1 Gabriel-Ulmer duality

as in [MP]: M. M. – Andy Pitts:

"Some results on locally finitely presentable categories", TAMS 1987, 473–496

2-categories LEX and LFC:

LEX: objects = 0-cells: cat's with finite limits

1-cells: functors preserving finite lim's

2-cells: all natural transformations

LFC: 0-cells: cat's with (small) limits & filtered colimits

1-cells: ... as before
2-cells:

Set, the category of (small) sets

is an object of both

LEX and LFC

and LEX-operations on Set

commute with LFC-operations on Set

[e.g.: a filtered colimit of pullback diagrams is a pullback diagram in Set]

This fact 'automatically' gives rise to a

2-adjunction

$$\mathrm{LEX}^{op} \underset{(-)^{\#}}{\overset{(-)^{*}}{\rightleftarrows}} \mathrm{LFC}$$

$$(-)^{*} \dashv (-)^{\#}$$

where: for $\mathbb{C} \in \mathrm{LEX}$:

$$\mathbb{C}^{*} = \mathrm{LEX}(\mathbb{C}; \mathrm{Set})$$

hom-category in the 2-cat LEX

and for $\mathbb{A} \in LFC$:

$$\mathbb{A}^{\#} = LFC(\mathbb{A}, Set)$$

with <u>unit</u> :

$$\eta_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{A}^{\#*}$$

and <u>counit</u>

$$\varepsilon_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}^{*\#}$$

both defined as <u>evaluations</u> ; e.g.

$$(X \in \mathbb{C}) \longmapsto^{\varepsilon_{\mathbb{C}}} [M \in LEX(\mathbb{C}, Set) \longmapsto M(X) \in Set]$$

Let : $Lex \subset LEX$: full sub-2-cat of LEX on (equivalent-to) <u>small</u> objects.

Theorem (Gabriel-Ulmer reformulated by [MP])

(1) For $\mathbb{C} \in Lex$, $\varepsilon_{\mathbb{C}} : \mathbb{C} \xrightarrow{\simeq} \mathbb{C}^{*\#}$ is an equivalence of categories

2) For any cat $\mathbb{A}$, $\mathbb{A} \simeq \mathbb{C}^*$ for some $\mathbb{C} \in \mathrm{Lex}$ iff

a) $\mathbb{A} \in L(FC)$

and

b) there is small $\mathbb{B}$ such that

$\mathbb{A} \simeq \mathrm{Flat}(\mathbb{B}) \underset{\mathrm{DEF}}{=}$ the free completion of $\mathbb{B}$ to a category with filtered colimits

[Denote the full sub-2-cat of LFC on objects the locally finitely presentable categories: those with a) and b)]

3) (Corollary) For $\mathbb{A} \in \mathrm{Lfp}$, $\eta_{\mathbb{A}} : \mathbb{A} \xrightarrow{\simeq} \mathbb{A}^{\#*}$ is an equivalence of categories. We obtain 2D duality:

$$\mathrm{Lex}^{op} \underset{(-)^{\#}}{\overset{(-)^*}{\rightleftarrows}} \mathrm{Lfp} \qquad \simeq_2$$

(2-equivalence)

A "model-theoretical" result in [MP]:

<u>Let:</u> $\mathbb{B} \in \mathrm{Lfp}$

$\mathbb{A} \in \mathrm{LFC}$ (!)

$F : \mathbb{A} \longrightarrow \mathbb{B}$: LFC-morphism

such that: F is <u>faithful</u>

&

F is <u>full</u> on isomorphisms:

$\begin{matrix} A_1 \\ A_2 \end{matrix}$ in $\mathbb{A}$, $\begin{tikzcd} FA_1 \arrow[d, "\cong"', "g"] \\ FA_2 \end{tikzcd}$ an iso in $\mathbb{B}$

$$\Longrightarrow$$

$\exists$: $\begin{tikzcd} A_1 \arrow[d, "f"', "\cong"] \\ A_2 \end{tikzcd}$ in $\mathbb{A}$ s.t. $\begin{tikzcd} FA_1 \arrow[d, "Ff"'] \\ FA_2 \end{tikzcd} = g$

<u>Then:</u> $\boxed{\mathbb{A} \in \mathrm{Lfp}}$

$\boxed{2}$ Gray categories

The following items:

0-cells : all 2-categories $\mathbb{X}$

1-cells : all 2-functors $\mathbb{X} \xrightarrow{F} \mathbb{A}$

2-cells : all pseudo-natural transformations

(pnt's)

$$\begin{tikzcd} \mathbb{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \Phi"] & \mathbb{A} \end{tikzcd}$$

3-cells : all modifications

$$\begin{tikzcd} \mathbb{X} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Phi\Downarrow \xrightarrow{M} \Downarrow\Psi"] & \mathbb{A} \end{tikzcd}$$

form a 3D category, a Gray-category; in fact the 'universal' one. We call it GRAY (with Steve Lack)

What is a Gray category?

A Gray-category has 0-, 1-, 2- and 3-cells.

It has "vertical" compositions such as

$$X \xrightarrow{F} Y \xrightarrow{G} Z \quad \longmapsto \quad X \xrightarrow{gf} Z$$

$$\begin{tikzcd} X \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\Phi"] \arrow[r, phantom, shift right=3ex, "\Downarrow\Psi"] & Y \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} X \arrow[r, bend left=40, "F"] \arrow[r, bend right=40, "H"'] \arrow[r, phantom, "\Downarrow \Psi\Phi"] & Y \end{tikzcd}$$

$$\begin{tikzcd}[column sep=large] X \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Phi\Downarrow \overset{M}{\Rrightarrow} \Downarrow \overset{N}{\Rrightarrow} \Downarrow\Gamma"] & Y \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd}[column sep=large] X \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Phi\Downarrow \overset{NM}{\Rrightarrow} \Downarrow\Gamma"] & Y \end{tikzcd}$$

and "whiskerings" such as

$$\begin{tikzcd} X \arrow[r, bend left=40, "F"] \arrow[r, bend right=40, "G"'] \arrow[r, phantom, "\Downarrow\Phi"] & Y \arrow[r, "H"] & Z \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} X \arrow[r, bend left=40, "HF"] \arrow[r, bend right=40, "HG"'] \arrow[r, phantom, "\Downarrow H\Phi"] & Z \end{tikzcd}$$

$$\begin{tikzcd}[column sep=large] X \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Phi\Downarrow \overset{M}{\Rrightarrow} \Downarrow\Psi"] & Y \arrow[r, "H"] & Z \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd}[column sep=huge] X \arrow[r, bend left=50, "HF"] \arrow[r, bend right=50, "HG"'] \arrow[r, phantom, "H\Phi\Downarrow \overset{HM}{\Rrightarrow} \Downarrow H\Psi"] & Z \end{tikzcd}$$

with corresponding <u>strict</u>

associativity and distributivity

laws such as

$$\begin{tikzcd}[row sep=large] X \arrow[r, shift left=9] \arrow[r, shift left=3] \arrow[r, shift right=3] \arrow[r, shift right=9] \arrow[r, phantom, shift left=6, "\Downarrow\Phi"] \arrow[r, phantom, "\Downarrow\Psi"] \arrow[r, phantom, shift right=6, "\Downarrow\Gamma"] & Y \end{tikzcd} \Rightarrow \Gamma(\Psi\Phi) = (\Gamma\Psi)\Phi$$

$$\begin{tikzcd} X \arrow[r, shift left=6] \arrow[r] \arrow[r, shift right=6] \arrow[r, phantom, shift left=3, "\Downarrow\Phi"] \arrow[r, phantom, shift right=3, "\Downarrow\Psi"] & Y \arrow[r, "H"] & Z \end{tikzcd} \Rightarrow (H\Psi)(H\Phi) = H(\Psi\Phi)$$

<u>PLUS</u>: for

$$\begin{tikzcd} X \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Phi\Downarrow"] & Y \arrow[r, bend left, "H"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\Psi\Downarrow"] & Z \end{tikzcd}$$

(<u>not</u> (directly) composable horizontally!)

we have 3-cell $\langle\Psi,\Phi\rangle$

as in

$$\begin{tikzcd}[column sep=huge, row sep=huge] HF \arrow[r, "\Psi F"] \arrow[d, "H\Phi"'] \arrow[dr, phantom, "\cong \Downarrow\langle\Psi,\Phi\rangle"] & IF \arrow[d, "I\Phi"] \\ HG \arrow[r, "\Psi G"'] & IG \end{tikzcd}$$

that is:

$$(I\,\Phi)(\Psi F) \xrightarrow[\langle \Psi, \Phi \rangle]{\cong} (\Psi G)(H\,\Phi)$$

with coherence conditions:

1) for

$$\begin{tikzcd} X \arrow[r, bend left=50] \arrow[r] \arrow[r, bend right=50] \arrow[r, phantom, shift left=2ex, "\Downarrow \Phi_1"] \arrow[r, phantom, shift right=2ex, "\Downarrow \Phi_2"] & Y \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \Psi"] & Z \end{tikzcd}$$

the resulting diagram conneting the 3-cells

$$\langle \Psi, \Phi_1 \rangle, \quad \langle \Psi, \Phi_2 \rangle, \quad \langle \Psi, \Phi_2 \Phi_1 \rangle$$

commutes

– and dually ...

2) for

$$\begin{tikzcd} X \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Phi_1 \Downarrow \xrightarrow{M} \Downarrow \Phi_2"] & Y \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \Psi"] & Z \end{tikzcd} :$$

'connect' 3-cells

$$M, \quad \langle \Psi, \Phi_1 \rangle, \quad \langle \Psi, \Phi_2 \rangle$$

– and dually.

[somewhat incomplete definition; there are identity cells ...]

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The usual definition of Gray-category:

take the ordinary, 1-category

$GRAY_{original}$

of 2-categories and 2-functors. It is a <u>closed</u> category with <u>internal hom</u>:

$[\mathbb{X}, \mathbb{A}]$ = the <u>2-cat</u> of all

2-functors $\mathbb{X} \xrightarrow{F} \mathbb{A}$

pnt'ls $\begin{tikzcd} \mathbb{X} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\Phi"] & \mathbb{A} \end{tikzcd}$

and modifications $\begin{tikzcd} \mathbb{X} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Phi\Downarrow \xrightarrow{M} \Downarrow\Psi"] & \mathbb{A} \end{tikzcd}$.

This makes $GRAY_{original}$ a

<u>symmetric</u> <u>closed</u> <u>monoidal</u> category

A Gray category, in the original sense, is a $(\mathrm{Gray}_{\text{original}}, \otimes, \mathbb{1}, [-,-])$ -<u>enriched</u> category. There is a bijection between Gray-categories of the Crans (unpacked?) kind, and thos of the original enriched kind – and, in this bijection, our GRAY above corresponds to $\mathrm{GRAY}_{\text{original}}$ as enriched over <u>itself</u>.

$\therefore$ Via $\mathcal{V}$-enriched category theory (Max Kelly's classic book), we have a Yoneda theory: for any Gray category $\mathbb{X}$,

$$\mathbb{X} \xrightarrow[\text{Yoneda}]{\qquad\qquad} \underbrace{\mathrm{GRAY}^{\mathbb{X}^{op}}}_{\text{a Gray category}}$$

'Naive' motivation: take

2-cats, 2-functors & pnt's :

$$\begin{tikzcd} X \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\Phi"] & Y \arrow[r, bend left, "H"] \arrow[r, bend right, "I"'] \arrow[r, phantom, "\Downarrow\Psi"] & Z \end{tikzcd}$$

and the resulting:

$$\begin{tikzcd} HF \arrow[r, "\Psi F"] \arrow[d, "H\Phi"'] & IF \arrow[d, "I\Phi"] \\ HG \arrow[r, "\Psi G"'] & IG \end{tikzcd}$$

This <u>would</u> commute if Φ, Ψ were ordinary natural transformations. <u>Instead</u>, what we have is, at each x in X:

$$\begin{tikzcd} H(F(x)) \arrow[r, "\Psi_{F(x)}"] \arrow[d, "H(\Phi_x)"'] \arrow[dr, phantom, "\cong\ \Downarrow \Psi_{(\Phi_x)}"] & I(F(x)) \arrow[d, "I(\Phi_x)"] \\ H(G(x)) \arrow[r, "\Psi_{G(x)}"'] & I(G(x)) \end{tikzcd}$$

.

$\Psi_{(\Phi_x)}$: the naturality isomorphism

for the pnt Ψ at the arrow

$F(x) \xrightarrow{\Phi_x} G(x)$ of the domain 2-cat Y.

We have a modification (M)

$$\begin{tikzcd} HF \arrow[r, "\Psi F"] \arrow[d, "H\Phi"'] \arrow[dr, phantom, "\cong \Downarrow M"] & IF \arrow[d, "I\Phi"] \\ HG \arrow[r, "\Psi G"'] & IG \end{tikzcd}$$

& coherence 1) and 2) are satisfied. as a consequence of properties of pnt's.

— this was a (partial) verification that GRAY is indeed an (unpacked) Gray cat.

$\boxed{3}$ (Further) 'categorifying' Gabriel-Ulmer

To LEX before, there corresponds the Gray category (2-LEX) : of all 2-cat's having finite (conical) pseudo limits and cotensors $X \pitchfork 2$

and to LFC, the Gray category

2-LFC, of 2-cats with small pseudo limits, cotensors $X \pitchfork 2$, and filtered 2-colimits.

In both cases, the 1-cells are the 2-functors preserving respective operations; the 2-, and 3-cells are all pnt's and modifications.

Cat: the 2-cat of all small cat's is in both 2-LEX and 2-LFC, with the 2-LEX op's commuting with the 2-LFC op's.

GET: strict 3-adjunction:

$$\begin{tikzcd} 2\text{-LEX}^{op} \arrow[r, shift left, "(-)^*"] \arrow[r, phantom, "\perp"] & 2\text{-LFC} \arrow[l, shift left, "(-)^\#"] \end{tikzcd} \qquad (*)$$

$\mathbb{C} \in 2\text{-LEX}$: $\quad \mathbb{C}^* = 2\text{-LEX}(\mathbb{C}, \mathrm{Cat})$

hom-2-cat in Gray-cat 2-LEX

$\mathbb{A} \in 2\text{-LFC}$: $\quad \mathbb{A}^\# = 2\text{-LFC}(\mathbb{A}, \mathrm{Cat})$

Explanation: In any closed symmetric monoidal category $\mathbb{V}$, with objects $S, X, Y, \cdots$

we have the adjunction

$$\begin{tikzcd} \mathbb{V}^{op} \arrow[r, shift left=2, "{[-,S]}"] \arrow[r, phantom, "\perp"] & \mathbb{V} \arrow[l, shift left=2, "{[-,S]}"] \end{tikzcd} \qquad (**)$$

depicted as:

$$\frac{\dfrac{\dfrac{X \longrightarrow [Y,S]}{X \otimes Y \longrightarrow S}}{Y \otimes X \longrightarrow S}}{Y \longrightarrow [X,S]}$$

2-LEX and 2-LFC are non-full sub-Gray-categories of GRAY itself.
The adjunction (*) is a <u>restriction</u> of (**).

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Let: 2-Lex : full sub-Gray-category of 2-LEX on the small objects.

Theorem (new?; 'promised' by Max Kelly in Blackwell-Kelly-Power)

(1) For $\mathbb{C} \in$ 2-Lex:

$$\varepsilon_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}^{*\#}$$

is a biequivalence of 2-categories.

(2) For any 2-cat $\mathbb{A}$, $\mathbb{A} \simeq_{bi} \mathbb{C}^*$ for some $\mathbb{C} \in$ 2-Lex (iff)

1) $\mathbb{A} \in$ 2-L(FC)

2) $\mathbb{A} \simeq_{bi}$ 2-Flat$(\mathbb{B}) \overset{\text{DEF}}{=}$ the free completion of $\mathbb{B}$ to a 2-category with filtered 2-colimits, for some small 2-cat $\mathbb{B}$.

2-Lfp : full sub-Gray-cat of 2-LFC on objects $\mathbb{A}$ satisfying 1) and 2).

3) (Corollary) For $\mathbb{A} \in$ 2-Lfp,

$$\eta_{\mathbb{A}} : \mathbb{A} \xrightarrow[\simeq_2]{} \mathbb{A}^{\#*}$$

is a biequivalence, and we get the <u>3D duality</u>

$$\text{2-Lex}^{op} \underset{(-)^{\#}}{\overset{(-)^*}{\rightleftarrows}} \text{2-Lfp} \quad (\simeq_3)$$

—·—

Essentially algebraic 2-category $\overset{\text{DEF}}{=}$ object of 2-Lfp.

<u>Same</u> as the 2-category of models of a small 2-limit <u>sketch</u>

with pnt's as morphisms (1-cells), and modifications as 2-cells.

Examples: the <u>first-order categorical doctrines</u> (A. Kock - G. Reyes: Handbook of Mathematical Logic)

Cart, Lex, Regular, Exact, Coherent, Pretopos, ...

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Model theory : half-theorem: not quite proved: the direct analog of the quoted result from [MP]

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$\boxed{4}$ Ingredients

4.1 Bi-initial object theorem

Let: $\mathbb{X}$: locally small 2-cat

$\mathcal{S}$: small set of objects of $\mathbb{X}$

which is <u>weakly initial</u>:

$\forall X \in \mathbb{X}.\ \exists S \in \mathcal{S}.\ \exists f : S \xrightarrow{f} X.$

Suppose: $\mathbb{X}$ has all small (conical! not "weighted"!) bilimits.

Then: $\mathbb{X}$ has a biinitial object

[A is <u>biinitial</u> means: for every $X \in \mathbb{X}$, $[A, X] \simeq \mathbb{1}$ = the terminal category]

The proof uses the

<u>Lemma</u> Let: $W \in \mathbb{X}$

$(W) \underset{\text{def}}{=}$ the full sub-2-cat of $\mathbb{X}$ on the single object W

$(W) \xrightarrow[\Gamma]{} \mathbb{X}$: inclusion

Suppose $E = \text{bilim}(\Gamma)$ exists in $\mathbb{X}$, with bilimiting cone

$$\begin{tikzcd} (W) \arrow[r, bend left, "\ulcorner E \urcorner"] \arrow[r, phantom, "\downarrow\pi"] \arrow[r, bend right, "\Gamma"'] & \mathbb{X} \end{tikzcd}.$$

(π is a pnt!)

<u>Then</u>: $\forall t : W \to E$, we have

$t \circ \pi_E \cong 1_E$ $\qquad$ $\begin{tikzcd} W \arrow[r, bend left, "t"] & E \arrow[l, bend left, "\pi_E"] \arrow[loop right, "1_E"] \end{tikzcd}$

4.2 Representability

$\mathbb{X}$: locally small 2-cat

$A \in \mathbb{X}$.

$\approx\!\!> \hom_{\mathbb{X}}(A,-) : \mathbb{X} \longrightarrow \mathrm{Cat}$

'representable' 2-functor.

Let: $\mathbb{X} \xrightarrow{F} \mathrm{Cat}$ any 2-functor

$A \in \mathbb{X}$, $a \in F(A)$ (object in cat. $F(A)$)

We have the ordinary nat. transf. (put, in part.)

$$e^{(A,a)} : \hom_{\mathbb{X}}(A,-) \longrightarrow F$$

$$\begin{tikzcd} \mathbb{X} \arrow[r, bend left, "\hom(A,-)"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow e^{(A,a)}"] & \mathrm{Cat} \end{tikzcd}$$

defined by:

$$e^{(A,a)}_X : \hom_{\mathbb{X}}(A,X) \longrightarrow FX$$

$$A \xrightarrow{f} X \longmapsto (Ff)(a) \ (\in FX)$$

$$FA \xrightarrow{Ff} FX$$

<u>Definition</u> $(A, a) = (A \in \mathbb{X}, a \in F(A))$

provides a <u>representation</u> for F iff$_{def}$

$e^{(A,a)}$ is a biequivalence (in the 2-cat

$\hom(\hom_{\mathbb{X}}(A, -), F)$), or equivalently, if

each $e^{(A,a)}_X$ is an equivalence of cat's $(X \in \mathbb{X})$.

<u>FACT</u> (Yoneda) F is representable $\overset{def}{\Longleftrightarrow}$ there is

A such that $F \simeq_2 \hom_{\mathbb{X}}(A, -)$

if and only if

there is a representation (A, a) for F

<u>Definition</u> For $\mathbb{X} \xrightarrow[F]{} \mathrm{Cat}$

$e\ell(F)$: 2-category

objects: $(X, x) : X \in \mathbb{X}, x \in F(X)$

arrows:

$$\frac{(X, x) \xrightarrow{(f, \theta)} (Y, y)}{X \xrightarrow{f} Y \quad \& \quad (Ff)(x) \xrightarrow[\cong]{\theta} y}$$

(in the cat Y)

2-cells:

$$\begin{tikzcd}(X,x) \arrow[r, bend left, "{(f,\theta)}"] \arrow[r, bend right, "{(g,\tau)}"'] \arrow[r, phantom, "\Downarrow\mu"] & (Y,y)\end{tikzcd}:$$

$$\begin{tikzcd}X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\mu\Downarrow"] & Y\end{tikzcd} \quad \& \quad \begin{tikzcd}(Ff)(x) \arrow[dr, "\theta"] \arrow[dd, "(F\mu)_x"'] & \\ \circ & y \\ (Fg)(x) \arrow[ur, "\tau"'] & \end{tikzcd}$$

Proposition Assume that $\mathbb{X}$ is groupoidal: all 2-cells in $\mathbb{X}$ are isomorphisms. (!)

Let: $F : \mathbb{X} \longrightarrow \mathrm{Cat}$.

Then: (A, a) is a biinitial object of $el(F)$ iff (A,a) provides a representation for F.

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With every 2-cat $\mathbb{X}$, we can associate the (non-full) subcategory $\mathbb{X}^{iso}$ of $\mathbb{X}$, with 0- and 1-cells the same as those of $\mathbb{X}$, and

and 2-cells the isomorphisms in $\mathbb{X}$.

We have the inclusion 2-functor

$$\mathbb{X}^{iso} \xrightarrow[I]{} \mathbb{X}.$$

Given $F : \mathbb{X} \to Cat$, we get

$$F \circ I : \mathbb{X}^{iso} \longrightarrow Cat$$

Proposition Suppose that (A, a) provides a representation for $F \circ I$. Suppose, moreover, that 1) the 2-cat $\mathbb{X}$ has cotensors $X \pitchfork 2$:

for every $X \in \mathbb{X}$, we have

$$\begin{tikzcd} X \pitchfork 2 \arrow[r, shift left=2, "\pi_0"] \arrow[r, shift right=2, "\pi_1"'] \arrow[r, phantom, "\Downarrow \pi_1^0"] & X \end{tikzcd}$$

(pseudo-) terminal among all

$$\begin{tikzcd} Y \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] \arrow[r, phantom, "\Downarrow p_1^0"] & X \end{tikzcd}$$

and 2) F preserves cotensors

Then (A, a) provides a representation for F itself.

Representability theorem

(analog of Mac Lane's identically named theorem (CWM p. 118))

Suppose $\mathbb{X}$ locally small

has all small bilimits

and cotensors $X \pitchfork 2$

Let: $F : \mathbb{X} \longrightarrow Cat$ 2-functor.

Then F is representable

iff

1) F preserves all bilimits and cotensors

and

2) $el(F)$ has a small weakly initial set of objects.