfrak{F}_i^{\mathcal{E}} \to \mathfrak{F}_{i+1}^{\mathcal{E}}$, so that we can form the space $\mathfrak{F}^{\mathcal{E}} = \cup \mathfrak{F}_i^{\mathcal{E}}$, the points of which are infinite sequences $(f_i, f_{i+1}, \dots)$ with $f_i \in \mathfrak{F}_i^{\mathcal{E}}$ and $\pi_{i+1,i}^*(f_i) = f_{i+1}$. This $\mathfrak{F}^{\mathcal{E}}$ is a $C^\infty(M)$-algebra. If the system $\mathcal{E}$ is not formally integrable, the set of infinite sequences can be void, and the algebra $\mathfrak{F}^{\mathcal{E}}$ can be trivial. To detect formal integrability, we investigate the finite level jets algebras $\mathfrak{F}_i^{\mathcal{E}}$ via the following algebraic approach.

Let $\mathcal{E}$ be defined by a collection $F = (F_1, \dots, F_r) \in \operatorname{diff}(\pi, \nu)$ of non-linear scalar differential operators of orders $k_1, \dots, k_r$ (can be repeated). Post-composition of our differential operator $F : C^\infty(\pi) \to C^\infty(\nu)$ with other non-linear differential operators $\square$ (composition from the left $\hat{\square} \circ F$) gives the following exact sequence of $C^\infty(M)$-modules

$$\operatorname{diff}(\nu, \mathbf{1}) \xrightarrow{F} \operatorname{diff}(\pi, \mathbf{1}) \to \mathfrak{F}^{\mathcal{E}} \to 0. \tag{4}$$

Denote $\mathcal{J}_t(F) = \langle \hat{\square}_i \circ F_i \,\big|\, \operatorname{ord} \square_i + k_i \le t, 1 \le i \le r \rangle \subset \operatorname{diff}_t(\pi, \mathbf{1})$ the submodule generated by $F_1, \dots, F_r$ and their total derivatives up to order

t. Then
$$\mathfrak{F}_i^{\mathcal{E}} = \operatorname{diff}_i(\pi, \mathbf{1})/\mathcal{J}_i(F_1, \ldots, F_r). \tag{5}$$

It is important that the terms of (4) are modules over the algebra of scalar $\mathscr{C}$-differential operators $\mathscr{C}\operatorname{Diff}(\mathbf{1},\mathbf{1})$, which are total derivative operators and have the following form in local coordinates [KLV]: $\Delta = \sum f_\sigma \mathcal{D}_\sigma$, with $f_\sigma \in C^\infty(J^\infty(M))$. We can identify $\mathscr{C}\operatorname{Diff}(\mathbf{1},\mathbf{1}) = \cup \mathfrak{F}_i^{\mathbf{1}} \otimes \operatorname{Diff}_j(\mathbf{1},\mathbf{1})$ with the twisted tensor product of the algebras $\mathfrak{F}^{\mathbf{1}} = C^\infty(J^\infty(M))$ and $\operatorname{Diff}(\mathbf{1},\mathbf{1})$ over the action
$$\hat{\Delta}: \ \mathfrak{F}_i^{\mathbf{1}} \to \mathfrak{F}_{i+j}^{\mathbf{1}} \quad \text{for} \quad \Delta \in \operatorname{Diff}_j(\mathbf{1},\mathbf{1}).$$
This $\mathscr{C}\operatorname{Diff}(\mathbf{1},\mathbf{1})$ is a non-commutative $C^\infty(M)$-algebra. We need a more general $\mathfrak{F}$-module of $\mathscr{C}$-differential operators $\mathscr{C}\operatorname{Diff}(\pi,\mathbf{1}) = \cup \mathscr{C}\operatorname{Diff}_i(\pi,\mathbf{1})$, where
$$\mathscr{C}\operatorname{Diff}_i(\pi,\mathbf{1}) = \mathfrak{F}_i \otimes_{C^\infty(M)} \operatorname{Diff}_i(\pi,\mathbf{1}).$$
Remark that $\mathscr{C}\operatorname{Diff}(\pi,\mathbf{1})$ is a filtered $\mathscr{C}\operatorname{Diff}(\mathbf{1},\mathbf{1})$-module.

Define now the filtered $\mathfrak{F}^{\mathcal{E}}$-module $\mathscr{C}\operatorname{Diff}^{\mathcal{E}}(\pi,\mathbf{1})$ with $\mathscr{C}\operatorname{Diff}_i^{\mathcal{E}}(\pi,\mathbf{1}) = \mathfrak{F}_i^{\mathcal{E}} \otimes \operatorname{Diff}_i(\pi,\mathbf{1})$. Since the module $\operatorname{Diff}(\pi,\mathbf{1})$ is projective and we can identify $\operatorname{diff}(\pi,\mathbf{1})$ with $\mathfrak{F}$, we have from (5) the following exact sequence
$$0 \to \mathcal{J}_i(F) \otimes \operatorname{Diff}_i(\pi,\mathbf{1}) \to \mathscr{C}\operatorname{Diff}_i(\pi,\mathbf{1}) \to \mathscr{C}\operatorname{Diff}_i^{\mathcal{E}}(\pi,\mathbf{1}) \to 0. \tag{6}$$

Similar modules can be defined for the vector bundle ν and they determine the $\mathfrak{F}^{\mathcal{E}}$-module $\mathcal{E}^* = \cup \mathcal{E}_i^*$ by the following sequence:
$$\mathscr{C}\operatorname{Diff}_i^{\mathcal{E}}(\nu,\mathbf{1}) \xrightarrow{\ell_F} \mathscr{C}\operatorname{Diff}_{i+k}^{\mathcal{E}}(\pi,\mathbf{1}) \to \mathcal{E}_{i+k}^* \to 0, \tag{7}$$
where $\ell : \operatorname{diff}(\pi,\nu) \to \mathfrak{F} \otimes_{C^\infty(M)} \operatorname{Diff}(\pi,\nu)$ is the operator of universal linearization [KLV], $\ell_F = \ell(F)$ (described in the next section).

This sequence is not exact in the usual sense, but it becomes exact in the following one. The space to the left is an $\mathfrak{F}_i^{\mathcal{E}}$-module, the middle term is an $\mathfrak{F}_{i+k}^{\mathcal{E}}$-module. The image $\ell_F(\mathscr{C}\operatorname{Diff}_i^{\mathcal{E}}(\nu,\mathbf{1}))$ is an $\mathfrak{F}_i^{\mathcal{E}}$-module, but we generate by it an $\mathfrak{F}_{i+k}^{\mathcal{E}}$-submodule in the middle term. With this understanding of the image the term $\mathcal{E}_{i+k}^*$ of (7) is an $\mathfrak{F}_{i+k}^{\mathcal{E}}$-module and the sequence is exact. In other words
$$\mathcal{E}_s^* = \mathscr{C}\operatorname{Diff}_s^{\mathcal{E}}(\pi,\mathbf{1})/(\mathfrak{F}_s^{\mathcal{E}} \cdot \operatorname{Im} \ell_F).$$

Sequences (7) are nested (i.e. their union is filtered) and so we have the sequence
$$\mathcal{E}_{s-1}^* \to \mathcal{E}_s^* \to \mathcal{F}g_s^* \to 0, \tag{8}$$
which becomes exact if we treat the image of the first arrow as the corresponding generated $\mathfrak{F}_s^{\mathcal{E}}$-module. Thus $\mathcal{F}g_s^*$ is an $\mathfrak{F}_s^{\mathcal{E}}$-module with support on $\mathcal{E}_s$ and its value at a point $x_s \in \mathcal{E}_s$ is dual to the s-symbol of the system $\mathcal{E}$:
$$(\mathcal{F}g_s^*)_{x_s} = g_s^*(x_s); \qquad g_s(x_s) = \operatorname{Ker}[T_{x_s}\pi_{s,s-1} : T_{x_s}\mathcal{E}_s \to T_{x_{s-1}}\mathcal{E}_{s-1}].$$

In general non-linear situation Definition 1 should be changed to

Definition 2. *System $\mathcal{E}$ is formally integrable if the maps $\pi_{i+1,i} : \mathcal{E}_{i+1} \to \mathcal{E}_i$ are submersions.*

Proposition 5. *A system $\mathcal{E}$ is formally integrable iff the modules $\mathcal{F}g_s^*$ are projective and the maps $\pi_{i+1,i}^* : \mathcal{E}_i^* \to \mathcal{E}_{i+1}^*$ are injective.*

Note that whenever prolongations $\mathcal{E}_{k+l}$ exist and k is the maximal order, the fibers of the projections $\pi_{t,s} : \mathcal{E}_t \to \mathcal{E}_s$ carry natural affine structures for $t > s \geq k$.

2.5. **Linearizations and evolutionary differentiations.** Consider a non-linear differential operator $F \in \mathrm{diff}_k(\pi,\nu)$ and two sections $s, h \in C^\infty(\pi)$ (we assume π to be a vector bundle, though it's not essential). Define

$$\ell_{F,s}(h) = \tfrac{d}{dt} F(s + th)|_{t=0}$$

This operator is linear in h and depends on its k-jets, so we have

$$\ell_{F,s} \in \mathrm{Hom}_{C^\infty(M)}(\mathcal{J}^k(\pi), C^\infty(\nu)) = \mathrm{Diff}_k(\pi,\nu).$$

Moreover value of this operator at $x \in M$ depends on k-jet of s, so that $\ell_{F,s} = j_k(s)^*(\ell_F)$. We will also denote $\ell_{F,x^k} = \ell_{F,s}$ for $x_k = [s]_x^k$. This dependence is however non-linear and we get $\ell_F \in \mathfrak{F} \otimes_{C^\infty(M)} \mathrm{Diff}(\pi,\nu)$.

In such a form this operator generalizes to the case of different orders.

Definition 3. *Operator $\ell : \mathrm{diff}(\pi,\nu) \to \mathscr{C}\,\mathrm{Diff}(\pi,\nu) = \mathfrak{F} \otimes_{C^\infty(M)} \mathrm{Diff}(\pi,\nu)$ is called the operator of linearization (universal linearization in* [KLV]*).*

It is instructive to note that whenever the evolutionary PDE (t being an extra variable)

$$\partial_t u = G(u), \quad G \in \mathrm{diff}(\pi,\pi),\ u \in C^\infty(\pi),$$

with initial condition $u(0) = s$ is solvable, then for each $x_k = [s]_x^k$ we get: $\ell_{F,x_k}(G) = \frac{d}{dt} F\bigl(u(t)\bigr)|_{t=0}$ for (any if non-unique) solution $u(t)$.

In canonical coordinates (trivializing ν) linearization of $F = (F_1, \dots, F_r)$ is $\ell_F = \ell(F) = (\ell(F_1), \dots, \ell(F_r))$ with

$$\ell(F_i) = \sum (\partial_{p_\sigma^j} F_i) \cdot \mathcal{D}_\sigma^{[j]},$$

where $\mathcal{D}_\sigma^{[j]}$ denotes the operator $\mathcal{D}_\sigma$ applied to the j-th component of the section from $C^\infty(\pi)$.

Recall that $\mathfrak{F}$ is an algebra of functions on $J^\infty\pi$ with usual multiplication and $\mathrm{diff}(\pi,\nu)$ is a lef $\mathfrak{F}$-module: $\mathfrak{F}_i \cdot \mathrm{diff}_k(\pi,\nu) \subset \mathrm{diff}_{\max\{i,k\}}(\pi,\nu)$. With respect to this structure the operator of linearization satisfies the Leibniz rule:

$$\ell_{H\cdot F} = \ell_H \cdot F + H \cdot \ell_F, \quad H \in \mathrm{diff}(\pi, \mathbf{1}),\ F \in \mathrm{diff}(\pi,\nu). \tag{9}$$

Since ℓ_F is a derivation in F, we can introduce the operator $\ni_G$ by the formula

$$\ni_G^\nu(F) = \ell_F(G), \quad F \in \mathrm{diff}(\pi,\nu),\ G \in \mathrm{diff}(\pi,\pi).$$

Definition 4. *The operator $\ni_G^\nu : \mathrm{diff}(\pi,\nu) \to \mathrm{diff}(\pi,\nu)$ is called the evolutionary differentiation corresponding to $G \in \mathrm{diff}(\pi,\pi)$.*

In canonical coordinates with $G = (G_1, \dots, G_m)$ the i-th component of the evolutionary differentiation equals

$$\mathcal{I}^{\nu}_{G;\,i} = \sum (\mathcal{D}_\sigma G_j) \cdot \partial_{p^j_\sigma}{}^{[i]},$$

where $\partial_{p^j_\sigma}{}^{[i]}$ denotes the operator $\partial_{p^j_\sigma}$ applied to the j-th component of the section from $C^\infty(\nu)$.

As a consequence of (9) evolutionary differentiations satisfy the Leibniz rule:

$$\mathcal{I}^{\nu}_G(H \cdot F) = \mathcal{I}^{\nu}_G(H) \cdot F + H \cdot \mathcal{I}^{\nu}_G(F), \quad H \in \mathrm{diff}(\pi, \mathbf{1}),\ F \in \mathrm{diff}(\pi, \nu). \tag{10}$$

Moreover since linear differential operators commute with $\frac{d}{dt}$, we get:

$$\hat{K} \circ \mathcal{I}^{\nu}_G = \mathcal{I}^{\xi}_G \circ \hat{K}, \quad \forall K \in \mathrm{Diff}(\nu, \xi). \tag{11}$$

Proposition 6. [KLV]. *$\mathbb{R}$-linear maps satisfying (10) and (11) are evolutionary differentiations and only them.*

Corollary 7. *The space $\mathfrak{Ev}(\pi, \nu) = \{\mathcal{I}^{\nu}_G : G \in \mathrm{diff}(\pi, \pi)\}$ for fixed vector bundles π, ν is a Lie algebra with respect to the commutator.*

Consider the surjective $\mathbb{R}$-linear map

$$\mathcal{I}^{\pi} : \mathrm{diff}(\pi, \pi) \to \mathfrak{Ev}(\pi, \pi), \qquad G \mapsto \mathcal{I}^{\pi}_G. \tag{12}$$

It is injective because $\mathcal{I}^{\pi}_G(\mathrm{Id}) = G$, and so can be used to introduce Lie algebra structure on $\mathrm{diff}(\pi, \pi)$, with respect to which (12) is an anti-isomorphism of Lie algebras:

$$\mathcal{I}^{\pi}_{\{F,G\}} = [\mathcal{I}^{\pi}_G, \mathcal{I}^{\pi}_F], \qquad F, G \in \mathrm{diff}(\pi, \pi).$$

Definition 5. *The bracket $\{F, G\}$ is called the higher Jacobi bracket.*

This bracket generalizes the Lagrange-Jacobi bracket from classical mechanics and contact geometry as well as Poisson bracket from symplectic geometry. It coincides with the commutator for linear differential operators.

We can calculate $\{F, G\} = \mathcal{I}^{\pi}_G(F) - \mathcal{I}^{\pi}_F(G) = \ell_F(G) - \ell_G(F)$. In canonical coordinates the bracket writes:

$$\{F, G\}_i = \sum \big(\mathcal{D}_\sigma(G_j) \cdot \partial_{p^j_\sigma} F_i - \mathcal{D}_\sigma(F_j) \cdot \partial_{p^j_\sigma} G_i\big).$$

2.6. **Brackets and multi-brackets of differential operators.** Let $\pi = m \cdot \mathbf{1}$ be the trivial bundle of rank m. Then linearization of the operator $F \in \mathrm{diff}(\pi, \mathbf{1})$ can be written in components: $\ell(F) = (\ell_1(F), \dots, \ell_m(F))$.

Multi-bracket of $(m + 1)$ differential operators F_i on π is another differential operator on π, given by the formula [KL$_4$]:

$$\{F_1, \dots, F_{m+1}\} = \frac{1}{m!} \sum_{\alpha \in S_m, \beta \in S_{m+1}} (-1)^\alpha \, (-1)^\beta \, \ell_{\alpha(1)}(F_{\beta(1)}) \circ \dots \circ \ell_{\alpha(m)}(F_{\beta(m)}) \left(F_{\beta(m+1)}\right).$$

When $m = 1$ we obtain the higher Jacobi bracket.

For linear vector differential operators $\nabla_i : m \cdot C^\infty_{\rm loc}(M) \to C^\infty_{\rm loc}(M)$, represented as rows $(\nabla_i^1, \dots, \nabla_i^m)$ of scalar linear differential operators, the multi-bracket has the form:

$$\{\nabla_1, \dots, \nabla_{m+1}\} = \sum_{k=1}^{m+1} (-1)^{k-1}\,\mathrm{Ndet}[\nabla_i^j]_{i\neq k}^{1\le j\le m} \circ \nabla_k,$$

where Ndet is a version of non-commutative determinant [KL$_4$].

If we interchange Ndet and ∇_k in the above formula, we obtain the opposite multi-bracket $\{F_1, \dots, F_{m+1}\}^\dagger$ (taking another representative for Ndet).

Theorem 8. [KL$_{10}$]. *Let $F_i \in \mathrm{diff}(\pi, \mathbf{1})$ be vector differential operators, $1 \le i \le m+2$, and let $F_{k,i}$ denote component i of F_k and $\{\cdots\}_i$ the i^{th} component of the multi-bracket. Then the multi-bracket and the opposite multi-bracket are related by the following identities (check means absence of the argument) for any $1 \le i \le m$:*

$$\sum_{k=1}^{m+2} (-1)^k \Big[\ell_{\{F_1,\dots,\check F_k,\dots,F_{m+2}\}_i^\dagger} F_k - \ell_{F_{k,i}} \{F_1, \dots, \check F_k, \dots, F_{m+2}\} \Big] = 0.$$

For $m = 1$ this formula becomes the standard Jacobi identity. In this case $F_i \in \mathrm{diff}(\mathbf{1}, \mathbf{1})$ are scalar differential operators, multi-bracket becomes the higher Jacobi bracket $\{F, G\}$ and we get:

$$\sum_{\text{cyclic}} \big(\ell_F\{G, H\} - \ell_{\{G,H\}} F \big) = \sum_{\text{cyclic}} \{F, \{G, H\}\} = 0.$$

Thus the multi-bracket identities could be considered as generalized Jacobi identities (but neither in the sense of Nambu, generalized Poisson, nor as SH-algebras [N, LS]). We called them non-commutative Plücker identities in [KL$_{10}$], because their symbolic analogs are precisely the standard Plücker formulas. Symbolic counterpart of the above identities can be interpreted as multi-version of the integrability of characteristics ([GQS, KL$_{10}$]).

Finally we give a coordinate representation of the introduced multi-bracket. As in the classical contact geometry there is a variety of brackets (see more in [KS]). The following is the multi-bracket analog of the Mayer bracket:

$$[F_1, \dots, F_{m+1}] = \frac{1}{m!} \sum_{\substack{\sigma\in S_{m+1}\\ \nu\in S_m}} \frac{\mathrm{sgn}(\sigma)}{\mathrm{sgn}(\nu)} \sum_{\substack{1\le i\le m\\ |\tau_i| = k_{\sigma(i)}}} \prod_{j=1}^m \frac{\partial F_{\sigma(j)}}{\partial p^{\nu(j)}_{\tau_j}}\, \mathcal{D}_{\tau_1+\dots+\tau_m} F_{\sigma(m+1)},$$

where $F_i \in \mathrm{diff}_{k_i}(m \cdot \mathbf{1}, \mathbf{1})$. For $m = 1$ this gives Mayer brackets instead of Jacobi brackets [KL$_1$]. We have ([KL$_{10}$]):

Proposition 9. *Restrictions of the two multi-brackets to the system $\mathcal{E} = \{F_1 = \dots = F_{m+1} = 0\}$ coincide:*

$$\{F_1, \dots, F_{m+1}\} \equiv [F_1, \dots, F_{m+1}] \bmod \mathcal{J}_{k_1+\dots+k_{m+1}-1}(F_1, \dots, F_{m+1}).$$

3. Formal theory of PDEs

3.1. **Symbolic systems.** Consider vector spaces T of dimension n and N of dimension m (usually over the field $\mathbb{R}$, but also possible over $\mathbb{C}$). The symmetric power $ST^*=\oplus_{i\geq 0}S^iT^*$ can be identified with the space of polynomials on T.

Spencer δ-complex is the graded de Rham complex of N-valued differential forms on T with polynomial coefficients:

$$0\to S^kT^*\otimes N\xrightarrow{\delta}S^{k-1}T^*\otimes N\otimes T^*\xrightarrow{\delta}\cdots\xrightarrow{\delta}S^{k-n}T^*\otimes N\otimes\Lambda^nT^*\to 0,$$

where $S^iT^*=0$ for $i<0$. By Poincaré lemma δ-complex is exact.

For a linear subspace $h\subset S^kT^*\otimes N$ its *first prolongation* is

$$h^{(1)}=\{p\in S^{k+1}T^*\otimes N\,|\,\delta p\in h\otimes T^*\}$$

Higher prolongations are defined inductively and satisfy $(h^{(k)})^{(l)}=h^{(k+l)}$.

Definition 6. Symbolic system *is a sequence of subspaces $g_k\subset S^kT^*\otimes N$ such that $g_{k+1}\subset g_k^{(1)}$, $k\geq 0$.*

If $\mathcal{E}$ is a system of PDEs of maximal order k and $x_k\in\mathcal{E}_k$, then the symbols of $\mathcal{E}$, namely $\{g_i(x_k)\}$ form a symbolic system.

We usually assume $g_0=N$ (if $g_0\subsetneqq N$ one can shrink N). With every such a system we associate its Spencer δ-complex of order k:

$$0\to g_k\xrightarrow{\delta}g_{k-1}\otimes T^*\xrightarrow{\delta}g_{k-2}\otimes\Lambda^2T^*\to\cdots\xrightarrow{\delta}g_{k-n}\otimes\Lambda^nT^*\to 0.$$

The cohomology group at the term $g_i\otimes\Lambda^jT^*$ is denoted by $H^{i,j}(g)$ and is called Spencer δ-cohomology.

When g is the symbolic system corresponding to a system of PDEs we denote the cohomology by $H^{i,j}(\mathcal{E};x_k)$ and often omit reference to the point.

Another way to deal with the system $g\overset{i}{\hookrightarrow}ST^*\otimes N$ is to consider its dual $g^*=\oplus g_k^*$, which is an epimorphic image of $ST\otimes N^*$ via the map i^*. The last space is naturally an ST-module and we can try to carry the module structure to g^* by the formula $w\cdot i^*(v)=i^*(w\cdot v)$, $w\in ST$, $v\in ST\otimes N^*$. Correctness of this operation has the following obvious meaning:

Proposition 10. *System $g\subset ST^*\otimes N$ is symbolic iff g^* is an ST-module.*

Orders of the system is the following set:

$$\operatorname{ord}(g)=\{k\in\mathbb{Z}_+\,|\,g_k\neq g_{k-1}^{(1)}\}.$$

Multiplicity of an order k is $m(k)=\dim g_{k-1}^{(1)}/g_k$ and this equals to the dimension of the Spencer δ-cohomology group $H^{k-1,1}(g)$.

Hilbert basis theorem implies finiteness of the set of orders.

Definition 7. *Call* formal codimension *of a symbolic system g the number of elements in* $\mathrm{ord}(g)$ *counted with multiplicities. In other words*

$$\mathrm{codim}(g)=\sum_{k=1}^{\infty}\dim H^{k-1,1}(g).$$

3.2. **Spencer δ-cohomology.** Let us show how to calculate the Spencer δ-cohomology in some important cases. Denote $m=\dim N$, $r=\mathrm{codim}(g)$ and $U=\mathbb{R}^r$. Then minimal resolution of the symbolic module starts as follows:

$$\cdots\to ST\otimes U^*\longrightarrow ST\otimes N^*\to g^*\to 0.$$

Definition 8. *Call a symbolic system g* generalized complete intersection *if the symbolic module satisfies:* $\operatorname{depth}\operatorname{ann}(g^*)\geq r-m+1$.

This condition will be interpreted for systems of PDEs in §3.6. It is a condition of general position for module g^* in the range $m<r<m+n$.

Any generalized complete intersection g is a Cohen-Macaulay system. By standard theorems from commutative algebra we have in fact equality for depth in the definition above.

Theorem 11. [KL$_{10}$]. *If g is a generalized complete intersection, then the only non-zero Spencer δ-cohomology are given by the formula:*

$$H^{*,j}(g)=\begin{cases} N & \text{for } j=0,\\ U & \text{for } j=1,\\ S^{j-2}N^*\otimes\Lambda^{m+j-1}U & \text{for } 2\leq j\leq r+1-m\leq n.\end{cases}$$

In this formula we suppressed bi-grading. If g corresponds to a system $\mathcal{E}$ of different orders PDEs, then $H^{*,j}$ is a sum of different cohomology spaces. They can be specified as follows (if there're several equal $H^{i,j}$ in the sum below, we count only one and the rest contributes to the growth of dimension):

$$H^{*,0}(g)=H^{0,0}(g),\qquad H^{*,1}(g)=\bigoplus_{i\in\mathrm{ord}(g)}H^{i-1,1}(g),$$
$$H^{*,2}(g)=\bigoplus_{i_1,\dots,i_{m+1}\in\mathrm{ord}(g)}H^{i_1+\dots+i_{m+1}-2,2}(g),$$
$$H^{*,3}(g)=\bigoplus_{i_1,\dots,i_{m+2}\in\mathrm{ord}(g)}H^{i_1+\dots+i_{m+2}-3,3}(g)\quad\text{etc}\dots$$

One of the most important techniques in calculating Spencer δ-cohomology of a symbolic system g comes from commutative algebra, because they $\mathbb{R}$-dualize to Koszul homology of the symbolic module g^* ([S$_2$]). In particular, homology calculus can be equivalently represented by calculating free resolvents of g^*, see [Gr].

However Spencer δ-cohomology are related to certain constructions specific to PDEs, which we are going to describe.

Having a symbolic system $g = \{g_l \subset S^l T^* \otimes N\}$ and a subspace $V^* \subset T^*$ we define another system $\tilde{g} = \{g_l \cap S^l V^* \otimes N\} \subset SV^* \otimes N$. This is a symbolic system, called the *V^*-reduction*.

It is important that such $\tilde{g}$ are precisely the symbolic systems corresponding to symmetry reductions, with respect to Lie group actions [AFT].

Theorem 12. [KL$_2$]. *Let g be a Cohen-Macaulay symbolic system and a subspace $V^* \subset T^*$ be transversal to the characteristic variety of g:*

$$\operatorname{codim}(\operatorname{Char}^{\mathbb{C}}(g) \cap \mathbb{P}^{\mathbb{C}} V^*) = \operatorname{codim} \operatorname{Char}^{\mathbb{C}}(g).$$

Then Spencer δ-cohomology of the system g and its V^*-reduction $\tilde{g}$ are isomorphic:

$$H^{i,j}(g) \simeq H^{i,j}(\tilde{g}).$$

Another important transformation is related to solving Cauchy problem for general PDEs. Let $W \subset T$. The following exact sequence allows to project along the subspace $V^* = \operatorname{ann}(W)$:

$$0 \to V^* \hookrightarrow T^* \to W^* \to 0$$

Applying this projection to the symbolic system g we get a new symbolic system $\bar{g}_k \subset S^k W^* \otimes N$, called W-restriction.

In order to describe the result we need to introduce some concepts.

The first is involutivity. With every symbolic system $g \subset ST^* \otimes N$ and any $k \geq 0$ we can relate the symbolic system $g^{|k\rangle}$, which is generated by all differential corollaries of the system deduced from the order k:

$$g_i^{|k\rangle} = \begin{cases} S^i T^* \otimes N, & \text{for } i < k; \\ g_k^{(i-k)}, & \text{for } i \geq k. \end{cases}$$

Note that g is a system of pure order k if and only if $g = g^{|k\rangle}$. In this case classical Cartan definition of involutivity can be equivalently expressed via vanishing of Spencer δ-cohomology (see Serre's letter in [GS]):

$$H^{i,j}(g) = 0 \quad \forall i \neq k-1.$$

For a system of different orders we have:

Definition 9. *A system g is involutive if all systems $g^{|k\rangle}$ are involutive.*

The number of conditions in this definition is not infinite, since only $k \in \operatorname{ord}(g)$ are essential. This general involutivity can still be expressed via vanishing of δ-cohomology for systems $g^{|k\rangle}$, but not for the system g ([KL$_9$]).

Let us denote

$$\Upsilon^{i,j} = \bigoplus_{r>0} S^r V^* \otimes \delta(S^{i+1-r} W^* \otimes \Lambda^{j-1} W^*) \otimes N, \quad \Theta^{i,j} = \bigoplus_{q>0} \Upsilon^{i,q} \otimes \Lambda^{j-q} V^*,$$

where δ is the Spencer operator. Let also $\Pi^{i,j} = \delta(S^{i+1} V^* \otimes N \otimes \Lambda^{j-1} V^*)$.

Call a subspace $V^* \subset T^*$ strongly non-characteristic for a symbolic system g if $g_k \cap V^* \cdot S^{k-1} T^* \otimes N = 0$ for $k = r_{\min}(g)$ the minimal order of the system.

Theorem 13. [KL$_9$]. *Let V^* be a strongly non-characteristic subspace for a symbolic system g. If g is involutive, then its W-restriction $\bar g$ is also involutive.*

Moreover the Spencer δ-cohomology of g and $\bar g$ are related by the formula:

$$H^{i,j}(g)\simeq\bigoplus_{q>0}H^{i,q}(\bar g)\otimes\Lambda^{j-q}V^*\oplus\delta^{i+1}_{r_{min}(g)}\cdot[\Theta^{i,j}\oplus\Pi^{i,j}]\oplus\delta^i_0\delta^j_0\cdot H^{0,0}(\bar g),$$

where δ^t_s is the Kronecker symbol.

If $\bar g$ is an involutive system of pure order $k=r_{min}(\bar g)=r_{max}(\bar g)$, then g is also an involutive system of pure order k and the above formula holds.

The first two parts of this theorem generalize previous results of pure first order by Quillen and Guillemin, see [Gu$_2$].

3.3. **Geometric structures.** These are given by specification of a Lie group G in light of Klein's Erlangen program [Kl], though prolongations usually make this into infinite-dimensional Lie pseudo-group, see [GS, SS, Ta] and also §4.5. Not going much into details, we consider calculation of Spencer δ-cohomology and restrict, for simplicity, to the first order structures.

They correspond to G-structures, discussed in [St]. More general cases are studied in [Gu$_1$, L$_1$]. Thus $g=g^{|1\rangle}$ is generated in order 1 with subspace $g_1=\mathfrak{g}\subset\mathrm{gl}(n)$ being a matrix Lie algebra, corresponding to G.

Respective system of PDEs describes equivalence of a geometric structure, governed by a Lie group G, to the standard flat model. PDEs describing automorphism groups can be investigated similarly.

As we shall see in the next section, the group $H^{*,2}(g)$ plays an important role in investigation of formal integrability. For geometric structures this is the space of curvatures/torsions. We shall illustrate this with three examples:

(1) Almost complex geometry: $\mathfrak{g}=\mathrm{gl}(\frac{n}{2},\mathbb{C})$. It is given by a tensor $J\in C^\infty(T^*M\otimes TM)$, $J^2=-\mathbf{1}$;
(2) Riemannian geometry: $\mathfrak{g}=\mathrm{so}(n)$. It is given by a tensor $q\in C^\infty(S^2T^*M)$, $q>0$;
(3) Almost symplectic geometry: $\mathfrak{g}=\mathrm{sp}(\frac{n}{2})$. It is given by a tensor $\omega\in C^\infty(\Lambda^2T^*M)$, $\omega^n\neq 0$.

In all three cases $T=T_xM=N$ and there is a linear structure J, q or ω respectively on T.

In the first case (T,J) is a complex space and we can identify $g_1=T^*\otimes_{\mathbb{C}}T$. The prolongations are $g_i=S^i_{\mathbb{C}}T^*\otimes_{\mathbb{C}}T$ (all tensor products over $\mathbb{C}$). The only non-zero Spencer δ-cohomology are:

$$H^{0,k}(g)=\Lambda^k_{\bar{\mathbb{C}}}T^*\otimes_{\bar{\mathbb{C}}}T,$$

which is the space of all skew-symmetric k-linear $\mathbb{C}$-antilinear T-valued forms on T. The system is involutive.

In the second system identification $T \overset{g}{\simeq} T^*$ yields $g_1 = \Lambda^2 T^*$. Since $g_2 = T^* \otimes \Lambda^2 T^* \cap S^2 T^* \otimes T^* = 0$, prolongations vanish $g_{1+i} = 0$ and the system is of finite type. The only non-zero Spencer δ-cohomology are:

$$H^{0,k}(g) = \Lambda^k T^* \otimes T, \quad H^{1,k} = \mathrm{Ker}(\delta : \Lambda^2 T^* \otimes \Lambda^k T^* \to T^* \otimes \Lambda^{k+1} T^*).$$

Thus g is not involutive. We can rewrite the cohomology in bi-grade $(1, 2)$ as $H^{1,2}(g) = \mathrm{Ker}(S^2 \Lambda^2 T^* \to \Lambda^4 T^*)$. Note that $H^{0,2}$ is the space of torsions and $H^{1,2}$ the space of curvature tensors.

In the last case we identify $T \overset{\omega}{\simeq} T^*$ and then get $g_1 = S^2 T^*$. Therefore prolongations $g_i = S^{i+1} T^*$ and the system is of infinite type. The only non-zero Spencer δ-cohomology are:

$$H^{0,k}(g) = \Lambda^{k+1} T^*.$$

The system is involutive.

3.4. **Cartan connection and Weyl tensor.** We define *regular system of PDEs* $\mathcal{E}$ of maximal order k as a submanifold $\mathcal{E}_k \subset J^k \pi$ co-filtered by $\mathcal{E}_l$, with $\mathcal{E}_i^{(1)} \supset \mathcal{E}_{i+1}$ and $\pi_i : \mathcal{E}_{i+1} \to \mathcal{E}_i$ being a bundle map, such that the symbolic system and the Spencer δ-cohomology form graded bundles over it. We define orders $\mathrm{ord}(\mathcal{E})$ of the system and its formal codimension $\mathrm{codim}(\mathcal{E})$ as these quantities for the symbolic system.

Cartan distribution on $\mathcal{E}_k$ is $\mathcal{C}_{\mathcal{E}_k} = \mathcal{C}_k \cap T\mathcal{E}_k$. *Cartan connection* on $\mathcal{E}_k$ is a horizontal subdistribution in it, i.e. a smooth family $H(x_k) \subset \mathcal{C}_{\mathcal{E}_k}(x_k)$, $x_k \in \mathcal{E}_k$, such that $d\pi_k : H(x_k) \to T_x M$ is an isomorphism. A Cartan connection yields the splitting $\mathcal{C}_{\mathcal{E}_k}(x_k) \simeq H(x_k) \oplus g_k(x_k)$ of the Cartan distribution into horizontal and vertical components.

Given a distribution Π on a manifold its *first derived differential system* $\partial\Pi$ is generated by the commutators of its sections. In the regular case it is a distribution and one gets the effective normal bundle $\nu = \partial\Pi/\Pi$. The *curvature* of Π is the vector-valued 2-form $\Xi_\Pi \in \Lambda^2 \Pi^* \otimes \nu$ given by the formula:

$$\Xi_\Pi(\xi, \eta) = [\xi, \eta] \bmod \Pi, \quad \xi, \eta \in C^\infty(\Pi)$$

(it is straightforward to check that Ξ_Π is a tensor).

The *metasymplectic structure* Ω_k on $J^k(\pi)$ is the curvature of the Cartan distribution [L$_1$, KL$_6$]. At a point x_k it is a 2-form on $\mathcal{C}_k(x_k)$ with values in the vector space $F_{k-1}(x_{k-1}) = T_{x_{k-1}}\left[\pi^{-1}_{k-1,k-2}(x_{k-2})\right] \simeq S^{k-1} T^*_x M \otimes N_x$.

To describe it fix a point $x_{k+1} \in J^{k+1}(\pi)$ over x_k and decompose $\mathcal{C}_k(x_k) = L(x_{k+1}) \oplus F_k(x_k)$. Then $\Omega_k(\xi, \eta) = 0$ if both ξ, η belong simultaneously either to $L(x_{k+1})$ or to $F_k(x_k)$. But if $\xi \in L(x_{k+1})$ corresponds to $X = d\pi_k(\xi) \in T_x M$ and $\eta \in F_k(x_k)$ corresponds to $\theta \in S^k T^*_x M \otimes N_x$, then the value of $\Omega_k(\xi, \eta)$ equals

$$\Omega_k(X, \theta) = \delta_X \theta \in S^{k-1} T^*_x M \otimes N_x,$$

where $\delta_X = i_X \circ \delta$ is the differentiation along X. The introduced structure does not depend on the point x_{k+1} determining the decomposition because the subspace $L(x_{k+1})$ is Ω_k-isotropic.

Restriction of the metasymplectic structure $\Omega_k \in F_{k-1} \otimes \Lambda^2 \mathcal{C}_k^*$ to the equation is the tensor $\Omega_{\mathcal{E}_k} \in g_{k-1} \otimes \Lambda^2 \mathcal{C}_{\mathcal{E}_k}^*$. Given a Cartan connection H we define its curvature at x_k to be $\Omega_{\mathcal{E}_k}|_{H(x_k)} \in g_{k-1} \otimes \Lambda^2 T_x^* M$. Considered as an element of the Spencer complex it is δ-closed and change of the Cartan connection effects in a shift by a δ-exact element.

The *Weyl tensor* $W_k(\mathcal{E}; x_k)$ of the PDEs system $\mathcal{E}$ is the δ-cohomology class $[\Omega_{\mathcal{E}_k}|_{H(x_k)}] \in H^{k-1,2}(\mathcal{E}; x_k)$ ([L$_1$]). For G-structures it coincides with the classical structural function [St]. For more general geometric structures it equals torsion/curvature tensor [G$_1$].

Prolongation $\mathcal{E}_{k+1} = \mathcal{E}_k^{(1)}$ is called regular if $\pi_{k+1,k} : \mathcal{E}_{k+1} \to \mathcal{E}_k$ is a bundle map. For regular systems a necessary and sufficient condition for regularity of the first prolongation is vanishing of the Weyl tensor: $W_k(\mathcal{E}) = 0$.

This gives the following criterion of formal integrability:

Theorem 14. *Let $\mathcal{E} = \{\mathcal{E}_l\}_{l=0}^k$ be a regular system of maximal order k. Then the system is formally integrable iff $W_i(\mathcal{E}) = 0$ for all $i \geq k$.*

Note that the number of conditions is indeed finite due to Poincaré δ-lemma: starting from some number i_0 all groups $H^{i,2}(\mathcal{E}) = 0$ for $i > i_0$ (see a bound for i_0 in [Sw$_1$]). This in fact was an original sufficient (cf. to necessary and sufficient in the above statement) criterion of formal integrability in [Go, S$_2$]: If all second cohomology groups $H^{i,2}$ vanish, $i \geq k$, then the regular system is formally integrable.

Tensor $W_k(\mathcal{E})$ plays a central role in equivalence problems [KL$_6$]. Calculating Weyl tensor is a complicated issue, see e.g. [KL$_1$, KL$_2$, KL$_3$], where it was calculated for complete intersection systems of PDEs. Let us perform calculation for the examples from §3.3.

(1) In this case $W_1 = N_J$ is the Nijenhuis tensor of the almost complex structure J. Its vanishing gives integrability condition of almost complex structure. The space $H^{0,2}(g) = \Lambda^2_{\bar{\mathbb{C}}} T^* \otimes_{\bar{\mathbb{C}}} T$ is the space of Nijenhuis tensors.

(2) Here Cartan connection is a linear connection ∇, preserving q. Tensor $W_1 = T_\nabla \in H^{0,2}$ is the torsion and its vanishing leads us to Levi-Civita connection ∇_q. The next Weyl tensor is the Riemannian curvature $R_q \in H^{1,2}$ and its vanishing yields flatness of the metric q.

(3) Curvature is $W_1 = d\omega \in H^{0,2}$. Thus integrability $W_1 = 0$ gives us symplectic structure ω.

Remark 4. *Looking at these examples we observe that searching for involutivity is sometimes superfluous: All three geometries are equally important and from the point of view of getting solutions one just studies formal integrability, which usually occurs at smaller number of prolongations than involutivity.*

Let us finish by mentioning without calculations that $W_k(\mathcal{E})$ equals the conformal Weyl tensor for the conformal Lie algebra $\mathfrak{g} = \mathrm{co}(n)$ and the Weyl projective tensor for the projective Lie algebra $\mathfrak{g} = \mathrm{sl}(n+1)$. Whence the name.

3.5. **Compatibility and solvability.** Investigation of overdetermined systems of PDEs begins with checking compatibility conditions. Riquier-Janet theory makes finding compatibility conditions algorithmic.

With Riquier approach [Ri] one expresses certain higher order derivatives via others (i.e. bring equations to the orthonomic form), differentiate PDEs and substitute the expressed quantities. If new equations arise, the system is called active, otherwise passive. In modern language we talk of formal integrability. Riquier's test on passivity allows to disclose the compatibility conditions.

Janet monomials [J] (and also Thomas's [To]) allow to check compatibility via cross differentiations of respective equations, which is determined by their differential monomials in certain ordering. To a large extent this can be seen as an origin of computer differential algebra (Gröbner bases etc).

Being algorithmic, these approaches are heavily calculational and so good luck in generators in $\mathcal{E}$ and coordinates in jet spaces plays a major role. On the contrary Cartan's theory [C$_2$] has a geometric base (Vessiot's dual approach [V] is of the same flavor) and so coupled with Spencer's homological technique [S$_1$] allows to calculate compatibility conditions in a visibly minimal number of steps.

Remark 5. *Original Cartan's approach aims though to involutivity, not just to formal integrability, cf. Remark 4. In this respect Riquier-Janet theory is more economic.*

Using the machinery of the previous section we can describe one step prolongation as follows. Assume $\mathcal{E}$ is a regular system of PDEs of maximal order k, which includes compatibility to order k. Then $W_k(\mathcal{E}) \in H^{k-1,2}(\mathcal{E})$ is precisely the obstruction to prolongation to $(k+1)$-st jets.

The number of compatibility conditions is $\dim H^{k-1,2}(\mathcal{E})$ (this quantity is constant along $\mathcal{E}$ due to regularity) and they are just components of the Weyl tensor $W_k(\mathcal{E})$. These latter are certain differential equations of ord $\leq k$.

If $W_k(\mathcal{E}) = 0$, the system can be prolonged to level $(k+1)$ and we get a system $\mathcal{E}_{k+1}$, with projections $\pi_{k+1,k} : \mathcal{E}_{k+1} \to \mathcal{E}_k$ being a vector bundle, so that we get new regular system and can continue prolongations. By Hilbert's theorem $H^{i,2}(\mathcal{E}) = 0$ starting from some number i_0. Then there's no more obstructions to compatibility and the system is formally integrable.

If the Weyl tensor is non-zero, we disclose new equations in the system $\mathcal{E}$, which are differential corollaries of ord $\leq k$, and so we change the system by adding them. The new system is

$$\tilde{\mathcal{E}} = \mathcal{E} \cap \pi_{k+1,k}(\mathcal{E}_k^{(1)}) = \{x_k \in \mathcal{E}_k : W_k(\mathcal{E}; x_k) = 0\}.$$

We restart investigation of formal integrability with this new system of equations. This approach is called prolongation-projection method.

The following statement, known as Cartan-Kuranishi theorem, states that we do not continue forever:

Theorem 15. *After a finite number of prolongations-projections system $\mathcal{E}$ will be transformed into a formally integrable system $\bar{\mathcal{E}} \subset \mathcal{E}$.*

This statement was formulated by Cartan in [C_2] without precise conditions. It was proved by Kuranishi [Kur_1] under suitable regularity assumptions (see also [Ma]), but essentially the proof was published long before by the Russian school [Ra, Fi].

With regularity assumptions we remove points, where the ranks of symbol bundles/δ-cohomology drop, together with their projections and prolongations. One hopes that most points will survive, so that the above theorem holds at a generic point.

While in general this is not known, it holds in some good situations. In algebraic case the result is due to Pommaret [Pom] and in analytic case due to Malgrange [M_2].

Note that we started with regular systems $\mathcal{E}$ of maximal order k, though one should start from $\mathcal{E}_1$. Arriving to $\mathcal{E}_l$ one can either add compatibility conditions or new equations of the system of order $l+1$. In the latter case the projection $\pi_{l+1,l} : \mathcal{E}_{l+1} \to \mathcal{E}_l$ is a fiber bundle (in regular case). The prolongation-projection method can be generalized to this situation and Theorem 15 holds in the same range of assumptions.

As a result of the method we get a minimal formally integrable sub-system $\bar{\mathcal{E}} \subset \mathcal{E}$. If it is non-empty the system $\mathcal{E}$ is called (formally) solvable. Indeed all (formal) solutions of $\mathcal{E}$ coincide with these of $\bar{\mathcal{E}}$. This alternative E.Cartan [C_2] characterized as follows: *"after a finite number of prolongations the system becomes involutive or contradictory"*.

3.6. **Formal integrability via multi-brackets and Massey product.** Though Weyl tensors $W_k(\mathcal{E})$ are precisely compatibility conditions, it is important to have a good calculational formula for the latter, at least for some classes of PDEs. The following is a wide class of systems, important in applications.

Let $\mathcal{E} \subset J^k(\pi)$ be a regular system of maximal order k, consisting of $r = \operatorname{codim}(\mathcal{E})$ differential equations on $m = \operatorname{rank}(\pi)$ unknown functions.

Definition 10. *System $\mathcal{E}$ is of* generalized complete intersection type *if*

(1) $m \le r < n + m$;
(2) *The characteristic variety has* $\dim_{\mathbb{C}} \mathrm{Char}^{\mathbb{C}}_{x_k}(\mathcal{E}) = n + m - r - 2$ *at each point* $x_k \in \mathcal{E}$ (*we assume* $\dim \emptyset = -1$);
(3) *The characteristic sheaf $\mathcal{K}$ over* $\mathrm{Char}^{\mathbb{C}}_{x_k}(\mathcal{E}) \subset \mathbb{P}^{\mathbb{C}}T^*$ *has fibers of dimension 1 everywhere.*

If $\mathcal{E}$ is a generalized complete intersection in this sense, then its symbolic system g is a generalized complete intersection in the sense of definition 8.

Note that vanishing of the multi-brackets due to the system is a necessary condition for formal integrability, because they belong to differential ideal of the system.

Theorem 16. [KL_4, KL_{10}]. *Consider a system of PDEs*

$$\mathcal{E}=\Bigl\{F_i\Bigl(x^1,\dots,x^n,u^1,\dots,u^m,\frac{\partial^{|\sigma|}u^j}{\partial x^\sigma}\Bigr)=0\,:\,1\le i\le r\Bigr\},\quad \operatorname{ord}(F_i)=k_i.$$

If $\mathcal{E}$ is a system of generalized complete intersection type, then it is formally integrable if and only if the multi-brackets vanish due to the system:

$$\{F_{i_1},\dots,F_{i_{m+1}}\}\ \mathrm{mod}\,\mathcal{J}_{k_{i_1}+\dots+k_{i_{m+1}}-1}(F_1,\dots,F_r)=0.$$

In particular, we get the following compatibility criterion for scalar PDEs:

Corollary 17. *Let $\mathcal{E}=\{F_1[u]=0,\dots,F_r[u]=0\}$ be a scalar system of complete intersection type, i.e. $r\le n$ and $\operatorname{codim}_{\mathbb{C}}\operatorname{Char}^{\mathbb{C}}(\mathcal{E})=r$. Then formal integrability expresses via Mayer-Jacobi brackets as follows:*

$$\{F_i,F_j\}=0\ \mathrm{mod}\,\mathcal{J}_{k_i+k_j-1}(F_1,\dots,F_r),\quad \forall\ 1\le i<j\le r.$$

This criterion is effective in the study of not only compatibility, but also solvability of systems of PDEs. Examples of applications are [GL, KL_{10}]. Moreover, since differential syzygy is provided explicitly, it is more effective than the method of differential Gröbner basis or its modifications [K].

Let now describe a sketch of the general idea how to investigate systems of PDEs $\mathcal{E}=\{F_1[u^1,\dots,u^m]=0,\dots,F_r[u^1,\dots,u^m]=0\}$ for compatibility.

Take a pair of equations F_i and F_j, $i<j$. Even though the system $\{F_i=0,F_j=0\}$ is underdetermined (for $m>1$) it can possess compatibility conditions $\Theta_{ij}=0$ of order t_{ij} (this actually means that after a change of coordinates this pair of PDEs will involve only one dependent function; but rigorously can be expressed only via non-vanishing second Spencer δ-cohomology). We denote $\Theta_{ij}^{\mathcal{E}}=\Theta_{ij}\ \mathrm{mod}\,\mathcal{J}_{t_{ij}}(F_1,\dots,F_r)$. Thus we get compatibility conditions $\Theta_{ij}^{\mathcal{E}}=0$ of orders $\tau_{ij}\le t_{ij}$.

Then we look to triples F_i,F_j,F_h with $i<j<h$, get in a similar way compatibility conditions $\Theta_{ijh}^{\mathcal{E}}=0$ of orders τ_{ijh} and so forth. In general we get "generalized s-brackets" $\Theta_{i_1\dots i_s}^{\mathcal{E}}$ of orders $\tau_{i_1\dots i_s}$ for all $2\le s\le r$ (see [KL_1] §3.2 for an example of 3-bracket in the case $n=k=2$, $m=1$, $r=3$).

The formula of the operator $\Theta_{i_1\dots i_s}^{\mathcal{E}}$ and the number $\tau_{i_1\dots i_s}$ ($i_1<\dots<i_s$) strongly depends on the type of the system and varies with the type of characteristic variety/symbolic module. For each type of normal form or singularity one gets own formulas. In the range $m\le r<m+n$ the generic condition is that all generalized s-brackets for $s\le m$ are void and the first obstruction to formal integrability are $(m+1)$ multi-brackets. Theorem 16 states that this will be the only set of compatibility conditions.

Remark 6. *Important case of an overdetermined system with $r=m$ constitute Einstein-Hilbert field equations* [Eb]. *They possess compatibility conditions, which hold identically, implying formal integrability of the system.*

Note that the idea of calculating successively 2-product for a pair, then 3-product for a triple (in the case it vanishes for all sub-pairs) etc is very similar to Massey products in topology and algebra. Resembling situation is observed in deformation theory of module structures[4].

We note however that in the context of PDEs the situation is governed by order. We examine the set $\{\tau_{ij}, \tau_{ijh}, \dots, \tau_{1\dots r}\}$ (some numbers can be omitted if respective $\Theta^{\mathcal{E}}_{i_1\dots i_s}$ are void) and take the minimal order. These compatibility conditions are investigated first. Being non-zero, they are added to $\mathcal{E}$ and one considers a new system $\tilde{\mathcal{E}}$ (which can be simpler with lower order compatibility conditions, cf. §3.5). If these compatibility conditions are satisfied, we take the next ones and so on. The procedure is finite in the same sense as in Cartan-Kuranishi prolongation-projection theorem.

3.7. **Integral Grassmanians revisited.** A system of PDEs $\mathcal{E} \subset J^1(E, m)$ is said to be *determined* if $\operatorname{codim} \mathcal{E} = m$ and $\operatorname{codim}_{\mathbb{C}} \operatorname{Char}^{\mathbb{C}}(\mathcal{E}) = 1$. We usually represent such systems as the kernel of a (non-linear) operator $F : C^\infty(\pi) \to C^\infty(\nu)$ with $\operatorname{rank} \pi = \operatorname{rank} \nu = m$.

In this section we restrict to the case $k = 1$ of first order equations. Let w_i^1 be Stiefel-Whitney classes of the tautological vector bundle over the Grassmanian $I(x_1)$, as before.

Theorem 18. [L$_2$]. *Let $\mathcal{E} \subset J^1(E, m)$ be a determined system such that the characteristic variety $\operatorname{Char}^{\mathbb{C}}_{x_1}(\mathcal{E})$ does not belong to a hyperplane for any $x_1 \in \mathcal{E}$. Then the embedding $I\mathcal{E}(x_1) \hookrightarrow I(x_1)$ induces an isomorphism of cohomology with $\mathbb{Z}_2$-coefficients up to dimension n in all cases except the following:*

1. *$m = 2, n \geq 3$. Then $H^*(I\mathcal{E}(x_1), \mathbb{Z}_2)$ is isomorphic to the algebra $\mathbb{Z}_2[w_1^1, \dots, w_n^1, U_{n-1}, \operatorname{Sq} U_{n-1}]$ up to dimension n, where U_{n-1} has dimension $n - 1$ and Sq is the Steenrod square.*
2. *$m = 3, n = 2$. Then $H^*(I\mathcal{E}(x_1), \mathbb{Z}_2)$ is isomorphic to the algebra $\mathbb{Z}_2[w_1^1, w_2^1, \rho_1, \dots, \rho_r]$ up to dimension 2, where dimensions of ρ_i equal 2 and r is a number of components of $\operatorname{Char}^{\mathbb{C}}(\mathcal{E}, x_1)$ with the fibers of the kernel sheaf $\mathcal{K}$ of dimension 1.*
3. *$m = n = 2$. Then $I\mathcal{E}(x_1)$ is diffeomorphic to the torus $S^1 \times S^1$ in hyperbolic case or to the complex projective line $\mathbb{CP}^1$ in elliptic case.*

This yields calculation of cohomology of integral Grassmanians for determined systems. Underdetermined systems can be treated similarly.

Finding cohomology of $I\mathcal{E}(x_k)$ in general overdetermined case seems to be a hopeless problem. However in many cases they stabilize after a sufficient number of prolongations. This constitutes a topological version of the Cartan-Kuranishi theorem:

Theorem 19. [L$_2$]. *Let $\mathcal{E}$ be a system of differential equations of pure first order $\mathcal{E}_1 \subset J^1(E, m)$. Suppose that it is formally integrable and characteristically regular and such that the characteristic variety $\operatorname{Char}^{\mathbb{C}}_{x_1}(\mathcal{E})$ does not belong to a hyperplane for any $x_1 \in \mathcal{E}_1$. Assume also that $\dim_{\mathbb{C}} \operatorname{Char}^{\mathbb{C}}(\mathcal{E}) > 0$.*

---

[4] We thank A. Laudal for a fruitful discussion on this topic.

Then the embeddings $I\mathcal{E}_l(x_l) \hookrightarrow I(x_l)$ induce an isomorphism in cohomology with $\mathbb{Z}_2$-coefficients up to dimension n for sufficiently large values of l.

Let $I^k(E,m) = \bigcup I(x_k)$ be the total space of all integral Grassmanians. Then any integral n-dimensional manifold $L \subset J^k(E,m)$ defines a *tangential map* $t_L : L \to I^k(E,m)$, where $t_L : L \ni x_k \longmapsto T_{x_k}L \in I(x_k)$. Each cohomology class $\kappa \in H^l(I^k(E,m),\mathbb{Z}_2)$ gives rise to a *characteristic class* $\kappa(L) = t_L^*(\kappa)$ on integral manifolds L. By Theorem 2 $H^*(I^k(E,m),\mathbb{Z}_2)$, considered as an algebra over $H^*(J^k(E,m),\mathbb{Z}_2)$, is generated by the Stiefel-Whitney classes $w_1^k,\dots,w_n^k$ of the tautological bundle over $I^k(E,m)$ up to dimension n. Moreover since the bundle $J^l(E,m) \to J^{l-1}(E,m)$ is affine for $l > 1$ and for $l = 1$ is the standard Grassmanian bundle, we get that $H^*(J^k(E,m),\mathbb{Z}_2) = H^*(J^1(E,m),\mathbb{Z}_2)$, considered as an algebra over $H^*(E,\mathbb{Z}_2)$, is generated by the Stiefel-Whitney classes $w_1,\dots,w_n$ of the tautological bundle over $J^1(E,m)$, provided that $\pi_1(E) = 0$.

Let $\mathcal{E}_k \subset J^k(E,m)$ be a formally integrable system of PDEs of maximal order k and $I\mathcal{E}_{k+l} = \bigcup I\mathcal{E}_{k+l}(x_{k+l}) \subset I^{k+l}(E,m)$ be a total space of all integral Grassmanians associated with l-th prolongation $\mathcal{E}_{k+l} = \mathcal{E}_k^{(l)}$. If the differential equation satisfies the conditions of Theorem 19, then for a sufficiently large l the cohomology $H^*(I\mathcal{E}_{k+l},\mathbb{Z}_2)$, considered as an algebra over $H^*(\mathcal{E}_{k+l},\mathbb{Z}_2)$, is generated by the Stiefel-Whitney classes $w_1^k,\dots,w_n^k$ of the tautological vector bundle up to dimension n. On the other hand all bundles $\pi_{k+l,k+l-1} : \mathcal{E}_{k+l} \to \mathcal{E}_{k+l-1}$ are affine for $l > 0$ and hence $H^*(\mathcal{E}_{k+l},\mathbb{Z}_2) = H^*(\mathcal{E}_k,\mathbb{Z}_2)$.

By a *Cauchy data* we mean an $(n-1)$-dimensional integral manifold $\Gamma \subset \mathcal{E}_{k+l}$ together with a section $\gamma : \Gamma \to I\mathcal{E}_{k+l}$ such that $T_{x_{k+l}}\Gamma \subset \gamma(x_{k+l})$ for all $x_{k+l} \in \Gamma$. A solution of the Cauchy problem is an integral n-dimensional submanifold $L \subset \mathcal{E}_{k+l}$ with boundary such that $\Gamma = \partial L$. Each cohomology class $\theta \in H^{n-1}(I\mathcal{E}_{k+l},\mathbb{Z}_2)$ defines a characteristic number $\theta(\Gamma) = \langle \gamma^*\theta, \Gamma\rangle$.

Theorem 20. *If the Cauchy problem (Γ,γ) has a solution, then the characteristic numbers $\theta(\Gamma)$ vanish for all $\theta \in H^{n-1}(I\mathcal{E}_{k+l},\mathbb{Z}_2)$.*

4. Local and global aspects

4.1. **Existence theorems.** System of PDEs $\mathcal{E}$ is called locally/globally integrable, if for its infinite prolongation $\mathcal{E}_\infty$ and any admissible jet $x_\infty \in \mathcal{E}_\infty$ there exists a local/global smooth solution $s \in C^\infty(\pi)$ with $[s]_x^\infty = x_\infty$ (this clearly can be generalized to more general spaces $J^k(E;m)$ of jets).

If the system $\mathcal{E}$ is of finite type and formally integrable, then it is locally integrable. Indeed, the Cartan distribution $\mathcal{C}_{\mathcal{E}_k}$ for k so large that $g_k = 0$ has rank n and is integrable by the Frobenius theorem. Its local integral leaves are solutions of the system $\mathcal{E}$.

For infinite type systems, $\dim g_k \nrightarrow 0$, formal integrability does not imply local integrability in general, some additional conditions should be assumed. One sufficient condition for existence of local solutions is analyticity as Cartan-Kähler theorem claims [Kah]:

Theorem 21. *Let a system $\mathcal{E}$ be analytic, regular and formally integrable. Then it is locally integrable, i.e. any admissible jet $x_\infty \in \mathcal{E}_\infty$ is the jet of a local analytic solution.*

This theorem is a generalization of Cauchy-Kovalevskaya theorem [Pe]. Other generalizations are known, see [ES]. In particular we should mention Ovsyannikov's theorem, according to which for a system $\mathcal{E}$, written in the orthonomic form

$$\frac{\partial^{k_i} u^i}{\partial t^{k_i}} = F_i(t, x, u, \mathcal{D}_\sigma u), \quad i = 1, \dots, m$$

(F_i does not contain derivatives of order higher than k_i and it is free of $\partial_t^{k_i} u^i$) it is enough to require analyticity only in $x = (x^1, \dots, x^{n-1})$ and continuity in t in order to get solution to the Cauchy initial value problem

$$\frac{\partial^l u^i}{\partial t^l}(0, x) = U_l^i(x), \quad l = 0, \dots, k_i - 1,\ i = 1, \dots, m. \tag{13}$$

Notice that the solution to a formally integrable system $\mathcal{E}$ in general form of Theorem 21 is given by a sequence of solutions of Cauchy problems, see [Kah, BCG^3]. The initial value problem is specified similar to (13), with prescribed collection of s_p functions of p arguments, ..., s_0 constants, where s_i are Cartan characters [C_2], see also §4.3. Thus one can, in principle, slightly relax analyticity conditions of Theorem 21 via Ovsyannikov's approach.

Also note that according to Holmgren's theorem ([Pe]) a local solution to the Cauchy problem for a formally integrable analytic system $\mathcal{E}$ is unique even if the (non-characteristic) initial data is only smooth. In general smooth case formal integrability implies local integrability only in certain cases.

One of such cases is when the system $\mathcal{E}$ is purely hyperbolic, i.e. when complex characteristics complexify the real ones: $\mathrm{Char}^{\mathbb{C}}(\mathcal{E}) = (\mathrm{Char}(\mathcal{E}))^{\mathbb{C}}$. In other words any real plane of complimentary dimension in $\mathbb{P}^{\mathbb{C}}T^*$ intersects $\mathrm{Char}(\mathcal{E})$ in $\deg \mathrm{Char}^{\mathbb{C}}(\mathcal{E})$ real different points.

More general case is represented by involutive hyperbolic systems, which are given by the condition that on each step of Cartan-Kähler method the arising determined systems are hyperbolic (these systems are non-unique, see [Y] for the precise definition) on the symbolic level.

Theorem 22. [Y]. *If an involutive hyperbolic system $\mathcal{E}$ is formally integrable and the Cauchy data is non-characteristic, there exists a local solution. In particular, if not all covectors are characteristic for $\mathcal{E}$, then there is a local solution through almost any admissible jet $x_\infty \in \mathcal{E}_\infty$.*

In particular one can solve locally the Cauchy problem for the second order hyperbolic quasi-linear systems of the type arising in general relativity [CB].

Another important example constitute purely elliptic systems, i.e. such $\mathcal{E}$ that the real characteristic variety is empty: $\mathrm{Char}(\mathcal{E}) = \emptyset$.

A system $\mathcal{E} = \{F_i(x, u, \mathcal{D}_\sigma u) = f_i(x)\}$ (independent and dependent variables x, u are multi-dimensional) is said to be of analytic type if locally near any point $x_k \in \mathcal{E}_k$ the (non-linear in general) differential operator $F = (F_i)_{i=1}^r$ is analytic in a certain chart (but the charts may overlap smoothly and $f = (f_i)_{i=1}^r$ is just smooth).

Theorem 23. [S_1, M_1]. *Elliptic formally integrable system $\mathcal{E}$ of analytic type is locally integrable.*

Spencer conjectured [S_2] that any formally integrable elliptic system is locally integrable[5], but this was not proved in the full generality.

A certain progress was due to the works of MacKichan [McK_1, McK_2] and Sweeney [Sw_2]. They studied solvability of the Neumann problem and related this to the δ-estimate on a linear operator Δ: For any large k in

$$0 \to g_{k+1} \xrightarrow{\delta} g_k \otimes T^* \xrightarrow{\delta} g_{k-1} \otimes \Lambda^2 T^* \text{ and any } \xi \in g_k \otimes T^* \cap \operatorname{Ker} \delta^*$$

(operator δ^* is conjugated to δ with respect to some Hermitian metrics on T^*, π, ν), we have $\|\delta\xi\| \geq \frac{k}{\sqrt{2}}\|\xi\|$. Their results imply (see [S_2], also [Tar]):

Theorem 24. *Let $\mathcal{E} = \operatorname{Ker}\Delta$ be a formally integrable system, with not all covectors being characteristic. Suppose that the operator Δ satisfies the δ-estimate. Then the system $\mathcal{E}$ is locally integrable.*

4.2. **Local solvability.** $\mathcal{E}$ is called solvable if we can guarantee existence of a local/global solution. Obviously one should first carry prolongation-projection method to get a maximal formally integrable system $\bar{\mathcal{E}} \subset \mathcal{E}$ (this is usually called bringing $\mathcal{E}$ to an involutive form, compare though with Remark 4), so we can assume that already $\mathcal{E}$ fulfills compatibility conditions.

In light of Cartan-Kähler theorem one would like to specify an admissible jet of solution. It is possible for hyperbolic systems and their generalizations (see Theorem 22), but not for all systems. However even the problem of finding some solution can be non-solvable.

Consider at first smooth linear differential operators. Let us restrict to $\mathbb{C}$-scalar PDEs, i.e. differential operators $\Delta = \Delta_1 + i\Delta_2 : C^\infty(M;\mathbb{C}) \to C^\infty(M;\mathbb{C})$ of order k (Δ_i are real, so one can think of determined real system $\mathcal{E}$ with $\operatorname{rank}\pi = 2$, but it is of special type: $\operatorname{codim}\mathrm{Char}(\mathcal{E}) = 2$).

The first example of operator of this type such that the corresponding PDE $\Delta(u) = f$ not locally solvable for some smooth $f \in C^\infty(M;\mathbb{C})$, was constructed by H. Levi [Lw][6].

This example was later generalized by Hörmander, Grushin and others. In fact, Hörmander found a necessary condition of solvability of for principal

[5]In fact, we cannot prescribe the value of jet of the solution, and so it's better to talk here of local solvability from the next section.

[6]His Δ was a very nice analytic operator of order 1, namely Cauchy-Riemann operator on the boundary of the pseudo-convex set $\{|z_1|^2 + 2\operatorname{Im} z_2 < 0\} \subset \mathbb{C}^2$.

type operators. Using the bracket approach of §3.6 we can formulate it as follows. Let $\mathcal{E}$ be the above PDE written as the real system $\{\Delta_1(u) = f_1, \Delta_2(u) = f_2\}$ ($\Delta_1 = \operatorname{Re}\Delta$, $\Delta_2 = \operatorname{Im}\Delta$ and similar for f) and let σ_s denote the symbol of order s. Then local solvability of a system $\mathcal{E}$ implies

$$\sigma_{2k-1}\Bigl(\{\Delta_1(u), \Delta_2(u)\} \bmod \mathcal{E}\Bigr) = 0. \tag{14}$$

Hörmander formulated his condition differently [Ho$_2$]. Namely denote $H = \sigma_k(\Delta_1)$, $F = \sigma_k(\Delta_2)$. Then the Poisson bracket $\{H, F\}$ vanishes on $\Sigma = \mathrm{Char}_{\mathrm{aff}}(\mathcal{E}) = \{H = 0, F = 0\} \subset T^*M$. In other words, whenever Σ is a submanifold, it is involutive[7].

The condition that the operator Δ has a principal type means that the Hamiltonian vector field X_H on $\{H = 0\}$ is not tangent to the fibers of the projection $T^*M \to M$ (it is possible to multiply Δ by a function $a \in C^\infty(M;\mathbb{C})$, so that a particular choice of H and F is not essential, but the condition involves linear combinations of dH and dF).

Trajectory of the vector field X_H are called bi-characteristics and considered on the invariant manifold $\{H = 0\}$ they are called null bi-characteristics.

Note that even when Σ is not a submanifold, condition (14) can be reformulated as follows

Along null bi-characteristics X_H function F vanishes to at least 2^{nd} order.

This condition was refined by Nirenberg-Treves to the following condition:

Along null bi-characteristics X_H function *F does not change its sign.* ($\mathcal{P}$)

If Σ is a submanifold in T^*M and dH, dF are independent on its normal bundle, this condition is equivalent to (14). In general it strictly includes (implies) Hörmander condition. Indeed if order of zero for F along null bi-characteristics is finite, it should be even.

It turns out that this new condition is not only necessary, but also sufficient for solvability ([NT] with the condition of order $k = 1$ or base dimension $n = 2$ or that the principal part is analytic; [BF] in general):

Theorem 25. *If Δ is of principal type and satisfies condition ($\mathcal{P}$), then for any smooth f linear PDE $\Delta(u) = f$ is locally solvable.*

This theorem was generalized to pseudo-differential operators (see [Le] for important partial cases and review; [De] in general), with condition ($\mathcal{P}$) being changed to a similar condition (Ψ). In such a form it is sometimes possible to give a sufficient condition for global solvability (see loc.cit).

In the second paper [NT] Nirenberg and Treves gave a vector version (determined system of special type with $\mathrm{rank}_{\mathbb{R}}(\pi) = 2m$) of the above theorem:

---

[7]Note that for real problems, when $F = 0$ and $\mathrm{Im}(u) = 0$ this condition is void. Indeed linear determined PDEs of principal type with real (nonconstant) coefficients in the principal part are locally solvable [Ho$_1$] (this also follows from Theorem 25).

Theorem 26. *If $\Delta = \Delta_1 + i\Delta_2 \in \mathrm{Diff}_k(m \cdot \mathbf{1}_{\mathbb{C}}, m \cdot \mathbf{1}_{\mathbb{C}})$ is a complex smooth operator of principal type and the homogeneous Hamiltonian $H + iF = \det[\sigma_\Delta]$ of order mk satisfies condition $(\mathcal{P})$, then the system of linear PDEs $\Delta(u) = f$ is locally solvable for any smooth vector-valued function $f \in C^\infty(M; \mathbb{C}^m)$.*

Note however that the principal type condition of [NT] is formulated so that multiple characteristics are excluded (this is equivalent to the claim that $\mathcal{K}$ is a 1-dimensional bundle over $\mathrm{Char}^{\mathbb{C}}(\mathcal{E})$), though in some cases this condition can be relaxed.

For general systems the solvability question is still open and one can be tempted to approach it via successive sequence of determined systems, like in Cartan-Kähler theorem (see Guillemin's normal forms in [Gu_2, BCG^3]).

Remark 7. *The solutions obtained via the above methods are usually distributions, though in some cases they can be proved to be smooth by using elliptic regularity or Sobolev's embedding theorem* [Ho_2, Pe]*. The methods can be generalized to weakly non-linear situations, but for strongly non-linear PDEs effects of multi-valued solutions require new insight* [KLR].

Finally let us consider an important case of evolutionary PDEs $u_t = L[u]$, where L is a non-linear differential operator involving only $\mathcal{D}_x$ differentiations, in the splitting of base coordinates $\mathbb{R} \times U = \{(t, x)\}$. The Cauchy problem for such systems is often posed on the characteristic submanifold $\Sigma^{n-1} = \{t = 0\}$, which contradicts the approach of Cartan-Kähler theorem.

Nevertheless in many cases it is possible to show that the solution exists. For instance, consider the system $\partial_t u = Au + F(t, x, u)$ with A being a determined linear differential operator on the space $\mathcal{W}$ of smooth vector-functions of x and F can be non-linear (usually of lower order). If the homogeneous linear system $\partial_t u = Au$ is solvable and e^{At} is a semi-group (on a certain Banach completion of $\mathcal{W}$), then provided that F is Lipschitz on $\mathcal{W}$, we can guarantee existence of a local solution to the initial value problem $u(0, x) = u_0(x)$ (in fact weak solutions; strong solutions are guaranteed if $\mathcal{W}$ can be chosen a reflexive Banach space [SY]).

This scheme works well for differential operators A with constant coefficients. Moreover, global solvability can be achieved. Consider, for instance a non-autonomous reaction-diffusion equation

$$\partial_t u = a\Delta u - f(t, u) + g(t, x),$$

where $x \in U \Subset \mathbb{R}^{n-1}$, $a \in \mathrm{Gl}_+(\mathbb{R}^m)$ is a positive constant matrix, Δ the Laplace operator and the functions f, g belong to certain Hölder spaces. The boundary behavior is governed by Dirichlet or Neumann or periodic conditions. Then provided that function f has a limited growth behavior at infinity (see [CV] for details) the initial problem $u(0, x) = u_0(x)$ for this system is globally solvable.

Similar schemes (with characteristic Cauchy problems) work also for PDEs involving higher derivatives in t, for example damped hyperbolic equation

[CV]. This allows to consider evolutionary PDEs as dynamical systems. In fact, bracket approach for compatibility and generalized Lagrange-Charpit method of §4.4 allows to establish and investigate finite-dimensional sub-dynamics, see [KL, LL].

4.3. **Dimension of the solutions space.** In his study of systems $\mathcal{E}$ of PDEs [C_2] (interpreted as exterior differential systems) Cartan constructed a sequence of numbers s_i, which are basic for his involutivity test. These numbers depend on the flag of subspaces one chooses for investigation of the system and so have no invariant meaning.

The classical formulation is that a general solution depends on s_p functions of p variables, s_{p-1} functions of $(p-1)$ variables, $\dots$, s_1 functions of 1 variable and s_0 constants (we adopt here the notations from [BCG^3]; in Cartan's notations [C_2] we should rather write s_p, s_p+s_{p-1}, $s_p+s_{p-1}+s_{p-2}$ etc). However as Cartan notices just after the formulation [C_2], this statement has only a calculational meaning.

Nevertheless two numbers are absolute invariants and play an important role. These are Cartan genre, i.e. the maximal number p such that $s_p\neq 0$, but $s_{p+1}=0$, and Cartan integer $\sigma=s_p$. As a result of Cartan's test a general solution depends on σ functions of p variables (and some number of functions of lower number of variables, but this number can vary depending on a way we parametrize the solutions).

Here in analytical category a general solution is a local analytic solution obtained as a result of application of Cartan-Kähler theorem and thus being parametrized by the Cauchy data. In smooth category one needs a condition to ensure existence of solutions with any admissible jet, see §4.1-4.2.

In general we can calculate these numbers in formal category. We call p *functional dimension* and σ *functional rank* of the solutions space $\operatorname{Sol}(\mathcal{E})$ [KL_5]. These numbers can be computed via the characteristic variety. If the characteristic sheaf $\mathcal{K}$ over $\operatorname{Char}^{\mathbb{C}}(\mathcal{E})$ has fibers of dimension k, then

$$p=\dim\operatorname{Char}^{\mathbb{C}}(\mathcal{E})+1,\ \ \sigma=k\cdot\deg\operatorname{Char}^{\mathbb{C}}(\mathcal{E}).$$

The first formula is a part of Hilbert-Serre theorem ([Ha]), while the second is more complicated. Actually Cartan integer σ was calculated in [BCG^3] in general situation and the formula is as follows.

Let $\operatorname{Char}^{\mathbb{C}}(g)=\cup_\epsilon\Sigma_\epsilon$ be the decomposition of the characteristic variety into irreducible components and $d_\epsilon=\dim\mathcal{K}_x$ for a generic point $x\in\Sigma_\epsilon$. Then

$$\sigma=\sum d_\epsilon\cdot\deg\Sigma_\epsilon.$$

The clue to this formula is commutative algebra. Namely *Hilbert polynomial* ([Ha]) of the symbolic module g^* equals

$$P_{\mathcal{E}}(z)=\sigma z^p+\dots$$

A powerful method to calculate the Hilbert polynomial is resolution of a module. In our case a resolution of the symbolic module g^* exists and it can be expressed via the Spencer δ-cohomology. Indeed, the Spencer cohomology of the symbolic system g is $\mathbb{R}$-dual to the Koszul homology of the module g^* and for algebraic situation this resolution was found in [Gr].

This yields the following formulae [KL$_5$]. Let

$$\binom{z+k}{k} = \frac{1}{k!}(z+1)\cdot(z+2)\cdots(z+k).$$

Denote $S_j(k_1,\dots,k_n) = \sum_{i_1<\dots<i_j} k_{i_1}\cdots k_{i_j}$ the j-th symmetric polynomial and let also

$$s_i^n = \frac{(n-i)!}{n!}\,S_i(1,\dots,n)$$

Thus

$$s_0^n = 1,\quad s_1^n = \frac{n+1}{2},\quad s_2^n = \frac{(n+1)(3n+2)}{4\cdot 3!},\quad s_3^n = \frac{n(n+1)^2}{2\cdot 4!},$$

$$s_4^n = \frac{(n+1)(15n^3+15n^2-10n-8)}{48\cdot 5!}\qquad \text{etc.}$$

If we decompose

$$\binom{z+n}{n} = \sum_{i=0}^{n} s_i^n \frac{z^{n-i}}{(n-i)!},$$

then we get the expression for the Hilbert polynomial

$$P_{\mathcal{E}}(z) = \sum_{i,j,q}(-1)^i h^{q,i} s_j^n \frac{(z-q-i)^{n-j}}{(n-j)!} = \sum_{k=0}^{n} b_k \frac{z^{n-k}}{(n-k)!},$$

where

$$b_k = \sum_{j=0}^{k}\sum_{q,i}(-1)^{i+j+k} h^{q,i} s_j^n \frac{(q+i)^{k-j}}{(k-j)!}.$$

Let us compute these dimensional characteristics p,σ for two important classes of PDEs.

If $\mathcal{E}$ is an involutive systems, then $H^{i,j}(\mathcal{E}) = 0$ for $i \notin \operatorname{ord}(\mathcal{E})-1$, $(i,j)\neq(0,0)$, and the above formula becomes more comprehensible.

Let us restrict for simplicity to the case of systems of PDEs $\mathcal{E}$ of pure first order. Then

$$P_{\mathcal{E}}(z) = h^{0,0}\binom{z+n}{n} - h^{0,1}\binom{z+n+1}{n+1} + h^{0,2}\binom{z+n+2}{n+2} - \dots$$
$$= b_1\frac{z^{n-1}}{(n-1)!} + b_2\frac{z^{n-2}}{(n-2)!} + \dots + b_0.$$

Vanishing of the first coefficient $b_0 = 0$ is equivalent to vanishing of Euler characteristic for the Spencer δ-complex, $\chi = \sum_i(-1)^i h^{0,i} = 0$, and this is equivalent to the claim that not all the covectors from ${}^{\mathbb{C}}T^*\setminus 0$ are characteristic for the system g.

The other numbers b_i are given by the above general formulas, though now they essentially simplify. For instance

$$b_1 = \tfrac{n+1}{2}\, b_0 - \sum (-1)^i h^{0,i} i = \sum (-1)^{i+1} i \cdot h^{0,i}.$$

If $\operatorname{codim} \operatorname{Char}^{\mathbb{C}}(\mathcal{E}) = n - p > 1$, then $b_1 = 0$ and in fact then $b_i = 0$ for $i < n - p$, but $b_{n-p} = \sigma$.

Theorem 27. [KL$_5$]. *If* $\operatorname{codim} \operatorname{Char}^{\mathbb{C}}(\mathcal{E}) = n - p$, *then the functional rank of the system equals*

$$\sigma = \sum_i (-1)^i h^{0,i} \frac{(-i)^{n-p}}{(n-p)!}.$$

One can extend the above formula for general involutive system and thus compute the functional dimension and functional rank of the solutions space (some interesting calculations can be found in classical works [J, C$_2$]).

Consider also an important partial case of Cohen-Macaulay systems:

Theorem 28. [KL$_{10}$]. *Let $\mathcal{E}$ be a formally integrable system of generalized complete intersection type with orders $k_1, \dots, k_r$. Then the space $\mathcal{S}ol_{\mathcal{E}}$ has formal functional dimension and rank equal respectively*

$$p = m + n - r - 1, \qquad \sigma = S_{r-m+1}(k_1, \dots, k_r).$$

4.4. **Integrability methods.** Most classical methods for integration of PDEs are related to symmetries ([Lie$_1$, G$_1$, F]).

A *symmetry* of a system $\mathcal{E}$ is a Lie transformation of $J^k\pi$, resp. $J^k(E, m)$, that preserves $\mathcal{E}$ (k is the maximal order of $\mathcal{E}$). Internal symmetry is a structural diffeomorphism of $\mathcal{E}$, i.e. a diffeomorphism of $\mathcal{E}_k$ (not necessary inducing diffeomorphisms of $\mathcal{E}_l$ for $l \le k$) that preserves the Cartan distributions $\mathcal{C}_{\mathcal{E}_k}$. In many important cases, the systems $\mathcal{E}$ are rigid [KLV], in which case internal and external symmetries coincide.

In practice the group (in fact, pseudo-group, see the next section) of symmetries $\operatorname{Sym}(\mathcal{E})$ is difficult to calculate and it is much easier to work with the corresponding Lie algebra of infinitesimal symmetries ([Lie$_2$, LE]). These are Lie vector fields X_φ on the space[8] $J^k\pi$, which are tangent to $\mathcal{E}$.

The generating function φ has order 0 or 1 in the classical case (point or contact transformations). Equation for φ to be a symmetry of a system $\mathcal{E} = \{F_\alpha = 0\}$ can be written in the form (for some differential operators Q_α):

$$X_\varphi(F_\alpha)|_{\mathcal{E}} = 0 \quad \Leftrightarrow \quad \ell_{F_\alpha}\varphi = \sum Q_\alpha F_\alpha.$$

Notice that when the system is scalar, i.e. $\pi = \mathbf{1}$, and $\deg F_\alpha = k_\alpha$, $\deg \varphi = \varkappa$, then the defining equations can be written in the form

$$\{F_\alpha, \varphi\} = 0 \bmod \mathcal{J}_{k_\alpha + \varkappa - 1}(\mathcal{E}). \tag{15}$$

When $\varphi \in \mathfrak{F}_i$, $i > 1$, the field X_φ does not define a flow on any finite jet-space, but rather on $J^\infty(\pi)$. If this flow leaves $\mathcal{E}_\infty$ invariant, then φ

[8] We take the affine chart to have formulas (2) representing X_φ.

(or X_φ) is called *higher symmetry* ([KLV]). Denoting by $\ell_F^{\mathcal{E}}$ the restriction $\ell_F|_{\mathcal{E}_\infty}$ we obtain the defining equations of higher symmetries:

$$\varphi \in \operatorname{sym}(\mathcal{E}) \quad \Leftrightarrow \quad \ell^{\mathcal{E}}_{F_\alpha}(\varphi) = 0.$$

Here $\operatorname{sym}(\mathcal{E}) = D_{\mathcal{C}}(\mathcal{E}_\infty)/\mathcal{C}D(\mathcal{E}_\infty)$ is the quotient of the Lie algebra $D_{\mathcal{C}}$ of all symmetries of the Cartan distribution $\mathcal{C}_{\mathcal{E}}$ on $\mathcal{E}_\infty$ by the space $\mathcal{C}D$ of trivial symmetries, tangent to the distribution $\mathcal{C}_{\mathcal{E}}$.

Conservation laws ω_ψ with generating function ψ are obtained from the dual equation

$$(\ell^{\mathcal{E}}_{F_\alpha})^*(\psi) = 0,$$

where Δ^* is the formally dual to an operator Δ.

Remark 8. *Both symmetries and conservation laws enter variational bicomplex or equivalently $\mathcal{C}$-spectral sequence for the system $\mathcal{E}$, see* [T, SCL, Kru, A] *and references therein.*

Notice that classical (point and contact) symmetries as well as classical conservation laws are widely used to find classes of exact solutions and partially integrate the system, see [CRC, Ol]. In fact, almost all known exact methods are based on the idea of symmetry or intermediate integral [G_2, F].

Due to Corollary 17 this also holds for higher symmetries/conservation laws. Indeed if $\mathcal{G} = \langle \varphi_1, \dots, \varphi_s \rangle \subset \operatorname{sym}(\mathcal{E})$ is a Lie subalgebra of symmetries of a compatible system $\mathcal{E}$, then the joint system $\tilde{\mathcal{E}} = \mathcal{E} \cap \{\varphi_1 = 0, \dots, \varphi_s = 0\}$, provided that regularity assumptions are satisfied, is compatible too.

Classical Lagrange-Charpit method [G_1, Gun] for first order PDEs consists in a special type overdetermination of the given system $\mathcal{E}$, so that the new system is again compatible[9]. *Generalized Lagrange-Charpit method* [KL_3] works for any system of PDEs and it also consists in overdetermination to a compatible system.

For systems of scalar PDEs it is often more convenient to impose additional equations $F_{r+1}, \dots, F_{r+s}$ to the system $\mathcal{E} = \{F_1 = 0, \dots, F_r = 0\}$, so that the joint system $\tilde{\mathcal{E}} = \{F_1 = 0, \dots, F_{r+s} = 0\}$ is of complete intersection type. Then if $\mathcal{E}$ is compatible, the compatibility of the sub-system $\tilde{\mathcal{E}} \subset \mathcal{E}$ can be expressed as follows (see Corollary 17):

$$\{F_i, F_j\} = 0 \bmod \mathcal{J}_{k_i+k_j}(\tilde{\mathcal{E}}) \quad \text{for} \quad 1 \le i \le r+s,\ r < j \le r+s.$$

Note that (15) is a particular case of these equations. For a system of vector PDEs ($\operatorname{rank} \pi > 1$) the corresponding situation, when the compatibility condition writes effectively, should be the generalized complete intersection (see Theorem 16), and then the conditions of generalized Lagrange-Charpit method can be written via multi-brackets.

Let us remark that intermediate integrals are partial cases of this approach (we called additional PDEs $F_{r+1} = 0, \dots, F_{r+s} = 0$ *auxiliary integrals* in

[9]This stays in contrast with the method of differential ansatz, where the additional equations are imposed with only condition that the joint system is solvable.

[KL$_2$]). More generally, most integrability schemes (Lax pairs, Sato theory, commuting hierarchies etc) are closely related to compatibility criteria.

For instance, Backlund transformations [PRS, IA] can be treated as follows. Let $\mathcal{E}_1 = \{F_1 = 0, \dots, F_r = 0\} \subset J^\infty(\pi_1)$ be a compatible system. Extend $\pi_1 \hookrightarrow \pi = \pi_1 \oplus \pi_2$ and let us impose new PDEs $\{F_{r+1} = 0, \dots, F_{r+s} = 0\}$, which are not auxiliary integrals in the sense that the joint system $\tilde{\mathcal{E}} = \{F_1 = 0, \dots, F_{r+s} = 0\}$ is not compatible. If the compatibility conditions modulo the system $\mathcal{E}_1$ are reduced to a compatible system $\mathcal{E}_2 \subset J^\infty(\pi_2)$, then any solution of $\mathcal{E}_1$ gives (families of) solutions of $\mathcal{E}_2$.

For the sin-Gordon equation $u_{xy} = \sin u$ the additional equations are $v_x = \sin w$, $w_y = \sin v$, $u = v + w$ and we get $\mathcal{E}_2 = \mathcal{E}_1$ for $z = u - v$; here $\pi_1 = \mathbf{1}$ (fiber coordinate u) and $\pi_2 = \mathbf{1}$ (fiber coordinate z).

Finally consider the classical Darboux method of integrability [D, G$_2$, AK]. It is applied to hyperbolic second-order PDEs $F = 0$ on the plane (if quasi-linear, then local point transformation brings it to the form $u_{xy} = f(x, y, u, u_x, u_y)$; in general denote the characteristic fields by X, Y), which by a sequence of Laplace transformations reduce to the trivial PDE $u_{xy} = 0$.

In this case the equation possesses a closed form general solution depending on two arbitrary functions of 1 variable. They are obtained via a pair of intermediate integrals $I_1 = 0$, $I_2 = 0$, such that the system $\{F = 0, I_1 = 0\}$ is compatible and has one common characteristic X, while the system $\{F = 0, I_2 = 0\}$ is compatible and has one common characteristic Y. All three equations are compatible as well (and this system is already free of characteristics, i.e. of finite type).

For Liouville equation $u_{xy} = e^u$ the pair of second order intermediate integrals is $I_1 = u_{xx} - \frac{1}{2}u_x^2 = f(x)$ and $I_2 = u_{yy} - \frac{1}{2}u_y^2 = g(y)$, i.e. we have $\mathcal{D}_y(I_1) = 0$ and $\mathcal{D}_x(I_2) = 0$ on $\mathcal{E}$.

Thus Darboux method can be treated as a particular case of generalized Lagrange-Charpit method, but in this case we relax the condition of complete intersection (for overdetermined system in dimension two this yields $\mathrm{Char}^{\mathbb{C}}(\mathcal{E}) = \emptyset$) to possibility of common characteristics (in this case criterion of Theorem 16 fails and compatibility conditions become of lower orders and simpler).

4.5. **Pseudogroups and differential invariants.** Let a group G act on the manifold E by diffeomorphisms. Its action lifts naturally to the jet-space $J^k(E, m)$. An important modification of this situation is when G acts by contact transformations on $J^1(E, 1)$.

A general G-representation via Lie transformations is a prolongation of one of these by the Lie-Backlund theorem, see §1.4. We also investigate group actions on differential equations $\mathcal{E}$. We again require that the group acts by symmetries of $\mathcal{C}_\mathcal{E}$, but now they need not to be external, and if the system is not rigid (§4.4), they may not to be prolongation of point of contact symmetries.

It is often assumed that G is a Lie group, because then one can exploit the formulas of §1.5 to lift transformations to the higher jets, without usage of the inverse function theorem.

A function I is called a *differential invariant* of order k with respect to the action of G, if it is constant on the orbits $G^k \cdot x_k \subset J^k(E,m)$ of the lifted action. For connected Lie groups this writes simpler: $\hat{X}(I) = 0$, $X \in \mathfrak{g}$, where $\mathfrak{g} = \mathrm{Lie}(G)$ is the corresponding Lie algebra.

Denote by $\mathcal{I}_k$ the algebra of differential invariants of order $\le k$. Then $\mathcal{I} = \cup \mathcal{I}_k$ is a filtered algebra, with the associated graded algebra $\mathfrak{O} = \oplus \mathfrak{O}^k$ called the algebra of covariants ([KL$_7$]). The latter plays an important role in setting a Spencer-type calculus for pseudo-groups ([KL$_8$]).

Similar to invariant functions there are defined invariant (multi-) vector fields, invariant differential forms, various invariant tensors, differential operators on jet-spaces etc.

Invariant differentiations play a special role in producing other differential invariants. Levi-Civita connection is one of the most known examples. Tresse derivatives are the very general class of such operations and they are defined as follows.

Suppose we have $n = \dim E - m$ differential invariants $f_1, \dots, f_n$ on $\mathcal{E}_k \subset J^k(E,m)$. Provided $\pi_{k+1,k}(\mathcal{E}_{k+1}) = \mathcal{E}_k$ we define the differential operator

$$\hat{\partial}_i : C^\infty(\mathcal{E}_k) \to C^\infty(\mathcal{E}'_{k+1}),$$

where $\mathcal{E}'_{k+1}$ is the open set of points $x_{k+1} \in \mathcal{E}_{k+1}$ with

$$df_1 \wedge \dots \wedge df_n|_{L(x_{k+1})} \ne 0. \tag{16}$$

We require that $\{f_i\}_{i=1}^n$ are such that $\mathcal{E}'_{k+1}$ is dense in $\mathcal{E}_{k+1}$. For the trivial equation $\mathcal{E}_i = J^i(E,m)$ this is always the case. But if the equation $\mathcal{E}$ is proper, this is a requirement of "general position" for it. Given condition (16) we write:

$$df|_{L(x_{k+1})} = \sum_{i=1}^n \hat{\partial}_i(f)(x_{k+1})\, df_i|_{L(x_{k+1})},$$

which defines the function $\hat{\partial}_i(f)$ uniquely at all the points $x_{k+1} \in \mathcal{E}'_{k+1}$. This yields an invariant differentiation $\hat{\partial}_i = \hat{\partial}/\hat{\partial} f_i : \mathcal{I}_k \to \mathcal{I}_{k+1}$. The expressions $\hat{\partial}_i(f) = \hat{\partial} f/\hat{\partial} f_i$ are called Tresse derivatives of f with respect to f_i ([KL$_8$]).

For affine charts $J^k(\pi) \subset J^k(E,m)$ this definition coincides with the classical one ([Tr, T, Ol]). Consider some examples of calculations of scalar differential invariants[10].

(1) *Diffeomorphisms of the projective line.*

1a. Left SL_2*-action.* For a diffeomorphism $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ and $g \in \mathrm{SL}_2(\mathbb{R})$ define the left action by $g(f) = g \circ f$. The corresponding Lie algebra $\mathfrak{g} = \mathrm{sl}(2)$

[10]Some of these facts are contained in classical textbooks. We obtained the formulas thanks to the wonderful Mapple-11 package DiffGeom by I.Anderson.

is generated by the vector fields $\langle\partial_u, u\partial_u, u^2\partial_u\rangle$ on $J^0(\mathbb{R})$. The algebra $\mathcal{I}$ of differential invariants is generated by x, the Schwartz derivative

$$j_3 = \frac{2p_1p_3 - 3p_2^2}{2p_1^2} \quad \text{and all total derivatives} \quad \mathcal{D}_x^k(j_3),\ k > 0.$$

1b. Right SL_2*-action.* The right action of $G = \mathrm{SL}_2(\mathbb{R})$ on $\mathbb{RP}^1$ is defined by the formula: $g(f) = f \circ g^{-1}$. The corresponding Lie algebra $\mathfrak{g} = \mathrm{sl}(2)$ is generated by the vector fields $\langle\partial_x, x\partial_x, x^2\partial_x\rangle$ on $J^0(\mathbb{R})$. The algebra $\mathcal{I}$ of differential invariants is generated by u, the inverse Schwartz derivative

$$J_3 = \frac{2p_1p_3 - 3p_2^2}{2p_1^4} \quad \text{and the Tresse derivatives} \quad \frac{\hat{\partial}^k}{\hat{\partial}u^k}(J_3),\ k > 0.$$

(2) *Curves in the classical plane geometries.*

2a. Metric plane. The Lie algebra of plane motions $\mathfrak{m}_2$ is generated by the vector fields $\langle\partial_x, \partial_u, x\partial_u - u\partial_x\rangle$ on the plane $\mathbb{R}^2 = J^0(\mathbb{R})$. There is an $\mathfrak{m}_2$-invariant differentiation (metric arc)

$$\nabla = \frac{1}{\sqrt{p_1^2+1}}\frac{d}{dx},$$

and the algebra of $\mathfrak{m}_2$-differential invariants is generated by the curvature

$$\kappa_2 = \frac{p_2}{(p_1^2+1)^{3/2}} \quad \text{and the derivatives} \quad \nabla^r\kappa_2,\ r > 0.$$

2b. Conformal plane. The Lie algebra of plane conformal transformations $\mathfrak{co}_2$ is generated by the vector fields $\langle\partial_x, \partial_u, x\partial_u - u\partial_x, x\partial_x + u\partial_u\rangle$ on the plane $\mathbb{R}^2 = J^0(\mathbb{R})$. There is a $\mathfrak{co}_2$-invariant differentiation (conformal arc)

$$\nabla = \frac{p_1^2+1}{p_2}\frac{d}{dx},$$

and the algebra of $\mathfrak{co}_2$-differential invariants is generated by the conformal curvature

$$\kappa_3 = \frac{p_1^2p_3 + p_3 - 3p_1p_2^2}{p_2^2} \quad \text{and the derivatives} \quad \nabla^r\kappa_3,\ r > 0.$$

2c. Symplectic plane. The Lie algebra of plane symplectic transformations $\mathfrak{sp}_2$ is generated by the vector fields $\langle\partial_x, \partial_u, x\partial_u, u\partial_x, x\partial_x - u\partial_u\rangle$ on the plane $\mathbb{R}^2 = J^0(\mathbb{R})$. There is an $\mathfrak{sp}_2$-invariant differentiation (symplectic arc)

$$\nabla = \frac{1}{\sqrt[3]{p_2}}\frac{d}{dx},$$

and the algebra of $\mathfrak{sp}_2$-differential invariants is generated by the symplectic curvature

$$\kappa_4 = \frac{3p_2p_4 - 5p_3^2}{3p_2^{8/3}} \quad \text{and the derivatives} \quad \nabla^r\kappa_4,\ r > 0.$$

2d. Affine plane. The Lie algebra of plane affine transformations $\mathfrak{a}_2$ is generated by the vector fields $\langle \partial_x, \partial_u, x\partial_u, u\partial_x, x\partial_x, u\partial_u \rangle$ on the plane $\mathbb{R}^2 = J^0(\mathbb{R})$. There is an $\mathfrak{a}_2$-invariant differentiation (affine arc)

$$\nabla = \frac{p_2}{\sqrt{3p_2p_4 - 5p_3^2}} \frac{d}{dx},$$

and the algebra of $\mathfrak{a}_2$-differential invariants is generated by the affine curvature

$$\kappa_5 = \frac{9p_2^2p_5 + 40p_3^3 - 45p_2p_3p_4}{9(3p_2p_4 - 5p_3^2)^{3/2}} \quad \text{and the derivatives} \quad \nabla^r\kappa_5,\ r > 0.$$

2e. Projective Plane. The Lie algebra of plane projective transformations $\mathfrak{sl}_3$ is generated by the vector fields $\langle \partial_x, \partial_u, x\partial_u, u\partial_x, x\partial_x, u\partial_u, x^2\partial_x + xu\partial_u, xu\partial_x + u^2\partial_u \rangle$ on the plane $\mathbb{R}^2 = J^0(\mathbb{R})$.

There are two relative differential invariants:

$$\Theta_3 = \frac{-9p_2^2p_5 + 45p_2p_3p_4 - 40p_3^3}{54p_2^3}, \quad \Theta_8 = 6\Theta_3\frac{d^2\Theta_3}{dx^2} - 7\left(\frac{d\Theta_3}{dx}\right)^2$$

of degrees 3 and 8, and of orders 5 and 7 respectively. There is also an $\mathfrak{sl}_3$-invariant differentiation (projective arc, or Study invariant differentiation)

$$\nabla = \frac{1}{\sqrt[3]{\Theta_3}} \frac{d}{dx},$$

and the algebra of $\mathfrak{sl}_3$-differential invariants is generated by the projective curvature

$$\kappa_7 = \frac{\Theta_8^3}{\Theta_3^8} \quad \text{and the derivatives} \quad \nabla^r\kappa_7,\ r > 0.$$

Pseudogroups are infinite-dimensional Lie groups, which can be obtained by integrating Lie equations [C_1, Eh, Kur_2, SS, KS]. Differential invariants and Tresse derivatives are defined for them in the same manner.

Theorem 29. *Algebra $\mathcal{I}$ of differential invariants of pseudogroup G action is finitely generated by algebraic operations and Tresse derivatives.*

This theorem (with a proper assumption of regularity) was formulated and sketched by A. Tresse [Tr], though important partial cases were considered before by S. Lie [Lie_1] (see also [H]). The proof for (finite-dimensional) Lie groups was given by Ovsiannikov [Ov], for pseudogroups acting on jet-spaces by Kumpera [Kum]. The general case of pseudogroups G acting on systems of PDEs $\mathcal{E}$ was completed in [KL_8].

Similar to Cartan-Kuranishi theorem one hopes that generic points of $\mathcal{E}$ are regular. This is possible to show in good (algebraic/analytic) situations.

Pseudogroups constitute a special class of Lie equations. With general approach of [KL_8] one does not require their local integrability from the beginning. It is important that passage from formal integrability to the local one is easier for pseudogroups compared to general systems of PDEs.

For instance, formally integrable transitive flat pseudogroups are locally integrable [BM, P].

Pseudogroups are basic for solution of equivalence problem. Pseudogroups are also fundamental for establishing special symmetric solutions of PDEs, they can be used to multiply transversal solutions and in some cases (if the pseudogroup is big enough) to integrate PDEs [Ov, KLV, CRC, Ol, SCL, AFT, KLR].

4.6. **Spencer D-cohomology.** The Spencer differential

$$D:\mathcal{J}^k(\pi)\otimes\Omega^l(M)\to\mathcal{J}^{k-1}(\pi)\otimes\Omega^{l+1}(M)$$

is uniquely defined by the following conditions:

(i) D is $\mathbb{R}$-linear and satisfies the Leibniz rule:

$$D(\theta\otimes\omega)=D(\theta)\wedge\omega+\pi_{k,k-1}(\theta)\otimes d\omega,\quad \theta\in\mathcal{J}^k(\pi),\ \omega\in\Omega^l(M).$$

(ii) The following sequence is exact:

$$0\to C^\infty(\pi)\xrightarrow{j_k}\mathcal{J}^k(\pi)\xrightarrow{D}\mathcal{J}^{k-1}(\pi)\otimes\Omega^1(M).$$

The latter operator can be described as follows. Let $x\in M$, $v\in T_xM$, $\theta\in\mathcal{J}^k(\pi)$ and $x_k=\theta(x)\in\pi_k^{-1}(x)$. Since $\tilde\theta=\pi_{k,k-1}(\theta)\in C^\infty(\pi_{k-1})$, the value $D_v\theta=i_v\circ D(\theta)\in J^{k-1}_x\pi$ equals $\rho^{\mathrm{v}}_{k-1}\circ(j_{k-1}\tilde\theta)_*(v)$, where

$$\rho^{\mathrm{v}}_{k-1}:T_{x_{k-1}}(J^{k-1}\pi)\simeq L(x_k)\oplus J^{k-1}_x\pi\to J^{k-1}_x\pi$$

is the projection to the second component (the splitting depends only on x_k). Thus $D(\theta)=0$ if and only if $\tilde\theta(M)$ is an integral manifold of the Cartan distribution on $J^{k-1}(\pi)$ and therefore has the form $j_{k-1}(s)$, which yields $\theta=j_k(s)$ for some $s\in C^\infty(\pi)$.

The above geometric description implies that the Spencer operator D is natural, $D\circ\pi_{k+1,k}=\pi_{k,k-1}\circ D$. Moreover let $\alpha:E_\alpha\to M_\alpha$ and $\beta:E_\beta\to M_\beta$ be two vector bundles and $\Psi:\alpha\to\beta$ be a morphism over a smooth map $\psi:M_\alpha\to M_\beta$, $\psi\circ\alpha=\beta\circ\Psi$, such that $\Psi_x:\alpha^{-1}(x)\to\beta^{-1}(\psi(x))$ are linear isomorphisms for all $x\in M$. Then Ψ generates a map of sections: $\Psi^*:C^\infty(\beta)\to C^\infty(\alpha)$, where $\Psi^*(h)(x)=\Psi_x^{-1}\bigl(h(\psi(x))\bigr)$. This in turn generates a map of k-jets: $\Psi_k^*:\mathcal{J}^k(\beta)\to\mathcal{J}^k(\alpha)$ and $\Psi_k^*\circ j_k=j_k\circ\Psi^*$. Then naturality of D means that

$$D\circ\Psi_k^*\otimes\psi^*=\Psi_{k-1}^*\otimes\psi^*\circ D.$$

The above properties of the Spencer differential yield $D^2=0$. Hence the following sequence is a complex:

$$0\to C^\infty(\pi)\xrightarrow{j_k}\mathcal{J}^k(\pi)\xrightarrow{D}\mathcal{J}^{k-1}(\pi)\otimes\Omega^1(M)\xrightarrow{D}\cdots\xrightarrow{D}\mathcal{J}^{k-n}(\pi)\otimes\Omega^n(M)\to 0.$$

It is called the first (naive) *Spencer complex*.

Let $\mathcal{E}=\{\mathcal{E}_k\subset J^k\pi\}$ be a system of linear PDEs. Assume that $\mathcal{E}$ is formally integrable. Then the 1-st Spencer complex can be restricted to $\mathcal{E}$, meaning that $D : \underline{\mathcal{E}_k} \to \underline{\mathcal{E}_{k-1}} \otimes \Omega^1(M)$, where $\underline{\mathcal{E}_k} = C^\infty(\pi_k|_{\mathcal{E}_k})$ denotes the space of sections (non-holonomic solutions of $\mathcal{E}_k$). The resulting complex

$$0 \to \underline{\mathcal{E}_k} \xrightarrow{D} \underline{\mathcal{E}_{k-1}} \otimes \Omega^1(M) \xrightarrow{D} \cdots \xrightarrow{D} \underline{\mathcal{E}_{k-n}} \otimes \Omega^n(M) \to 0 \tag{17}$$

is called the first Spencer complex associated with the system $\mathcal{E}$.

The exact sequences $0 \to g_k \to \mathcal{E}_k \to \mathcal{E}_{k-1} \to 0$ induce the exact sequences of the Spencer complexes and this together with δ-lemma shows that the cohomology of the 1st Spencer complex are stabilizing for sufficiently large k. The stable cohomology are called *Spencer D-cohomology* of $\mathcal{E}$ and they are denoted by $H^i_D(\mathcal{E})$, $i = 0, 1, \dots, n$.

Remark that $H^0_D(\mathcal{E}) = \mathrm{Sol}(\mathcal{E})$ is the space of global smooth solutions of $\mathcal{E}$. Other cohomology group $H^i_D(\mathcal{E})$ describe the solutions spaces of the systems of PDEs corresponding to the place i of the Spencer complex and $H^*_D(\mathcal{E})$ is a module over the de Rham cohomology of the base $H^*(M)$.

Due to the summands g_k the first complex is not formally exact (=exact on the level of formal series). The construction of the second (sophisticated) Spencer complex amends this feature. This 2nd complex is defined as follows.

Pick a vector bundle morphism $\Theta : \mathcal{E}_k \to \mathcal{E}_{k+1}$ that is right-inverse to the projection $\pi_{k+1,k}$: $\pi_{k+1,k} \circ \Theta = \mathrm{id}$. Let $D_\Theta = D \circ \Theta : \underline{\mathcal{E}_k} \otimes \Omega^i(M) \to \underline{\mathcal{E}_k} \otimes \Omega^{i+1}(M)$. Another right-inverse $\Theta' : \mathcal{E}_k \to \mathcal{E}_{k+1}$ gives:

$$D_\Theta - D_{\Theta'} : \underline{\mathcal{E}_k} \otimes \Omega^i(M) \to \delta(g_{k+1} \otimes \Omega^i(M)).$$

Therefore for the quotient $C^i_k = \mathcal{E}_k \otimes \Lambda^i T^*M / \delta(g_{k+1} \otimes \Lambda^{i-1} T^*M)$ the factor-operators (denoted by the same letter D) are well-defined and they constitute the factor complex

$$0 \to \underline{C^0_k} \xrightarrow{D} \underline{C^1_{k-1}} \xrightarrow{D} \cdots \xrightarrow{D} \underline{C^n_{k-n}} \to 0,$$

which is called the 2nd Spencer complex. Its cohomology stabilize for sufficiently large k and coincide with stable cohomology of the 1st Spencer complex. Moreover the second Spencer D-complex is formally exact [S$_2$].

Another approach to the Spencer D-cohomology is via the compatibility complex. Let $\Delta_1 : C^\infty(\pi_1) \to C^\infty(\pi_2)$ be a differential operator. Denote by $\Delta_2 : C^\infty(\pi_2) \to C^\infty(\pi_3)$ its compatibility operator, i.e. $\Delta_2 \circ \Delta_1 = 0$ and $\mathrm{Im}[\psi^{\Delta_1}_\infty : J^\infty(\pi_1) \to J^\infty(\pi_2)] = \mathrm{Ker}[\psi^{\Delta_2}_\infty : J^\infty(\pi_2) \to J^\infty(\pi_3)]$.

Denoting Δ_3 the compatibility operator for the operator Δ_2 and so on we get the compatibility complex

$$C^\infty(\pi_1) \xrightarrow{\Delta_1} C^\infty(\pi_2) \xrightarrow{\Delta_2} C^\infty(\pi_3) \xrightarrow{\Delta_3} \cdots .$$

Existence of such complexes was proved by Kuranishi (also Goldschmidt, see [S$_2$] and references therein) whenever $\mathcal{E} = \mathrm{Ker}(\Delta_1)$ is formally integrable. Moreover any two such formally exact complexes are homotopically equivalent. Hence the 2nd Spencer D-complex provides us with an explicit construction of such a complex.

However the Spencer D-complexes are not necessary minimal in the sense that ranks of the bundles π_i can be reduced. Important method for constructing minimal compatibility complexes comes from resolutions in commutative algebra. In such a form they can even be generalized to the non-linear situation. For (non-linear) systems of generalized complete intersection type the compatibility complexes were constructed in [KL_{10}].

Remark 9. *Since cohomology of a compatibility complex equal $H^*_D(\mathcal{E})$, this gives a way to calculate non-linear Spencer D-cohomology. To define the non-linear version of Spencer D-complex one can use the machinery of §2.4.*

4.7. **Calculations of Spencer cohomology.** Consider some examples.

(1) If $\mathcal{E} = \mathrm{Ker}\big[\Delta : C^\infty(\pi) \to C^\infty(\pi)\big]$ is a determined system of PDEs, $\mathrm{Char}^{\mathbb{C}}(\Delta) \neq \mathbb{P}^{\mathbb{C}}T^*$, then
$$H^0_D(\mathcal{E}) = \mathrm{Ker}(\Delta) = \mathrm{Sol}(\mathcal{E}),\ H^1_D(\mathcal{E}) = \mathrm{Coker}(\Delta) \simeq \mathrm{Ker}(\Delta^*).$$
(2) Spencer D-cohomology of a system $\mathcal{E}$ of PDEs, defined by the de Rham differential $d : C^\infty(M) \to \Omega^1(M)$, coincide with the de Rham's cohomology of the base manifold: $H^*_D(\mathcal{E}) = H^*_{dR}(M)$.
(3) Let $\nabla : C^\infty(\pi) \to C^\infty(\pi) \otimes \Omega^1(M)$ be a flat connection. Then the Spencer cohomology of the corresponding system $\mathcal{E}$ coincide with the de Rham cohomology of M with coefficients in π: $H^*_D(\mathcal{E}) = H^*_\nabla(\pi)$.
(4) Let M be a complex manifold and π a holomorphic vector bundle over it. Denote by $\Omega^{p,q}(\pi)$ the (p,q)-forms on M with values in π. Then the Spencer D-cohomology of the Cauchy-Riemann equation given by the operator $\bar\partial : \Omega^{p,0}(\pi) \to \Omega^{p,1}(\pi)$ are the Dolbeault cohomology $H^*_{\bar\partial}(M, \Omega^p(\pi))$.
(5) Let $\mathcal{E}$ be a formally integrable system of finite type. Then $\pi_{k+1,k} : \mathcal{E}_{k+1} \to \mathcal{E}_k$ are isomorphisms for large k. Thus the Spencer differential $D : \mathcal{E}_k \simeq \mathcal{E}_{k+1} \to \mathcal{E}_k \otimes \Omega^1(M)$ defines a flat (Cartan) connection ∇ in the vector bundle π_k and the Spencer cohomology equal the de Rham cohomology of this connection: $H^*_D(\mathcal{E}) = H^*_\nabla(\mathcal{E}_k)$.

Finally consider the calculations of Spencer cohomology using the technique of spectral sequences. We will investigate a formally integrable system $\mathcal{E} = \{\mathcal{E}_k \subset J^k\pi\}$ of linear PDEs of maximal order l in a bundle $\pi : E \to M$.

Assume that the base manifold M is itself a total space of a fibre bundle $\varkappa : M \to B$. We say that $\varkappa$ is a *noncharacteristic bundle* if all fibres $F_b = \varkappa^{-1}(b)$, $b \in B$, are strongly noncharacteristic for $\mathcal{E}$ in the sense of §3.2.

A vector field X on M is said to be *vertical* if $\varkappa_*(X) = 0$. A differential form $\theta \in \mathcal{E}_i \otimes \Omega^r(M)$ is called *q-horizontal* if $X_1 \wedge \dots \wedge X_{q+1} \rfloor \theta = 0$ for any vertical vector fields $X_1, \dots, X_{q+1}$ on M. Denote by $\underline{\mathcal{E}_i} \otimes \Omega^r_q(M)$ the module of q-horizontal elements with $\mathcal{E}_i$-values.

Let $F_{p,q} = \underline{\mathcal{E}_{l-p-q}} \otimes \Omega^{p+q}_q(M)$. Then $\{F_{p,q}\}$ gives a filtration of Spencer complex (17) and $D(F_{p,q}) \subset F_{p,q+1}$. Denote by $\{E^{p,q}_r,\ d^{p,q}_r : E^{p,q}_r \to E^{p+r,q-r+1}_r\}$ the spectral sequence associated with this filtration.

In order to describe the spectral sequence we assume that $\varkappa$ is a noncharacteristic bundle and consider the restriction $\bar{\pi}_b : E_b \to F_b$ of the bundle π to a fibre F_b. Denote the respective restrictions of $\pi_i : \mathcal{E}_i \to M$ to F_b by $\overline{\mathcal{E}_{i;b}}$ (cf. §3.2 for restrictions of symbolic systems). They satisfy the condition $\overline{\mathcal{E}_{i+1;b}} \subset \overline{\mathcal{E}_{i;b}}^{(1)}$. Due to Cartan-Kuranishi prolongation theorem there exists a number i_0 such that $\overline{\mathcal{E}_{i+1;b}} \subset \overline{\mathcal{E}_{i;b}}^{(1)}$ for $i \geq i_0$.

We call system $\overline{\mathcal{E}}(b) = \{\overline{\mathcal{E}_{i;b}} \subset J^i(\bar{\pi}_b)\}$ the restriction of $\mathcal{E}$ to the fibre F_b. By Theorem 13 involutivity of $\mathcal{E}$ implies involutivity of $\overline{\mathcal{E}} = \overline{\mathcal{E}}(b)$ for all $b \in B$ (in fact, the theorem concerns only symbolic levels, while the claim involves restrictions of the Weyl tensors). Similar, $\overline{\mathcal{E}}$ is formally integrable provided that $\mathcal{E}$ is formally integrable (but for the needs of Spencer D-cohomology we can restrict to systems $\overline{\mathcal{E}^{|i\rangle}}$ for $i \geq i_0$).

The following theorem is a generalization of the classical Leray-Serre theorem into the context of Spencer cohomology.

Theorem 30. [L_3, LZ_2]. *Let $\mathcal{E}$ be a formally integrable system of linear PDEs on a bundle π over M and let $\varkappa : M \to B$ be a noncharacteristic bundle. Assume that the Spencer D-cohomology $H_D^*\big(\overline{\mathcal{E}}(b)\big)$ form a smooth vector bundle over B. Then the above spectral sequence $E_r^{p,q}$ converges to the Spencer D-cohomology $H_D^*(\mathcal{E})$ and the first terms of it equal:*

(0) $E_0^{p,q} = F_{p,q}/F_{p+1,q-1} \simeq \mathcal{E}_{l-p-q} \otimes_{C^\infty(M)} [\Omega^q(\varkappa) \otimes_{C^\infty(B)} \Omega^p(B)]$, *where* $\Omega^q(\varkappa) = \Omega^q(M)/\Omega^q_{q-1}(M)$ *is a module of totally vertical q-forms;*

(1) $E_1^{p,q} \simeq H_D^q(\overline{\mathcal{E}}) \otimes \Omega^p(B)$, *the differential* $d_1^{0,q} : H_D^q(\overline{\mathcal{E}}) \to H_D^q(\overline{\mathcal{E}}) \otimes \Omega^1(B)$ *is a flat connection ∇ on the bundle of Spencer cohomology $H_D^q(\overline{\mathcal{E}})$;*

(2) $E_2^{p,q} \simeq E_2^{p,q} = H_\nabla^p\big(B, H_D^q(\overline{\mathcal{E}})\big)$, *i.e. the usual ∇-de Rham cohomology with coefficients in the sheaf of sections of Spencer D-cohomology.*

Assuming that the Spencer cohomology $H_D^q(\mathcal{E})$ are finite dimensional we define the Euler characteristic $\chi(\mathcal{E})$ as

$$\chi(\mathcal{E}) = \sum_{i=0}^{n} (-1)^i \dim H^i(\mathcal{E}).$$

Then the above theorem shows that $\chi(\mathcal{E}) = \chi(\overline{\mathcal{E}}) \cdot \chi_B$, where χ_B is the Euler characteristic of B.

Remark 10. *Borel theorem on computation of cohomology of homogeneous spaces together with Leray-Serre spectral sequence constitute the base for computations of de Rham cohomology of smooth manifolds. Borel theorem was generalized to the context of Spencer cohomology in* [LZ_1], *when the symmetry group was assumed compact.*

Note that δ-estimate from §4.1 guarantees local exactness of the Spencer complex ([Sw_2, McK_1, McK_2], Theorem 24 is a partial case). Thus Spencer D-cohomology is the cohomology of the base M with coefficients in the sheaf $\mathrm{Sol}_{\mathrm{loc}}(\mathcal{E})$.

Finite-dimensionality of $H^*(\mathcal{E})$ can be guaranteed if the system $\mathcal{E}$ is elliptic and the manifold M is compact. Another situations is the generalization of the above construction, when the manifold M is foliated (not necessary fibered) and the leaves wind over the manifold densely.

Finally we remark that vanishing of the Spencer cohomology $H^q_D(\mathcal{E}) = 0$ means global solvability of the PDEs corresponding to the operator D at the q-th place of the Spencer complex, provided that compatibility conditions are satisfied.

References

[A] I. M. Anderson, *The variational bicomplex*, Prepr. Utah State Univ. (1989), `http://www.math.usu.edu/~fg_mp`.

[AFT] I. M. Anderson, M. E. Fels, C. G. Torre, *Group invariant solutions in mathematical physics and differential geometry*, The geometrical study of differential equations, Contemp. Math. **285**, A.M.S. (2001), 51–60.

[AK] I. M. Anderson, N. Kamran, *The variational bicomplex for hyperbolic second-order scalar partial differential equations in the plane*, Duke Math. J. **87**, no. 2 (1997), 265–319.

[BF] R. Beals, C. Fefferman, *On local solvability of linear partial differential equations*, Ann. of Math. (2) **97** (1973), 482–498.

[BH] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, Cambridge, U.K. (1993)

[BR] D. A. Buchsbaum, D. S. Rim, *A generalized Koszul complex, II. Depth and multiplicity*, Trans. A.M.S. **111** (1964), 197–224.

[BCG^3] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, P. A. Griffiths, *Exterior differential systems*, MSRI Publications **18**, Springer-Verlag (1991).

[BM] C. Buttin, P. Molino, *Theoreme general d'equivalence pour les pseudogroupes de Lie plats transitifs*, J. Diff. Geometry, **9** (1974), 347–354.

[C_1] E. Cartan, *Sur la structure des groupes infinis de transformations*, in *Oeuvres completes*, II **2**, 571–714, Gautier-Villars, Paris (1953).

[C_2] E. Cartan, *Les systèmes différentiels extérieurs et leurs applications géométriques* (French), Actualités Sci. Ind. **994**, Hermann, Paris (1945).

[CV] V. V. Chepyzhov, M. I. Vishik, *Attractors for equations of mathematical physics*, A.M.S. Colloquium Publications **49**, Providence, RI (2002).

[CB] Y. Choquet-Bruhat (Foures-Bruhat), *Theoreme d'existence pour certains systemes d'equations aux derivees partielles non lineaires* (French), Acta Math. **88**, 141–225 (1952).

[CRC] W. F. Ames, R. L. Anderson, V. A. Dorodnitsyn, E. V. Ferapontov, R. K. Gazizov, N. H. Ibragimov, S. R. Svirshchevskiĭ, *CRC handbook of Lie group analysis of differential equations. Vol. 1. Symmetries, exact solutions and conservation laws*, CRC Press, Boca Raton (1994).

[D] G. Darboux, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, I–IV partie. Réimpr. de la edition de 1887-1915-1894-1896. Chelsea Publishing Co., Bronx, N.Y. (1972).

[De] N. Dencker, *The resolution of the Nirenberg-Treves conjecture*, Ann. of Math. (2) **163** (2006), no. 2, 405–444.

[E] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag (1995).

[Eh] C. Ehresmann, *Introduction à la théorie des structures infinitésimales et des pseudogroupes de Lie* (French), Colloque de topologie et géométrie différentielle, no. 11, 16 pp. Strasbourg (1952).

[Eb] D. W. Ebner, *How Hilbert has found the Einstein equations before Einstein and forgeries of Hilbert's page proofs*, ArXive: physics/0610154.

[ES] Yu. Egorov, M. A. Shubin, *Linear PDEs. Foundations of the classical theory* (transl. from the 1988 Russian original), Partial differential equations I, Encyclopaedia Math. Sci. **30**, Springer (1992).

[F] A.R. Forsyth, *Theory of differential equations*, Vol.1-6. Cambridge University Press (1900–1906).

[Fi] S. P. Finikov, *Cartan's Method of Exterior Forms in Differential Geometry* (Russian), OGIZ, Moscow-Leningrad (1948).

[GL] V. Goldberg, V. Lychagin, *On the Blaschke conjecture for 3-webs*, J. Geom. Anal. **16** (2006), no. 1, 69–115.

[Go] H. Goldschmidt, *Integrability criteria for systems of nonlinear partial differential equations*, J. Diff. Geom., **1**(3) (1967), 269–307.

[G_1] E. Goursat, *Lecons sur l'intégration des équations aux dérivées partielles du premier ordere*, Hermann, Paris (1891)

[G_2] E. Goursat, *Lecons sur l'intégration des équations aux dérivées partielles du second ordere*, II, Hermann, Paris (1898)

[Gr] M. L. Green, *Koszul cohomology and geometry*, in: "Lectures on Riemann surfaces", World Sci. Publ., 177–200 (1989).

[Gun] N. M. Gunter, *Integration of PDEs of the first order*, ONTI (Russian) Leningrad-Moscow (1934).

[GS] V. Guillemin, S. Sternberg, *An algebraic model of transitive differential geometry*, Bull. A.M.S., **70** (1964), 16–47.

[GQS] V. Guillemin, D. Quillen, S. Sternberg, *The integrability of characteristics*, Comm. Pure Appl. Math. **23**, no.1 (1970), 39–77.

[Gu_1] V. Guillemin, *The integrability problem for G-structures*, Trans. A.M.S., **116** (1965), 544-560.

[Gu_2] V. Guillemin, *Some algebraic results concerning the characteristics of overdetermined partial differential equations*, Amer. Journal of Math. **90** (1968), 270-284.

[H] M. Halpen, *Sur les invariants differentiels*, Thèse, Paris (1878).

[Ha] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag (1977).

[Ho_1] L. Hörmander, *On the theory of general partial differential operators*, Acta Math. **94** (1955), 161–248.

[Ho_2] L. Hörmander, *Linear partial differential operators*, 3rd edition, Springer-Verlag (1969).

[IA] N. H. Ibragimov, R. L. Anderson, *Lie-Backlund transformations in applications*, SIAM Studies in Applied Mathematics **1**, SIAM, Philadelphia (1979).

[J] M. Janet, *Leçons sur les systèms d'équations aux derivées partielles*, Gauthier-Villers, Paris (1929).

[Kah] E. Kähler, *Einfürhung in die Theorie der Systeme von Differentialgleichungen*, B.G. Teubner, Leipzig (1934).

[Kl] F. Klein, *Vergleichende Betrachtungen uber neuere geometrische Forschungen*, Erlangen, Andreas Deichert, 1872.

[KLV] I. S. Krasilschik, V. V. Lychagin, A. M. Vinogradov, *Geometry of jet spaces and differential equations*, Gordon and Breach (1986).

[K] B. S. Kruglikov, *Note on two compatibility criteria: Jacobi-Mayer bracket vs. differential Gröbner basis*, Lobachevskii Journ. Math. **23** (2006), 57–70.

[KL] B. Kruglikov, O. Lychagina, *Finite dimensional dynamics for Kolmogorov-Petrovsky-Piskunov equation*, Lobachevskii J. Math. **19** (2005), 13–28.

[KL_1] B. S. Kruglikov, V. V. Lychagin, *Mayer brackets and solvability of PDEs – I*, Diff. Geom. and its Appl. **17** (2002), 251–272.

[KL$_2$] B. S. Kruglikov, V. V. Lychagin, *Mayer brackets and solvability of PDEs – II*, Trans. Amer. Math. Soc. **358**, no.3 (2005), 1077–1103.

[KL$_3$] B. S. Kruglikov, V. V. Lychagin, *A compatibility criterion for systems of PDEs and generalized Lagrange-Charpit method*, A.I.P. Conference Proceedings, *Global Analysis and Applied Mathematics: International Workshop on Global Analysis*, **729**, no. 1 (2004), 39–53.

[KL$_4$] B. S. Kruglikov, V. V. Lychagin, *Multi-brackets of differential operators and compatibility of PDE systems*, Comptes Rendus Math. **342**, no. 8 (2006), 557–561.

[KL$_5$] B. S. Kruglikov, V. V. Lychagin, *Dimension of the solutions space of PDEs*, ArXive e-print: math.DG/0610789; In: *Global Integrability of Field Theories*, Proc. of GIFT-2006, Ed. J.Calmet, W.Seiler, R.Tucker (2006), 5–25.

[KL$_6$] B. S. Kruglikov, V. V. Lychagin, *On equivalence of differential equations*, Acta et Comment. Univ. Tartuensis Math. **3** (1999), 7–29.

[KL$_7$] B. S. Kruglikov, V. V. Lychagin, *Transitive and transversal actions of pseudogroups on submanifolds*, Russian Math. (Izv. VUZ, Matematika) **48**, no. 11, 30-44 [engl. 27-40] (2004).

[KL$_8$] B. S. Kruglikov, V. V. Lychagin, *Invariants of pseudogroup actions: Homological methods and Finiteness theorem*, Int. J. Geomet. Meth. Mod. Phys. **3**, no. 5 & 6 (2006), 1131–1165.

[KL$_9$] B. S. Kruglikov, V. V. Lychagin, *Spencer delta-cohomology, restrictions, characteristics and involutive symbolic PDEs*, ArXive: math.DG/0503124; to appear in Acta Math. Applicandae.

[KL$_{10}$] B. S. Kruglikov, V. V. Lychagin, *Compatibility, multi-brackets and integrability of systems of PDEs*, prepr. Univ. Tromsø 2006-49; ArXive: math.DG/0610930.

[Kru] D. Krupka, *Global variational principles: foundations and current problems*, Global analysis and applied mathematics, 3–18, AIP Conf. Proc. **729**, Amer. Inst. Phys., NY (2004).

[Kum] A. Kumpera, *Invariants differentiels d'un pseudogroupe de Lie. Part I*, J. Differential Geometry **10** (1975), no. 2, 289–345; *Part II*, ibid. **10** (1975), no. 3, 347–416.

[KS] A. Kumpera, D. Spencer, *Lie equations. Volume 1: General theory*, Princeton University Press and University Tokyo Press (1972).

[Kur$_1$] M. Kuranishi, *On E. Cartan's prolongation theorem of exterior differential systems*, Amer. J. Math. **79** (1957), 1–47.

[Kur$_2$] M. Kuranishi, *On the local theory of continuous infinite pseudogroups. I-II.* Nagoya Math. J., **15** (1959), 225–260; **19** (1961), 55–91.

[KLR] A. Kushner, V. Lychagin, V. Roubtsov, *Contact geometry and non-linear differential equations*, Cambridge University Press (2006).

[Le] N. Lerner, *When is a pseudo-differential equation solvable?* Ann. Inst. Fourier **50**, no. 2 (2000), 443–460.

[Lw] H. Lewy, *An example of a smooth linear partial differential equation without solution*, Ann. of Math. (2) **66** (1957), 155–158.

[Lie$_1$] S. Lie, *Gesammelte Abhandlungen*, Bd. 1-4, Teubner, Leipzig-Kristiania (1929).

[Lie$_2$] S. Lie, *Theorie der Transformationsgruppen* (Zweiter Abschnitt, unter Mitwirkung von Prof.Dr.Friederich Engel), Teubner, Leipzig (1890).

[LE] S. Lie, F. Engel, *Theorie der Transformationsgruppen*, vol. II *Begründungstransformationen*, Leipzig, Teubner (1888-1893).

[L$_1$] V. V. Lychagin, *Homogeneous geometric structures and homogeneous differential equations*, in A. M. S. Transl., *The interplay between differential geometry and differential equations*, V. Lychagin Eds., ser. 2, **167** (1995), 143–164.

[L$_2$] V. V. Lychagin, *Geometric theory of singularities of solutions of nonlinear differential equations* (Russian), Itogi Nauki i Tekhniki, Problems in geometry **20**, 207–247, VINITI, Moscow (1988); Engl. transl.: J. Soviet Math. **51** (1990), no. 6, 2735–2757.

[L_3] V. V. Lychagin, *Differential Operators on fiber bundles*, Uspekhi Mat. Nauk; Engl.Transl.: Russian Math. Surveys **40**, no. 2 (1985), 187–188.

[LL] V. Lychagin, O. Lychagina, *Finite dimensional dynamics for evolutionary equations*, Nonlinear Dynamics, DOI 10.1007/s11071-006-9049-5 (2006).

[LZ_1] V. V. Lychagin, L. Zilbergleit, *Spencer Cohomology and Symmetry Group*, Acta Appl. Math. **41**, no.1-3 (1995), 227–245.

[LZ_2] V. V. Lychagin, L. Zilbergleit, *Spencer cohomology*, Geometry in partial differential equations, World Sci. Publ. (1994), 215–252.

[LS] T. Lada, J. Stasheff, *Introduction to SH Lie algebras for physicists*, Int. J. Theor. Phys., **32** (1993), no.7, 1087–1103.

[M_1] B. Malgrange, *Equations de Lie. I-II.*, J. Differential Geometry, **6** (1972), 503–522; **7** (1972), 117–141.

[M_2] B. Malgrange, *Systèmes différentiels involutifs* (French), Panoramas et Syntheses, Paris (2005).

[Ma] Y. Matsushima, *On a theorem concerning the prolongation of a differential system*, Nagoya Math. J. **6**, 1–16 (1953).

[McK_1] B. MacKichan, *A generalization to overdetermined systems of the notion of diagonal operators. I. Elliptic operators*, Acta Math. **126** (1971), 83–119.

[McK_2] B. MacKichan, *A generalization to overdetermined systems of the notion of diagonal operators. II. Hyperbolic operators*, Acta Math. **134**, no. 3-4 (1975), 239–274.

[N] Y. Nambu, *Generalized Hamiltonian dynamics*, Phys. Rev. D **7** (1973), 2405–2412.

[NT] L. Nirenberg, J. F. Treves, *On local solvability of linear partial differential equations. Part I. Necessary conditions*, Comm. Pure Appl. Math. **23** (1970), 1–38; *Part II. Sufficient conditions*, ibid. **23** (1970), 459–509; correction ibid. **24** (1971), no. 2, 279–288.

[Ol] P. Olver, *Applications of Lie groups to differential equations*, Graduate Texts in Mathematics, **107**, Springer-Verlag, New York (1986).

[Ov] L. V. Ovsiannikov, *Group analysis of differential equations*, Russian: Nauka, Moscow (1978); Engl. transl.: Academic Press, New York (1982).

[Pe] I. G. Petrovsky, *Lectures on partial differential equations*, 7th ed., MGU (Russian), Moscow (1984); Engl. transl.: Dover (1991).

[P] A. S. Pollack, *The integrability problem for pseudogroup structures*, J. Diff. Geometry, **9** (1974), 355–390.

[Pom] J.-F. Pommaret, *Differential Galois theory*, Gordon and Breach (1978).

[PRS] F. A. E. Pirani, D. C. Robinson, W. F. Shadwick, *Local jet bundle formulation of Backlund transformations. With applications to nonlinear evolution equations*, Mathematical Physics Studies **1**, Reidel Publ.Co. (1979).

[Ra] P. K. Rashevsky, *Geometric theory of PDEs*, OGIZ (Russian) Moscow-Leningrad (1947).

[Ri] C. Riquier, *Les systèmes d'équations aux derivées partielles*, Gauthier-Villars, Paris (1910).

[SS] I. M. Singer, S. Sternberg, *On the infinite groups of Lie and Cartan*, J. d'Analyse Math., **15** (1965), 1–114.

[S_1] D. Spencer, *Deformation of structures on manifolds defined by transitive pseudogroups*, Part I, Ann. of Math. (2), **76** (1962), 306–398; Part II, ibid. 399–445.

[S_2] D. C. Spencer, *Overdetermined systems of linear partial differential equations*, Bull. Amer. Math. Soc., **75** (1969), 179–239.

[St] S. Sternberg, *Lectures on Differential Geometry*, Prentice-Hall, Englewood Cliffs, 1964.

[Sw_1] W. J. Sweeney, *The δ-Poincaré estimate*, Pacific J. Math. **20** (1967), 559–570.

[Sw_2] W. J. Sweeney, *A uniqueness theorem for the Neumann problem*, Ann. of Math. (2) **90** (1969), 353–360.

[SCL] A. V. Bocharov, V. N. Chetverikov, V. N.; S. V. Duzhin, N. G. Khor'kova, I. S. Krasil'shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky, A. M. Vinogradov, *Symmetries and conservation laws for differential equations of mathematical physics*, Transl. Math. Monograph **182**, A.M.S., Providence (1999).

[SY] G. R. Sell, Y. You, *Dynamics of evolutionary equations*, Appl. Math. Sci. **143**, Springer (2002).

[Ta] N. Tanaka, *On differential systems, graded Lie algebras and pseudogroups*, Jour. Math. Kyoto Univ., **10** (1970), 1–82.

[Tar] N. N. Tarkhanov, *The parametrix method in the theory of differential complexes* (Russian), Nauka, Novosibirsk (1990); Engl. transl.: *Complexes of differential operators*, Mathematics and its Applications **340**, Kluwer (1995).

[T] T. V. Thomas, *The differential invariants of generalized spaces*, Cambridge, The University Press (1934).

[To] J. Thomas, *Differential systems*, American Mathematical Society (1937).

[Tr] A. Tresse, *Sur les invariants differentiels des groupes continus de transformations*, Acta Math. **18** (1894), 1–88.

[Tu] W. M. Tulczyjew, *The Euler-Lagrange resolution. Differential geometrical methods in mathematical physics*, Lecture Notes in Math. **836** (1980), pp. 22–48.

[V] E. Vessiot, *Sur une théorie nouvelle des problèmes généraux d'integration*, Bull. Soc. Math. France **52** (1924), 336–395.

[Y] D. Yang, *Involutive hyperbolic differential systems*, Mem. Amer. Math. Soc. **68**, no. 370 (1987).

Department Mathematics and Statistics, Faculty of Science, University of Tromsø, Tromsø 90-37, Norway.

E-mail address: `kruglikov@math.uit.no`, `lychagin@math.uit.no`.