EQUIVARIANT AND NONEQUIVARIANT MODULE SPECTRA

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Abstract. Let G be a compact Lie group, let R_G be a commutative algebra over the sphere G-spectrum S_G, and let R be its underlying nonequivariant algebra over the sphere spectrum S. When R_G is split as an algebra, as holds for example for $R_G = MU_G$, we show how to "extend scalars" to construct a split R_G-module $M_G = R_G \wedge_R M$ from an R-module M. We also show how to compute the coefficients M_*^G in terms of the coefficients R_*^G, R_*, and M_*. This allows the wholesale construction of highly structured equivariant module spectra from highly structured nonequivariant module spectra. In particular, it applies to construct MU_G-modules from MU-modules and therefore gives conceptual constructions of equivariant Brown-Peterson and Morava K-theory spectra.

1. Introduction

We enrich the theory of highly structured modules over highly structured ring spectra that was developed in [4] by showing how to construct highly structured equivariant modules from highly structured nonequivariant modules. Throughout, we let G be a compact Lie group. As pointed out in its introduction, although [4] is written nonequivariantly, its theory applies verbatim equivariantly; an equivariant exposition will be given in [10]. The equivariant ring spectra we are interested in are the algebras over the equivariant sphere spectrum S_G. These are essentially equivalent to the earlier E_∞ ring G-spectra of [9, VII§2]. The prime examples are S_G itself and the spectrum MU_G of stabilized equivariant complex cobordism; MU_G is proven to be a commutative S_G-algebra in [8]. In [5], Elmendorf and I showed how to construct examples from nonequivariant S-algebras. For finite groups G, a great many other examples are known.

In [4], we gave new constructions of the Brown-Peterson spectra BP, the connective and periodic Morava K-theory spectra $k(n)$ and $K(n)$, and all of the other spectra that are usually constructed from MU by means of the Baas-Sullivan theory of manifolds with singularities by killing some of the generators of MU_* and inverting others. The new constructions give all of these spectra module structures over the commutative S-algebra MU. We also analyzed their multiplicative structure as "MU-ring spectra", which are the analogs in the derived category of MU-modules of classical ring spectra in the stable homotopy category.

In [8], a localization and completion theorem is proven that relates M_*^G to $M_*(BG)$ and M_G^* to $M^*(BG)$ for any split MU_G-module M_G with underlying nonequivariant MU-module M. To apply this theorem to a given MU-module M, such as $M = BP$, $M = k(n)$, or $M = K(n)$, we must be able to construct a split MU_G-module M_G with underlying MU-module M. We shall accomplish this by means of a kind of "extension of scalars" functor that transforms MU-modules into split MU_G-modules. This construction commutes with smash products and so carries MU-ring spectra to MU_G-ring G-spectra.

We let $\mathscr{M}_R$ denote the category of modules over an S-algebra R and let $G\mathscr{M}_{R_G}$ denote the category of modules over an S_G-algebra R_G. We let $\mathscr{D}_R$ and $G\mathscr{D}_{R_G}$ denote the respective derived categories; these are obtained from the respective homotopy categories by adjoining inverses to the weak equivalences or, equivalently, by passing to approximations by weakly equivalent cell modules. As usual, we write E^G_* for the homotopy groups of the G-fixed point spectrum of a G-spectrum E_G.

Theorem 1.1. *There is a monoidal functor $MU_G \wedge_{MU} (?) : \mathscr{M}_{MU} \longrightarrow G\mathscr{M}_{MU_G}$. If M is a cell MU-module, then $M_G \equiv MU_G \wedge_{MU} M$ is split as an MU_G-module with underlying nonequivariant MU-module M. The functor $MU_G \wedge_{MU} (?)$ induces a derived monoidal functor $\mathscr{D}_{MU} \longrightarrow G\mathscr{D}_{MU_G}$. Therefore, if M is an MU-ring spectrum, then $MU_G \wedge_{MU} M$ is an MU_G-ring G-spectrum. Moreover, there is a strongly convergent natural spectral sequence*

$$\mathrm{Tor}^{MU_*}_{p,q}(MU^G_*, M_*) \Longrightarrow M^G_{p+q}$$

of differential MU_*-modules. In particular, if G is Abelian, then

$$M^G_* \cong MU^G_* \otimes_{MU_*} M_*.$$

The term "split as a module" will be given a precise meaning below. The last statement holds since MU^G_* is a free MU_*-module when G is Abelian [2]. This is true in a few other cases, but little is known in general about the algebraic structure of MU^G_*. A special case of the theorem solves a problem raised by Carlsson [1]:

"Define and compute equivariant Morava K-theory spectra."

There is nothing in the literature on this subject. Theorem 1.1 allows the definitions

$$k(n)_G = MU_G \wedge_{MU} k(n) \quad \text{and} \quad K(n)_G = MU_G \wedge_{MU} K(n),$$

and it computes the coefficients when G is Abelian. Similarly, we can define equivariant Brown-Peterson spectra by

$$BP_G = MU_G \wedge_{MU} BP,$$

and so on for all of the other standard MU-modules in current use. We point out an anomaly in one of the few familiar cases: the equivariant form $k_G = MU_G \wedge_{MU} k$ of the MU-module k that represents connective K-theory cannot represent the usual "connective equivariant K-theory" since, as observed by Greenlees, the latter theory does not take values in modules over the $RO(G)$-graded ring MU^G_*. We do not know whether or not the equivariant form $MU_G \wedge_{MU} K$ of periodic K-theory that we construct represents equivariant K-theory, but we conjecture that it does.

The theorem is a special case of one that applies to all S_G-algebras that are "split as algebras", in a sense that we shall make precise below.

Theorem 1.2. *Let R_G be a commutative S_G-algebra and assume that R_G is split as an algebra with underlying nonequivariant S-algebra R. Then there is a monoidal functor $R_G \wedge_R (?) : \mathscr{M}_R \longrightarrow G\mathscr{M}_{R_G}$. If M is a cell R-module, then $M_G \equiv R_G \wedge_R M$ is split as an R_G-module with underlying nonequivariant R-module M. The functor $R_G \wedge_R (?)$ induces a derived monoidal functor $\mathscr{D}_R \longrightarrow G\mathscr{D}_{R_G}$. Therefore, if M is an R-ring spectrum, then $R_G \wedge_R M$ is an R_G-ring G-spectrum. Moreover, there is a strongly convergent natural spectral sequence*

$$\mathrm{Tor}^{R_*}_{p,q}(R^G_*, M_*) \Longrightarrow M^G_{p+q}$$

of differential R_*-modules. In particular, if either R^G_* or M_* is a flat R_*-module, then

$$M^G_* \cong R^G_* \otimes_{R_*} M_*.$$

To justify the application to MU_G, we need to know that it is split as an algebra. This is a special case of a general criterion for R_G to be split as an algebra. The notion of a global $\mathscr{I}_*$-FSP was defined in [8], and it was shown there that the sphere and cobordism functors provide examples.

Theorem 1.3. *If T is a global $\mathscr{I}_*$-FSP, then its associated commutative S_G-algebra is split as an algebra for every compact Lie group G.*

Although the basic idea and construction predate the writing of [5], this paper is best understood as a sequel to that one, and we shall freely use its notations and results. The reader is referred to [9, 6, 10] for the relevant background on equivariant stable homotopy theory.

2. Change of universe and operadic smash products

The functor $R_G \wedge_R M$ that we shall construct depends on an extension of the operadic smash product

$$\wedge_{\mathscr{L}} : G\mathscr{S}U[\mathbb{L}] \times G\mathscr{S}U[\mathbb{L}] \longrightarrow G\mathscr{S}U[\mathbb{L}]$$

of [4, Ch I] that incorporates the change of universe functors

$$I^U_{U'} : G\mathscr{S}U'[\mathbb{L}'] \longrightarrow G\mathscr{S}U[\mathbb{L}]$$

of [5, 2.1]. Recall from [5, 1.1] that the functors $I^U_{U'}$ are monoidal equivalences of categories.

Definition 2.1. Let U, U', and U'' be G-universes. For an $\mathbb{L}'$-spectrum M and an $\mathbb{L}''$-spectrum N, define an $\mathbb{L}$-spectrum $M \wedge_{\mathscr{L}} N$ by

$$M \wedge_{\mathscr{L}} N = I^U_{U'} M \wedge_{\mathscr{L}} I^U_{U''} N.$$

Obviously, the formal properties of this product can be deduced from those of the functors $I^U_{U'}$ together with those of the operadic smash product for the fixed universe U. In particular, since the functor $I^U_{U'}$ takes $S_{U'}$-modules to S_U-modules and the smash product over S_U is the restriction to S_U-modules of the smash product over $\mathscr{L}$, we have the following observation. Here S_U denotes the sphere G-spectrum indexed on U.

Lemma 2.2. *The functor $\wedge_{\mathscr{L}} : G\mathscr{S}U'[\mathbb{L}'] \times G\mathscr{S}U''[\mathbb{L}''] \longrightarrow G\mathscr{S}U[\mathbb{L}]$ restricts to a functor*

$$\wedge_{S_U} : G\mathscr{M}_{S_{U'}} \times G\mathscr{M}_{S_{U''}} \longrightarrow G\mathscr{M}_{S_U}.$$

There is an alternative description of this product that makes its structure more apparent. It depends on the following generalization of [4, I.5.4], which in fact is implied by that result; compare [5, 2.2].

Lemma 2.3. *Assume given universes U, U', U'', U'_r for $1 \leq r \leq i$, and U''_s for $1 \leq s \leq j$, where $i \geq 1$ and $j \geq 1$. Then the following diagram is a split coequalizer of spaces and therefore a coequalizer of G-spaces; the maps γ are given by sums and compositions of linear isometries.*

$$\begin{array}{c} \mathscr{I}(U'\oplus U'',U)\times\mathscr{I}(U',U')\times\mathscr{I}(U'',U'')\times\mathscr{I}(\oplus_{r=1}^{i}U'_r,U')\times\mathscr{I}(\oplus_{s=1}^{j}U''_s,U'') \\ \Big\downarrow \gamma\times\mathrm{id}\ \Big\downarrow \mathrm{id}\times\gamma \\ \mathscr{I}(U'\oplus U'',U)\times\mathscr{I}(\oplus_{r=1}^{i}U'_r,U')\times\mathscr{I}(\oplus_{s=1}^{j}U''_s,U'') \\ \Big\downarrow \gamma \\ \mathscr{I}((\oplus_{r=1}^{i}U'_r)\oplus(\oplus_{s=1}^{j}U''_s),U). \end{array}$$

Lemma 2.4. *There is a natural isomorphism*

$$M\wedge_{\mathscr{L}}N\longrightarrow\mathscr{I}(U'\oplus U'',U)\ltimes_{\mathscr{I}(U',U')\times\mathscr{I}(U'',U'')}M\wedge N,$$

where $\wedge$ on the right is the external smash product $G\mathscr{S}U'\times G\mathscr{S}U''\longrightarrow G\mathscr{S}(U'\oplus U'')$.

Proof. Expanding definitions, we see that $M\wedge_{\mathscr{L}}N$ is

$$\mathscr{I}(U\oplus U,U)\ltimes_{\mathscr{I}(U,U)\times\mathscr{I}(U,U)}[(\mathscr{I}(U',U)\ltimes_{\mathscr{I}(U',U')}M)\wedge(\mathscr{I}(U'',U)\ltimes_{\mathscr{I}(U'',U'')}N)].$$

Formal properties of the twisted half-smash product allow us to rewrite this as

$$[\mathscr{I}(U\oplus U,U)\times_{\mathscr{I}(U,U)\times\mathscr{I}(U,U)}\mathscr{I}(U',U)\times\mathscr{I}(U'',U)]\ltimes_{\mathscr{I}(U',U')\times\mathscr{I}(U'',U'')}M\wedge N.$$

The previous lemma gives a homeomorphism

$$\mathscr{I}(U\oplus U,U)\times_{\mathscr{I}(U,U)\times\mathscr{I}(U,U)}\mathscr{I}(U',U)\times\mathscr{I}(U'',U)\longrightarrow\mathscr{I}(U'\oplus U'',U)$$

of G-spaces over $\mathscr{I}(U'\oplus U'',U)$, and the conclusion follows. □

Similarly, as in the proof of [4, I.5.5 and I.5.6], Lemma 2.3 implies the following associativity property of our generalized operadic smash products and therefore, upon restriction, of our generalized smash products over sphere G-spectra.

Lemma 2.5. *Let $M\in G\mathscr{S}U'[\mathbb{L}']$, $P\in G\mathscr{S}U''[\mathbb{L}'']$, and $N\in G\mathscr{S}U'''[\mathbb{L}''']$. Then both $(M\wedge_{\mathscr{L}'}P)\wedge_{\mathscr{L}}N$ and $M\wedge_{\mathscr{L}}(P\wedge_{\mathscr{L}'''}N)$ are canonically isomorphic to*

$$\mathscr{I}(U'\oplus U''\oplus U''',U)\ltimes_{\mathscr{I}(U',U')\times\mathscr{I}(U'',U'')\times\mathscr{I}(U''',U''')}M\wedge P\wedge N,$$

which in turn is canonically isomorphic to

$$I^U_{U'}M\wedge_{\mathscr{L}}I^U_{U''}P\wedge_{\mathscr{L}}I^U_{U'''}N.$$

Using change of universe explicitly or, via the previous lemmas, implicitly, we can define modules indexed on one universe over algebras indexed on another.

Definition 2.6. Let $R\in G\mathscr{M}_{S_{U''}}$ be an $S_{U''}$-algebra and let $M\in G\mathscr{M}_{S_{U'}}$. Say that M is a right R-module if it is a right $I^{U'}_{U''}R$-module, and similarly for left modules.

It is quite clear how one must define smash products over R in this context.

Definition 2.7. Let $R\in G\mathscr{M}_{S_{U''}}$ be an $S_{U''}$-algebra, let $M\in G\mathscr{M}_{S_{U'}}$ be a right R-module and let $N\in G\mathscr{M}_{S_{U'''}}$ be a left R-module. Define

$$M\wedge_R N=I^U_{U'}M\wedge_{I^U_{U''}R}I^U_{U'''}N.$$

Here we have used that $I^U_{U''}\cong I^U_{U'}I^{U'}_{U''}$ and that $I^U_{U'}M$ is therefore an $I^U_{U''}R$-module, and similarly for N. Expanding definitions and using the associativity isomorphism of Lemma 2.5, we obtain the following more explicit description.

Lemma 2.8. *$M \wedge_R N$ is the coequalizer displayed in the diagram*

$$\begin{array}{c} \mathscr{I}(U' \oplus U'' \oplus U''', U) \ltimes_{\mathscr{I}(U',U') \times \mathscr{I}(U'',U'') \times \mathscr{I}(U''',U''')} M \wedge R \wedge N \\ \downarrow\downarrow \\ \mathscr{I}(U' \oplus U''', U) \ltimes_{\mathscr{I}(U',U') \times \mathscr{I}(U''',U''')} M \wedge N \\ \downarrow \\ M \wedge_R N, \end{array}$$

where the parallel arrows are induced by the actions of R on M and on N.

Evidently these smash products inherit good formal properties from those of the smash products of R-modules studied in [4]. Similarly, their homotopical properties can be deduced from the homotopical properties of the smash product of R-modules and the homotopical properties of the $I_{U'}^{U}$, which were studied in [5].

3. S_G-algebras and their underlying S-algebras

We are concerned with genuine G-spectra and their comparison with naive G-spectra. Recall that these are indexed respectively on a complete G-universe U and its G-fixed point universe U^G. We write S_G for the sphere G-spectrum indexed on U and S for the sphere spectrum indexed on U^G. We regard nonequivariant spectra such as S as G-spectra with trivial G-action. We have the forgetful change of universe functor $i^* : G\mathscr{S}U \longrightarrow G\mathscr{S}U^G$ obtained by forgetting those indexing spaces of U that are not contained in U^G. The underlying nonequivariant spectrum E of a G-spectrum E_G is defined to be i^*E_G, with its action by G ignored. Said another way, let $U^\#$ denote U with its action by G ignored and let $E_G^\#$ denote the nonequivariant spectrum indexed on $U^\#$ that is obtained from E_G by forgetting the action of G. Then $E = i^*E_G^\#$.

The G-fixed point spectrum of E_G is obtained by taking the spacewise fixed points of i^*E_G. We say that E_G is split if there is a map $E \longrightarrow (E_G)^G$ of spectra indexed on U^G whose composite with the inclusion of $(E_G)^G$ in E is an equivalence. As observed in [7, 0.4], E_G is split if and only if there is a map of G-spectra $i_*E \longrightarrow E_G$ that is a nonequivariant equivalence, where $i_* : G\mathscr{S}U^G \longrightarrow G\mathscr{S}U$ is the left adjoint of i^*. In either form, the notion of a split G-spectrum is essentially a homotopical one. More precisely, it is a derived category notion: its purpose is to allow the comparison of equivariant and nonequivariant homology and cohomology theories, which are defined on derived categories. Thus we could have used weak equivalences in the definitions just given, and we shall use the term equivalence to mean weak equivalence of underlying spectra or G-spectra in what follows; we will add the adjective "weak" in cases where we would not expect to have a homotopy equivalence in general.

We must modify these definitions in the context of highly structured ring and module spectra. This point was left implicit in both [6] and [8], where reference was made to "the underlying S-algebra R of an S_G-algebra R_G": in fact there is no obvious way to give $R = i^*R_G^\#$ a structure of S-algebra. The point becomes clear when one thinks back to the underlying E_∞ ring structures. We are given G-maps

$$\mathscr{I}(U^j, U) \ltimes R_G^j \longrightarrow R_G, \tag{3.1}$$

and there is no obvious way to obtain induced nonequivariant maps

$$\mathscr{I}((U^G)^j, U^G) \ltimes R^j \longrightarrow R. \tag{3.2}$$

We therefore think of $i^*R_G^\#$ as only prescribing the appropriate weak homotopy type of the underlying S-algebra of an S_G-algebra R_G.

Definition 3.3. An underlying S-algebra of an S_G-algebra R_G is an S-algebra whose underlying spectrum is weakly equivalent to $i^*R_G^\#$.

There are several natural ways to construct such an S-algebra. First, forgetting the G-actions and regarding the maps (3.1) as maps of nonequivariant spectra, we obtain a nonequivariant $S_U^\#$-algebra $R_G^\#$ indexed on $U^\#$ from our equivariant S_U-algebra R_G. We may choose a nonequivariant linear isomorphism $f : U^G \longrightarrow U^\#$. By conjugation of linear isometries by f, we obtain an isomorphism between the nonequivariant linear isometries operads of U^G and of $U^\#$, and we see immediately that $f^*R_G^\#$ is an S_U^G-algebra. By Lemma 3.4 below, it is weakly equivalent to $i^*R_G^\#$.

Second, instead of making an arbitrary choice of an isomorphism f, we can follow the philosophy of [4] and consider the twisted function spectrum $F[\mathscr{I}(U^G, U^\#), R_G^\#)$; Lemma 3.4 shows that $F[\mathscr{I}(U^G, U^\#), R_G^\#)$ is also weakly equivalent to $i^*R_G^\#$. As in [4, I.7.5], one can construct a weakly equivalent operadic modification of the spectrum $F[\mathscr{I}(U^G, U^\#), R_G^\#)$ that is an S-algebra. However, there is a simpler way to arrive at the cited operadic modification: it turns out to be given by a functor that is right adjoint to the functor $I_{U^G}^{U^\#} : \mathscr{S}U^G[\mathbb{L}] \longrightarrow \mathscr{S}U^\#[\mathbb{L}]$, and [5, 2.3] shows that the right adjoint of $I_{U^G}^{U^\#}$ is $I_{U^\#}^{U^G}$. By [5, 1.1], $I_{U^\#}^{U^G} R_G^\#$ is an S-algebra. By Lemma 3.4 and the specialization to $G = e$ of Lemma 3.5 below, $I_{U^\#}^{U^G} R_G^\#$ is weakly equivalent to $i^*R_G^\#$.

Lemma 3.4. *For nonequivariant spectra $F \in \mathscr{S}U^\#$, there are natural weak equivalences between i^*F and f^*F and between i^*F and $F[\mathscr{I}(U^G, U^\#), F)$.*

Proof. Choose a path $h : I \longrightarrow \mathscr{I}(U^G, U^\#)$ connecting i to f. For spectra $E \in \mathscr{S}U^G$, there result natural maps $i_*E \longrightarrow h \ltimes E \longleftarrow f_*E$ in $\mathscr{S}U^\#$, and these are homotopy equivalences if E is tame, for example if E has the homotopy type of a CW spectrum [4, I.2.5]. Similarly, the natural map $i_*E \longrightarrow \mathscr{I}(U^G, U^\#) \ltimes E$ is a homotopy equivalence when E is tame. Conjugation from left to right adjoints gives the conclusions since simple diagram chases show that the conjugate natural maps induce isomorphisms of homotopy groups. □

Lemma 3.5. *Let $f : U \longrightarrow U'$ be an isomorphism of G-universes. Then there are natural isomorphisms*

$$f_*E \cong I_U^{U'} E \quad \text{and} \quad f^*E' \cong I_{U'}^U E'$$

for $E \in G\mathscr{S}U[\mathbb{L}]$ and $E' \in G\mathscr{S}U'[\mathbb{L}']$.

Proof. Regard f as a G-map from a point into $\mathscr{I}(U, U')$. Then the following composite is a homeomorphism of G-spaces over $\mathscr{I}(U, U')$:

$$\{*\} \times \mathscr{I}(U, U) \xrightarrow{f \times \mathrm{id}} \mathscr{I}(U, U') \times \mathscr{I}(U, U) \xrightarrow{\circ} \mathscr{I}(U, U').$$

By [9, VI.3.1(iii)], there results a natural isomorphism

$$f_*(\mathscr{I}(U, U) \ltimes E) \cong \mathscr{I}(U, U') \ltimes E.$$

Passing to coequalizers, we obtain

$$f_*E \cong f_*(\mathscr{I}(U,U) \ltimes_{\mathscr{I}(U,U)} E) \cong \mathscr{I}(U,U') \ltimes_{\mathscr{I}(U,U)} E \equiv I_U^{U'} E.$$

Since $f^* = f_*^{-1}$, the second isomorphism follows from the first. □

We can now define the terms "split as an algebra" and "split as a module" that appear in Theorem 1.2.

Definition 3.6. A commutative S_G-algebra R_G is split as an algebra if there is a commutative S-algebra R and a map $\eta : I_{U^G}^U R \longrightarrow R_G$ of S_G-algebras such that η is a (nonequivariant) equivalence of spectra and the natural map $\alpha : i_*R \longrightarrow I_{U^G}^U R$ is an (equivariant) equivalence of G-spectra.

Since the composite $\eta \circ \alpha : i_*R \longrightarrow R_G$ is a nonequivariant equivalence and the natural map $R \longrightarrow i^*i_*R$ is a weak equivalence (provided that R is tame, [9, II.1.8] and [4, I.2.5]), R is weakly equivalent to $i^*R_G^{\#}$. Of course, R_G is split as a G-spectrum with splitting map $\eta \circ \alpha$. By abuse, we refer to R as "the" rather than "an" underlying nonequivariant S-algebra of R_G.

Definition 3.7. Let R_G be a commutative S_G-algebra that is split as an algebra with underlying S-algebra R and let M_G be an R_G-module. Regard M_G as an $I_{U^G}^U R$-module by pullback along η. Then M_G is split as an R_G-module if there is an R-module M and a map $\chi : I_{U^G}^U M \longrightarrow M_G$ of $I_{U^G}^U R$-modules such that χ is a (nonequivariant) equivalence of spectra and the natural map $\alpha : i_*M \longrightarrow I_{U^G}^U M$ is an (equivariant) equivalence of G-spectra.

Again, M is weakly equivalent to $i^*M_G^{\#}$ and M_G is split as a G-spectrum with splitting map $\chi \circ \alpha : i^*M \longrightarrow M_G$. By abuse, we call M the underlying nonequivariant R-module of M_G.

The ambiguity that we allow in the notion of an underlying object is quite useful: it allows us to arrange the condition on α in the definitions if we have succeeded in arranging the condition on η. The proof of this depends on the closed model category structures on all categories in sight that is given in [4, VII§4].

Lemma 3.8. *Let R_G be a commutative S_G-algebra and M_G be an R_G-module.*
(i) Suppose given a commutative S-algebra R' and a map $\eta' : I_{U^G}^U R' \longrightarrow R_G$ of S_G-algebras such that η' is a (nonequivariant) equivalence of spectra. Let $\gamma : R \longrightarrow R'$ be a weak equivalence of S-algebras, where R is a q-cofibrant commutative S-algebra, and define $\eta = \eta' \circ I_{U^G}^U \gamma : I_{U^G}^U R \longrightarrow R_G$. Then R_G is split as an algebra with underlying nonequivariant S-algebra R and splitting map η.
(ii) Suppose given an R'-module M' and a map $\chi' : I_{U^G}^U M' \longrightarrow M_G$ of $I_{U^G}^U R'$-modules such that χ' is a (nonequivariant) equivalence of spectra. Regard M' as an R-module by pullback along γ, let $\vartheta : M \longrightarrow M'$ be a weak equivalence of R-modules, where M is a q-cofibrant R-module, and define $\chi = \chi' \circ I_{U^G}^U \vartheta : I_{U^G}^U M \longrightarrow M_G$. Then M_G is split as an R_G-module with underlying nonequivariant R-module M and splitting map χ.

Proof. It is immediate from [5, 1.4] that $\alpha : i_*R \longrightarrow I_{U^G}^U R$ and $\alpha : i_*M \longrightarrow I_{U^G}^U M$ are equivalences of G-spectra. Thus we need only observe that, ignoring the G-action, the maps $I_{U^G}^{U^{\#}} \gamma$ and $I_{U^G}^{U^{\#}} \vartheta$ are weak equivalences since it is immediate from the case $G = e$ of Lemma 3.5 that the functor $I_{U^G}^{U^{\#}}$ preserves weak equivalences. □

Proof of Theorem 1.2. As observed in [4, VII.1.3], the splitting map $\eta : I_{U^G}^U R \longrightarrow R_G$ of our given split commutative S_G-algebra R_G is the unit of a structure of $I_{U^G}^U R$-algebra on R_G. The composite

$$R_G \wedge_{S_G} I_{U^G}^U R \xrightarrow{\mathrm{id} \wedge \eta} R_G \wedge_{S_G} R_G \xrightarrow{\phi} R_G$$

gives R_G a structure of right R-module in the sense prescribed in Definition 2.6, and R_G is an (R_G, R)-bimodule with left action of R_G induced by the product ϕ of R_G. Therefore, for an R-module M, we can take $U = U'$ and $U^G = U'' = U'''$ in Definition 2.7 and define

$$M_G = R_G \wedge_R M.$$

Clearly M_G is an R_G-module with action induced by the left action of R_G on itself.

The functor $I_{U^G}^U : \mathscr{M}_R \longrightarrow G\mathscr{M}_{I_{U^G}^U R}$ is monoidal by [5, 1.3]. The functor

$$R_G \wedge_{I_{U^G}^U R} (?) : G\mathscr{M}_{I_{U^G}^U R} \longrightarrow G\mathscr{M}_{R_G}$$

is monoidal since $R_G \cong R_G \wedge_{R_G} R_G$ and

$$(R_G \wedge_{R_G} R_G) \wedge_{I_{U^G}^U R} (M \wedge_{I_{U^G}^U R} N) \cong (R_G \wedge_{I_{U^G}^U R} M) \wedge_{R_G} (R_G \wedge_{I_{U^G}^U R} N)$$

by a comparison of coequalizer diagrams. Therefore the functor $R_G \wedge_R (?)$ is monoidal.

As observed in Lemma 3.8, we can assume that our given underlying nonequivariant S-algebra R is q-cofibrant as an S-algebra. Let M be a cell R-module. Then M is q-cofibrant and, by [5, 1.4], $\alpha : i_* M \longrightarrow I_{U^G}^U M$ is an equivalence. To prove that M_G is split as an R_G-module, define

$$\chi = \eta \wedge \mathrm{id} : I_{U^G}^U M \cong I_{U^G}^U R \wedge_{I_{U^G}^U R} I_{U^G}^U M \longrightarrow R_G \wedge_{I_{U^G}^U R} I_{U^G}^U M = M_G.$$

Clearly χ is a map of $I_{U^G}^U R$-modules, and we must prove that it is an equivalence of spectra. Recall from [4, III§1] that we have a free functor F_R from spectra to R-modules given by

$$F_R X = R \wedge_S (S \wedge_{\mathscr{L}} \mathbb{L}X) \cong R \wedge_{\mathscr{L}} \mathbb{L}X;$$

here $\mathscr{L}$ and $\mathbb{L}$ refer to the universe U^G, but we have a similar free functor F_{R_G} from G-spectra to R_G-modules based on use of the linear isometries operad for U, and similarly for $I_{U^G}^U R$. If we forget about G-actions and compare definitions, we find by use of an isomorphism $f : U^G \longrightarrow U^\#$ that, nonequivariantly,

$$I_{U^G}^{U^\#} F_R X \cong F_{I_{U^G}^{U^\#} R} X.$$

Recalling the definition of cell R-modules from [4, III.2.1], we see that cell R-modules are built up via pushouts and sequential colimits from the free R-modules generated by sphere spectra and their cones. The functor $I_{U^G}^U$ is a left adjoint, whether we interpret it equivariantly or nonequivariantly. We conclude that this functor carries cell R-modules to $I_{U^G}^U R$-modules that, with G-actions ignored, are nonequivariant cell $I_{U^G}^{U^\#} R$-modules. Now [4, III.3.8] gives that χ is an equivalence of spectra since η is an equivalence of spectra.

The passage to derived categories is immediate, modulo one slight subtlety: our functor on modules was constructed as the composite of two functors, but, as we saw in [5, §3], it does not follow that the induced functor on derived categories factors as a composite. The solution is simple: we ignore the intermediate category $G\mathscr{D}_{I_{U^G}^U R}$, as it is of no particular interest to us. Since cell approximation of R-modules commutes up to equivalence with smash products, passage to derived categories preserves smash products.

Finally, we must construct the spectral sequence claimed in Theorem 1.2. By definition, $R_*^G = \pi_*((R_G)^G)$, where $(R_G)^G = (i^*R_G)^G$. Similarly,

$$M_*^G = \pi_*((R_G \wedge_R M)^G), \quad \text{where} \quad (R_G \wedge_R M)^G = (i^*(R_G \wedge_R M))^G.$$

Assuming as we may that all given algebras and modules are suitably cofibrant, we obtain the same groups if we replace the functor i^* by the more structured change of universe functor $I_U^{U^G}$. In view of the following lemma, the desired spectral sequence is a special case of the spectral sequence

$$\mathrm{Tor}_{*,*}^{R_*}(M_*, N_*) \Longrightarrow \pi_*(M \wedge_R N)$$

constructed in [4, IV.4.1] for an S-algebra R and R-modules M and N. □

Lemma 3.9. *Let R_G be split as an algebra with underlying S-algebra R. For R-modules M, there is a natural isomorphism of R-modules*

$$(I_U^{U^G} R_G)^G \wedge_R M \cong (I_U^{U^G}(R_G \wedge_R M))^G.$$

Proof. Passage to G-fixed points commutes with smash products in the category of naive G-spectra. The same is true in the category $G\mathscr{M}_S$ of equivariant S-modules and therefore in the the category $G\mathscr{M}_R$ for any G-trivial S-algebra R. In our situation, G acts trivially on both R and M, and it follows that

$$(I_U^{U^G} R_G)^G \wedge_R M \cong ((I_U^{U^G} R_G) \wedge_R M)^G. \tag{3.10}$$

Since $R \cong I_U^{U^G} I_{U^G}^U R$, $M \cong I_U^{U^G} I_{U^G}^U M$, and the functor $I_U^{U^G}$ is monoidal, we have

$$(I_U^{U^G} R_G) \wedge_R M \cong I_U^{U^G}(R_G \wedge_{I_{U^G}^U R} I_{U^G}^U M) \equiv I_U^{U^G}(R_G \wedge_R M). \tag{3.11}$$

We obtain the desired isomorphism by composing (3.10) with the isomorphism obtained from (3.11) by passage to G-fixed point spectra. □

4. Global $\mathscr{I}_*$-functors and split S_G-algebras

We must prove Theorem 1.3. The notion of a global $\mathscr{I}_*$-FSP, or $\mathscr{G}\mathscr{I}_*$-FSP, was defined in [8, §§5,6]. We shall only sketch the definition here, referring the reader to [8] for more details. In fact, we only need a tiny fraction of the structure that is present on S_G-algebras that arise from $\mathscr{G}\mathscr{I}_*$-FSP's. In what follows, we could work with either real or complex inner product spaces, and of course the complex case is the one relevant to complex cobordism; see [8, 6.5].

Let $\mathscr{G}\mathscr{I}_*$ be the category of pairs (G, V) consisting of a compact Lie group G and a finite dimensional G-inner product space V; the morphisms $(\alpha, f) : (G, V) \longrightarrow (G', V')$ consist of a homomorphism $\alpha : G \longrightarrow G'$ of Lie groups and an α-equivariant linear isomorphism $f : V \longrightarrow V'$. Let $\mathscr{G}\mathscr{T}$ be the category of pairs (G, X), where G is a compact Lie group and X is a based G-space; the morphisms $(\alpha, f) : (G, X) \longrightarrow (G', X')$ consist of a homomorphism $\alpha : G \longrightarrow G'$ and an α-equivariant based map $f : X \longrightarrow X'$. Let $S^\bullet : \mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ be the functor that sends a pair (G, V) to the based G-space S^V.

A $\mathscr{G}\mathscr{I}_*$-FSP T is a continuous functor $T : \mathscr{G}\mathscr{I}_* \longrightarrow \mathscr{G}\mathscr{T}$ over the category $\mathscr{G}$ of compact Lie groups together with continuous natural transformations

$$\eta : S^\bullet \longrightarrow T \quad \text{and} \quad \omega : T \wedge T \longrightarrow T \circ \oplus$$

such that the appropriate unity, associativity, and commutativity diagrams commute. Since T is a functor over $\mathscr{G}$, we may write $T(G,V) = (G,TV)$, and we require that

$$T(\alpha, \mathrm{id}) = (\alpha, \mathrm{id}) : (G, TV') \longrightarrow (G', TV') \tag{4.1}$$

for a homomorphism $\alpha : G \longrightarrow G'$ and a G'-inner product space V' regarded by pullback along α as a G-inner product space. Henceforward, we abbreviate notation by writing $T(G,V) = TV$ on objects. For each object (G,V), we are given a G-map

$$\eta : S^V \longrightarrow TV.$$

For each pair of objects (G,V) and (G',V'), we are given a $G \times G'$-map

$$\omega : TV \wedge TV' \longrightarrow T(V \oplus V');$$

by pullback along the diagonal, we regard ω as a G-map when $G = G'$.

We insert some observations that show the power of condition (4.1) and are at the heart of our work. Let e denote the trivial group and let $\iota : e \longrightarrow G$ and $\varepsilon : G \longrightarrow e$ be the unique homomorphisms.

Lemma 4.2. *If V has trivial G-action, then TV also has trivial G-action. For a general G-inner product space V, if $V^{\#}$ denotes V regarded as an e-space, then $TV^{\#}$ is the space TV with its action by G ignored.*

Proof. For the first statement, the functor T carries the morphism $(\varepsilon, \mathrm{id}) : (G,V) \longrightarrow (e,V)$ of $\mathscr{G}\mathscr{I}_*$ to the morphism $(\varepsilon, \mathrm{id})$ of $\mathscr{G}\mathscr{T}$, so that the identity map on TV must be ε-equivariant. For the second statement, the functor T carries the morphism $(\iota, \mathrm{id}) : (e,V) \longrightarrow (G,V)$ to the identity map on the space TV. □

For a compact Lie group G and a G-universe U, we obtain a G-prespectrum $T_{(G,U)}$ indexed on U with Vth G-space TV. The structural maps are given by the composites

$$TV \wedge S^{W-V} \xrightarrow{\mathrm{id} \wedge \eta} TV \wedge T(W-V) \xrightarrow{\omega} TW$$

for $V \subset W$. Write $R_{(G,U)}$ for the G-spectrum $LT_{(G,U)}$, where L is the spectrification functor of [9, I.2.2].

Now suppose given G-universes U and U'. Then there is a canonical map of G-spectra indexed on U'

$$\zeta : \mathscr{I}(U,U') \ltimes R_{(G,U)} \longrightarrow R_{(G,U')}. \tag{4.3}$$

Indeed, if $f : U \longrightarrow U'$ is a linear isometry and V is an indexing space in U, then the maps $Tf : TV \longrightarrow Tf(V)$ specify a map of prespectra $T_{(G,U)} \longrightarrow f^*T_{(G,U')}$ indexed on U. By adjunction, Tf gives a map of prespectra $\zeta(f) : f_*T_{(G,U)} \longrightarrow T_{(G,U')}$ indexed on U'; see [9, p.58]. We record the following observation for later reference.

Lemma 4.4. *If $f : U \longrightarrow U'$ is an isomorphism, then $\zeta(f) : f_*T_{(G,U)} \longrightarrow T_{(G,U')}$ is an isomorphism; if f is a G-map, then $\zeta(f)$ is a G-map.*

Intuitively, the twisted half-smash product $\mathscr{I}(U,U') \ltimes R_{(G,U)}$ is obtained by gluing together the spectrifications $f_*R_{(G,U)}$ of the $f_*T_{(G,U)}$, and the maps $\zeta(f)$ glue together to give the G-map ζ. This sort of argument first appeared in [11, IV.1.6, IV.2.2], before the twisted half-smash product was invented, and it was formalized in current terminology in [9, VI.2.17].

Fixing G and U, a precisely similar argument, formalized in [9, VI.5.5, VII.2.4, and VII.2.6], shows that the maps

$$\xi_j(f) : TV_1 \wedge \cdots TV_j \xrightarrow{\omega} T(V_1 \oplus \cdots \oplus V_j) \xrightarrow{Tf} Tf(V_1 \oplus \cdots \oplus V_j)$$

for linear isometries $f : U^j \longrightarrow U$ give rise to maps

$$\xi_j : \mathscr{L}(j) \ltimes (R_{(G,U)})^j \longrightarrow R_{(G,U)}$$

that give $R_{(G,U)}$ a structure of $\mathscr{L}$ G-spectrum. When the universe U is complete, so that its linear isometries operad $\mathscr{L}$ is an E_∞ operad of G-spaces, this means that $R_{(G,U)}$ is an E_∞ ring G-spectrum. The map ξ_1 gives $R_{(G,U)}$ an action of $\mathscr{L}(1) = \mathscr{I}(U,U)$, and functoriality implies that ξ_2 factors through the coequalizer that defines the operadic smash product, giving

$$\xi : R_{(G,U)} \wedge_{\mathscr{L}} R_{(G,U)} = \mathscr{L}(2) \ltimes_{\mathscr{L}(1)^2} (R_{(G,U)})^2 \longrightarrow R_{(G,U)}. \tag{4.5}$$

As explained in [4, II.3.3 and II§4], the maps ξ_j for $j \geq 3$ can be reconstructed from the maps for $j = 1$ and $j = 2$.

Lemma 4.6. *The map ζ of (4.3) factors through the coequalizer to give a map*

$$\zeta : I_U^{U'} R_{(G,U)} = \mathscr{I}(U, U') \ltimes_{I(U,U)} R_{(G,U)} \longrightarrow R_{(G,U')},$$

and ζ is a map of $\mathscr{L}'$ G-spectra, where $\mathscr{L}'$ is the linear isometries operad of U'.

Proof. The factorization is clear from functoriality. To check that ζ is a map of $\mathscr{L}'$ G-spectra, we must show that the following diagram commutes:

$$\begin{array}{ccc}
\mathscr{L}'(2) \ltimes_{\mathscr{L}'(1)^2} (I_U^{U'} R_{(G,U)})^2 & \xrightarrow{\mathrm{id} \ltimes \zeta^2} & \mathscr{L}'(2) \ltimes_{\mathscr{L}'(1)^2} (R_{(G,U')})^2 \\
\downarrow \xi & & \downarrow \xi \\
I_U^{U'} R_{(G,U)} & \xrightarrow[\zeta]{} & R_{(G,U')}.
\end{array}$$

Using Lemma 2.3 to identify the upper left corner of the diagram and chasing through the definitions, we see that both composites coincide with the following one:

$$\begin{array}{c}
\mathscr{I}(U \oplus U, U') \ltimes_{\mathscr{I}(U,U)^2} (R_{(G,U)})^2 \\
\downarrow \mathrm{id} \ltimes \omega \\
\mathscr{I}(U \oplus U, U') \ltimes_{\mathscr{I}(U \oplus U, U \oplus U)} R_{(G, U \oplus U)} \\
\downarrow \zeta \\
R_{(G,U)};
\end{array}$$

here ω is induced by passage to spectra from the evident map of prespectra. □

Returning to our fixed G and a complete G-universe U, we consider $R_{(e,U^G)}$ and $R_{(G,U)}$. We deduce from Lemma 4.2 that

$$R_{(e,U^G)} = R_{(G,U^G)} \quad \text{and} \quad R^{\#}_{(G,U)} = R_{(e,U^{\#})}. \tag{4.7}$$

That is, $R_{(G,U^G)}$ is $R_{(e,U^G)}$ regarded as a G-trivial G-spectrum indexed on the G-trivial universe U^G, and $R_{(G,U)}$ regarded as a nonequivariant spectrum indexed on $U^{\#}$ is $R_{(e,U^{\#})}$

The first of these equalities allows us to specialize the map ζ to obtain a map of E_∞ ring G-spectra

$$\zeta : I^U_{U^G} R_{(e,U^G)} = \mathscr{I}(U^G, U) \ltimes_{I(U^G,U^G)} R_{(G,U^G)} \longrightarrow R_{(G,U)}. \tag{4.8}$$

The second of these equalities allows us to identify the target of the underlying map $\zeta^\#$ of nonequivariant spectra with $R_{(e,U^\#)}$.

Lemma 4.9. *The map $\zeta^\#$ is an isomorphism of spectra.*

Proof. Choose an isomorphism $f : U^G \longrightarrow U^\#$. It is immediate that the composite

$$f_* R_{(e,U^G)} \xrightarrow{\cong} I^{U^\#}_{U^G} R_{(e,U^G)} \xrightarrow{\zeta^\#} R_{(e,U^\#)}$$

coincides with the isomorphism $\zeta(f)$ of Lemma 4.4; here the unlabelled isomorphism is given by the case $G = e$ of Lemma 3.5. □

To pass to S_G-algebras, we let R be the S-algebra $S \wedge_{\mathscr{L}} R_{(e,U^G)}$ and R_G be the S_G-algebra $S_G \wedge_{\mathscr{L}} R_{(G,U)}$ (where $\mathscr{L}$ refers respectively to U^G and to U). By [5, 2.4], we have an isomorphism of S_G-algebras

$$I^U_{U^G} R \cong S_G \wedge_{\mathscr{L}} I^U_{U^G} R_{(e,U^G)},$$

and this allows us to define an isomorphism of S_G-algebras

$$\eta = \mathrm{id} \wedge \zeta : I^U_{U^G} R \longrightarrow R_G. \tag{4.10}$$

At this level of generality, we cannot expect to prove that $\alpha : i_* R \longrightarrow I^U_{U^G} R$ is an equivalence of G-spectra, although it seems plausible that this holds in the examples of interest. However, we can appeal to q-cofibrant approximation, as in Lemma 3.8, to complete the proof of Theorem 1.3, thereby losing that η is an isomorphism in order to make sure that α is an equivalence.

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