INFINITE LOOP SPACE THEORY REVISITED

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Just over two years ago I wrote a summary of infinite loop space theory [37]. At the time, there seemed to be a lull in activity, with little immediately promising work in progress. As it turns out, there has been so much done in the interim that an update of the summary may be useful.

The initial survey was divided into four chapters, dealing with additive infinite loop space theory, multiplicative infinite loop space theory, descriptive analysis of infinite loop spaces, and homological analysis of infinite loop spaces. We shall devote a section to developments in each of these general areas and shall also devote a section to the newly evolving equivariant infinite loop space theory.

Two of the biggest developments will hardly be touched on here however. I ended the old survey with the hope that "much new information will come when we learn how the rich space level structures described here can effectively be exploited for calculations in stable homotopy theory." This hope is being realized by work in two quite different directions.

As discussed in [37, §4], the approximation theorem to the effect that $\Omega^n\Sigma^n X$ is a group completion of the simple combinatorial space C_nX plays a central role in the general theory. I stated there that "homotopical exploitation of the approximation theorem has barely begun." This is no longer the case. Such exploitation is now one of the more active areas of homotopy theory, recent contributions having been made by Mahowald, Brown and Peterson, R. Cohen, Sanderson and Koschorke, Caruso and Waner, and F. Cohen, Taylor, and myself. I plan to summarize the present state of the art in [42], and will content myself here with a remark in section two and a brief discussion of the equivariant approximation theorem in section five.

Second, the notion of E_∞ ring spectrum discussed in [37, §11] led to a simpler homotopical notion of H_∞ ring spectrum. This concept is really part of stable homotopy theory as understood classically, rather than part of infinite loop space theory, and seems to be basic to that subject. An introduction and partial summary of results based on this concept are given in [39]. A complete treatment will appear in the not too distant future [5]; meanwhile, the main results are available in the theses of Bruner [4], Steinberger [60], Lewis [28], and McClure [44].

I must end the introduction on a less sanguine note. Even in this short report, I shall have to mention a disconcertingly large number of published errors, both theoretical and calculational, both mine and those of many others. I do not know whether to ascribe this to carelessness, the complexity of the subject, or simple human blindness. Certainly the lesson is that an attitude of extreme skepticism is warranted towards any really difficult piece of work not supported by total detail. This pertains particularly to some of the embryonic theories discussed in sections two and five.

§1. *Additive infinite loop space theory*

The first change to be celebrated is in the state of the art of exposition. In an attempt to make the subject accessible to beginners, Frank Adams has written a truly delightful tract [1]. Anyone wishing a painless introduction, in particular to the various approaches to the recognition principle, is urged to read it.

In Adams' survey, there is a little of the flavor of competition between these approaches, and I was perhaps the worst offender in spreading this atmosphere. The point is that the black boxes for constructing spectra out of space level data looked so drastically different that it was far from obvious to me that they would produce equivalent spectra from the same data.

A major advance in the last two years is that we now have such a uniqueness theorem. There is only one infinite loop space machine, but there are various ways to construct it.

The first uniqueness theorem of this sort is due to Fiedorowicz [12], who axiomatized the passage from rings to the spectra of algebraic K-theory. (Actually, there are $\lim^1$ problems associated with getting the pairing he needs on the Gersten-Wagoner spectra; the argument in [13] is wrong, for the silly but substantive reason that η on page 165 fails to be a natural transformation.) Fiedorowicz' idea is based on the following simple, but extremely fruitful, observation which is at the heart of all the spectrum level uniqueness theorems discussed below. Let X be a bispectrum, namely a sequence of spectra $X_i = \{X_{i,j}\}$ and equivalences of spectra $X_i \to \Omega X_{i+1}$. Then the $0^{\underline{th}}$ spectrum $X_0 = \{X_{0,j}\}$ is equivalent to the spectrum $\{X_{i,0}\}$. Here spectra are at least Ω-spectra; one has variants depending on what category one is working in [43, App. A].

Thomason and I used this idea to axiomatize infinite loop space machines [43], and I want to say just enough about our work to explain precisely what such a gadget is.

Consider topological categories with objects the based sets $\underline{n} = \{0,1,\ldots,n\}$. Let F be the category of finite based sets; its objects are the $\underline{n}$ and its morphisms are all functions which take 0 to 0. Inside F, we have the subcategory Π consisting of the injections and projections, namely those morphisms $f:\underline{m} \to \underline{n}$ such that $f^{-1}(j)$ has at most one element for $1 \leq j \leq n$. We say that G is a category of operators if it contains Π and maps to F; we say that G is an E_∞ category if the map to F is an equivalence. Let T be the category of based spaces. A G-space is a functor $G \to T$, written $\underline{n} \to X_n$ on objects, such that the n projections $\underline{n} \to \underline{1}$ induce an equivalence $X_n \to X_1^n$ for $n \geq 0$ (and a technical cofibration condition is satisfied).

An infinite loop space machine E is an E_∞ category G and a functor E from G-spaces to spectra together with a natural group completion $\iota:X_1 \to E_0X$. Thus π_0E_0X is the universal group associated to the monoid π_0X_1 and, for any commutative coefficient ring, H_*E_0X is obtained from the Pontryagin ring H_*X_1 by localizing at $\pi_0X_1 \subset H_0X_1$.

With just this one axiom, we prove that any two infinite loop space machines defined on G-spaces are naturally equivalent. Actually, we prove the uniqueness theorem for F-spaces and deduce it for G-spaces by use of a functor from G-spaces to F-spaces suitably inverse to the pullback functor the other way. The proof for F-spaces proceeds by comparing any given machine to Segal's original machine [50]. An E_∞ operad (as in [37, §2]) gives rise to an E_∞ category G. May's original machine [35,36] was only defined on those G-spaces with X_n actually equal to X_1^n. We generalize its domain of definition to all G-spaces and so conclude that the May and Segal machines are equivalent. Any other machine which really is a machine must be equivalent to these.

I have also given an addendum [40] asserting the uniqueness of infinite loop space machines defined on permutative categories, the point being that there are several quite different ways of passing from such categories to the domain data (G-spaces) of infinite loop space theory.

Due to work of Thomason [64], we now have a much better understanding of this passage, together with a more general class of morphisms to which it can be applied. (On objects, restriction to permutative categories is harmless; see [37, §8].) Some discussion may be worthwhile, since I for one find the ideas illuminating. Given a permutative category $(A,\Box,*,c)$, so that $\Box:A\times A \to A$ is an associative product with unit $*$ and natural commutativity isomorphism c, one's first attempt to get into the domain of an infinite loop space machine is to try to write down a functor $\mathcal{F} \to \mathrm{Cat}$ with $n^{\underline{th}}$ category precisely A^n. In detail, for a morphism $f:\underline{m} \to \underline{n}$ in $\mathcal{F}$, one defines a functor $f_*:A^m \to A^n$ by

$$f_*(A_1,\ldots,A_m) = (B_1,\ldots,B_n),\ B_k = \coprod_{f(j)=k} A_j,$$

on objects and morphisms. Due to permutations, these functors fail to define a functor $\mathcal{F} \to \mathrm{Cat}$, but it is a simple matter to use c to write down natural transformations $c(f,g):(fg)_* \to f_*g_*$. Upon writing out the formal properties satisfied by these data, one sees that one has a sort of system category theorists have known about for years, and have called a lax functor (up to opposite conventions on the c(f,g), hence the term op-lax in [64]). Ross Street [63] provides not just one but two ways of constructing an associated functor $\mathcal{F} \to \mathrm{Cat}$. Either way, the $n^{\underline{th}}$ category is equivalent to A_1^n and we obtain an $\mathcal{F}$-space upon application of the classifying space functor B. A third way of getting such a functor is due to Segal [50] and explained in detail in [40]. Street [63] developed a notion of lax natural transformation between lax functors and showed that such things induce actual natural transformations under either of his constructions. Upon application of B, we deduce that lax natural transformations induce maps of $\mathcal{F}$-spaces. This allows morphisms $F:\mathcal{A} \to \mathcal{B}$ with coherent natural transformations $FA \square FB \to F(A \square B)$ which need not be isomorphisms; neither Segal's construction nor my passage from permutative categories to E_∞ spaces is functorial with respect to such lax morphisms.

I should add that these observations are not the main thrust of Thomason's work in [64], his primary purpose being to show that B converts homotopy colimits of categories, suitably defined, to homotopy colimits of spaces. (A detailed categorical study of this comparison has since been given by Gray [18].) Thomason [65] later used this result, or rather its spectrum level version, to deduce some very interesting spectral sequences involving the algebraic K-theory of permutative categories.

Before leaving the additive theory, I want to say a bit about two more uniqueness theorems. The first reconciles two natural ways of looking at the stable classifying spaces of geometric topology. Consider Top for definiteness; needless to say, the argument is general. One can form $\mathrm{BTop} = \varinjlim \mathrm{BTop}(n)$. This is an $\mathcal{L}$-space, where $\mathcal{L}$ is the linear isometries operad; see [37, §7]. On the other hand, one can regard $\coprod \mathrm{Top}(n)$ as a permutative category. There result two spectra, and I proved in [41] that the first is in fact the connected cover of the second. While this may seem plausible enough, the lack of obvious technical relationship between the linear isometries data and the permutative data makes the proof one of the more difficult in the subject. With this result, the foundations seem to be complete; any two machine-built spectra which ought to be equivalent are equivalent.

The last uniqueness theorem I want to mention concerns A_∞ spaces (see [35, §3]) rather than E_∞ spaces and is due to Thomason [66]. In [35, p. 134], I gave two machines for constructing a classifying space, or delooping, functor on $\mathcal{C}$-spaces X, where $\mathcal{C}$ is an A_∞ operad. One can either form a bar construction $B(S^1, C\times C_1, X)$ directly or replace X by an equivalent monoid B(M,C,X) and take the classical classifying space of the latter. The second approach is more or less obviously equivalent to the delooping machines for A_∞ spaces of Boardman and Vogt [3] and Segal [50]. When X is an E_∞ space regarded as an A_∞ space by neglect of structure, one is looking at first deloopings in the May and Segal machines respectively, hence the two are equivalent by the spectrum level uniqueness theorem. In general, the total lack of commutativity in the situation, with the concomitant lack of the simple group completion notion, makes the consistency much harder. Thomason has given a quite ingenious proof that these two deloopings are always equivalent. The result gains interest from work to be mentioned in the next section.

§2. *Multiplicative infinite loop space theory*

Here the most significant development has been that mentioned in the introduction, the invention and exploitation of H_∞ ring spectra. As discussed in [37, §11], E_∞ ring spectra are defined in terms of actions by an E_∞ operad $\mathcal{G}$ on spectra. H_∞ ring spectra are defined in the stable category, without reference to operads, but are really given in terms of actions up to homotopy by E_∞ operads. While H_∞ ring spectra are much more amenable to homotopical analysis, E_∞ ring spectra are of course still essential to the infinite loop space level applications for which they were designed (see [37, §10-14]). In particular, there is no H_∞ analog of the recognition principle which allows one to construct E_∞ ring spectra from E_∞ ring spaces. (I must report that the passage from bipermutative categories to E_∞ ring spaces in [36, VI §4], despite being intuitively obvious, is blatantly wrong; a correct treatment will be given in [5].)

Another significant development has been the appearance of interesting examples of E_n and H_n ring spectra and of E_n ring spaces for $1 \leq n < \infty$. The definitional framework is exactly the same as when $n = \infty$, except that now $\mathcal{G}$ is not an E_∞ operad but an E_n operad, so that its $j^{\underline{th}}$ space has the Σ_j-equivariant homotopy type of the configuration space of j-tuples of distinct points of R^n.

Lewis [5, 28] has shown that if X is an n-fold loop space and $f:X \to BO$ is an n-fold loop map, then the resulting Thom spectrum Mf is an E_n ring spectrum; if BO is replaced by BF, one at least gets an H_n ring spectrum.

E_n ring spaces have appeared, totally unexpectedly, in connection with the analysis of the multiplicative properties of the generalized James maps

$$j_q : C_nX \to Q(C^+_{n,q} \wedge_{\Sigma_q} X^{(q)})$$

used by Cohen, Taylor and myself [9] to stably split C_nX. The product over $q \geq 0$ of the targets is an E_n ring space, and the map j with components j_q is "exponential" in the sense that it carries the additive E_n action on C_nX to the new multiplicative E_n action on the product. In principle, this completely determines the homological behavior of the James maps. I shall say more about this in [42], but it will be some time before details appear.

Another recent development concerns A_∞ ring spaces, or E_1 ring spaces in the language above. These are rings up to all higher coherence homotopies. I have constructed the algebraic K-theory of an A_∞ ring space R as follows [38] (modulo some annoying corrections necessary in the combinatories, which will be supplied in [5]). We form the space M_nR of (n×n)-matrices with coefficients in R. Writing down the ordinary matrix product, but with the additions and multiplications involved parametrized by the given operad actions, we construct an A_∞ operad $\mathcal{H}_n$ which acts on M_nR. We then construct morphisms of operads $\mathcal{H}_{n+1} \to \mathcal{H}_n$ such that the usual inclusion $M_nR \to M_{n+1}R$ is an $\mathcal{H}_{n+1}$-map, where M_nR is an $\mathcal{H}_{n+1}$-space by pullback. We next form pullback diagrams of $\mathcal{H}_n$-spaces

$$\begin{array}{ccc} FM_nR & \longrightarrow & M_nR \\ \downarrow & & \downarrow \\ GL(n,\pi_0R) & \longrightarrow & M_n(\pi_0R). \end{array}$$

Thus FM_nR is the space of invertible components in M_nR. We have a classifying space functor B_n on $\mathcal{H}_n$-spaces for each n (indeed, as discussed in the previous section, a choice of equivalent functors). We let KR be the plus construction on the telescope of the spaces B_nFM_nR and define $K_*R = \pi_*KR$. Various basic properties of KR are proven in [38]; for example, if $FR = FM_1R$ is the unit space of R, then the inclusion of monomial matrices in FM_nR yields a natural map $Q_0(B_1FR \amalg \{0\}) \to KR$.

If R is a discrete ring, this is Quillen's K_*R. If R is a topological ring, it is Waldhausen's [67]. In these cases, KR is an infinite loop space [38, 10.12]. I have several more or less rigorous unpublished proofs that KR is always a first loop space, but I could easily write a disquisition on how not to prove that KR is an infinite loop space in general. The latter failures are joint work with Steiner and Thomason, but Steiner still has one promising idea that has yet to be shot down. Certainly the infinite deloopability of KR is a deep theorem if it is true.

While various other A_∞ ring spaces are known, the motivation comes from Waldhausen's work [67] connecting the Whitehead groups for stable PL concordance to algebraic K-theory. For a based space X, $Q(\Omega X \amalg \{0\})$ is an A_∞ ring space and we define $AX = KQ(\Omega X \amalg \{0\})$, this being one of Waldhausen's proposed definitions of the algebraic K-theory of a space. We also define $A(X;Z) = K\tilde{N}(\Omega X \amalg \{0\})$, where $\tilde{N}(\Omega X \amalg \{0\})$ is the free topological Abelian group generated by ΩX or, equivalently, the realization $|Z[GSX]|$ of the integral group ring of the Kan loop group on the total singular complex of X. In [38], I constructed a rational equivalence $AX \to A(X;Z)$.

Waldhausen [68] constructed another functor, call it WX, and established a natural fibration sequence with total space $WX \times Z$, fibre a homology theory (as a functor of X), and base space $Wh^{PL}(X)$. (As far as I know, proofs are not yet available. However, Steinberger and others have checked out the indications of proof in [68] and in Waldhausen's lectures. The connection with concordance groups depends on a stability claim of Hatcher [19], the published proof of which is definitely incorrect; Hatcher and Igusa (and I am told Burghelea) assure us that there is an adequate correct claim, but no proof has yet been given.) Waldhausen also claims a rational equivalence $WX \to A(X;Z)$, and it is on the basis of this claim that all calculational applications proceed. I have not yet seen or heard any convincing indications of proof. Clearly it suffices to show that WX and AX are equivalent, and this would be a deep and satisfying theorem even if an alternative argument were available. Steinberger is working towards this result and seems to be reasonably close to a proof.

There has been one other recent development of considerable interest. Woolfson has given a Segal style treatment of parts of multiplicative infinite loop space theory. His paper [72] is devoted to a theory analogous to the E_∞ ring theory summarized in [37, §12]. His paper [73] is devoted to a reformulation in his context of the orientation theory discussed in [37, §14] and to a proof of Nishida's nilpotency theorem along lines proposed by Segal [52]. (I have not read [72] or the first half of [73] for details, but the passage from particular bipermutative categories to hyper Γ-spaces sketched in [72] is unfortunately just as blatantly wrong as my passage from bipermutative categories to E_∞ ring spaces in [36]; as stated before, a correct treatment of this point will appear in [5]. The second half of [73] cannot be recommended; the proof of Theorem 2.2 is incorrect, and the argument as a whole is much harder than that based on the simpler homotopical notion of an H_∞ ring spectrum [5, 39].)

This theory raises further uniqueness questions of the sort discussed in the first section, and these have been considered by Thomason. The conclusion seems to be that there probably exists an appropriate theory but that the details would be so horrendous that it would not be worth developing unless a commanding need arose.

Incidentally, a Segal type approach to the construction and infinite delooping of KR was one of the failures mentioned above.

§3. *Descriptive analysis of infinite loop spaces*

The deepest new result under this heading is the proof of the infinite loop version of the complex Adams conjecture. When localized away from r, the composite $BU \xrightarrow{\psi^r - 1} BU \xrightarrow{j} BSF$ is not just null homotopic as a map of spaces but as a map of infinite loop spaces. That is, the associated composite map of spectra is null homotopic. This was originally announced by Friedlander and Seymour [17]. Their proposed proofs proceeded along wholly different lines. That of Seymour was based on Snaith's assertion [56, 4.1] that Seymour's bundle theoretical model [54] for the fibre JU(r) of ψ^r-1 could be constructed in a more economical way. Snaith's assertion is now known to be false*, and this line of proof is moribund. (The error also makes [56, §4-7] and [57] much less interesting.) I have been carefully checking Friedlander's proof. It is an enormously impressive piece of mathematics, and I am convinced that it is correct. It will appear in [16], in due course.

The infinite loop Adams conjecture, when combined with earlier results and the uniqueness theorem for the stable classifying spaces of geometric topology discussed in section one, largely completes the program of analyzing these infinite loop spaces at odd primes. The grand conclusion is stated in the introduction of [41]. One essential

* See Seymour and Snaith, these Proceedings.

ingredient was the work of Madsen, Snaith, and Tornehave [32], and complete new proofs of their results have been given by Adams [1, §6].

There remain interesting problems at $p = 2$. Here no lifting $\alpha:BSO \to F/O$ of $\psi^r - 1:BSO \to BSO$, $r \equiv \pm 3 \bmod 8$, can even be an H-map. An analysis of the homological behavior of one choice of α has been given by Brumfiel and Madsen [6] and the deviation from additivity of another choice has been studied in detail by Tornehave [67]. A provocative formulation of a possible 2-primary infinite loop version of the real Adams conjecture has been given by Miller and Priddy [46], although we have not the slightest idea of how their conjectures might be proven. Similarly, Madsen [31, 2.9] has made some very interesting conjectures about the infinite loop structure of F/Top at $p = 2$, but again there are no proofs in sight.

One very satisfying result along these lines has been given by Priddy [47]. Using the transfer and homology calculations, he has shown that, at the prime 2, SF is a direct factor (up to homotopy) in $QB(\Sigma_2 \int \Sigma_2)$ and F/O is a direct factor in QBO(2). The first assertion is a deeper multiplicative analog of the Kahn-Priddy theorem, their proof of which has just recently appeared [24,25]. That result gave that, at any prime p, Q_0S^0 is a direct factor in $QB\Sigma_p$. It is natural to conjecture that SF is also a direct factor in $QB(\Sigma_p \int \Sigma_p)$ for $p > 2$. However, because of the problems explained in [8, II §6], a proof along Priddy's lines would be much more difficult. There are three other splittings of this general nature that should be mentioned. Segal [49] proved that BU is a direct factor in QBU(1) and Becker [2] proved that BSp is a direct factor in QBSp(1) and BO is a direct factor in QBO(2). Snaith [59] rederived these last splittings and used them to deduce stable decompositions of the classifying spaces BG for G = U(n), Sp(n), or O(2n).

In my original survey, I neglected to mention Segal's paper [51]. Let $A = \{A_q | q \geq 0\}$ be a graded commutative ring. Then $\times_{q\geq 0} K(A_q,q)$ is a ring space with unit space $A_0^* \times (\times_{q\geq 1} K(A,q))$ and special unit space $\times_{q\geq 1} K(A,q)$. Segal proved that these unit spaces are infinite loop spaces. Steiner [61,62] later gave an improved argument which showed that these infinite loop structures are functorial in A and used the functoriality to prove certain splittings of these infinite loop spaces in case A is p-local, such splittings having been conjectured by Segal. (I find the earlier of Steiner's proofs the more convincing.) Snaith [58] showed that the total Stiefel-Whitney and Chern classes $\sum_{q\geq 1} w_q:BO \to \times_{q\geq 1} K(Z_2,q)$ and $\sum_{q\geq 1} c_q:BU \to \times_{q\geq 1} K(Z,2q)$ fail to commute with transfer. However, this does not disprove Segal's conjecture [51, p. 293] about these classes. Segal was quite careful to avoid such transfer pathologies by asking if the map $\coprod_{n\geq 0} BO(n) \to (\times_{q\geq 1} K(Z_2,q))\times Z$ specified by $(\sum_{q\geq 1} w_q)\times\{n\}$ on $BO(n)$ extends to an infinite loop map $BO\times Z \to (\times_{q\geq 1} K(Z_2,q))\times Z$ for a suitable infinite loop structure on the target, and similarly for the Chern classes. (I am told that this has now been proven by a student of Segal's. See [36, Remarks VIII.1.4], interpreting the remarks additively rather than multiplicatively, for a discussion of the relationship of the transfer for $BO\times Z$ to that for $BO = BO\times\{0\}$.)

The last, but by no means least, piece of progress to be reported in this area is the complete analysis by Fiedorowicz and Priddy [15] of the infinite loop spaces associated to the classical groups of finite fields and their relationship to the image of J spaces obtained as fibres of maps $\psi^r-1:BG \to BG'$ for stable classical groups G and G'. While this is an extraordinarily rich area of mathematics, the grand conclusion is that there is a one-to-one correspondence, realized by infinite loop equivalences coming from Brauer lifting of modular representations, between these two kinds of infinite loop spaces. In a sequel, Fiedorowicz [14] considers the uniqueness of the localizations at p prime to r of the infinite loop spaces JG(r) obtained with G = G' above being O, U, or Sp. In particular cases of geometric interest, the problem is not hard [56, §3], but the general answer is most satisfactory: $JG(r)_p$ and $JH(s)_p$ are equivalent as infinite loop spaces if and only if they have abstractly isomorphic homotopy groups.

§4. *Homological analysis of infinite loop spaces*

Probably the biggest development under this heading is again the work of Fiedorowicz and Priddy [15] just cited. They give an exhaustive analysis of the homologies, with their homology operations, of the various image of J. spaces. Amusingly, some of the most useful formulae, in particular for the real image of J spaces at the prime 2, are wholly inaccessible without the connection with finite groups. Their work also includes complete information on the homology and cohomology of all of the various classical groups of finite fields (away from the characteristic).

In [8, II §13], I used these calculations to study the Bockstein spectral sequences in the fibration sequence $B\,\mathrm{Coker}\,J \to BSF \to BJ_\otimes$

at p = 2. I would like to record one inconsequential error; [8, II.13.7] should read $\tilde{Q}^{2i+2} x_{(i,i)} \equiv x_{(2i+1,2i+1)}$ mod #-decomposables, the line of proof being as indicated but with due regard to the middle term of the mixed Cartan formula. In [8, II.13.8], the error term $\tilde{Q}^{4i}\beta\sigma_*\tilde{y}_{(2i,2i-1)}$ should therefore be $\sigma_*\tilde{y}_{(4i-1,4i-1)}$ rather than zero. No further changes are needed. (Another inconsequential error occurs in [8, III App]; Cohen has published the required corrections in [7, App].)

Incidentally, Madsen's assertion [31, 3.5], which is stated without proof, can be read off immediately from the calculations of [8, II §13]. This result plays a key role in Madsen's very interesting theorem that a kO-orientable spherical fibration ξ over X admits a topological reduction if and only if certain characteristic classes $\tau_i(\xi) \in H^{2^i-1}(X;Z_2)$ are zero. In other words, the obstruction to kO-orientability is not only the sole obstruction to reducibility away from 2 (as discussed in [37,§§14 and 18]), it is also a large part of the obstruction at p = 2.

In my original survey, I did not do justice to the work of Hodgkin and Snaith [22,55] on the mod p K-theory of infinite loop spaces in general and of those infinite loop spaces of greatest geometric interest in particular. In [37, §17], I did sketch their proof of the key fact that K_*(Coker J) = 0, and they have since published a very readable account [23] of this and related calculations.

I should mention one subterranean set of calculations. A reasonably good understanding of the Adams spectral sequence converging to π_*MSTop at p > 2 now exists. Two preprints, by Mann and Milgram [33] and Ligaard and myself [30], gave partial and complete information respectively on $H^*(MSTop;Z_p)$ as an A-module. This material is also in Ligaard's thesis [29], and he did much further work with me on the calculation of E_2. In my archives, I have nearly complete information on E_2, with descriptions as matric Massey products of all generators of $E_2^{s,t}$ for s > 0. I also have a thorough analysis of the differentials coming from $\pi_*MSO \to \pi_*MSTop$ and from the Bockstein spectral sequence of BCoker J [8, II.10.7], this being an elaboration of exploratory calculations in an undistributed preprint by Mann and Milgram. Mann has in his archives a calculation of a key piece of the spectral sequence from these differentials. However, a complete calculation of all of $E_r^{s,*}$ for s > 0 is out of reach algebraically, and we have very little control over the huge amount of noise in $E_2^{0,*}$. Milgram has in his archives a very nice geometrical argument to show that some of this noise does in fact survive to E_∞. Altogether though, we are very far from a complete determination of π_*MSTop, and the interest of all four parties seems to have flagged.

The work reported so far was already well under way when my earlier survey was written. There are two major later homological developments to report. The first is both negative and positive. In [10], Curtis claimed to prove that the mod 2 Hurewicz homomorphism for QS^0 annihilated all elements of π_*^s except the Hopf maps and, where present, the Arf invariant maps. The assertion may or may not be true, but Wellington's careful analysis [71] makes clear that we are very far from a proof by any known techniques. On the positive side, Wellington's work gives a good hold on the global structure of the cohomology of iterated loop spaces. In principal, this is a dualization problem from the homology calculations of Cohen [8, III]. The latter give $H_*(\Omega_0^n\Sigma^n X;Z_p)$ explicitly as an algebra and with precise recursive formulae for the coproduct and action by the Steenrod algebra A (see [37, §24]). Wellington proves that $H^*(\Omega_0^n\Sigma^n X;Z_p)$ is isomorphic as an algebra to the universal enveloping algebra of a certain Abelian restricted Lie algebra M_n^*X. While M_n^*X admits an A-action with respect to which its enveloping algebra is a free A-algebra, the isomorphism does not preserve the A-module structures. With this as his starting point, Wellington gives a detailed analysis of the problem of determining the A-annihilated primitive elements in $H_*(\Omega_0^n\Sigma^n X;Z_p)$, the main technique being a method for computing Steenrod operations in M_nX by use of the differential structure of the Λ-algebra.

The last homological development I wish to report concerns the relationship between the homology of infinite loop spaces and the homology of spectra. Let $X = E_0$ be the zero$^{\underline{th}}$ space of a spectrum $E = \{E_i\}$. In [35, p. 155-156], I pointed out that my two-sided bar construction gave spectral sequences $\{{}^iE^rX\}$ such that ${}^iE^2X$ is a well-defined computable functor of the R-algebra H_*X, where R is the Dyer-Lashof algebra, and $\{{}^iE^rX\}$ converges to H_*E_i. I specifically asked for a precise description of ${}^iE^2X$ as some homological functor of X, but I never pursued the point.

Much later, but independently, Miller [45] used resolution techniques to construct a spectral sequence $\{E^rX\}$ converging from a suitable functor of the R-algebra H_*X to H_*E. More importantly he developed techniques allowing explicit computation of E^2X in favorable cases and studied the behavior of the Steenrod operations in the spectral sequence. In particular, he showed that the spectral sequence collapses for $E = K(Z,0)$.

A little later, Kraines independently rediscovered this spectral sequence. I shall only say a little about his joint work with Lada on this subject, since their paper also appears in this volume [26]. They give a very pretty spectrum-level version of my two-sided bar construction, thus obtaining a most satisfactory geometric construction of Miller's algebraic spectral sequence. Among other things, the close connection with the geometry allows them to use the spectral sequence to disprove the long discredited conjecture that a representable functor with a transfer extends to a cohomology theory. More applications will surely appear, and further study of this spectral sequence is bound to be profitable.

§5. *Equivariant infinite loop space theory*

One of the most fashionable activities in modern topology is to take one's favorite theory, put an action of a compact Lie group G on all spaces in sight, and ask how much of the theory remains valid. Much less ambitiously, one might restrict G to be finite.

For the homotopy theorist, the first thing one wants is a thorough study of G-CW complexes. This we now have in the full generality of compact Lie groups, the relevant theory having been initiated by Matumota [34] and completed by Waner [69]. Any G-space is weakly G-equivalent to a G-CW complex and a weak G-equivalence between G-CW complexes is a G-equivalence. Actually, once ordinary CW-theory is developed properly, these and other standard results present little difficulty. Much more deeply, all of Milnor's basic theorems about spaces of the homotopy type of CW-complexes generalize to G-CW complexes; see Waner [69].

The next thing one wants is a good theory of G-bundles and G-fibrations (with some other structural group, A say, in the bundle case), including classification theorems for bundles or fibrations over G-CW complexes. This too we now have in the full generality of compact Lie groups, the bundle theory having been supplied by Segal [48], Tom Dieck [11], and Lashof and Rothenberg [27] and the fibration theory having been supplied by Waner [69], with addenda by Hauschild [21].

One is then led to ask if the resulting stable K-theories extend to cohomology theories. In the bundle case, as Segal has explained [48], one can generalize Bott periodicity. In the fibration case, and in the case of topological rather than linear bundles, one is inexorably led to develop G-infinite loop space theory.

I am quite confident that the eventual state of the art will precisely parallel the situation sketched in the first section. There will be two main approaches to the recognition principle, namely a G-Segal machine and a G-May machine, and there will be a uniqueness theorem on G-infinite loop space machines which ensures an equivalence between them. However, work in this direction is still in its infancy, and full details are not yet in place. It may well be necessary to restrict to finite groups, and we do so in the following discussion.

The present situation is this. I am in possession of three unpublished manuscripts, by Segal [53], Hauschild [21], and Waner [70], all of which I received within a month of the present writing (October 1978). In the first, Segal sketches a G-Segal machine, and I have little doubt that any missing details can be filled in. The other two give a G-May machine. In the latter approach, as I long ago explained to both authors, modulo a few technical points which turn out to be a bit tricky but not particularly difficult, it is formal to reduce the G-recognition principle to the stable G-approximation theorem.

Unstably, the G-approximation theorem asserts the existence of a natural "G-group completion" $C(V,X) \to \Omega^V\Sigma^V X$ for based G-spaces X, where V is a G-representation, Ω^V and Σ^V are the loops and suspension associated to the one-point compactification of V, and C(V,X) is the G-space of finite unordered subsets of V with labels in X. More precisely, $C(V,X) = \coprod_{j\geq 0} F(V,j)\times_{\Sigma_j} X^j/(\approx)$, where F(V,j) is the configuration space of j-tuples of distinct points of V and the equivalence relation encodes basepoint identifications. In the stable version, one takes colimits over G-representations V contained in a G-space R^∞ which contains each irreducible representation infinitely often. Hauschild [20] has published an argument for the stable theorem in the special case $X = S^0$, and Segal's manuscript [53] sketches an argument for the sharper unstable result, also for $X = S^0$. The bulk of Waner's manuscript [70] is devoted to a proof of the stable theorem for general X and the main part of Hauschild's manuscript [21] is devoted to a proof of the unstable result for general X. The various arguments are quite complicated and, at this writing, I cannot claim to fully understand any of them. However, I am reasonably sure that the union of Hauschild [20] and Waner [70] does include a complete proof of the stable theorem.

In any case, granting the stable G-approximation theorem, we have the G-recognition principle in a form applicable to G-E_∞ spaces and can apply it to the classifying spaces for stable spherical G-fibrations and topological G-bundles. Thus the relevant K-theories extend to G-cohomology theories. It is to be expected that this will be a powerful tool for the study of the equivariant Adams conjecture, this application being work in progress by Waner.

In connection with the G-approximation theorem, it is worth remarking that the paper by Cohen, Taylor, and myself [9] on the splitting of spaces of the same general form as C(V,X) above applies virtually verbatim with G-actions put in everywhere. There are evident notions of G-coefficient systems $\mathcal{C}$, G-Π spaces $\underline{X}$, and a resulting general construction of $C\underline{X}$ as in [9, §1-2]. The maps for the approximation theorem, but not the approximation theorem itself, can be used precisely as in [9] to obtain stable splittings of such G-spaces $C\underline{X}$, provided only that each $\mathcal{C}_j$ is Σ_j-free. That is, the suspension G-spectrum of $C\underline{X}$ is weakly G-equivalent to the wedge of the suspension G-spectra of the successive filtration quotients $\mathcal{C}_j^+ \wedge_{\Sigma_j} X^{(j)}$.

In fact, as we intend to make precise elsewhere, the whole argument of [9] is so formal that it can be carried out in an axiomatic setting of general topological categories with suitable extra structure. Indeed, the whole framework of definitions exploited in the study of iterated loop spaces can be set up in such a setting, and it can be expected that the resulting theory will find many future applications.

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