

ASL Annual Meeting, March 2010, Washington D.C.

1 Gabriel-Ulmer duality

as in [MP]: M.M. - Andy Pitts:

"Some results on locally finitely presentable categories", TAMS 1987, 473-496

2-categories LEX and LFC:

LEX: objects = 0-cells: cat's with finite limits

1-cells: functors preserving finite lim's

2-cells: all natural transformations

LFC: 0-cells: cat's with

(small) limits & filtered colimits

1-cells:

... as before

2-cells:

Set, the category of (small) sets

is an object of both

LEX and LFC

and LEX-operations on Set

commute with LFC-operations on Set

[e.g.: a filtered colimit of pullback diagrams is a pullback diagram in Set]

This fact 'automatically' gives rise to a

2-adjunction

$$\text{LEX}^{\text{op}} \begin{array}{c} \xrightarrow{(-)^*} \\ \xleftarrow{(-)^\#} \end{array} \text{LFC}$$

$$(-)^* \dashv (-)^\#$$

where:

for

$\mathbb{C} \in \text{LEX}$:

\mathbb{C}^*

$= \text{LEX}(\mathbb{C}; \text{Set})$

hom-category
in the 2-cat
LEX

and for $A \in \text{LFC}$:

3

$$A^\# = \text{LFC}(A, \text{Set})$$

with unit:

$$\eta_A : A \longrightarrow A^{\#\ast}$$

and counit

$$\epsilon_C : C \longrightarrow C^{\#\ast}$$

both defined as evaluations; e.g.

$$(X \in C) \xrightarrow{\epsilon_C} [M \in \text{LEX}(C, \text{Set})$$

$$\longmapsto M(X) \in \text{Set}]$$

Let: $\text{Lex} \subset \text{LEX}$: full sub-2-cat
of LEX on (equivalent to) small objects.

Theorem (Gabriel-Ulmer reformulated by [MP])

(1) For $C \in \text{Lex}$, $\epsilon_C : C \xrightarrow{\cong} C^{\#\ast}$

is an equivalence of categories

(2) For any cat A , $A \simeq C^*$ 4

for some $C \in \text{Lex}$ (iff)

a) $A \in \text{L(FC)}$

and

b) there is small B such that

$A \simeq \text{Flat}(B) \stackrel{\text{DEF}}{=} \text{the free completion}$
of B to a category with filtered colimits

[Denote the full sub-2-cat of LFC on
objects the \checkmark locally finitely presentable categories:

those with a) and b)]

(3) (Corollary) For $A \in \text{Lfp}$,

$\eta_A : A \xrightarrow{\simeq} A^{\#*}$ is an equivalence

of categories. We obtain 2D duality:

$$\text{Lex}^{\text{op}} \begin{array}{c} \xrightarrow{(-)^*} \\ \xleftarrow{(-)^\#} \end{array} \text{Lfp}$$

(2-equivalence)

A "model-theoretical" result in [MP]: 5

Let: $B \in Lfp$

$A \in LFC$ (!)

$F: A \rightarrow B$: LFC-morphism

such that: F is faithful

& F is full on isomorphisms:

A_1
 A_2 in A , $FA_1 \cong \downarrow g$ an iso in B
 FA_2

\Rightarrow

\exists : A_1 $f \downarrow \cong$ in A s.t. FA_1 $Ff \downarrow = g$
 A_2 FA_2

Then: $A \in Lfp$

2 Gray categories

6

The following items:

0-cells : all 2-categories

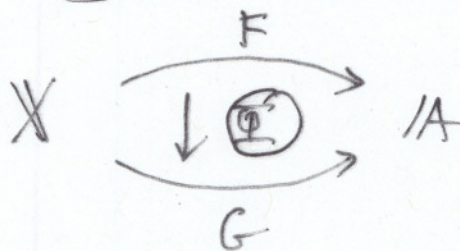


1-cells : all 2-functors

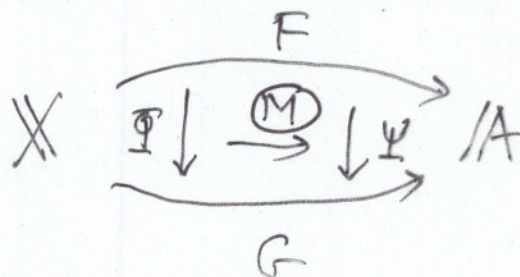


2-cells : all pseudo-natural transformations

(pnt's)



3-cells : all modifications



form a 3D category, a Gray-category;
in fact the 'universal' one. We call
it GRAY (with Steve Lack)

What is a Gray category?

A Gray-category has 0-, 1-, 2- and 3-cells.

It has "vertical" compositions such as

$$X \xrightarrow{F} Y \xrightarrow{G} Z \quad \mapsto \quad X \xrightarrow{\textcircled{GF}} Z$$

$$X \begin{array}{c} \xrightarrow{F} \\ \downarrow \Phi \\ \xrightarrow{G} \\ \downarrow \Psi \\ \xrightarrow{H} \end{array} Y \quad \mapsto \quad X \begin{array}{c} \xrightarrow{F} \\ \downarrow \textcircled{\Psi\Phi} \\ \xrightarrow{H} \end{array} Y$$

$$X \begin{array}{c} \xrightarrow{F} \\ \downarrow \Phi \rightarrow \downarrow \Psi \\ \downarrow \Psi \rightarrow \downarrow \Gamma \\ \xrightarrow{G} \end{array} Y \quad \mapsto \quad X \begin{array}{c} \xrightarrow{F} \\ \downarrow \Phi \rightarrow \downarrow \Psi \\ \downarrow \Psi \rightarrow \downarrow \Gamma \\ \xrightarrow{G} \end{array} Y$$

and "whiskerings" such as

$$X \begin{array}{c} \xrightarrow{F} \\ \downarrow \Phi \\ \xrightarrow{G} \end{array} Y \xrightarrow{H} Z \quad \mapsto \quad X \begin{array}{c} \xrightarrow{HF} \\ \downarrow \textcircled{H\Phi} \\ \xrightarrow{HG} \end{array} Z$$

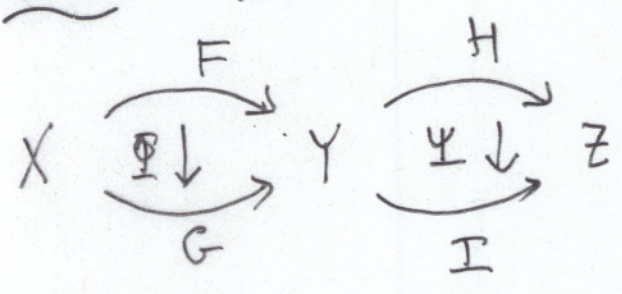
$$X \begin{array}{c} \xrightarrow{F} \\ \downarrow \Phi \rightarrow \downarrow \Psi \\ \downarrow \Psi \rightarrow \downarrow \Gamma \\ \xrightarrow{G} \end{array} Y \xrightarrow{H} Z \quad \mapsto \quad X \begin{array}{c} \xrightarrow{HF} \\ \downarrow \Phi \rightarrow \downarrow \Psi \\ \downarrow \Psi \rightarrow \downarrow \Gamma \\ \xrightarrow{HG} \end{array} Z$$

with corresponding strict associativity and distributivity laws such as

$$\begin{array}{c}
 X \xrightarrow{\quad} Y \\
 \downarrow \Phi \\
 \xrightarrow{\quad} \\
 \downarrow \Psi \\
 \xrightarrow{\quad} \\
 \downarrow \Gamma \\
 \xrightarrow{\quad}
 \end{array}
 \Rightarrow \Gamma(\Psi\Phi) \cong (\Gamma\Psi)\Phi$$

$$\begin{array}{c}
 X \xrightarrow{\quad} Y \xrightarrow{H} Z \\
 \downarrow \Phi \\
 \xrightarrow{\quad} \\
 \downarrow \Psi \\
 \xrightarrow{\quad}
 \end{array}
 \Rightarrow (H\Psi)(H\Phi) \cong H(\Psi\Phi)$$

Plus: for



(not (directly) comparable horizontally!)

we have 3-cell $\langle \Psi, \Phi \rangle$

as in

$$\begin{array}{ccc}
 HF & \xrightarrow{\Psi F} & IF \\
 \downarrow H\Phi & \cong & \downarrow I\Phi \\
 HG & \xrightarrow{\Psi G} & IG
 \end{array}$$

that is:

$$(I \Phi)(\Psi F) \xrightarrow[\langle \Psi, \Phi \rangle]{\cong} (\Psi G)(H \Phi)$$

with coherence conditions:

1) for

$$\begin{array}{ccc}
 X & \xrightarrow{\quad} & Y & \xrightarrow{\quad} & Z \\
 & \downarrow \Phi_1 & & \downarrow \Psi & \\
 & \xrightarrow{\quad} & & & \\
 & \downarrow \Phi_2 & & & \\
 & \xrightarrow{\quad} & & &
 \end{array}$$

the resulting diagram connecting the 3-cells

$$\langle \Psi, \Phi_1 \rangle, \langle \Psi, \Phi_2 \rangle, \langle \Psi, \Phi_2 \Phi_1 \rangle$$

commutes
- and dually -

2) for-

$$\begin{array}{ccc}
 X & \xrightarrow{\quad} & Y & \xrightarrow{\quad} & Z \\
 & \Phi_1 \downarrow \xrightarrow{M} \downarrow \Phi_2 & & \downarrow \Psi & \\
 & \xrightarrow{\quad} & & &
 \end{array}$$

'connect' 3-cells

$$M, \langle \Psi, \Phi_1 \rangle, \langle \Psi, \Phi_2 \rangle$$

- and dually.

[somewhat incomplete definition;
there are identity cells ...]

The usual definition of Gray-category:

take the ordinary, 1-category

GRAY_{original}

of 2-categories and 2-functors. It is

a closed category with internal hom:

$[X, A] =$ the 2-cat of all

2-functors $X \xrightarrow{F} A$

nat'l's $X \begin{array}{c} \xrightarrow{F} \\ \downarrow \Phi \\ \xrightarrow{G} \end{array} A$

and modifications

$X \begin{array}{c} \xrightarrow{F} \\ \Phi \downarrow \quad \rightarrow \quad \downarrow \Psi \\ \xrightarrow{G} \end{array} A.$

This makes GRAY_{original} a

Symmetric closed monoidal category

A Gray category, in the original sense, 11

is a $(\text{Gray}_{\text{original}}, \otimes, \mathbb{1}, [-, -])$ -enriched

category. There is a bijection between

Gray-categories of the Crans (unpacked?)
kind, and the original enriched kind
that of

- and, in this bijection, our GRAY
above corresponds to $\text{GRAY}_{\text{original}}$ as
enriched over itself.

∴ Via V -enriched category theory

(Max Kelly's classic book), we have

a Yoneda theory: for any Gray category X ,

$$X \xrightarrow{\text{Yoneda}} \text{GRAY}^{X^{\text{op}}}$$

a Gray category

'Naive' motivation: take

2-cats, 2-functors & nat's :

$$\begin{array}{ccccc}
 & & F & & H \\
 & \curvearrowright & \rightarrow & \curvearrowright & \\
 X & & & & Y & & & & Z \\
 & \downarrow \Phi & & \downarrow \Psi & \\
 & \curvearrowright & & \curvearrowright & \\
 & & G & & I
 \end{array}$$

and the resulting:

$$\begin{array}{ccc}
 HF & \xrightarrow{\Psi F} & IF \\
 H\Phi \downarrow & & \downarrow I\Phi \\
 HG & \xrightarrow{\Psi G} & IG
 \end{array}$$

This would commute if Φ, Ψ were ordinary natural transformations. Instead, what we have is, at each x in X :

$$\begin{array}{ccccc}
 H(F(x)) & \xrightarrow{\Psi_{F(x)}} & & & I(F(x)) \\
 H(\Phi_x) \downarrow & \cong \Downarrow \Psi_{(\Phi_x)} & & & \downarrow I(\Phi_x) \\
 H(G(x)) & \xrightarrow{\Psi_{G(x)}} & & & I(G(x))
 \end{array}$$

$\Psi_{(\Phi_x)}$: the naturality isomorphism

for the pnt Ψ at the arrow

$F(x) \xrightarrow{\Phi_x} G(x)$ of the domain 2-cat \mathcal{Y} .

We have a modification (M)

$$\begin{array}{ccc} HF & \xrightarrow{\Psi F} & IF \\ H\Phi \downarrow & \cong \Downarrow (M) & \downarrow I\Phi \\ HG & \xrightarrow{\Psi G} & IG \end{array}$$

& coherence 1) and 2) are satisfied. as a consequence of properties of pnt's.

- this was a (partial) verification that GRAY is indeed an (unpacked) Gray cat.

[3] (Further) 'categorifying' Gabriel-Ulmer

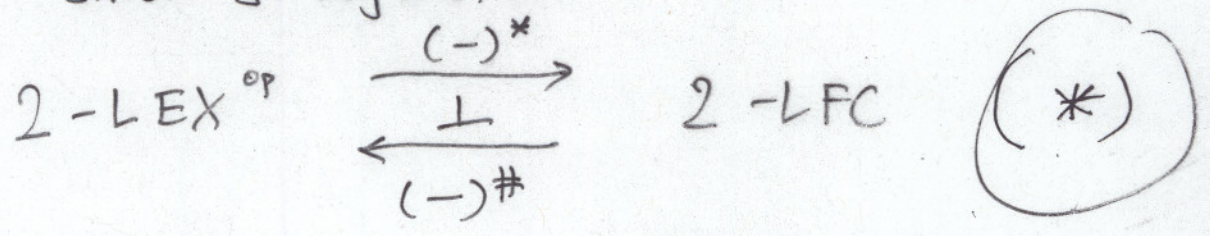
To LEX before, there corresponds the Gray category (2-LEX) : of all 2-cat's having finite (conical) pseudo limits and cotensors $X \pitchfork 2$

and to LFC, the Gray category
 2-LFC, of 2-cats with small pseudolimits,
 cotensors $X \pitchfork 2$, and filtered 2-colimits.

In both cases, the 1-cells are the 2-functors
 preserving respective operations; the 2-, and 3-cells
 are all pts and modifications.

Cat: the 2-cat of all small cat's is
 in both 2-LEX and 2-LFC, with the
 2-LEX op's commuting with the 2-LFC op's.

GET: strict 3-adjunction:



$C \in 2\text{-LEX} : C^* = 2\text{-LEX}(C, \text{Cat})$

↖ hom-2-cat
 in Gray-cat
 2-LEX

$A \in 2\text{-LFC} : A^\# = 2\text{-LFC}(A, \text{Cat})$

Explanation: In any closed symmetric monoidal
 category \mathbb{X} , with objects S, X, Y, \dots

We have the adjunction

15

$$\mathbb{V}^{op} \begin{array}{c} \xrightarrow{[-, S]} \\ \perp \\ \xleftarrow{[-, S]} \end{array} \mathbb{V} \quad (**)$$

depicted as:

$$\frac{X \longrightarrow [Y, S]}{Y \longrightarrow [X, S]}$$

2-LEX and 2-LFC are non-full sub-Gray-categories of GRAY itself.

The adjunction (*) is a restriction of (**).

Let: 2-Lex : full sub-Gray-category of 2-LEX on the small objects.

~~Theorem~~ (new?; 'promised' by Max Kelly in Blackwell-Kelly-Power)

(1) For $\mathbb{C} \in 2\text{-Lex}$:

$$E_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}^{*\#}$$

is a biequivalence of 2-categories.

2) For any 2-cat \mathcal{A} , $\mathcal{A} \simeq_{hi} \mathbb{C}^*$ for some $\mathbb{C} \in 2\text{-Lex}$ iff

1) $\mathcal{A} \in 2\text{-LFC}$

2) $\mathcal{A} \simeq_{hi} 2\text{-Flat}(\mathbb{B}) \stackrel{\text{DEF}}{=} \text{the free completion of } \mathbb{B} \text{ to a 2-category with filtered 2-colimits, for some small 2-cat } \mathbb{B}.$

2-Lfp : full sub-Gray-cat of 2-LFC on objects \mathcal{A} satisfying 1) and 2).

3) (Corollary) For $\mathcal{A} \in 2\text{-Lfp}$,

$$\eta_{\mathcal{A}} : \mathcal{A} \xrightarrow{\simeq_2} \mathcal{A}^{\#*}$$

is a biequivalence, and we get the 3D duality

$$2\text{-Lex}^{op} \begin{array}{c} \xrightarrow{(-)^*} \\ \simeq_3 \\ \xleftarrow{(-)^\#} \end{array} 2\text{-Lfp}$$

Essentially algebraic 2-category $\stackrel{\text{DEF}}{=} \text{object of } 2\text{-Lfp}.$

Same as the 2-category of models of a small 2-limit sketch

with pnt's as morphisms (1-cells), and modifications as 2-cells.

Examples: the first-order categorical doctrines

(A. Kock - G. Reyes: Handbook of Mathematical Logic)

Cart, Lex, Regular, Exact, Coherent, Pretopos, ...

Model theory : half-theorem : not quite proved : the direct analog of the quoted result from [MP]

4 Ingredients

4.1 Bi-initial object theorem

Let: \mathcal{X} : locally small 2-cat

\mathcal{Y} : small set of objects of \mathcal{X}

which is weakly initial:

$$\forall X \in \mathcal{X}. \exists S \in \mathcal{Y}. \exists f : S \xrightarrow{f} X.$$

Suppose: \mathcal{X} has all small (conical! not "weighted") bilimits.

Then: \mathcal{X} has a biinitial object

[A is biinitial means: for every $X \in \mathcal{X}$, $[A, X] \simeq \mathbb{1}$ = the terminal category]

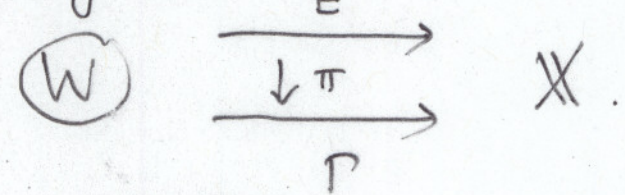
The proof uses the

Lemma Let: $W \in \mathcal{X}$

(W) $\stackrel{\text{def}}{=} \text{the full sub-2-cat of } \mathcal{X} \text{ on the single object } W$

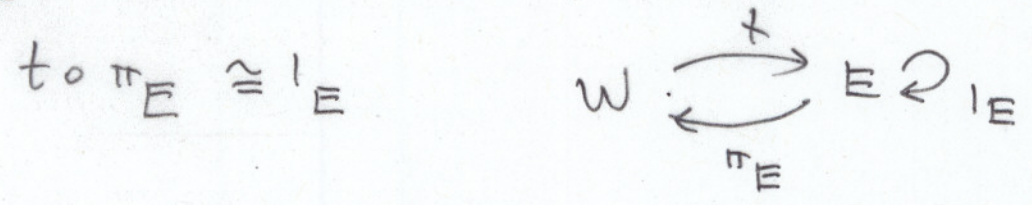
(W) $\xrightarrow{P} \mathcal{X}$: inclusion

Suppose $E = \text{bilim}(P)$ exists in \mathcal{X} , with bilimiting cone τ_E



(π is a pnt!)

Then: $\forall t : W \rightarrow E$, we have



4.2 Representability

\mathcal{X} : locally small 2-cat

$A \in \mathcal{X}$.

$\approx \Rightarrow \text{hom}_{\mathcal{X}}(A, -) : \mathcal{X} \longrightarrow \text{Cat}$
 'representable' 2-functor.

Let: $\mathcal{X} \xrightarrow{F} \text{Cat}$ any 2-functor
 $A \in \mathcal{X}$, $a \in F(A)$ (object in cat. $F(A)$)

We have the ordinary nat. transf. (put, in part.)

$$e^{(A,a)} : \text{hom}_{\mathcal{X}}(A, -) \longrightarrow F$$

$$\mathcal{X} \begin{array}{c} \xrightarrow{\text{hom}(A, -)} \\ \downarrow e^{(A,a)} \\ \xrightarrow{F} \end{array} \text{Cat}$$

defined by:

$$e_{\mathcal{X}}^{(A,a)} : \text{hom}_{\mathcal{X}}(A, X) \longrightarrow FX$$

$$A \xrightarrow{f} X \longmapsto (Ff)(a) (\in FX)$$

$$\begin{array}{ccc} & \searrow & \swarrow \\ & FA & \xrightarrow{Ff} FX \end{array}$$

Definition $(A, a) = (A \in \mathbb{X}, a \in F(A))$

provides a representation for F if def $e^{(A, a)}$ is a biequivalence (in the 2-cat $\text{hom}(\text{hom}_{\mathbb{X}}(A, -), F)$), or equivalently, if each $e_X^{(A, a)}$ is an equivalence of cats ($\mathbb{X} \in \text{Cat}$).

FACT (Yoneda) F is representable $\stackrel{\text{def}}{\iff}$ there is A such that $F \cong_2 \text{hom}_{\mathbb{X}}(A, -)$
if and only if

there is a representation (A, a) for F .

Definition For $\mathbb{X} \xrightarrow{F} \text{Cat}$

$\text{el}(F) : 2\text{-category}$

objects: $(X, x) : X \in \mathbb{X}, x \in F(X)$

arrows: $(X, x) \xrightarrow{(f, \theta)} (Y, y)$

$X \xrightarrow{f} Y \quad \& \quad (Ff)(x) \xrightarrow[\cong]{\theta} y$

(in the cat Y)

2-cells:

$$(X, x) \begin{array}{c} \xrightarrow{(f, \theta)} \\ \downarrow \textcircled{A} \\ \xrightarrow{(g, \tau)} \end{array} (Y, y):$$

$$X \begin{array}{c} \xrightarrow{f} \\ \textcircled{A} \downarrow \\ \xrightarrow{g} \end{array} Y \quad \& \quad \begin{array}{ccc} (Ff)(x) & & \\ & \searrow \theta & \\ & & y \\ (Fg)(x) & \nearrow \tau & \end{array}$$

Proposition Assume that \mathbb{X} is groupoidal.
 all 2-cells in \mathbb{X} are isomorphisms. !

Let: $F: \mathbb{X} \rightarrow \text{Cat}$.

Then: (A, a) is a biinitial object of $el(F)$ iff (A, a) provides a representation for F .

With every 2-cat \mathbb{X} , we can associate the (non-full) subcategory \mathbb{X}^{iso} of \mathbb{X} , with 0- and 1-cells the same as those of \mathbb{X} , and

and 2-cells the isomorphisms in \mathbb{X} .

We have the inclusion 2-functor

$$\mathbb{X}^{\text{iso}} \xrightarrow{\quad \text{I} \quad} \mathbb{X}.$$

Given $F: \mathbb{X} \rightarrow \text{Cat}$, we get

$$F \circ \text{I}: \mathbb{X}^{\text{iso}} \longrightarrow \text{Cat}$$

Proposition Suppose that (A, a)

provides a representation for $F \circ \text{I}$.

Suppose, moreover, that 1) the 2-cat \mathbb{X}

has cotensors $X \pitchfork 2$:

for every $X \in \mathbb{X}$, we have

$$X \pitchfork 2 \begin{array}{c} \xrightarrow{\pi_0} \\ \downarrow \pi_1^0 \\ \xrightarrow{\pi_1} \end{array} X$$

(pseudo-) terminal among all

$$Y \begin{array}{c} \xrightarrow{p_0} \\ \downarrow p_1^0 \\ \xrightarrow{p_1} \end{array} X$$

and 2) F preserves cotensors

Then (A, a) provides a representation
for F itself.

Representability theorem

(analog of Mac Lane's identically named theorem CWM p. 118)

Suppose \mathcal{X} locally small

has all small bilimits
and cotensors $X \pitchfork 2$

Let: $F: \mathcal{X} \rightarrow \text{Cat}$ 2-functor.

Then F is representable

(iff)

1) F preserves all bilimits and cotensors

and

2) $\text{el}(F)$ has a small weakly initial set of objects.